THE STABILITY OF A ROTATING VISCOUS JET*

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1. Introduction. The purpose of this paper is to discuss the stability of a column of homogeneous viscous liquid rotating as a rigid body. We shall show that the motion will be stable for perturbations of azimuthal wave number s and axial wave number ζ if and only if

$$T \geq \rho a^3\omega^2(a^2\zeta^2 + s^2 - 1)^{-1}, \tag{1}$$

where T is the surface tension, ρ the density of the liquid, a the radius of the column, and ω its angular velocity. This condition is to hold, provided that the right-hand-side of (1) is positive; when the right-hand-side of (1) is negative (i.e. when $s = 0$ and $a\zeta < 1$), the motion is always unstable.

Special results on this problem which are to be found in the literature are all immediately deducible from (1). Rayleigh [1] discussed axisymmetric disturbances of a non-rotating column (i.e., $\omega = s = 0$) and found that the equilibrium was stable or unstable according as $a\zeta >$ or < 1. While his result was established in the first place for a non-viscous liquid, he later [2] extended it to liquids of very high viscosity. Much more recently, Hocking [3] has considered the effect of rotation on the stability of a column to axisymmetric disturbances and has shown that, in the limiting case of very high viscosity, the necessary and sufficient condition for stability is that $T \geq \rho a^3\omega^2(a^2\zeta^2 - 1)^{-1}$. Another particular problem considered by Hocking was that of disturbances which are confined to cross-sectional planes, and are the same in all such planes, i.e., both ζ and the axial velocity component are zero. For the special cases of very high and very low viscosity, he showed that the condition for stability was $T \geq \rho a^3\omega^2(s^2 - 1)^{-1}$. This result was shown [4] to hold quite generally for all values of the viscosity. It should be noted that, for plane disturbances, we need consider only $s \geq 2$.

It is of interest that the critical value of the surface tension defined by (1) is independent of the coefficient of viscosity.

2. Equations of motion. The physical constants in our problem are: ω, the angular velocity of the rotating column; a, the radius of the unperturbed column; ρ, the density of the liquid; μ, its viscosity, which enters only in the combination $\nu \equiv \mu/\rho$ (kinematic viscosity); T, the surface tension. We make use of cylindrical coordinates, and denote velocities of the fluid in the r-,ϕ- and z-directions by: u, v, and w, respectively. In these coordinates, the Navier-Stokes equations are

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$$\left.\begin{aligned}
\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial r} + \frac{v}{r}\frac{\partial u}{\partial \phi} + w\frac{\partial u}{\partial z} - \frac{v^2}{r} &= -\frac{1}{\rho}\frac{\partial p}{\partial r} + \nu\left[\nabla^2 u - \frac{u}{r^2} - \frac{2}{r^2}\frac{\partial v}{\partial \phi}\right],\\
\frac{\partial v}{\partial t} + u\frac{\partial v}{\partial r} + \frac{v}{r}\frac{\partial v}{\partial \phi} + w\frac{\partial v}{\partial z} + \frac{uv}{r} &= -\frac{1}{r\rho}\frac{\partial p}{\partial \phi} + \nu\left[\nabla^2 v - \frac{v}{r^2} + \frac{2}{r^2}\frac{\partial u}{\partial \phi}\right],\\
\frac{\partial w}{\partial t} + u\frac{\partial w}{\partial r} + \frac{v}{r}\frac{\partial w}{\partial \phi} + w\frac{\partial w}{\partial z} &= -\frac{1}{\rho}\frac{\partial p}{\partial z} + \nu[\nabla^2 w)] ,
\end{aligned}\right\} \tag{2}$$

where

$$\nabla^2 \equiv \frac{\partial^2}{\partial r^2} + \frac{1}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \phi^2} + \frac{\partial^2}{\partial z^2},$$

while the continuity equation is written as

$$\frac{1}{r}\frac{\partial}{\partial r}(ru) + \frac{1}{r}\frac{\partial v}{\partial \phi} + \frac{\partial w}{\partial z} = 0. \tag{3}$$

The solution of the equation for unperturbed flow is given by

$$u = 0 = w, \qquad v = r\omega, \qquad p = \tfrac{1}{2}\rho\omega^2 r^2 + \text{const},$$

and the constant is fixed by the requirement that at the boundary of the cylinder $p(a) = T/a$. Thus, $p = \frac{1}{2}\rho\omega^2(r^2 - a^2) + T/a$.

