Mixmaster: Fact and Belief

J. Mark Heinzle*
Gravitational Physics, Faculty of Physics,
University of Vienna, A-1090 Vienna, Austria
and
Mittag-Leffler Institute of the Royal Swedish Academy of Sciences
S-18260 Djursholm, Sweden

Claes Uggla†
Department of Physics,
University of Karlstad, S-651 88 Karlstad, Sweden

Abstract

We consider the dynamics towards the initial singularity of Bianchi type IX vacuum and orthogonal perfect fluid models with a linear equation of state. Surprisingly few facts are known about the 'Mixmaster' dynamics of these models, while at the same time most of the commonly held beliefs are rather vague. In this paper, we use Mixmaster facts as a base to build an infrastructure that makes it possible to sharpen the main Mixmaster beliefs. We formulate explicit conjectures concerning (i) the past asymptotic states of type IX solutions and (ii) the relevance of the Mixmaster/Kasner map for generic past asymptotic dynamics. The evidence for the conjectures is based on a study of the stochastic properties of this map in conjunction with dynamical systems techniques. We use a dynamical systems formulation, since this approach has so far been the only successful path to obtain theorems, but we also make comparisons with the 'metric' and Hamiltonian 'billiard' approaches.

*Electronic address: `mark.heinzle@univie.ac.at`
†Electronic address: `claes.uggla@kau.se`

1 Introduction

Today, Bianchi type IX enjoys an almost mythical status in general relativity and cosmology, which is due to two commonly held beliefs: (i) Type IX dynamics is believed to be essentially understood; (ii) Bianchi type IX is believed to be a role model that captures the generic features of generic spacelike singularities. However, we will illustrate in this paper that there are reasons to question these beliefs.

The idea that type IX is essentially understood is a misconception. In actuality, surprisingly little is known, i.e., proved, about type IX asymptotic dynamics; at the same time there exist widely held, but rather vague, beliefs about Mixmaster dynamics, oscillations, and chaos, which are frequently mistaken to be facts. There is thus a need for clarification: What are the known facts and what is merely believed about type IX asymptotics? We will address this issue in two ways: On the one hand, we will discuss the main rigorous results on Mixmaster dynamics, the 'Bianchi type IX attractor theorem', and its consequences; in particular, we will point out the limitations of these results. On the other hand, we will provide the infrastructure that makes it possible to sharpen commonly held beliefs; based on this framework we will formulate explicit refutable conjectures.

Historically, Bianchi type IX vacuum and orthogonal perfect fluid models entered the scene in the late sixties through the work of Belinskii, Khalatnikov and Lifshitz[1] [1, 2] and Misner and Chitré [3, 4, 5, 6]. BKL attempted to understand the detailed nature of singularities and were led to the type IX models via a rather convoluted route, while Misner was interested in mechanisms that could explain why the Universe today is almost isotropic. BKL and Misner independently, by means of quite different methods, reached the conclusion that the temporal behavior of the type IX models towards the initial singularity can be described by sequences of anisotropic Kasner states, i.e., Bianchi type I vacuum solutions. These sequences are determined by a discrete map that leads to an oscillatory anisotropic behavior, which motivated Misner to refer to the type IX models as Mixmaster models [3, 4]. This discrete map, the Kasner map, was later shown to be associated with stochasticity and chaos [7, 8, 9], a property that has generated considerable interest—and confusion, see, e.g., [10, 11, 12, 13, 14, 15, 16, 17, 18, 19, 20] and references therein. A sobering thought: All claims about chaos in Einstein's equations rest on the (plausible) belief that the Kasner map actually describes the asymptotic dynamics of Einstein's equations; as will be discussed below, this is far from evident (despite being plausible) and has not been proved so far.

More than a decade after BKL's and Misner's investigations a new development took place: Einstein's field equations in the spatially homogeneous (SH) case were reformulated in a manner that allowed one to apply powerful dynamical systems techniques [21, 22, 23]; gradually a picture of a hierarchy of invariant subsets emerged where monotone functions restricted the asymptotic dynamics to boundaries of boundaries, see [10] and references therein. Based on work reviewed and developed in [10] and by Rendall [24], Ringström eventually produced the first major proofs about asymptotic type IX dynamics [25, 26]. This achievement is remarkable, but it does not follow that all questions are settled. On the contrary, so far nothing is rigorously known, e.g., about dynamical chaotic properties (although there are good grounds for beliefs), nor has the role of type IX models in the context of generic singularities been established [27, 28, 29, 30].

