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TWIST-SPUN TORUS KNOTS

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ABSTRACT. Zeeman has shown that the complement of a twist-spun knot fibres over the circle. He also proves that the group of the 5-twist-spun trefoil is just the direct product of the fundamental group of the fibre with the integers. We generalise this by showing that, for torus knots, the group of the twist-spun knot is such a direct product whenever the fibre is a homology sphere. This then suggests the question (asked by Zeeman for the case of the 5-twist-spun trefoil) as to whether there is a corresponding product structure in the geometry. We answer in the negative.

1. Let $K\subset S^3$ be a knot, and M_k its k-fold branched cyclic covering. Then if $K_k\subset S^4$ is the k-twist-spin of K, the main theorem of [5] states that S^4-K_k is a fibre bundle over S^1, with fibre homeomorphic to M_k-*, the punctured M_k. In particular, there is an exact sequence

$$1 \to \pi_1(M_k - *) \to \pi_1(S^4 - K_k) \to Z \to 1$$

which gives $\pi_1(S^4 - K_k)$ as a semidirect product of $\pi_1(M_k-*)\simeq\pi_1(M_k)$ and Z.

In [5], Zeeman discusses in detail the 5-twist-spun trefoil, and shows that in this case $\pi_1(S^4-K_5)$ is actually isomorphic to the direct product $\pi_1(M_5)\times Z$.

THEOREM 1. *Let K be a torus knot. Then $\pi_1(S^4-K_k)\simeq\pi_1(M_k)\times Z$ if, and only if, M_k is a homology 3-sphere.*

PROOF. The necessity of the condition that M_k be a homology 3-sphere is easily established by noting that $\pi_1(S^4-K_k)\simeq\pi_1(M_k)\times Z$ implies $H_1(S^4-K_k)\simeq H_1(M_k)\oplus Z$. But $H_1(S^4-K_k)\simeq Z$, by Alexander duality, and hence $H_1(M_k)=0$.

Now there is a well-known formula [2] which gives the order of $H_1(M_k)$ in terms of the Alexander polynomial of K, and it is not hard to show from this that if K is the torus knot of type m, n, then $H_1(M_k)=0$ if, and only if, $(k, mn)=1$.[2]

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[2] (a, b) will always denote the h.c.f. of a and b.

It is also well known that $\pi_1(S^3-K)$ has the presentation $(x, y: x^m=y^n)$. (Note that abelianisation takes $x\mapsto t^n$, $y\mapsto t^m$, where t is a generator of $H_1(S^3-K)\cong \boldsymbol{Z}$.)

Now let $a\in\pi_1(S^3-K)$ be a meridian element such that $a\mapsto t$ under abelianisation. Then a can be expressed in terms of x and y, $a=a(x, y)$, say (the precise expression will not matter), and an application of van Kampen's theorem shows that [5], [6]

$$G_k \cong |x, y, a: x^m = y^n, a = a(x, y), a^k x = x a^k, a^k y = y a^k|,$$

where we are writing G_k for $\pi_1(S^4-K_k)$.

If we denote the centre of G_k by $C(G_k)$, then clearly the elements a^k and $x^m=y^n=u$, say, both lie in $C(G_k)$, and hence $u^r a^{sk}\in C(G_k)$ for all integers r, s.

Note also that if $\alpha: G_k\to \boldsymbol{Z}$ is abelianisation, $\alpha(u^r a^{sk})=t^{rmn+sk}$. Now if $(k, mn)=1$, there exist integers r, s such that $rmn+sk=1$. Then $\alpha(u^r a^{sk})=t$, and the splitting $\beta: \boldsymbol{Z}\to G_k$ defined by $\beta(t)=u^r a^{sk}$ gives an isomorphism $G_k\cong\pi_1(M_k)\times \boldsymbol{Z}$ as required.

2. Theorem 1 shows that if K is the torus knot of type m, n, and $(k, mn)=1$, then

$$\pi_1(S^4 - K_k) \cong \pi_1((M_k - *) \times S^1).$$

A natural question (asked by Zeeman [5, Question 2] for the case of the 5-twist-spun trefoil) is therefore the following:

Is $S^4-K_k\cong(M_k-*)\times S^1$?

(Note that the answer is affirmative if $k=1$, for then M_k is just S^3, and K_k is unknotted.)