For the perturbed flow, we take our functions in the form:

$$\left.\begin{aligned}
u &= U^*(r, z)\exp[i(s\phi - \sigma t)]\\
v &= r\omega + V^*(r, z)\exp[i(s\phi - \sigma t)]\\
w &= W^*(r, z)\exp[i(s\phi - \sigma t)]\\
p &= \tfrac{1}{2}\rho\omega^2(r^2 - a^2) + \frac{T}{a} + \rho q^*(r, z)\exp[i(s\phi - \sigma t)],
\end{aligned}\right\} \tag{4}$$

and assume that U^*, V^*, W^*, q^*, as well as their derivatives, are small quantities whose squares may be neglected. For convenience, we shall at times use the symbol χ for $\exp[i(s\phi - \sigma t)]$. We have used starred letters U^*, etc. for convenience of notation when we subsequently go over to Fourier transforms.

The Navier-Stokes and continuity equations may then be written

$$\left.\begin{aligned}
i(s\omega - \sigma)U^* - 2\omega V^* &= -q_r^* + \nu\left[U_{rr}^* + \frac{1}{r}U_r^* - \frac{s^2+1}{r^2}U^* + U_{zz}^* - \frac{2is}{r^2}V^*\right]\\
i(s\omega - \sigma)V^* + 2\omega U^* &= -\frac{is}{r}q^* + \nu\left[V_{rr}^* + \frac{1}{r}V_r^* - \frac{s^2+1}{r^2}V^* + V_{zz}^* + \frac{2is}{r^2}U^*\right]\\
i(s\omega - \sigma)W^* &= -q_z^* + \nu\left[W_{rr}^* + \frac{1}{r}W_r^* - \frac{s^2}{r^2}W^* + W_{zz}^*\right]
\end{aligned}\right\} \tag{5}$$

$$U_r^* + \frac{1}{r}U^* + \frac{is}{r}V^* + W_z^* = 0, \tag{6}$$

where subscripts indicate differentiation. The necessary and sufficient condition for stability is that $\mathcal{I}(\sigma) \leq 0$, $\mathcal{I}(\sigma)$ denoting the imaginary part of σ.

3. The boundary conditions. (a) We shall assume that the perturbed boundary is the surface

$$\Sigma = r - a - H^*(z) \exp [i(s\phi - \sigma t)] = 0.$$

Since the surface moves with the fluid, we have

$$\frac{D\Sigma}{Dt} = \left(\frac{\partial}{\partial t} + u \frac{\partial}{\partial r} + \frac{v}{r} \frac{\partial}{\partial \phi} + w \frac{\partial}{\partial z}\right) \cdot \Sigma = 0.$$

Applying this condition to the perturbed functions, and neglecting terms of order higher than the first, we are left with

$$i(s\omega - \sigma)H^*(z) = U^*(a, z). \tag{7}$$

(b) The *normal stress condition* states that, at the boundary, the normal stress, $p_{nn} = p - \mu e_{nn}$, must be balanced by the normal force due to surface tension, $T(1/R_1 + 1/R_2)$, where R_1 and R_2 are the principal radii of curvature of the perturbed boundary. The mean curvature $(1/R_1 + 1/R_2)$ is obtained by elementary methods of differential geometry (see [5], Chap. III), in terms of the first and second derivatives of the vector

$$\mathbf{r} = ([a + H^*\chi] \cos \phi, [a + H^*\chi] \sin \phi, z).$$

We find that

$$\frac{1}{R_1} + \frac{1}{R_2} = -\frac{1}{a} + \chi\left[H^*_{zz} + \frac{1 - s^2}{a^2} H^*\right]. \tag{8}$$

