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COMMON OPERATOR PROPERTIES OF THE LINEAR OPERATORS RS AND SR

BRUCE A. BARNES

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Abstract. Let S and R be bounded linear operators defined on Banach spaces, $S\colon X \to Y$, $R\colon Y \to X$. When $\lambda \neq 0$, then the operators $\lambda - SR$ and $\lambda - RS$ have many basic operator properties in common. This situation is studied in this paper.

Introduction

It is a well-know and useful result that when A and B are elements of a Banach algebra, then

$$\sigma(AB)\backslash\{0\} = \sigma(BA)\backslash\{0\}$$

([BD, Prop. 6, p. 16], [R, Lemma (1.4.17)], [P, Prop. 2.1.8]). Here $\sigma(A)$ denotes the spectrum of A. The case where the Banach algebra is $B(X)$, the algebra of all bounded linear operators on a Banach space X, is of special interest.

More generally, let both X and Y be Banach spaces, and let $S\colon X \to Y$ and $R\colon Y \to X$ be bounded linear operators. Again, it is known that

$$\sigma(SR)\backslash\{0\} = \sigma(RS)\backslash\{0\}.$$

Here $RS \in B(X)$ and $SR \in B(Y)$. In this paper we study this situation, showing that, in fact, for $\lambda \neq 0$, $\lambda - SR$ and $\lambda - RS$ have many basic operator properties in common (for example: $\lambda - SR$ has closed range if and only if $\lambda - RS$ has closed range). Throughout we assume that X, Y, S, and R are as stated above.

For $T \in B(X)$, let $\mathcal{N}(T)$ denote the null space of T, and let $\mathcal{R}(T)$ denote the range of T.

1. Spectrum

Let A and B be elements of a ring with unit I. We recall some notation: $A \circ B = A + B - AB$; $I - (A \circ B) = (I - A)(I - B)$. When $A \circ B = B \circ A = 0$, then B is the unique element with this property. In this case we write $B = A^q$. Thus, $(I - A)(I - A^q) = (I - A^q)(I - A) = I$, and so $(I - A)^{-1} = I - A^q$.

We have the following known basic computation (which holds in a ring with unit). Of course, the computation holds with the roles of R and S reversed.

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Proposition 1 (The basic computation)**.** *For any $W \in B(X)$, let*

$$V = S(W - I)R \in B(Y).$$

(1) $(SR) \circ V = S((RS) \circ W)R$; *and*
(2) $V \circ (SR) = S(W \circ (RS))R$.

We verify (1):

$$\begin{aligned}(SR) \circ V &= SR + S(W - I)R - SRS(W - I)R = S[I + (W - I) - RS(W - I)]R \\ &= S[W + RS - RSW]R = S[(RS) \circ W]R.\end{aligned}$$

Suppose that $I - RS$ is invertible in $B(X)$, and set $W = (RS)^q \in B(X)$. Thus as remarked above, $(RS) \circ W = W \circ (RS) = 0$. Define $V \in B(Y)$ as in Proposition 1, so by that result, $(SR) \circ V = V \circ (SR) = 0$. Therefore $V = (SR)^q$, and $I - SR$ is invertible in $B(Y)$. From this it follows that:

$$\sigma(SR)\backslash\{0\} = \sigma(RS)\backslash\{0\}.$$

We will show later in this section that similar equalities hold for all the usual parts of the spectrum. Note that Proposition 1 also implies:

$I - SR$ has a left (right) inverse if and only if $I - RS$ has a left (right) inverse.

Proposition 2. (1) $S(\mathcal{N}(I - RS)) = \mathcal{N}(I - SR)$;
(2) $\mathcal{N}(S) \cap \mathcal{N}(I - RS) = \{0\}$.