THEOREM 2. *Let K be a torus knot, and suppose $k>1$. Then $S^4-K_k\not\cong (M_k-*)\times S^1$.*

PROOF. If $(k, mn)\neq 1$, then of course the manifolds are distinguished by their fundamental group, by Theorem 1.

To prove the result for the interesting case $(k, mn)=1$, we proceed as follows.

Write H_k for $\pi_1((M_k-*)\times S^1)$, and let G_k be as before, so that we have $G_k\cong H_k\cong\pi_1(M_k)\times \boldsymbol{Z}$.

It is clear that if $P(H_k)$ denotes the set of peripheral elements of H_k (see [3]), then $P(H_k)=1\times \boldsymbol{Z}$. In particular, $P(H_k)\subset C(H_k)$.

Now $S^4-K_k\cong(M_k-*)\times I/h$ (i.e., $(M_k-*)\times I$ with $(M_k-*)\times 0$ and $(M_k-*)\times 1$ identified via the homeomorphism $(x, 0)\mapsto(h(x), 1)$), where $h: M_k\to M_k$ is a homeomorphism such that $h(*)=*$ (in fact, h is the canonical covering homeomorphism of M_k [5]). Inclusion induces an isomorphism $G_k=\pi_1((M_k-*)\times I/h)\cong\pi_1(M_k\times I/h)$; let $\gamma\in G_k$ be the element corresponding to that represented by $*\times I/h$. Then clearly $\gamma\in P(G_k)$.

Moreover, if $\xi \in \pi_1(M_k - *) \subset \pi_1((M_k - *) \times I/h)$, we have the formula $\gamma^{-1}\xi\gamma = h_\#(\xi)$. In fact, if $(x_i : r_j)$ is a presentation of $\pi_1(M_k - *)$, G_k has the presentation

$$(x_i, \gamma : r_j, \gamma^{-1}x_i\gamma = h_\#(x_i)).$$

In particular, if $\gamma \in C(G_k)$, we would have $h_\# =$ the identity.

But sewing back the knot K_k (to get S^4) has the effect of adding the relation $\gamma = 1$ to G_k, and therefore $|x_i : r_j, x_i = h_\#(x_i)|$ has to be the trivial group. If $h_\# =$ the identity, however, this group is just $\pi_1(M_k)$, so if we show that $\pi_1(M_k) \neq 1$, we are through. For we will then have proved that there exists an element of $P(G_k)$ (namely γ) which does not lie in $C(G_k)$, whereas $P(H_k) \subset C(H_k)$. Hence $(M_k - *) \times I/h$ and $(M_k - *) \times S^1$ cannot be homeomorphic.

Lemma. *If M_k is the k-fold branched cyclic covering of a torus knot, and $k > 1$, then $\pi_1(M_k) \neq 1$.*

Proof. Recalling the exact sequence in §1, it will be sufficient to prove that $G_k \not\cong \mathbf{Z}$.

Let K be the torus knot of type m, n. Its group $\pi_1(S^3 - K) \cong |x, y : x^m = y^n|$, and if r, s are integers such that $rm + sn = 1$ (such integers exist since $(m, n) = 1$), then $y^r x^s$ is a meridian element (see for example [**4**]). It follows that G_k has the presentation

$$(x, y : x^m = y^n, (y^r x^s)^k x = x(y^r x^s)^k, (y^r x^s)^k y = y(y^r x^s)^k),$$

and hence maps homomorphically onto the group

$$|x, y : x^m = y^n = (y^r x^s)^k = 1|.$$

Sending $x \mapsto u^n$ and $y \mapsto v^m$ then defines a homomorphism from $|x, y : x^m = y^n = (y^r x^s)^k = 1|$ onto $|u, v : u^m = v^n = (vu)^k = 1|$. (The map is a homomorphism since the relations $u^{nm} = 1$, $v^{mn} = 1$, $(v^{mr} u^{ns})^k = 1$ are all consequences of the relations $u^m = 1$, $v^n = 1$, $(vu)^k = 1$, and it is onto since $u = u^{ns}$, $v = v^{mr}$.)

But this latter group is just the polyhedral group (m, n, k) (see [**1**]), which is nonabelian unless $m = 1$, $n = 1$, or $k = 1$, or $m = n = k = 2$, cases which are not relevant here. Hence G_k is nonabelian.

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