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ON WEIGHTED INTEGRABILITY OF TRIGONOMETRIC SERIES AND L^1-CONVERGENCE OF FOURIER SERIES

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ABSTRACT. A result concerning integrability of $f(x)L(1/x)(g(x)L(1/x))$, where $f(x)(g(x))$ is the pointwise limit of certain cosine (sine) series and $L(\cdot)$ is slowly vary in the sense of Karamata [5] is proved. Our result is an excluded case in more classical results (see [4]) and also generalizes a result of G. A. Fomin [1]. Also a result of Fomin and Telyakovskii [6] concerning L^1-convergence of Fourier series is generalized. Both theorems make use of a generalized notion of quasi-monotone sequences.

1. Introduction. A classical problem in the theory of trigonometric series concerns sufficient conditions in terms of the coefficients $\{a(n)\}$ for the Fourier character of cosine series

$$\frac{a(0)}{2} + \sum_{n=1}^{\infty} a(n)\cos nx \tag{1.1}$$

and the conjugate or sine series

$$\sum_{n=1}^{\infty} a(n)\sin nx. \tag{1.2}$$

All known results employ conditions which imply that the null sequence $\{a(n)\}$ is of bounded variation ($\sum_{n=1}^{\infty}|\Delta a(n)| < \infty$, $\Delta a(n) = a(n) - a(n+1)$, and $a(n) = o(1)$ $(n \to \infty)$). This further implies that the pointwise limit of (1.1) and (1.2) exist on $(0, \pi]$; these are denoted $f(x)$ and $g(x)$, respectively. Consequently, for the Fourier character of (1.1) or (1.2) it is necessary and sufficient that f or g be Lebesgue integrable on $(0, \pi]$. A recent result in this direction is the following, due to Fomin (see also [2, 3]).

THEOREM 1.1. *Let $a(n) = o(1)$ $(n \to \infty)$, and for some $p > 1$ let*

$$\sum_{n=1}^{\infty}\left(\frac{\sum_{k=n}^{\infty}|\Delta a(k)|^p}{n}\right)^{1/p} < \infty.$$

Then

(i) $f \in L^1(0, \pi]$, *and*

(ii) $g \in L^1(0, \pi]$ *if and only if* $\sum_{n=1}^{\infty}|a(n)|/n < \infty$.

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Many authors have varied the point of view of the above problem by considering weighted integrability of the sum functions (see the monograph of Boas **[4]** for a survey). These results give criteria for the integrability of $x^{-\gamma}f(x)L(1/x)$ and $x^{-\gamma}g(x)L(1/x)$, where $\gamma > 0$ and $L(\cdot)$ is a slowly varying function in the sense of Karamata **[5]**. In §3 an integrability result for $f(x)L(1/x)$ and $g(x)L(1/x)$ is proved, generalizing Theorem 1.1 in the case of cosine series and giving a restricted generalization in the case of sine series.

In the final section a generalization of the following theorem due to Fomin and Telyakovskii **[6]** (see also **[7]**) is proved. For succinct formulation the nth partial sums of the Fourier series of $f \in L^1(0, \pi]$ are denoted $S_n(f) = S_n(f, x)$. Also recall (Szasz **[8]**) that a null sequence $\{a(n)\}$ is said to be quasi-monotone if, for some $\alpha > 0$, $a(n)/n^\alpha \downarrow$ for $n \geqslant n_0(\alpha)$.

THEOREM 1.2. *Let* (1.1) *be the Fourier series of* $f \in L^1(0, \pi]$ *with quasi-monotone coefficients. Then* $\|S_n(f) - f\| = o(1)$ $(n \to \infty)$ *if and only if* $a(n)\lg n = o(1)$ $(n \to \infty)$.

An analogous result holds for sine series. Both our results make use of a generalization of quasi-monotone sequence developed in §2.

2. Preliminaries. A positive measurable function $L(u)$ is said to be slowly varying in the sense of Karamata **[5]** if, for $\lambda > 0$,

$$\lim_{u \to +\infty} \frac{L(\lambda u)}{L(u)} = 1. \tag{2.1}$$

Karamata **[5]** proved that (2.1) holds uniformly for λ contained in a bounded closed interval. Slowly varying sequences are defined analogously: a positive sequence $\{l(n)\}$ is said to be slowly varying if, for $\lambda > 0$,

$$\lim_{n \to +\infty} \frac{l([\lambda n])}{l(n)} = 1. \tag{2.2}$$

The class of slowly varying functions (sequences) is denoted by SV(**R**) (SV(N)).