Proof. Statement (2) clearly holds. Assume $x \in \mathcal{N}(I - RS)$, so $RSx = x$. Then $SRSx = Sx$, and thus, $S(\mathcal{N}(I - RS)) \subseteq \mathcal{N}(I - SR)$. To verify the opposite inclusion, suppose $y \in \mathcal{N}(I - SR)$. Arguing as above, we have

$$R(\mathcal{N}(I - SR)) \subseteq \mathcal{N}(I - RS).$$

Therefore, $Ry \in \mathcal{N}(I - RS)$. Then $y = SRy \in S(\mathcal{N}(I - RS))$. This proves (1).

We use $\sigma_p, \sigma_{ap}, \sigma_r$, and σ_c to denote the point, approximate point, residual, and continuous spectrum, respectively.

Theorem 3. (1) $\sigma(RS)\backslash\{0\} = \sigma(SR)\backslash\{0\}$;
(2) $\sigma_p(RS)\backslash\{0\} = \sigma_p(SR)\backslash\{0\}$;
(3) $\sigma_{ap}(RS)\backslash\{0\} = \sigma_{ap}(SR)\backslash\{0\}$;
(4) $\sigma_r(RS)\backslash\{0\} = \sigma_r(SR)\backslash\{0\}$;
(5) $\sigma_c(RS)\backslash\{0\} = \sigma_c(SR)\backslash\{0\}$.

Proof. As noted previously, (1) follows from Proposition 1. Also, (2) is an immediate corollary of Proposition 2 (1).

Now assume $\lambda \in \sigma_{ap}(RS)\backslash\{0\}$. This means there exists $\{x_n\} \subseteq X$, $\|x_n\| = 1$ for all n, and $\|(\lambda - RS)x_n\| \to 0$. Therefore, $\|(\lambda - SR)(Sx_n)\| = \|S(\lambda - RS)x_n\| \to 0$. Also, $\|Sx_n\|$, $n \geq 1$, is bounded away from zero, for if not, $\|Sx_{n_k}\| \to 0$ for some subsequence $\{x_{n_k}\}$ of $\{x_n\}$. But then,

$$|\lambda| = |\lambda|\|x_{n_k}\| \leq \|(\lambda - RS)x_{n_k}\| + \|RSx_{n_k}\| \to 0,$$

a contradiction. This proves $\lambda \in \sigma_{ap}(SR)$.

(4): $\lambda \in \sigma_r(RS)\backslash\{0\}$ means exactly that $\lambda \neq 0$, $\lambda \notin \sigma_p(RS)$, and $\mathcal{R}(\lambda - RS)^- \neq X$. We use T' to denote the adjoint of an operator T. We have $\mathcal{N}(\lambda - S'R') = \mathcal{N}(\lambda - (RS)') \neq \{0\}$. By Proposition 2 (1), $\mathcal{N}(\lambda - (SR)') = \mathcal{N}(\lambda - R'S') \neq \{0\}$. Therefore $\mathcal{N}(\lambda - SR)^- \neq Y$, and so $\lambda \in \sigma_r(SR)$.

(5):

$$\begin{aligned}\sigma_c(RS)\backslash\{0\} &= \sigma(RS)\backslash[\sigma_p(RS)\cup\sigma_r(RS)\cup\{0\}]\\ &= \sigma(SR)\backslash[\sigma_p(SR)\cup\sigma_r(SR)\cup\{0\}] = \sigma_c(SR)\cup\{0\}.\end{aligned}$$

2. Closed range

Recall that $I-RS$ has pseudoinverse (or generalized inverse) $I-W$ means that

$$(I-RS)(I-W)(I-RS) = (I-RS).$$

The existence of a pseudoinverse for $I-RS$ implies that $\mathcal{R}(I-RS)$ is closed (and more) [TL, Theorem 12.9, p. 251].

