PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 104, Number 3, November 1988

ANALYTICITY OF HOMOLOGY CLASSES

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(Communicated by Haynes R. Miller)

ABSTRACT. Let W be a real analytic manifold and $\{\alpha\} \in H_p(W, Z_2)$. We shall say that $\{\alpha\}$ is analytic if there exists a compact analytic subset S of W, such that: $\{\alpha\} = \{$fundamental class of $S\}$. The purpose of this short paper is to prove

THEOREM 1. *Let W be a paracompact real analytic manifold; then any homology class $\{\alpha\} \in H_p(W, Z_2)$ is analytic.*

We remember that a similar result does not hold in the real algebraic case (see [**1**]).

1. Definitions and well-known facts. Let V, W be two differentiable (i.e. C^∞) manifolds; then, on the set $M(V, W)$ of differentiable maps $f\colon V \to W$, we shall consider the Whitney topology (see [**2**, p. 42]). In the following we shall use the known result: if $f \in M(V, W)$, then there exists a neighborhood U, in the C^0 topology, of f such that any $g \in U$ is homotopic to f (see [**8**]). By real algebraic variety we shall mean: affine real algebraic variety. A regular algebraic variety shall be called: algebraic manifold. An algebraic map is the restriction of a rational regular map.

In the following we shall need

LEMMA 1. *Let $V \subset \mathbf{R}^n$, $W \subset \mathbf{R}^q$ be two real algebraic manifolds and $V \xrightarrow{\varphi} W$ a differentiable map. If V is compact and boarding to ϕ, then, for any $\varepsilon > 0$, there exists an algebraic submanifold $V' \subset \mathbf{R}^{n+q}$, an analytic isomorphism $V' \xrightarrow{\pi} V$ and an algebraic map $\varphi'\colon V' \to W$ such that*

(i) $\delta(\varphi(x), \varphi' \circ \pi^{-1}(x)) < \varepsilon$, $x \in V$,

(ii) $\delta'((d\varphi)(v), (d(\varphi' \circ \pi^{-1}))(v)) < \varepsilon$

for any tangent vector v, to V in x, where δ, δ' are two metrics on $\mathbf{R}^q$ and on the Grassmannian manifold.

PROOF. See [**3**].

LEMMA 2. *Let $V \subset \mathbf{R}^n$, $W \subset \mathbf{R}^q$ be two real algebraic manifolds and $\varphi\colon V \to W$ an algebraic map. Let us suppose that V is irreducible and there exists a Zariski open set $V' \subset V$ with the property: φ is injective on V'. Under these hypotheses, if T is the Zariski closure of $\varphi(V)$ in W, we have*

(i) $T \supset \varphi(V)$, $\dim T = \dim V$; *$T - \varphi(V)$ is contained in an algebraic set S,* $\dim S < \dim V$,

(ii) *{fundamental class of T}* $= \varphi_*$ *{fundamental class of V}.*

Received by the editors November 22, 1986.
1980 *Mathematics Subject Classification* (1985 *Revision*). Primary 55M57; Secondary 14G58.

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0002-9939/88 \$1.00 + \$.25 per page

PROOF. (i) is proved in [**4**, Lemma 1.1]. (ii) follows from the definition of the fundamental class (see [**5**]) and the proof of the first part of the lemma.

Now let $\varphi\colon V \to W$ be a differentiable map between differentiable manifolds. Let us denote by $\mathscr{E}(V)_x$, $\mathscr{E}(W)_{\varphi(x)}$ the stalks of the sheaves of the differentiable functions on V, W. We recall

DEFINITION. φ is called *finite*, in the point x, if $\mathscr{E}(V)_x$ is a finite $\varphi^*(\mathscr{E}(W)_{\varphi(x)})$ module.

We have

LEMMA 3. *Let* $\dim V \leq \dim W$ *and let us suppose that* V *is compact. Then the set of differentiable maps that are finite in any point is an open dense subset of* $M(V,W)$.

PROOF. See [**6**, p. 96] (see also [**2**, p. 169]).

2. The proof of the theorem. Now let us suppose that W is a compact real analytic manifold and $\{\alpha\} \in H_p(W, Z_2)$. It is known (see [**7**]) that we may suppose W is a real algebraic manifold. Moreover, there exists a compact differentiable manifold V and a differentiable map $\varphi\colon V \to W$ such that $\{\alpha\} = \varphi$ (fundamental class of V) (see [**8**]). By Lemma 1 we may suppose there exists an algebraic manifold $\hat{V} = V' \cup V''$ and an algebraic map $\hat{\varphi}\colon \hat{V} \to W$ such that:

(i) V' and V'' are diffeomorphic to V,

(ii) $\{\alpha\} = \hat{\varphi}_*$ (fundamental class of V') $= \hat{\varphi}$ (fundamental class of V''),

(iii) $\hat{\varphi}|_{V'}$ is in general position with respect to $\hat{\varphi}(V'')$.