e_{nn} is the component of the stress-tensor, referred to an orthogonal coordinate system where $\mathbf{n}$ is the normal to the perturbed surface. (The other two directions may be denoted by ϕ' and $\mathbf{z}'$.) However, all the components of the stress-tensor, referred to the coordinate system of the unperturbed surface, are already themselves quantities of the first order. Hence, to the approximation which we are using, $e_{nn} \simeq e_{rr}$ (and similarly, $e_{nz'} \simeq e_{rz}$, $e_{n\phi'} \simeq e_{r\phi}$). For e_{rr} we have

$$e_{rr} = 2 \frac{\partial u}{\partial r} = 2U^*_r\chi,$$

so that

$$p_{nn} \simeq \frac{\rho}{2} \omega^2(r^2 - a^2) + \frac{T}{a} + \rho q^*\chi - 2\mu U^*_r\chi = (\rho\omega^2 H^* a + \rho q^* - 2\mu U^*_r)\chi + \frac{T}{a}.$$

Thus we obtain for the normal stress condition at $r = a$

$$T\left[H^*_{zz} + \frac{1 - s^2}{a^2} H^*\right] = 2\mu U^*_r - \rho q^* - \rho\omega^2 a H^*. \tag{9}$$

In view of (7), we may rewrite this condition as:

$$\frac{1}{i(s\omega - \sigma)} \left[T\left(U^*_{zz} + \frac{1 - s^2}{a^2} U^*\right) + \rho\omega^2 a U^*\right] - 2\mu U^*_r + \rho q^* = 0. \tag{10}$$

(c) The *tangential shear conditions* require that

$$e_{n\phi'} = e_{nz'} = 0.$$

As remarked above, we may instead require

$$e_{n\phi} = e_{nz} = 0,$$

to the order of approximation which we are using throughout.
Since

$$e_{r\phi} = r\,\frac{\partial(v/r)}{\partial r} + \frac{1}{r}\frac{\partial u}{\partial \phi} \quad \text{and} \quad e_{rz} = \frac{\partial u}{\partial z} + \frac{\partial w}{\partial r},$$

it follows that the conditions on our functions at $r = a$ are

$$\frac{is}{r}\,U^* + V_r^* - \frac{1}{r}\,V^* = 0 \tag{11}$$

$$U_z^* + W_r^* = 0. \tag{12}$$

4. Fourier transforms. We now set $U(r, \zeta) = 1/(2\pi)^{1/2}\int_{-\infty}^{\infty} U^*(r, z)e^{-i\zeta z}\,dz$, etc., and apply the operator $1/(2\pi)^{1/2}\int_{-\infty}^{\infty}\cdots\exp(-i\zeta z)\,dz$ to our equations and boundary conditions. The effect of this Fourier transformation is to replace U^*, V^*, etc. in the equations by U, V, etc.; U_z^* is replaced by $i\zeta U$ and U_{zz}^* by $-\zeta^2 U$, where the functions U, V, W, Q now depend on r, ζ. We shall keep ζ fixed throughout, and thus our functions depend only on the variable r. (This is equivalent to considering a purely sinusoidal disturbance, of wave number ζ, instead of one which depends upon z in a general manner.)

The new form of the equations of motion is now

$$\left.\begin{aligned}
i(s\omega - \sigma)U - 2\omega V &= -q_r + \nu\left[U_{rr} + \frac{1}{r}\,U_r - \frac{s^2+1}{r^2}\,U - \zeta^2 U - \frac{2is}{r^2}\,V\right]\\
i(s\omega - \sigma)V + 2\omega U &= -\frac{is}{r}\,q + \nu\left[V_{rr} + \frac{1}{r}\,V_r - \frac{s^2+1}{r^2}\,V - \zeta^2 V + \frac{2is}{r^2}\,U\right]\\
i(s\omega - \sigma)W &= -i\zeta q + \nu\left[W_{rr} + \frac{1}{r}\,W_r - \frac{s^2}{r^2}\,W - \zeta^2 W\right]\\
U_r + \frac{1}{r}\,U + \frac{is}{r}\,V + i\zeta W &= 0.
\end{aligned}\right\} \tag{13}$$