Theorem 4. *$I-RS$ has a pseudoinverse if and only if $I-SR$ has a pseudoinverse.*

Proof. Assume $I-RS$ has pseudoinverse $I-W$ (as above). Let $V = S(W-I)R$. From Proposition 1 we have

$$\begin{aligned}(I-SR)(I-V) &= I-(SR)\circ V = I + S[-(RS)\circ W]R\\ &= I + S[(I-RS)(I-W)]R - SR.\end{aligned}$$

Thus,

$$\begin{aligned}&(I-SR)(I-V)(I-SR)\\ &\quad = I + S[(I-RS)(I-W)]R - SR - SR - S[(I-RS)(I-W)]RSR + SRSR\\ &\quad = I - 2SR + SRSR + S[(I-RS)(I-W)(I-RS)]R\\ &\quad = I - 2SR + SRSR + S[I-RS]R = I - SR.\end{aligned}$$

Note that the argument in the proof of Theorem 4 is completely algebraic, so the result holds in any ring with unit.

Let $T \in B(X)$. Equip $X/\mathcal{N}(T)$ with the usual quotient norm. A standard condition which is equivalent to $\mathcal{R}(T)$ being closed is:

there exists a bounded linear operator $M\colon \mathcal{R}(T) \to X/\mathcal{N}(T)$ such that

$$MTx = x + \mathcal{N}(T) \quad \text{for all } x \in X.$$

Theorem 5. *$\mathcal{R}(I-RS)$ is closed if and only if $\mathcal{R}(I-SR)$ is closed.*

Proof. Assume $I-RS$ has closed range Z. Write $(I-RS)^{\sim}\colon X/\mathcal{N}(I-RS) \to Z$ where $(I-RS)^{\sim}(x+\mathcal{N}(I-RS)) = (I-RS)x$. There exists a bounded linear operator

$$L\colon Z \to X/\mathcal{N}(I-RS)$$

such that for all $x \in X$,

$$L(I-RS)^{\sim}(x+\mathcal{N}(I-RS)) = x + \mathcal{N}(I-RS).$$

Define $\widetilde{S}\colon X/\mathcal{N}(I-RS) \to Y/\mathcal{N}(I-SR)$ by

$$\widetilde{S}(x+\mathcal{N}(I-RS)) = Sx + \mathcal{N}(I-SR).$$

Note that $\widetilde{S}$ is well-defined by Proposition 2. Also, it is straightforward to check that $\widetilde{S}$ is continuous.

Now define $M\colon \mathcal{R}(I-SR) \to Y/\mathcal{N}(I-SR)$ by $M(w) = w + \mathcal{N}(I-SR) + \widetilde{S}LRw$. Note here that $w = y - SRy$ for some y, so $Rw = (I-RS)Ry \in \mathcal{R}(I-RS)$. Thus, LRw is well-defined.

Also, since by definition M is an algebraic combination of continuous maps, M is continuous. Finally,

$$\begin{aligned} M(y - SRy) &= y - SRy + \mathcal{N}(I - SR) + \widetilde{S}LR(y - SRy) \\ &= y - SRy + \mathcal{N}(I - SR) + \widetilde{S}Ry = y + \mathcal{N}(I - SR). \end{aligned}$$

Therefore $\mathcal{R}(I - SR)$ is closed.

3. Fredholm properties

An operator $T \in B(X)$ is semi-Fredholm if $\mathcal{R}(T)$ is closed and either $\mathrm{nul}(T) = \dim(\mathcal{N}(T))$ or $\mathrm{nul}(T')$ is finite (as before, T' is the adjoint of T); see [CPY, 1.3 and Chapter 4]. We use the notation:

$$\begin{aligned} \Phi^+ &= \{T \in B(X) \colon \mathcal{R}(T) \text{ is closed and } \mathrm{nul}(T) < \infty\}; \\ \Phi^- &= \{T \in B(X) \colon \mathcal{R}(T) \text{is closed and } \mathrm{nul}(T') < \infty\}; \\ \Phi &= \Phi^+ \cap \Phi^-. \end{aligned}$$

Recall, when T is semi-Fredholm, then $\mathrm{ind}(T) = \mathrm{nul}(T) - \mathrm{nul}(T')$.

