PROCEEDINGS OF THE
AMERICAN MATHEMATICAL SOCIETY
Volume 32, Number 2, April 1972

INJECTIVE HULLS OF CERTAIN S-SYSTEMS OVER A SEMILATTICE

C. S. JOHNSON, JR. AND F. R. McMORRIS

ABSTRACT. We construct, in the category of S-systems over a semilattice, the injective hulls of S-systems which are homomorphic images of S-subsystems of S.

1. **Introduction.** In [1] Berthiaume showed that injective hulls exist in the category of S-systems (or S-sets) over a semigroup S. In that paper he also showed that if S is a chain then the injective hull of S itself is its Dedekind-MacNeile completion. In the present paper we consider the case where S is a semilattice and construct the injective hulls of S-systems which are homomorphic images of S-subsystems of S (or, in the notation of [3], S-systems which are in $\boldsymbol{HS}(S)$). We do this by adapting the techniques used by Bruns and Lakser in [2] to construct injective hulls in the category of semilattices. We obtain as corollaries Berthiaume's result for chains, a characterization of injective cyclic S-systems over a semilattice, and the result that a semilattice S is injective in the category of semilattices if and only if it is injective in the category of S-systems.

2. **Preliminaries.** Let S be a semigroup. A (right) S-*system* is a set M equipped with a map (written multiplicatively) from $M\times S$ to M such that $m(s_1s_2)=(ms_1)s_2$ for all $m\in M$ and all $s_1, s_2\in S$. If one thinks of each element of S as inducing a unary operation on an S-system M, then M is a finitary algebra and all the notions of universal algebra are available. Thus if M and N are S-systems we have $A\subseteq M$ is an S-subsystem of M if and only if $AS\subseteq A$, $\phi: M\to N$ is a homomorphism if and only if $\phi(ms)=\phi(m)s$ for all $m\in M$ and all $s\in S$, and an equivalence relation $\sim$ on M is a congruence relation if and only if $m_1\sim m_2$ implies $m_1s\sim m_2s$ for all $s\in S$. Unless otherwise stated, all algebraic notions will be in this category. *We will assume throughout that the semigroup S is a semilattice* (i.e., commutative and idempotent).

LEMMA 1. *If an S-system M has the property that $MS=M$, then it is partially ordered by the rule $m_1\leqq m_2$ if and only if $m_1=m_2s$ for some $s\in S$.*

Presented to the Society, June 1, 1971; received by the editors June 1, 1971.
AMS 1970 *subject classifications*. Primary 20M99; Secondary 16A52, 18B99.
Key words and phrases. S-systems, injective S-systems.

PROOF. For each $m \in M$ we have $m=m_1 s$ for some $m_1 \in M$ and $s \in S$ (since $MS=M$), and hence $m=ms$, so $m \leqq m$. If $m_1 \leqq m_2$ and $m_2 \leqq m_1$ we have $s_1, s_2 \in S$ such that $m_1=m_2 s_1$ and $m_2=m_1 s_2$. Now $m_1=m_2 s_1=(m_1 s_2)s_1= m_1(s_2 s_1)=(m_2 s_1)(s_2 s_1)=m_2(s_1 s_2 s_1)=m_2(s_1 s_2)=(m_2 s_1)s_2=m_1 s_2=m_2$. Transitivity is obvious.

Notice that if S has an identity and M is a unitary S-system, then $MS= M$ and Lemma 1 applies.

When we are dealing with a partial order on an S-system we will use the symbols "$\bigvee$" and "$\bigwedge$" to denote least upper bounds and greatest lower bounds, respectively.

We will refer to the partial order of Lemma 1 as the *natural partial order* on M.

If an S-system M is partially ordered in some way and if $A \subseteq M$ is such that $\bigvee A$ exists, we will say that $\bigvee A$ is *S-distributive* if and only if, for each $s \in S$, $\bigvee \{as \mid a \in A\}$ exists and equals $(\bigvee A)s$.