The boundary conditions become (all functions taken at $r = a$)

$$\left.\begin{aligned}
\frac{U}{i(s\omega - \sigma)}\left\{\zeta^2 T + \frac{s^2-1}{a^2}\,T - \rho a\omega^2\right\} - \rho q + 2\mu U_r &= 0\\
isU - V + V_r a &= 0\\
i\zeta U + W_r &= 0.
\end{aligned}\right\} \tag{14}$$

Dimensionless quantities can be introduced, which also simplify the equations. We define

$$R \equiv \zeta r,$$

and then

$$U_r = \zeta\,\frac{dU}{dR} \equiv \zeta U';$$

$$\alpha \equiv \frac{i(s\omega - \sigma)}{\nu\zeta^2},$$

$$\omega_0 \equiv \frac{\omega}{\nu\zeta^2},$$

and

$$Q \equiv \frac{q}{\nu\zeta}.$$

We further observe that it is useful to introduce the combinations $X \equiv U + iV$ and $Y \equiv U - iV$, in terms of which the first two equations of motion are recombined, to give one equation which involves only X and Q, while the other involves only Y and Q.

The final form of the equations of motion, in terms of the newly defined R, α, ω_0, X and Y, becomes

$$(1 + \alpha + 2i\omega_0)X = \left[-Q' + \frac{s}{R}Q\right] + \left[X'' + \frac{1}{R}X' - \frac{(s+1)^2}{R^2}X\right], \tag{A}$$

$$(1 + \alpha - 2i\omega_0)Y = \left[-Q' - \frac{s}{R}Q\right] + \left[Y'' + \frac{1}{R}Y' - \frac{(s-1)^2}{R^2}Y\right], \tag{B}$$

$$(1 + \alpha)W = -iQ + \left[W'' + \frac{1}{R}W' - \frac{s^2}{R^2}W\right], \tag{C}$$

$$-2iW = \left[X' + \frac{s+1}{R}X\right] + \left[Y' - \frac{s-1}{R}Y\right]. \tag{D}$$

The boundary conditions become ($R \equiv R_{r=a} = a$)

$$0 = 2U' - Q + U\frac{R_0\omega_0^2}{\alpha}\{M[R_0^2 + s^2 - 1] - 1\} \tag{E}$$

$$0 = isU - V + R_0V' \tag{F}$$

$$0 = W' + iU, \tag{G}$$

where all functions are taken at $r = a$, M being defined as the dimensionless quantity $T/\rho\omega^2a^3$.

5. Solution of the transformed equations of motion. The differential operator which appears in (A), (B) and (C), i.e.,

$$\left(\frac{d}{dR^2} + \frac{1}{R}\frac{d}{dR} - \frac{m^2}{R^2}\right),$$

suggests that we might try a solution in terms of Bessel functions. We start with (C), and assume $W = J_s(kR)$, where k is a constant to be determined. It follows that

$$Q = Q_0J_s(kR),$$

where

$$Q_0 = i(1 + \alpha + k^2) \equiv i\psi.$$

Hence

$$-Q' + \frac{s}{R}Q = ik\psi\left\{-\frac{d}{d(kR)}[J_s(kR)] + \frac{sJ_s(kR)}{kR}\right\} = ik\psi J_{s+1}(kR)$$

and similarly,

$$-Q' - \frac{s}{R}Q = -ik\psi J_{s-1}(kR).$$

Using these in equations (A) and (B) respectively, we see that (A) may be satisfied by setting

$$X = X_0 J_{s+1}(kR),$$

and (B) by setting

$$Y = Y_0 J_{s-1}(kR).$$

In particular, we have from (A)

$$X_0(1 + \alpha + 2i\omega_0) = ik\psi - X_0 k^2, \quad \text{or} \quad X_0 = \frac{ik\psi}{\psi + 2i\omega_0}$$

and from (B)

$$Y_0(1 + \alpha - 2i\omega_0) = -ik\psi - Y_0 k^2 \quad \text{or} \quad Y_0 = \frac{-ik\psi}{\psi - 2i\omega_0}.$$