In **[9]** Karamata introduced regularly varying sequences: a positive sequence $\{r(n)\}$ is said to be regularly varying if, for $\lambda > 0$ and some $\alpha > 0$,

$$\lim_{n \to \infty} \frac{r([\lambda n])}{r(n)} = \lambda^\alpha.$$

The class of such is denoted by RV(N). Regularly varying sequences are characterized **[9]** in form as follows: $\{r(n)\} \in \mathrm{RV}(N)$ if and only if $r(n) = n^\alpha l(n)$, for some $\alpha > 0$ and some $\{l(n)\} \in \mathrm{SV}(N)$.

A null sequence $\{a(n)\}$ is said to be *regularly varying quasi-monotone* if for some $\{r(n)\} \in \mathrm{RV}(N)$, $a(n)/r(n)\downarrow$ for $n \geqslant n_0$. The class of such sequences is denoted RQM and properly contains quasi-monotone sequences. We can now give the following generalization of the Cauchy condensation test.

LEMMA 2.1. *Let* $\{a(n)\} \in \mathrm{RQM}$. *Then the series* $\sum_{n=1}^{\infty} a(n)l(n)$ *and the series* $\sum_{n=0}^{\infty} 2^n a(2^n)l(2^n)$ *are equiconvergent for every* $\{l(n)\} \in \mathrm{SV}(N)$.

PROOF. For sufficiently large k, $a(k)/k^{\alpha}l_1(k)\downarrow$, where $\alpha > 0$ and $\{l_1(k)\} \in \mathrm{SV}(N)$. Consequently, for sufficiently large n,

$$\frac{a(2^{n+1})}{2^{\alpha(n+1)}l_1(2^{n+1})}\sum_{k=2}^{2^{n+1}-1} k^{\alpha}l_1(k)l(k) \leqslant \sum_{k=2^n}^{2^{n+1}-1} a(k)l(k)$$
$$\leqslant \frac{a(2^n)}{2^{\alpha n}l_1(2^n)}\sum_{k=2^n}^{2^{n+1}-1} k^{\alpha}l_1(k)l(k),$$

and so,

$$\frac{1}{2^{\alpha}}\frac{a(2^{n+1})}{l_1(2^{n+1})}\sum_{k=2^n}^{2^{n+1}-1} l_1(k)l(k) \leqslant \sum_{k=2^n}^{2^{n+1}-1} a(k)l(k)$$
$$\leqslant 2^{\alpha}\frac{a(2^n)}{l_1(2^n)}\sum_{k=2^n}^{2^{n+1}-1} l_1(k)l(k).$$

Since $\{l_1(k)l(k)\} \in \mathrm{SV}(N)$, the aforementioned uniform nature of (2.1) or (2.2) gives

$$\sum_{k=2^n}^{2^{n+1}-1} l_1(k)l(k) \sim 2^n l_1(2^n)l(2^n) \qquad (n \to \infty),$$

from which the conclusion follows.

Another basic property of slowly varying functions is the asymptotic relation [5]

$$u^{\alpha}\max_{u\leqslant s<\infty} s^{-\alpha}L(s) \sim L(u) \qquad (u \to \infty),$$

for any $\alpha > 0$. The following lemma resembles a classical Abelian theorem [10].

LEMMA 2.2. *Let $\{l(n)\} \in \mathrm{SV}(N)$ and let $\{m_k\}_0^{\infty}$ be a positive sequence such that, for some $0 < \alpha < 1$,*

$$\sum_{k=n}^{\infty}\frac{1}{m_k^{1-\alpha}} = O\left(\frac{1}{m_n^{1-\alpha}}\right) \qquad (n \to \infty).$$

Then

$$\sum_{k=0}^{\infty}\frac{l(m_k)}{m_k} < \infty \tag{2.3}$$

and

$$\sum_{k=n}^{\infty}\frac{l(m_k)}{m_k} = O\left(\frac{l(m_n)}{m_n}\right) \qquad (n \to \infty).$$

