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A SIMPLE PROOF OF A THEOREM OF ALBERT

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ABSTRACT. A simple proof is given of the following theorem of Albert: An associative division algebra of degree 4 over its center is of order 4 in the Brauer group if and only if it cannot be written as a tensor product of quaternion algebras.

The following theorem is stated without proof in [**3**, Theorem 11.2], where the reader is referred to [**1**] and [**2**].

THEOREM 1. *Let $\mathscr{D}/F$ be a central division algebra of degree 4. $\mathscr{D}$ has order 4 in the Brauer group if and only if it cannot be written as a tensor product of quaternion algebras.*

In this note we show that this follows easily from other (by no means trivial) results of Albert to be found in [**3**]. Albert's original proof is involved and not readily accessible to the modern reader. First recall that a finite dimensional division algebra has an involution of the first kind if and only if it is of order 1 or 2 in the Brauer group [**3**, Theorem 10.19]. Since $\mathscr{D}$ has order 2 or 4 in the Brauer group, Theorem 1 is equivalent to

THEOREM 2. *A central division algebra $\mathscr{D}/F$ of degree 4 has an involution of the first kind if and only if $\mathscr{D}=\mathscr{Q}_1 \otimes_F \mathscr{Q}_2$, $\mathscr{Q}_i$ quaternion algebras over F.*

PROOF. Let $\mathscr{D}/F$ be a central division algebra of degree 4. Then $\mathscr{D}$ contains a separable quadratic field extension E/F [**3**, Theorem 11.9]. Therefore $E=F(x)$ for some $x \in \mathscr{D}$ such that $x^2-x+\beta=0$, $\beta \in F$. By the Skolem-Noether theorem the automorphism of E/F mapping x into $1-x$ can be extended to an automorphism σ of $\mathscr{D}/F$ and $a^\sigma=c^{-1}ac$, $\forall a \in \mathscr{D}$, for some fixed $c \in \mathscr{D}$. Assume that $\mathscr{D}$ has an involution of the first kind. Then $\mathscr{D}$ has an involution * of the first kind such that * is the identity on E [**3**, Theorem 10.15]. We wish to construct an involution of the first kind which maps x into $1-x$. Consider the map $\varphi:\mathscr{D}\to\mathscr{D}$ given by $a^\varphi=c^{-1}a^*c$, $a \in \mathscr{D}$. If $c^*=\pm c$ this is an involution. Suppose

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that $c^* \neq -c$. Then $(c+c^*)^* = c+c^* \neq 0$ and $c+c^* = (1+c^*c^{-1})c$. Since

$$\begin{aligned} c^*c^{-1}x(c^*c^{-1})^{-1} &= c^*c^{-1}xc(c^*)^{-1} = c^*(1-x)(c^*)^{-1} \\ &= (c^{-1}(1-x)^*c)^* = (c^{-1}(1-x)c)^* \\ &= x^* = x, \end{aligned}$$

c^*c^{-1} commutes with x. So, letting $b=c$ if $c^* = -c$, $b=c+c^*$ otherwise, $a^\tau = b^{-1}a^*b$, $a \in \mathscr{D}$ defines an involution on $\mathscr{D}$ such that $x^\tau = 1-x$. Since E is τ stable, so is $C(E)$, the centralizer of E. Thus $C(E)/E$ is a quaternion algebra with involution τ of the second kind. Therefore $C(E) = \mathscr{Q}_1 \otimes_F E$, $\mathscr{Q}_1$ a central quaternion algebra over F [**3**, Theorem 10.21]. Then $\mathscr{D} = \mathscr{Q}_1 \otimes_F \mathscr{Q}_2$, where $\mathscr{Q}_2 = C(\mathscr{Q}_1)$ the centralizer of $\mathscr{Q}_1$. Since $\mathscr{Q}_2$ is a quaternion algebra over F we have proved that if $\mathscr{D}$ has an involution of the first kind then $\mathscr{D}$ is a tensor product of quaternion algebras. The converse is obvious.

Professor Tamagawa has obtained another proof of this theorem in his study of Clifford algebras.

References

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