Recall the following definitions in a category of algebras: An algebra C is *injective* if and only if every homomorphism from a subalgebra A of an algebra B into C has an extension to all of B. An extension C of an algebra A is *essential* if and only if any homomorphism from C to an algebra B, whose restriction to A is one-to-one, is itself one-to-one. An *injective hull* of an algebra is an essential, injective extension.

LEMMA 2. *Let C be an S-system which is partially ordered in such a way that $c= \bigvee \{cs \mid s \in S\}$ for each $c \in C$. If C is a complete lattice in which arbitrary joins are S-distributive, then C is injective.*

PROOF. Let A be an S-subsystem of an S-system B and let $\phi: A \to C$ be a homomorphism. Define $\phi^*: B \to C$ by

$$\phi^*(b) = \bigvee \{\phi(a) \mid a \in A,\ a = bs \text{ for some } s \in S\}.$$

If $b \in A$, then

$$\phi^*(b) = \bigvee \{\phi(bs) \mid s \in S\} = \bigvee \{\phi(b)s \mid s \in S\} = \phi(b)$$

and thus ϕ^* extends ϕ. If $s_0 \in S$ it is easy to see that $\{as_0 \mid a \in A,\ a=bs$ for some $s \in S\}=\{a \mid a \in A,\ a=bs_0 s$ for some $s \in S\}$. Thus

$$\begin{aligned}\phi^*(b)s_0 &= (\bigvee \{\phi(a) \mid a \in A,\ a = bs \text{ for some } s \in S\})s_0 \\ &= \bigvee \{\phi(a)s_0 \mid a \in A,\ a = bs \text{ for some } s \in S\} \\ &= \bigvee \{\phi(as_0) \mid a \in A,\ a = bs \text{ for some } s \in S\} \\ &= \bigvee \{\phi(a) \mid a \in A,\ a = bs_0 s \text{ for some } s \in S\} = \phi^*(bs_0).\end{aligned}$$

We will call a subset A of a poset C *join-dense* in C if and only if $c= \bigvee \{a \in A \mid a \leqq c\}$ for each $c \in C$. If A and C are also S-systems we will say that *S-distributive joins in A are preserved in C* if and only if $a= \bigvee_C B$

whenever $B \subseteq A$ and $a = \bigvee_A B$ is S-distributive. We will call a map ϕ on a poset P *decreasing* if and only if $\phi(a) \leqq a$ for all $a \in P$.

LEMMA 3. *Let C be an S-system which is partially ordered in such a way that the unary operations induced by S preserve the order and are decreasing. Let A be an S-subsystem of C and suppose that for each $a \in A$ there is an $s_a \in S$ such that, for each $c \in C$, $c \wedge a$ exists and equals cs_a. If A is join-dense in C and if S-distributive joins in A are preserved in C, then C is an essential extension of A.*

PROOF. Let $\phi: B \to C$ be a homomorphism with $\phi|_A$ one-to-one. If ϕ is not one-to-one there exist elements $a, b \in C$ with $a \neq b$ and $\phi(a) = \phi(b)$. Since A is join-dense in C we may suppose there exists $u \in A$ with $u \leqq b$ and $u \nleqq a$. We have $\phi(a \wedge u) = \phi(as_u) = \phi(a)s_u = \phi(b)s_u = \phi(bs_u) = \phi(b \wedge u) = \phi(u)$. Now suppose $s \in S$ and let $M = \{(u \wedge x)s \mid x \leqq a, x \in A\}$. If we show that $us = \bigvee_A M$ we will have shown (considering the special case $s = s_u$) that $u = \bigvee_A \{u \wedge x \mid x \leqq a, x \in A\}$ and is an S-distributive join. Hence $u = \bigvee_C \{u \wedge x \mid x \leqq a, x \in A\} \leqq a$, a contradiction. Since $u \wedge x \leqq u$ implies $(u \wedge x)s \leqq us$, it is clear that us is an upper bound for M. Let $v \in A$ be another upper bound for M with $v \neq us$. Since meets exist in A we may further assume that $v < us$. If $c \in A$ and $c \leqq (u \wedge a)s$ we have $c \leqq us$ and $c = us \wedge c = uss_c = us_c s = (u \wedge c)s$ with $c \leqq as \leqq a$. Hence we can again use the fact that A is join-dense in C and obtain

$$(u \wedge a)s = \bigvee_C \{(u \wedge x)s \mid x \leqq a, x \in A\} = \bigvee_C M \leqq v.$$