PROOF. For $N > n$,

$$\sum_{k=n}^{N}\frac{l(m_k)}{m_k} \leqslant \left(\sup_{k\geqslant n} m_k^{-\alpha}l(m_k)\right)\sum_{k=n}^{\infty}\frac{1}{m_k^{1-\alpha}}.$$

Consequently (2.3) holds, and

$$\sum_{k=n}^{\infty} \frac{l(m_k)}{m_k} \leqslant A\Big(m_n^{\alpha} \sup_{k \geqslant n} m_k^{-\alpha} l(m_k)\Big)\frac{1}{m_n},$$

where A is an absolute constant. This completes the proof.

3. Weighted integrability theorem. We prove the following theorem.

THEOREM 3.1. *Let* $L \in \mathrm{SV}(\mathbf{R})$ *such that* $L(u) \to \infty$ $(u \to \infty)$, *let* $a(n) = o(1)$ $(n \to \infty)$, *and for some* $p > 1$, *let*

$$\sum_{n=1}^{\infty} L(n)\left(\frac{\sum_{k=n}^{\infty}|\Delta a(k)|^p}{n}\right)^{1/p} < \infty. \tag{3.1}$$

Then (i) $f(x)L(1/x) \in L^1(0, \pi]$, *and*

(ii) *if* $\{|a(n)|\} \in$ RQM, *then* $g(x)L(1/x) \in L^1(0, \pi]$ *if and only if*

$$\sum_{n=1}^{\infty} \frac{|a(n)|}{n} L(n) < \infty. \tag{3.2}$$

PROOF. Applying Lemma 2.1 and the methods of [2], the series in (3.1) is equiconvergent with

$$\sum_{n=0}^{\infty} 2^n L(2^n)\left(\frac{1}{2^n} \sum_{k=2^n}^{2^{n+1}-1} |\Delta a(k)|^p\right)^{1/p}. \tag{3.3}$$

By Jensen's inequality, (3.1) implies $\sum_{n=1}^{\infty}|\Delta a(n)|L(n) < \infty$, so that $\{a(n)\}$ is of bounded variation. Also, we may suppose $1 < p \leqslant 2$, a necessary technicality. We prove (ii); (i) is similar. Summation by parts yields the pointwise limit

$$g(x) = \sum_{n=1}^{\infty} \Delta a(n) \tilde{D}_n(x), \qquad x \in (0, \pi],$$

where

$$\tilde{D}_n(x) = \frac{\cos(x/2) - \cos(n + 1/2)x}{2\sin(x/2)}$$

is the conjugate Dirichlet kernel. Letting $a(0) = 0$ and

$$\overline{D}_n(x) = -\frac{\cos(n + 1/2)x}{2\sin(x/2)},$$

we may write

$$g(x) = \sum_{n=0}^{\infty} \Delta a(n) \overline{D}_n(x), \qquad x \in (0, \pi].$$

The result will be obtained by means of the following estimate: for $N = 1, 2, \ldots,$

$$\int_{\pi 2^{-(N+1)}}^{\pi} |g(x)| L\left(\frac{1}{x}\right) dx = \sum_{n=0}^{N} |a(2^n)| \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L\left(\frac{1}{x}\right) \frac{dx}{x} + O\left(\sum_{n=0}^{\infty} 2^n L(2^n)\left(\frac{1}{2^n} \sum_{n=0}^{\infty} |\Delta a(k)|^p\right)^{1/p}\right), \tag{3.4}$$

the O-term being uniform with respect to N. It follows that $g(x)L(1/x) \in L^1(0, \pi]$ if and only if

$$\sum_{n=0}^{\infty} |a(2^n)| \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L\left(\frac{1}{x}\right) \frac{dx}{x} < \infty. \tag{3.5}$$

A change of variables gives

$$\int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L\left(\frac{1}{x}\right) \frac{dx}{x} = \int_{\pi}^{2\pi} L(2^n u) \frac{du}{u} \sim (\lg 2) L(2^n) \qquad (n \to \infty). \tag{3.6}$$

