Critical Exponents of the Four-State Potts Model

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Abstract

The critical exponents of the four-state Potts model are directly derived from the exact expressions for the latent heat, the spontaneous magnetization, and the correlation length at the transition temperature of the model. PACS numbers: 05.50.+q, 05.70.-a, 64.60.Cn, 75.10.Hk

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The q-state Potts model [1,2] is a generalization of the Ising model that is the twostate Potts model. Although the Potts model has not been solved exactly, there have been several exact results at the critical point for this model in two dimensions. In 1952 Potts [1] conjectured the exact critical temperatures of his model on the square lattice for all q by a Kramers-Wannier [3] type duality argument. In 1971 Temperley and Lieb [4] showed that the Potts model can be expressed as a staggered six-vertex model. Following the equivalence [5] between the Potts model and a staggered six-vertex model, in 1973 Baxter [6] calculated the free energy of the Potts model at the critical temperature, and showed that the model has a continuous phase transition for $q \leq 4$, and has a first-order phase transition (i.e. has latent heat) for q > 4. In 1979 den Nijs [7] conjectured the thermal scaling exponent for $q \leq 4$ by considering relation between the eight-vertex model and the Potts model. In 1980 Nienhuis et al. [8] and Pearson [9] conjectured independently the magnetic scaling exponent for $q \leq 4$ from numerical results. In 1981 Black and Emery [10] showed the den Nijs conjecture to be asymptotically exact by using the Coulomb-gas representation [11] of the Potts model and renormalization-group methods. In 1982 Baxter [12] calculated the spontaneous magnetization of the model at the transition point for q > 4. In 1983 den Nijs [13] verified a conjecture for the magnetic scaling exponent for $q \leq 4$ from the scaling behavior of the correlation function in the Coulomb-gas representation. In 1984 Dotsenko [14] again verified the conjectures for the thermal and magnetic scaling exponents for $q \leq 4$ using conformal field theory. Recently Buffernoir and Wallon [15] obtained an exact expression for the correlation length of the Potts model at the critical temperature for q > 4 by using Temperley-Lieb algebra [5] and a Bethe ansatz [16]. In this paper we derive the critical exponents of the four-state Potts model directly from the three main exact results of the Potts model which are Baxter's calculation of the latent heat and the spontaneous magnetization and Buffernoir and Wallon's calculation of the correlation length.

The Hamiltonian for the q-state Potts model on the isotropic square lattice is

$$\mathcal{H} = -J \sum_{\langle i,j \rangle} \delta(\sigma_i, \sigma_j), \tag{1}$$

where

$$\delta(\sigma_i, \sigma_j) = \begin{cases} 1, & \text{if } \sigma_i = \sigma_j, \\ 0, & \text{if } \sigma_i \neq \sigma_j, \end{cases}$$

and $\sigma_i, \sigma_j = 1, 2, ..., q$. The latent heat at the critical temperature T_c is given by [6]

$$L = 2J(1+q^{-\frac{1}{2}})\tanh\frac{\theta}{2}\prod_{n=1}^{\infty}(\tanh n\theta)^2, q > 4,$$
(2)

where θ is defined by

$$2\cosh\theta = q^{\frac{1}{2}}.$$

The zero-field (spontaneous) magnetization at T_c is [12]

$$M_0 = \prod_{n=1}^{\infty} \frac{1 - x^{2n-1}}{1 + x^{2n}}, q > 4,$$
(3)

where $x = e^{-2\theta}$. The correlation length, ξ , at T_c is given by [15]

$$\xi^{-1} = 2\ln\frac{\cosh\frac{3}{2}v}{\cosh\frac{1}{2}v} + 4\sum_{n=1}^{\infty}\frac{(-1)^n}{n}e^{-2nv}(\sinh nv)(\tanh 2nv), q > 4,\tag{4}$$

where v is defined by

$$2\cosh v = (2+q^{\frac{1}{2}})^{\frac{1}{2}}.$$

The behaviors of the latent heat, the zero-field magnetization, and the correlation length near the limit q = 4 are expressed by the following:

$$L \sim 2\pi J (1 + q^{-\frac{1}{2}}) e^{-\frac{\pi^2}{2}(q-4)^{-\frac{1}{2}}},$$
(5)

$$M_0 \sim 2e^{-\frac{\pi^2}{8}(q-4)^{-\frac{1}{2}}},$$
 (6)

and

$$\xi^{-1} \sim \frac{8}{\sqrt{2}} e^{-\pi^2 (q-4)^{-\frac{1}{2}}}.$$
(7)

All three expressions have the same limiting behavior.

In the spirit of scaling theory, the singular behavior in the latent heat and the spontaneous magnetization can be expressed in terms of the correlation length:

$$L \sim \xi^{-\frac{1}{2}},\tag{8}$$

and

$$M_0 \sim \xi^{-\frac{1}{8}}.$$
 (9)

Note that in (8) the latent heat and in (9) the spontaneous magnetization vanish as $q-4 \rightarrow 0^+$. The four-state Potts model can be considered the critical end point of a sequence of models (q > 4) with finite latent heats and finite spontaneous magnetizations. From standard scaling theory [17], the internal energy per site is

$$e(t, h = 0) = \xi^{-d + \frac{1}{\nu}} f_t(t\xi^{\frac{1}{\nu}}, h = 0),$$

where the subscript, t, means differentiation with respect to the reduced temperature. The latent heat is given by

$$L = \xi^{-d + \frac{1}{\nu}} \Delta f_t,$$

where $\Delta f_t = f_t(0^+, 0) - f_t(0^-, 0)$. Near the second-order transition point, i.e., near the limit q = 4, we have

$$L \sim \xi^{-d+\frac{1}{\nu}}.\tag{10}$$

Comparing (8) and (10), we obtain

$$d - \frac{1}{\nu} = \frac{1}{2}.$$
 (11)

Similarly, for the spontaneous magnetization per spin

$$M_0 = \xi^{-d+y_h} f_h(0,0)$$

= $\xi^{-\frac{\beta}{\nu}} f_h(0,0).$ (12)

Comparing this with (9) we see that

$$\frac{\beta}{\nu} = \frac{1}{8}.\tag{13}$$

For d = 2, from (11) and (13), the exact values for the scaling and critical exponents of the four-state Potts model are $y_t = \frac{3}{2}$, $y_h = \frac{15}{8}$, $\alpha = \frac{2}{3}$, $\beta = \frac{1}{12}$, $\gamma = \frac{7}{6}$, $\delta = 15$, $\nu = \frac{2}{3}$, and $\eta = \frac{1}{4}$ in agreement with values [18] derived for $q \leq 4$ from the Coulomb-gas representation [10,11,13] and conformal field theory [14].

The critical properties of the four-state Potts model have been studied extensively as the limiting case of a sequence $(q \leq 4)$ of models with continuous phase transitions. As is often the case, the limit of such a sequence, q = 4, exhibits strong corrections to scaling. It is, therefore, of interest to approach this problem from the opposite side, and regard the four-state Potts model as the limit of a sequence of models (q > 4) with a discontinuous, or first-order, transition. As $q \to 4^+$, the latent heat and spontaneous magnetization at T_c vanish, and the correlation length diverges. We have shown that by applying simple scaling arguments to exact calculations of L, M_0 , and ξ at T_c , one can derive the exact critical exponents and that they agree with those obtained for $q \leq 4$.

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- [18] See, for example, R. J. Baxter in Ref. [2], p. 352; F. Y. Wu in Ref. [2], p. 261. In Table V of p. 261, $\eta = \frac{1}{2}$ for q = 4 is a misprint.