Now we have

$$\begin{aligned}\phi(us) &= \phi(u)s = \phi(a \wedge u)s = \phi((a \wedge u)s) = \phi((a \wedge u)s \wedge v) = \phi((a \wedge u)ss_v) \\ &= \phi(a \wedge u)ss_v = \phi(u)ss_v = \phi(uss_v) = \phi(us \wedge v) = \phi(v),\end{aligned}$$

a contradiction. This establishes the fact that $us = \bigvee_A M$ and finishes the proof.

3. **Injective hulls.** Let M be an S-system such that $MS = M$. Recall that, by Lemma 1, M is partially ordered by the rule $m_1 \leqq m_2$ if and only if $m_1 = m_2 s$ for some $s \in S$. Following Bruns and Lakser we will call a subset N of M *admissible* if and only if $\bigvee N$ exists and is S-distributive, and we will call N a *D-ideal* if and only if $y \in N$ and $x \leqq y$ imply $x \in N$ (i.e., $NS \subseteq N$) and N is closed under S-distributive joins (i.e., $A \subseteq N$ and A admissible implies $\bigvee A \in N$). Now $I_D(M)$, the set of all D-ideals of M, is closed under arbitrary intersections and is thus a complete lattice under set inclusion. An obvious modification of the proof of [2, Lemma 3] shows that the join operation in $I_D(M)$ is given by

$$\bigvee \{A_i \mid i \in I\} = \{\bigvee N \mid N \subseteq \bigcup \{A_i \mid i \in I\},\ N \text{ admissible}\}.$$

It is easy to show that if N is a D-ideal of M then $Ns=\{ns|s\in S\}$ is also a D-ideal and that $Ns = N\cap Ms$. Thus $I_D(M)$ is a complete lattice in which arbitrary joins are S-distributive. Notice that $mS=\{x\in M|x\leqq m\}$, that these principal ideals are clearly D-ideals and that $m\mapsto mS$ is an embedding of M in $I_D(M)$. Now, considering M as an S-subsystem of $I_D(M)$, notice that S-distributive joins in M are preserved in $I_D(M)$.

It is clear that S itself is an S-system and we now restrict our attention to $\boldsymbol{HS}(S)$, that is, to S-systems which are of the form $A/\sim$ where A is an S-subsystem of S and $\sim$ is a congruence relation on A. Notice that A is an ideal of S and $\sim$ is a semigroup congruence on A (since we have assumed S to be commutative) and thus $A/\sim$ is a semilattice as well as an S-system. It is easy to see that $(A/\sim)S=A/\sim$ and that the partial order on $A/\sim$ as a semilattice coincides with the natural partial order of Lemma 1.

THEOREM. *If $M\in \boldsymbol{HS}(S)$, then $I_D(M)$ is the injective hull of M.*

PROOF. $M=A/\sim$ where $A\subseteq S$ is an ideal and $\sim$ is a congruence relation on A. Denoting arbitrary elements of $A/\sim$ by $[x]$ with $x\in A$, we have that $[a]S=Ma$ since $[a]s=[as]=[asa]=[as]a$ and $[x]a=[xa]=[ax]=[a]x$. Since a D-ideal N is the join of the principal ideals it contains we have

$$\begin{aligned} N &= \bigvee \{[a]S \mid [a] \in N\} = \bigvee \{N \cap Ma \mid [a] \in N\} \\ &\subseteq \bigvee \{N \cap Ms \mid s \in S\} = \bigvee \{Ns \mid s \in S\} \subseteq N. \end{aligned}$$

Thus $N=\bigvee \{Ns|s\in S\}$ for each $N\in I_D(M)$ so the hypotheses of Lemma 2 are satisfied and $I_D(M)$ is injective. Since the unary operations in $I_D(M)$ are given by $Ns=N\cap Ms$, for each $s\in S$, it is apparent that they preserve the order and are decreasing and that for each $[a]\in M$ we have $Na=N\cap Ma=N\cap [a]S$. Thus, by identifying M with the S-subsystem of $I_D(M)$ consisting of the principal order ideals of M, we see that the hypotheses of Lemma 3 are satisfied and that $I_D(M)$ is an essential extension of M.