Thus, the series in (3.5) is equiconvergent with $\sum_{n=0}^{\infty} |a(2^n)| L(2^n)$, completing the proof by Lemma 2.1, provided we verify (3.4). For $N = 1, 2, \ldots,$

$$\begin{aligned} &\left| \int_{\pi 2^{-(N+1)}}^{\pi} |g(x)| L\left(\frac{1}{x}\right) dx - \sum_{n=0}^{N} \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} \left| \sum_{k=0}^{2^n - 1} \Delta a(k) \bar{D}_k(x) \right| L\left(\frac{1}{x}\right) dx \right| \\ &\qquad \leqslant \sum_{n=0}^{N} \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} \left| \sum_{k=2^n}^{\infty} \Delta a(k) \bar{D}_k(x) \right| L\left(\frac{1}{x}\right) dx. \end{aligned} \tag{3.7}$$

Denote the right side by I_N; applying Hölder's inequality ($1/p + 1/q = 1$), followed by the Riesz **[11]** extension of the Hausdorff-Young theorem, one obtains

$$\begin{aligned} I_N &\leqslant \frac{1}{2} \sum_{n=0}^{N} \left(\int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L^p\left(\frac{1}{x}\right) \frac{dx}{\sin^p(x/2)} \right)^{1/p} \left\| \sum_{k=2^n}^{\infty} \Delta a(k) \cos\left(k + \frac{1}{2}\right) x \right\|_q \\ &\leqslant A_p \sum_{n=0}^{N} \left(\int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L^p\left(\frac{1}{x}\right) \frac{dx}{x^p} \right)^{1/p} \left(\sum_{k=2^n}^{\infty} |\Delta a(k)|^p \right)^{1/p}, \end{aligned} \tag{3.8}$$

where $\|\cdot\|_q$ is the $L^q(0, \pi]$-norm, and A_p is a constant dependent only on p. As in (3.6),

$$\int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L^p\left(\frac{1}{x}\right) \frac{dx}{x^p} \leqslant B 2^{n(p-1)} L^p(2^n),$$

where B is an absolute constant. From (3.8),

$$I_N \leqslant A_p \sum_{n=0}^{N} 2^n L(2^n) \left(\frac{1}{2^n} \sum_{k=2^n}^{\infty} |\Delta a(k)|^p \right)^{1/p},$$

where B has been absorbed into A_p. Returning to (3.7), we have

$$\begin{aligned} \int_{\pi 2^{-(N+1)}}^{\pi} |g(x)| L^p\left(\frac{1}{x}\right) dx &= \sum_{n=0}^{N} \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} \left| \sum_{k=0}^{2^n - 1} \Delta a(k) \bar{D}_k(x) \right| L\left(\frac{1}{x}\right) dx \\ &\quad + O\left(\sum_{n=0}^{\infty} 2^n L(2^n) \left(\frac{1}{2^n} \sum_{k=2^n}^{\infty} |\Delta a(k)|^p \right)^{1/p} \right), \end{aligned} \tag{3.9}$$

uniformly in N. Denote the first term on the right side by J_N. Applying the uniform estimate

$$|\bar{D}_n(x) + 1/x| \leqslant A(n+1), \qquad x \in (0, \pi],$$

A being an absolute constant, we have

$$\left| J_N - \sum_{n=0}^{N} \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} \left| \sum_{k=0}^{2^n-1} \Delta a(k) \right| L\left(\frac{1}{x}\right) \frac{dx}{x} \right| \tag{3.10}$$
$$\leqslant A \sum_{n=0}^{N} \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} \sum_{k=0}^{2^n-1} |\Delta a(k)|(k+1) L\left(\frac{1}{x}\right) dx.$$

Again similar to (3.6),

$$\int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L\left(\frac{1}{x}\right) dx \sim \frac{L(2^n)}{2^{n+1}} = O\left(\frac{L(2^n)}{2^n}\right) \qquad (n \to \infty).$$

Consequently, denoting the right side of (3.10) by J_N' and absorbing all absolute constants into A, we get

$$J_N' \leqslant A \sum_{n=0}^{N} \frac{L(2^n)}{2^n} \sum_{k=0}^{2^n-1} |\Delta a(k)|(k+1)$$
$$\leqslant A \sum_{k=0}^{2^N-1} |\Delta a(k)|(k+1) \sum_{n=[\lg_2(k+1)]}^{N} \frac{L(2^n)}{2^n},$$