Moreover, by Lemma 3, we may suppose

(iv) $\hat{\varphi}$ is finite in any $x \in \hat{V}$.

From Lemma 2, we may finally assume that

(v) if T is the Zariski closure of $\hat{\varphi}(\hat{V})$ in W, then $T - \hat{\varphi}(\hat{V})$ is contained in an algebraic set S such that $\dim S < \dim V = p$.

Let now $\tilde{\hat{\varphi}}\colon \tilde{\hat{V}} \to \tilde{W}$ be a complexification of $\hat{\varphi}$, such $\tilde{\hat{\varphi}}$ exists (see [**9**]), and we may suppose $\tilde{\hat{\varphi}}$ is finite in any point of $\tilde{\hat{V}}$, because the finiteness is an open condition (see [**2**, p. 168]). We shall suppose that $\tilde{\hat{V}} = \tilde{V}' \amalg \tilde{V}''$. The map $\tilde{\hat{\varphi}}$ is finite, hence the image of any analytic germ $\tilde{\hat{V}}_y$ is the germ of a complex analytic set of $\tilde{W}$, see [**10**, p. 162].

Let us now remark that the above facts imply that:

(a) $T =$ real part of the closure in the Zariski topology of $\tilde{\hat{\varphi}}(\tilde{\hat{V}})$,

(b) $d\tilde{\hat{\varphi}}$ has maximum rank on an open dense set of $\tilde{\hat{V}}$.

We deduce that, for any $x \in T$, we have three disjoint analytic irreducible germs of T_x:

(1) germs image of $\tilde{V}'$, of dimension p,

(2) germs image of $\tilde{V}''$, of dimension p,

(3) germ image of $\tilde{\hat{\varphi}}^{-1}(S)$, of dimension lower than p.

This proves that $T' = \hat{\varphi}(V') \cup S$ is an analytic subset of W and, clearly,

$$\{\alpha\} = \{\text{fundamental class of } T'\}.$$

In fact, for any $y \in T$, the germ $Y_y = |\bigcup_i \widetilde{\varphi}(\widetilde{V}'_{y_i})|_R$ is real analytic, where $\bigcup y_i = \widetilde{\varphi}^{-1}(y) \cap \widetilde{V}'$, $|Z|_R =$ real part of Z.

So $Y_y \cup S_y$ is real analytic and clearly $Y_y \cup S_y = T'_y$. In fact, in any point, T' is the union of a finite set of irreducible germs of T. So the theorem is proved under the hypothesis: W is compact. In the general case, let us remark that we may take a representative element α of $\{\alpha\}$ contained in a relatively compact open set U of W. We can now see U, up to analytic isomorphism, as an open set of a compact analytic manifold Z (take the unique analytic structure on the double of U). We can now prove the analyticity of $\{\alpha\}$ in Z and this implies, clearly, the analyticity of $\{\alpha\}$ in W. The theorem is proved.

BIBLIOGRAPHY

1. R. Benedetti and M. Dedo, *Counterexamples to representing homology classes by real algebraic varieties up to homeomorphisms*, Compositio Math. **53** (1984).
2. M. Golubisky and V. Guillemin, *Stable mappings and their singularities*, Springer-Verlag, Berlin and New York, 1973.
3. A. Tognoli, *Algebraic approximation of manifolds and spaces*, Sem. Bourbaki **548** (1979–80).
4. R. Benedetti and A. Tognoli, *On real algebraic vector bundles*, Bull. Sci. Math. (2) **104** (1980).
5. A. Borel and A. Haefliger, *La classe d'homologie fondamentale d'une espace analytique*, Bull. Soc. Math. France **89** (1961).
6. A. Andreotti and P. Holm, *Quasianalytic and parametric spaces*, Real and Complex Singularities, Noordhoff, Oslo, 1976.
7. A. Tognoli, *Su una congettura di Nash*, Ann. Scuola Norm. Sup. Cl. Sci. Pisa (3) **27** (1973).
8. R. Thom, *Quelques propriétés globales des variétés différentiables*, Comment. Math. Helv. **28** (1954).
9. A. Tognoli, *Algebraic geometry and Nash functions*, Institutiones **3** (1978),.
10. L. Kaup and B. Kaup, *Holomorphic functions of several variables*, De Gruyter, 1983.

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