COROLLARY 1. *If $M\in \boldsymbol{HS}(S)$, then M is injective if and only if it is a complete lattice in which arbitrary joins are S-distributive.*

PROOF. M is injective if and only if the embedding $m\mapsto mS$ of M in $I_D(M)$ is onto. This is true precisely when every D-ideal of M is principal. Clearly this is the case when M is a complete lattice in which arbitrary joins are S-distributive. Conversely, if every D-ideal is principal, then the partial ordering of $I_D(M)$ by set inclusion (under which $I_D(M)$ is a complete lattice with S-distributive joins) coincides with its natural partial order as an S-system, i.e., $m_1S\subseteq m_2S$ if and only if $m_1=m_2s$ for some $s\in S$. Since in this case M is isomorphic to $I_D(M)$, M is also a complete lattice with S-distributive joins.

COROLLARY 2 (BERTHIAUME). *If S is a chain, then its injective hull is its Dedekind-MacNeile completion.*

PROOF. If S is a chain, then every order ideal is a D-ideal and hence $I_D(S)$ is the Dedekind-MacNeile completion.

COROLLARY 3. *A semilattice S is injective in the category of semilattices if and only if it is injective in the category of S-systems.*

PROOF. By Corollary 1, S is injective in the category of S-systems if and only if it is a complete lattice with the property that $(\bigvee M)\wedge s = \bigvee \{m\wedge s \mid m\in M\}$ for all $s\in S$, $M\subseteq S$. By [**2**, Theorem 1] these properties characterize injectivity in the category of semilattices.

COROLLARY 4. *A cyclic S-system is injective if and only if it is a complete lattice (in its natural partial order) in which arbitrary joins are S-distributive.*

PROOF. If M is a cyclic S-system, then $M=xS$ for some $x\in M$. It is clear that $MS=M$, so M has a natural partial order (Lemma 1). Define a congruence relation on S by $s_1\sim s_2$ if and only $xs_1=xs_2$. The map $xs\mapsto[s]$ is an isomorphism between M and $S/\sim$ and hence $M\in \boldsymbol{HS}(S)$ and Corollary 1 applies.

COROLLARY 5. *Let M be an S-system such that $MS=M$. If for each $m\in M$ there exists an $s\in S$ such that $mS=Ms$, then M is injective if and only if it is a complete lattice in which arbitrary joins are S-distributive.*

PROOF. Define a congruence relation on S by $s_1\sim s_2$ if and only if $Ms_1=Ms_2$. The map $m\mapsto[s]$, where $mS=Ms$, is an isomorphism between M and an S-subsystem of $S/\sim$. Since $\boldsymbol{SH}(S)\subseteq \boldsymbol{HS}(S)$ by [**3**, Theorem 1, p. 152], $M\in\boldsymbol{HS}(S)$ and Corollary 1 applies.

REFERENCES

1. P. Berthiaume, *The injective envelope of S-sets*, Canad. Math. Bull. **10** (1967), 261–273. MR **35** #4321.

2. G. Bruns and H. Lakser, *Injective hulls of semilattices*, Canad. Math. Bull. **13** (1970), 115–118.

3. G. Grätzer, *Universal algebra*, Van Nostrand, Princeton, N.J., 1968. MR **40** #1320.

DEPARTMENT OF MATHEMATICS, BOWLING GREEN STATE UNIVERSITY, BOWLING GREEN, OHIO 43403

Current address (McMorris): Biomathematics Program, Box 5457, North Carolina State University, Raleigh, North Carolina 27607