$[\lg_2(k+1)]$ denoting the greatest integer in the base two logarithm of $k+1$. Appealing to Lemma 2.2 one obtains

$$J_N' \leqslant A \sum_{k=0}^{2^N-1} |\Delta a(k)|(k+1) \sum_{n=[\lg_2(k+1)]}^{\infty} \frac{L(2^n)}{2^n}$$
$$\leqslant A \sum_{k=0}^{2^N-1} |\Delta a(k)|(k+1) \frac{L(k+1)}{k+1}$$
$$\leqslant A \sum_{n=0}^{\infty} 2^n L(2^n) \left(\frac{1}{2^n} \sum^{\infty} |\Delta a(k)|^p \right)^{1/p}.$$

Returning to (3.10), we get

$$J_N = \sum_{n=0}^{N} |a(2^n)| \int_{\pi 2^{-(n+1)}}^{\pi 2^{-n}} L\left(\frac{1}{x}\right) \frac{dx}{x} + O\left(\sum_{n=0}^{\infty} 2^n L(2^n) \left(\frac{1}{2^n} \sum_{k=2n}^{\infty} |\Delta a(k)|^p \right)^{1/p} \right),$$

which concludes the proof of (3.4).

4. L^1-convergence of Fourier series. In this section we prove the following theorem concerning L^1-convergence of Fourier cosine series, an analogous result holds for Fourier sine series.

THEOREM 4.1. *Let* (1.1) *be the Fourier series of some* $f \in L^1(0, \pi]$ *with* $\{a(n)\} \in$ RQM. *Then* $\|S_n(f) - f\| = o(1)$ $(n \to \infty)$ *if and only if* $a(n) \lg n = o(1)$ $(n \to \infty)$.

PROOF. Let $\sigma_n(f) = \sigma_n(f, x)$ denote the $(C, 1)$ means of the Fourier cosine series of f. Summation by parts yields

$$\begin{aligned} S_n(f, x) - \sigma_n(f, x) &= \frac{1}{n+1} \sum_{k=1}^{n} k a(k) \cos kx \\ &= \frac{1}{n+1} \sum_{k=1}^{n-1} \Delta(k a(k)) \left[D_k(x) - \frac{1}{2} \right] \\ &\quad + \frac{n}{n+1} a(n) \left[D_n(x) - \frac{1}{2} \right], \end{aligned}$$

where

$$D_n(x) = \frac{1}{2} + \sum_{k=0}^{n} \cos kx = \frac{\sin(n + 1/2)x}{2\sin(x/2)}$$

is the Dirichlet kernel. Rearranging terms gives the following useful identity:

$$\begin{aligned} S_n(f, x) - \sigma_n(f, x) &= \frac{1}{n+1} \sum_{k=1}^{n-1} k \Delta a(k) D_k(x) \\ &\quad - \frac{1}{n+1} \sum_{k=0}^{n-1} a(k+1) D_k(x) + a(n) D_n(x). \end{aligned}$$

Applying the L^1-norm and using the well-known estimate

$$\|D_n\| = (2/\pi)\lg n + O(1) \qquad (n \to \infty),$$

we get

(4.1)

$$\begin{aligned} \|S_n(f) - \sigma_n(f)\| \leqslant \frac{2}{\pi} \cdot \frac{1}{n+1} \sum_{k=0}^{n-1} k |\Delta a(k)| \lg k + \frac{2}{\pi} \frac{1}{n+1} \sum_{k=0}^{n-1} |a(k+1)| \lg k \\ + \frac{2}{\pi} a(n) \lg n + o(1) \qquad (n \to \infty). \end{aligned}$$

For sufficiency, the hypothesis $a(n)\lg n = o(1)$ $(n \to \infty)$ implies that the second and third terms on the right side are $o(1)$ $(n \to \infty)$. Hence, we must show that

$$\frac{1}{n+1} \sum_{k=1}^{n} k |\Delta a(k)| \lg n = o(1) \qquad (n \to \infty). \tag{4.2}$$