The solution is then

$$\begin{aligned} W &= J_s(kR), \\ Q &= i\psi J_s(kR), \\ X &= \frac{ik\psi}{\psi + 2i\omega_0} J_{s+1}(kR), \\ Y &= \frac{-ik\psi}{\psi - 2i\omega_0} J_{s-1}(kR). \end{aligned}$$

These functions satisfy $(A) - (C)$ for all values of k. Substituting them in (D), we get

$$-2i = k(X_0 - Y_0) = ik^2\psi \frac{2\psi}{\psi^2 + 4\omega_0^2} = \frac{2i\psi^2(\psi - 1 - \alpha)}{\psi^2 + 4\omega_0^2}.$$

This is a cubic equation for ψ

$$\psi^3 - \alpha\psi^2 + 4\omega_0^2 = 0. \tag{15}$$

For each root $\psi_j (j = 1, 2, 3)$ we obtain a value of $k_j = (\psi_j - 1 - \alpha)^{1/2}$. Thus we have three independent systems of solutions deriving from

$$W_1 = J_s(k_1R), \qquad W_2 = J_s(k_2R), \quad \text{and} \quad W_3 = J_s(k_3R).$$

The general solution will be a linear combination of these

$$W = \lambda_1 J_s(k_1R) + \lambda_2 J_s(k_2R) + \lambda_3 J_s(k_3R),$$

with a corresponding form for Q, U, V. The boundary conditions (E), (F), (G) will now constitute a set of three homogeneous equations in the $\lambda_j (j = 1, 2, 3)$, so that a non-trivial solution can exist if and only if the determinant of the coefficients of λ_j vanishes. Thus we obtain the secular equation

$$\begin{vmatrix} 2U_1' - Q_1 + \beta U_1 & isU_1 - V_1 + R_0V_1' & W_1' + iU \\ 2U_2' - Q_2 + \beta U_2 & isU_2 - V_2 + R_0V_2' & W_2' + iU \\ 2U_3' - Q_3 + \beta U_3 & isU_3 - V_3 + R_0V_3' & W_3' + iU \end{vmatrix} = 0 \tag{16}$$

where we have used the abbreviation

$$\beta \equiv \frac{R_0\omega_0^2}{\alpha}\{M[R_0^2 + s^2 - 1] - 1\}. \tag{17}$$

6. Conditions for Stability. The relevant physical variables are M, R_0, ω_0, s and so also, by (17), is $\alpha\beta$. We can rewrite (16) as

$$N + \beta D = 0, \tag{18}$$

where

$$N \equiv \begin{vmatrix} 2U_1' - Q_1 & isU_1 - V_1 + R_0V_1' & W_1' + iU_1 \\ 2U_2' - Q_2 & isU_2 - V_2 + R_0V_2' & W_2' + iU_2 \\ 2U_3' - Q_3 & isU_3 - V_3 + R_0V_3' & W_3' + iU_3 \end{vmatrix} \tag{19}$$

and

$$D \equiv \begin{vmatrix} U_1 & R_0V_1' - V_1 & W_1' \\ U_2 & R_0V_2' - V_2 & W_2' \\ U_3 & R_0V_3' - V_3 & W_3' \end{vmatrix} \tag{20}$$

The determinants N, D depend on σ via the k_i's. Equation (17) may be again rewritten as

$$\frac{\alpha N}{D} = -R_0\omega_0^2[M(R_0^2 + s^2 - 1) - 1] \tag{21}$$

from which σ is to be determined. A set of physical variables will correspond to stable motion if and only if all the roots of (21) satisfy $\mathcal{I}(\sigma) \leq 0$. Since $\alpha = i(s\omega - \sigma)/\nu\zeta^2$, this is the same as $\mathcal{R}(\alpha) \leq 0$.