Theorem 6. *$I - RS \in \Phi$ (Φ^+, Φ^-) if and only if $I - SR \in \Phi$ (Φ^+, Φ^-). Furthermore, when $I - RS \in \Phi$, then* $\mathrm{ind}(I - RS) = \mathrm{ind}(I - SR)$.

Proof. The first statement follows directly from Theorem 5 and Proposition 2. Also, when $I - RS \in \Phi$, Proposition 2 implies that $\mathrm{nul}(I - RS) = \mathrm{nul}(I - SR)$ and $\mathrm{nul}(I - RS)') = \mathrm{nul}(I - S'R') = \mathrm{nul}(I - R'S') = \mathrm{nul}(I - (RS)')$. Thus, $\mathrm{ind}(I - RS) = \mathrm{ind}(I - SR)$.

4. Functional calculus

In this section we derive a useful relationship between the holomorphic functional calculi of RS and SR.

Theorem 7. *Let $g(\lambda)$ be a holomorphic function on some open set U such that $\sigma(SR) \cup \{0\} \subseteq U$. Let $f(\lambda) = \lambda g(\lambda)$. Then $f(SR) = Sg(RS)R$.*

Proof. Set $\Gamma = \sigma(SR) \cup \{0\}$, and let γ be a cycle which is contained in $U \backslash \Gamma$ with $\mathrm{ind}_\gamma(z) = 1$ for all $z \in \Gamma$, and $\mathrm{ind}_\gamma(z) = 0$ for all $z \notin U$. For $\lambda \neq 0$, we have

$$\lambda^{-1}[I - (\lambda^{-1}SR)^q] = (\lambda - SR)^{-1}. \tag{1}$$

Also, by Proposition 1 with $W = (\lambda^{-1}RS)^q$,

$$(\lambda^{-1}SR)^q = -\lambda^{-1}S[I - (\lambda^{-1}RS)^q]R. \tag{2}$$

Therefore,

$$
\begin{aligned}
f(SR) &= (2\pi i)^{-1} \int_\gamma f(\lambda)(\lambda - SR)^{-1}\, d\lambda \\
&= (2\pi i)^{-1} \left[\int_\gamma \lambda^{-1} f(\lambda)\, d\lambda - \int_\gamma f(\lambda)\lambda^{-1}(\lambda^{-1}SR)^q\, d\lambda \right] \quad \text{(by (1))} \\
&= (2\pi i)^{-1} \left[0 + \int_\gamma g(\lambda)\lambda^{-1} S[I - (\lambda^{-1}RS)^q] R\, d\lambda \right] \\
&\qquad\qquad\qquad\qquad\qquad\qquad \text{(by (2), and since } f(0) = 0) \\
&= S \left[(2\pi i)^{-1} \int_\gamma g(\lambda)(\lambda - RS)^{-1}\, d\lambda \right] R = Sg(RS)R \quad \text{(by (1))}.
\end{aligned}
$$

Corollary 8. *Let f and g be as above. Set $R_1 = g(RS)R$. Then $f(SR) = SR_1$, and since $f(\lambda) = g(\lambda)\lambda$, $f(RS) = g(RS)RS = R_1 S$.*

Therefore the results of this paper apply to $f(SR)$ and $f(RS)$.

5. Poles

Let $\lambda_0 \neq 0$ be an isolated point of $\sigma(RS)$. We adopt the notation and terminology in [TL, pp. 328–331]. In particular, for $n \neq 1$, let

$$
f_{-n}(\lambda) = \begin{cases} (\lambda - \lambda_0)^{n-1} & \text{if } |\lambda - \lambda_0| < r; \\ 0 & \text{if } |\lambda - \lambda_0| > 2r. \end{cases}
$$