Since $\{a(n)\} \in \mathrm{RQM}$, for some $\alpha > 0$ and some $\{l(n)\} \in \mathrm{SV}(N)$, we have $a(n)/n^{\alpha} l(n) \downarrow$. This implies that

$$a(n+1) \leqslant (1 + \alpha/n) \frac{l(n+1)}{l(n)} a(n),$$

without loss of generality, for all n. Consequently,

$$\Delta a(n) + \left[\left(1 + \frac{\alpha}{n}\right) \frac{l(n+1)}{l(n)} - 1 \right] a(n) \geqslant 0$$

and, finally,

$$|\Delta a(n)| \leqslant \Delta a(n) + 2 \left[\left(1 + \frac{\alpha}{n}\right) \frac{l(n+1)}{l(n)} \right] a(n). \tag{4.3}$$

We apply (4.3) to estimate the expression in (4.2); i.e.,

$$\frac{1}{n+1}\sum_{k=1}^{n} k|\Delta a(k)|\lg n \leqslant \frac{1}{n+1}\sum_{k=1}^{n} k\Delta a(k)\lg k + \frac{2}{n+1}\sum_{k=1}^{n}\left(1+\frac{\alpha}{k}\right)\frac{l(k+1)}{l(k)}a(k)\lg k.$$

The second term is $o(1)$ $(n \to \infty)$ since $\{l(n)\} \in \mathrm{SV}(N)$ and $a(n)\lg n = o(1)$ $(n \to \infty)$. For the first we apply summation by parts:

$$\frac{1}{n+1}\sum_{k=1}^{n} k\Delta a(k)\lg k = \frac{1}{n+1}\sum_{k=1}^{n-1} k\lg\left(1+\frac{1}{k}\right)a(k+1) + \frac{1}{n+1}\sum_{k=1}^{n-1} a(k+1)\lg(k+1) - \frac{n}{n+1}a(n+1)\lg n.$$

The first term is $o(1)$ $(n \to \infty)$ since $\lg(1+1/n) \approx 1/n$ $(n \to \infty)$; the second and third terms are $o(1)$ $(n \to \infty)$ since $a(n)\lg n = o(1)$ $(n \to \infty)$. This concludes the proof of sufficiency. For necessity we use the known estimate [**6**]

$$\|S_n(f) - f\| \geqslant \sum_{k=1}^{n} \frac{a(n+k)}{k}.$$

From the fact that $\{a(n)\} \in \mathrm{RQM}$ we obtain the inequality

$$\begin{aligned}(4.4)\qquad \sum_{k=1}^{n}\frac{a(n+k)}{k} &= \sum_{k=1}^{n}\frac{a(n+k)}{(n+k)^{\alpha}l(n+k)}\cdot\frac{(n+k)^{\alpha}l(n+k)}{k}\\ &\geqslant \frac{a(2n)}{(2n)^{\alpha}l(2n)}\sum_{k=1}^{n}\frac{(n+k)^{\alpha}l(n+k)}{k}\\ &\geqslant \left(\frac{n+1}{2n}\right)^{\alpha}\frac{a(2n)}{l(2n)}\sum_{k=1}^{n}\frac{l(n+k)}{k}.\end{aligned}$$

The asymptotic relation $l(k) \sim k^{\beta}[\sup_{n\geqslant k} n^{-\beta}l(n)]$ $(k \to \infty)$, gives for large n,

$$\begin{aligned}\sum_{k=n+1}^{2n}\frac{l(k)}{k-n} &\approx \sum_{k=n+1}^{2n}\frac{k^{\beta}\sup_{m\geqslant k} m^{-\beta}l(m)}{k-n}\\ &\geqslant \left[\sup_{m\geqslant 2n} m^{-\beta}l(m)\right]\sum_{k=n+1}^{2n}\frac{k^{\beta}}{k-n} \geqslant (n+1)^{\beta}\left[\sup_{m\geqslant 2n} m^{-\beta}l(m)\right]\sum_{k=1}^{n}\frac{1}{k}\\ &= \left(\frac{n+1}{2n}\right)^{\beta}(2n)^{\beta}\left[\sup_{m\geqslant 2n} m^{-\beta}l(m)\right]\sum_{k=1}^{n}\frac{1}{k} \sim \frac{1}{2^{\beta}}l(2n)\lg n.\end{aligned}$$

Returning to (4.4) concludes the proof.

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