However, finding all the roots of the transcendental equation is a hopeless task. Instead, we invert the problem. In principle, such an inversion implies that we should scan the half-plane $\mathcal{R}(\alpha) > 0$, and for each point determine all sets of the physical variables to which this value of α corresponds. The aggregate of all such sets of physical variables would then define precisely the aggregate of unstable motions. In practice, the following procedure was adopted. For fixed R_0, ω_0, s we wrote $\alpha = \gamma + i\delta$, with $\gamma > 0$. Each pair (γ, δ) determined, by (21) and (17), a value of $M \equiv M(\alpha)$. In general this value was complex, and hence without physical meaning. However, for each fixed γ there was found to exist a unique $\delta(\gamma) = \delta(\gamma, R_0, \omega_0, s)$ such that M is real. This special value of M was denoted by $M(\gamma)$.

For typical sets R_0, ω_0, s the functions $\delta(\gamma)$, $M(\gamma)$ were computed for a range of values of γ. It transpired that

(i) for $s = 0$, $R_0 < 1$, $M(\gamma)$ increased idefinitely with γ for $\gamma > 0$;

$$\delta(\gamma) = 0.$$

(ii) for all other cases, $M(\gamma)$ decreased as γ increased;

$\delta(\gamma)$ increased from zero as γ increased.

Both of these results concerning $M(\gamma)$ could have been predicted on physical grounds. It was pointed out by Rayleigh [1] that a small perturbation with $s = 0$, $R_0 < 1$ actually produces a decrease in the total surface area and hence releases surface energy. In case (i), then, the surface tension has a destabilizing effect so that one might expect that larger values of T would correspond to larger values of $\mathfrak{s}(\sigma)$, i.e., $M(\gamma)$ increases with γ. In all other cases, however, the effect of surface tension is in the direction of stabilizing the motion. Hence one might expect that larger values of T, and so of M, should correspond to smaller values of $\mathfrak{s}(\sigma)$, and hence of γ.

Concentrating, therefore, on case (ii), we see that if we let $\alpha = \gamma + i\delta$ tend to zero through those values which give a physically meaningful value of M, then M attains its highest value at $\alpha = 0$. We deduce that the necessary and sufficient condition for stability is that

$$M \geq \lim_{\alpha \to 0} M(\alpha) \equiv M_0 . \tag{22}$$

As to the value of this limit, we found from the computation that N/D remained finite as $\alpha \to 0$, and so

$$\lim_{\alpha \to 0} \alpha \frac{N}{D} = 0. \tag{23}$$

Hence, by (21),

$$M_0 = \frac{1}{R_0^2 + s^2 - 1} = \frac{1}{a^2\zeta^2 + s^2 - 1}, \tag{24}$$

recalling the definition of R_0. Relations (22) and (24) imply (1).

For the limiting case $R_0 \ll 1$, one can obtain adequate approximations to N and D by expanding the Bessel functions. Equation (21) for α then becomes algebraic, and its roots can be determined quite simply. This was done, and condition (1) for stability was confirmed. An interesting special case of this is $s = 1$, where the condition becomes $M \geq R_0^{-2}$. For $s > 1$, we find $M \geq (s^2 - 1)^{-1} + 0(R_0^2)$. These results are of course consistent with (1).

All of the computational work referred to above was executed on WEIZAC, the digital electronic computer of the Weizmann Institute of Science.

Note added in proof. (a) It should be pointed out that the stability condition (1) is unaffected if the unperturbed motion has a steady axial component W_0 in addition to the rotational velocity. It is easily seen from (2) that the only change necessary in (13) is to replace the expression $(s\omega - \sigma)$ in all three of its occurrences by $(s\omega - \sigma + \zeta W_0)$, and hence a redefinition of α in Sec. 4 as

$$\alpha = \frac{i}{\nu\zeta^2}(s\omega - \sigma + \zeta W_0).$$

Since $R(\alpha)$ is unaffected by this change the stability condition is unaltered.

(b) The fact noted in Sec. 6 that $\lim_{\gamma \to 0} \delta(\gamma) = 0$ is an example of the Pellew-Southwell principle of exchange of stability [see Proc. Roy. Soc. **A 176**, 312 (1940)].

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