(Here $r > 0$ is chosen so that $(\sigma(RS) \cup \{0\}) \backslash \{\lambda_0\} \subseteq \{\lambda\colon |\lambda - \lambda_0| > 2r.\}$ Let $B_n(RS) = f_{-n}(RS)$, and note that $B_1(RS)$ is the spectral projection corresponding to the spectral set $\{\lambda_0\}$. We use the same notation relative to SR; $B_n(SR) = f_{-n}(SR)$. Define

$$
h(\lambda) = \begin{cases} \lambda^{-1} & \text{if } |\lambda - \lambda_0| < r; \\ 0 & \text{if } |\lambda - \lambda_0| > 2r. \end{cases}
$$

We have $f_{-n}(\lambda) = (\lambda f_{-n}(\lambda) h(\lambda))$, which gives:

(1) $B_n(SR) = SRB_n(SR)h(SR)$; and

(2) $B_1(SR) = S[B_1(RS)h(RS)]R$.

((2) follows by applying Theorem 7.)

By definition λ_0 is a pole of order p of the resolvent of RS if $B_p(RS) \neq 0$, and $B_n(RS) = 0$ for all $n > p$ [TL, p. 330].

Theorem 9. *An isolated point $\lambda_0 \neq 0$ of $\sigma(RS)$ is a pole of order p of the resolvent of RS if and only if λ_0 is a pole of order p of the resolvent of SR. Furthermore, $\mathcal{R}(B_1(RS))$ is finite dimensional if and only if $\mathcal{R}(B_1(SR))$ is finite dimensional.*

Proof. We use the notation introduced above. Assume $B_n(RS) = 0$ for some $n > 1$. By Theorem 7 with $g(\lambda) = f_{-n}(\lambda)$, $f(\lambda) = \lambda g(\lambda)$, it follows that $SRB_n(SR) = S[B_n(RS)]R = 0$. By (1) it follows that $B_n(SR) = 0$. This argument establishes that $B_n(RS) = 0$ if and only if $B_n(SR) = 0$. The statement of the theorem concerning poles follows from this.

The second statement of the theorem follows from (2), since if $\mathcal{R}(B_1(RS))$ is finite dimensional, then $B_1(SR) = S[B_1(RS)h(RS)]R$ has finite dimensional range.

Recall that the smallest integer $p \geq 0$ such that $\mathcal{N}(T^p) = \mathcal{N}(T^{p+1})$ is called the ascent of the operator T (the ascent of T is infinite if $\mathcal{N}(T^n) \neq \mathcal{N}(T^{n+1})$ for all $n \geq 0$) [TL, Section V6]. The property that $\lambda_0 - T$ has finite ascent is closely connected to λ_0 being a pole of the resolvent of T; see [TL, Section V10].

Let $n \geq 0$ be an integer. There exists U_n such that

$$(I - SR)^{n+1} = I - SU_n; \qquad (I - RS)^{n+1} = I - U_n S.$$

In fact, by direct computation, $U_n = \sum_{k=1}^{n+1} (-1)^{k-1} \binom{n+1}{k} R(SR)^{k-1}$ works.

Proposition 10. *$I - RS$ has finite ascent p if and only if $I - SR$ has finite ascent p.*

Proof. Suppose, for some $n \geq 0$, $\mathcal{N}((I-RS)^n) = \mathcal{N}((I-RS)^{n+1})$. By the existence of U_k as indicated above, Proposition 2 applies to $(I - SR)^{k+1}$ for all $k \geq 0$. Thus,

$$\begin{aligned} \mathcal{N}((I - SR)^{n+1}) &= \mathcal{N}(I - SU_n) = S(\mathcal{N}(I - U_n S)) = S(\mathcal{N}((I - RS)^{n+1})) \\ &= S(\mathcal{N}((I - RS)^n)) = \mathcal{N}((I - SR)^n). \end{aligned}$$

This implies that $I - RS$ has ascent p.

6. Examples, applications

In this section we look at several situations where the results of the previous sections apply.

Example 11. Assume that the Banach space X is continuously embedded as a subspace of a Banach space Y. Assume that $T \in B(X)$ has an extension $\overline{T} \in B(Y)$. In [B1] operator properties of T and $\overline{T}$ are studied with the hypothesis that $\overline{T}(Y) \subseteq X$. All of the main results of [B1, §2] (and more!) can be derived from results in this paper. For let $S\colon X \to Y$ be the continuous embedding, $S(x) = x$. Let $R\colon Y \to X$ be the bounded operator, $R(y) = \overline{T}(y) \in X$. Then $T = RS$ and $\overline{T} = SR$. Therefore in this situation the results of the previous sections apply to T and $\overline{T}$.

Example 12. Let H be a Hilbert space. Assume $S\colon X \to H$ and $R\colon H \to X$ have the special property that SR is selfadjoint. Then $T = RS$ has many of the operator properties of a selfadjoint operator. Exactly this situation is studied in [B2].

In particular, suppose $X = H$, $R \geq 0$, and $S = S^*$. Then an operator of the form SR is called symmetrizable. The operator SR has many operator properties in common with the selfadjoint operator $R^{\frac{1}{2}} S R^{\frac{1}{2}}$.

Example 13. Let (Ω, μ) be a σ-finite measure space. Let $K(x, t)$ be a kernel on $\Omega \times \Omega$ with the property

$$k(x) = \operatorname*{ess\,sup}_{t \in \Omega} |K(x, t)| \in L^1(\mu).$$

The linear integral operator

$$T_K(f)(x) = \int_\Omega K(x, t) f(t) \, d\mu(t), \qquad f \in L^1(\mu),$$

is an operator in $B(L^1(\mu))$. In this case T_K is a Hille-Tamarkin operator, $T_K \in H_{11}$; see [J, Sections 11.3 and 11.5].

We may assume that $k(x)$ is everywhere defined and nonnegative. Define

$$J(x,t) = \begin{cases} k(x)^{-\frac{1}{2}} K(x,t) & \text{if } k(x) > 0; \\ 0 & \text{if } k(x) = 0. \end{cases}$$

Since

$$\operatorname*{ess\,sup}_{t\in\Omega} |J(x,t)|^2 = k(x) \in L^1(\mu), \qquad t \in \Omega,$$

the integral operator, $T_J\colon L^1 \to L^2$, is in the Hille-Tamarkin class H_{21}. Also, define

$$H(x,t) = J(x,t)k(t)^{\frac{1}{2}}.$$

Since $|H(x,t)|^2 \le k(x)k(t)$, it follows that T_H is a Hilbert-Schmidt operator on $L^2(\mu)$.

Now consider the operators $S\colon L^2 \to L^1$ and $R\colon L^1 \to L^2$ given by

$$S(f) = k^{\frac{1}{2}} f \quad (f \in L^2); \qquad R(g) = T_J(g) \quad (g \in L^1).$$

Then $SR = T_K$ and $RS = T_H$. We summarize:

Theorem 14. *Let $T_K\colon L^1 \to L^1$ be a Hille-Tamarkin operator in class H_{11}. Then there exist bounded operators $S\colon L^2 \to L^1$, $R\colon L^1 \to L^2$ such that $T_K = SR$ and RS is a Hilbert-Schmidt operator.*

Corollary 15. *Let $T_K\colon L^1 \to L^1$ be a Hille-Tamarkin operator in class H_{11}. Then T_K^2 is compact, and the nonzero eigenvalues of T_K (counted according to multiplicities) form a square summable sequence.*

Proof. Let $T_K = SR$ with RS Hilbert-Schmidt. Then $T_K^2 = S(RS)R$, so T_K^2 is compact. Also, the sequence of nonzero eigenvalues (counted according to multiplicities) of T_K and RS are the same by Proposition 2 and Theorem 3. This sequence is square summable by [Rg, Corollary 2.3.6, p. 89].

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Department of Mathematics, University of Oregon, Eugene, Oregon 97403

E-mail address: `barnes@math.uoregon.edu`