The outline of the paper is as follows. In Section 2 we briefly describe the Hubble-normalized dynamical systems approach and establish the connection with the metric approach. For simplicity we restrict ourselves to the vacuum case and the so-called orthogonal perfect fluid case, i.e., the fluid flow is orthogonal w.r.t. the SH symmetry surfaces; furthermore, we assume a linear equation of state. In Section 3 we discuss the levels of the Bianchi type IX so-called

[1] We will refer to the authors and their work as BKL.

Lie contraction hierarchy of subsets, where we focus on the Bianchi type I and type II subsets. In Section 4 we present the results of the local analysis of the fixed points of the dynamical system and discuss the stable and unstable subsets of these points which are associated with non-generic asymptotically self-similar behavior. Section 5 is devoted to a study of the network of sequences of heteroclinic orbits (heteroclinic chains) that is induced by the dynamical system on the closure of the Bianchi type II vacuum boundary of the type IX state space (which we refer to as the Mixmaster attractor subset). These sequences of orbits are associated with the Mixmaster map, which in turn induces the Kasner map and thus the Kasner sequences. We analyze the properties of non-generic Kasner sequences and discuss the stochastic properties of generic sequences. In Section 6 we discuss the main 'Mixmaster facts': Ringström's 'Bianchi type IX attractor theorem' [25, 26], Theorem 6.1, and a number of consequences that follow from Theorem 6.1 and from the results on the Mixmaster/Kasner map. In addition, we introduce and discuss the concept of 'finite Mixmaster shadowing'. In the subsection 'Attractor beliefs' of Section 7 we formulate two conjectures that reflect commonly held beliefs about type IX asymptotic dynamics and list some open issues that are directly connected with these conjectures. In the subsection 'Stochastic beliefs' we address the open question of which role the Mixmaster/Kasner map and its stochastic properties actually play in type IX asymptotic dynamics. This culminates in the formulation, and discussion, of two 'stochastic' conjectures. In Section 8 we present the Hamiltonian billiard formulation, see [5, 6] or [31]; we demonstrate that this approach yields a 'dual' formulation of the asymptotic dynamics. We point out that the billiard approach is a formidable heuristic picture, but fails to turn beliefs into facts. We conclude in Section 9 with a discussion of the main themes of this paper. Throughout this paper we use units so that $c = 1$ and $8\pi G = 1$, where c is the speed of light and G the gravitational constant.

2 Basic equations

We consider vacuum or orthogonal perfect fluid SH Bianchi type IX models (i.e., the fluid 4-velocity is assumed to be orthogonal to the SH symmetry surfaces) with a linear equation of state; we require the energy conditions (weak/strong/dominant) to hold, i.e., $\rho > 0$ and

$$-\tfrac{1}{3} < w < 1\,, \tag{1}$$

where $w = p/\rho$, and where ρ and p are the energy density and pressure of the fluid, respectively. By (1) we exclude the special cases $w = -\frac{1}{3}$ and $w = 1$, where the energy conditions are only marginally satisfied.[2]

As is well known, see, e.g., [10, 26] and references therein, for these models there exists a symmetry-adapted (co-)frame $\{\hat{\boldsymbol{\omega}}^1, \hat{\boldsymbol{\omega}}^2, \hat{\boldsymbol{\omega}}^3\}$,

$$d\hat{\boldsymbol{\omega}}^1 = -\hat{n}_1\, \hat{\boldsymbol{\omega}}^2 \wedge \hat{\boldsymbol{\omega}}^3\,, \quad d\hat{\boldsymbol{\omega}}^2 = -\hat{n}_2\, \hat{\boldsymbol{\omega}}^3 \wedge \hat{\boldsymbol{\omega}}^1\,, \quad d\hat{\boldsymbol{\omega}}^3 = -\hat{n}_3\, \hat{\boldsymbol{\omega}}^1 \wedge \hat{\boldsymbol{\omega}}^2\,, \tag{2a}$$

with $\hat{n}_1 = 1$, $\hat{n}_2 = 1$, $\hat{n}_3 = 1$, such that the type IX metric takes the form

$${}^4\mathbf{g} = -dt \otimes dt + g_{11}(t)\, \hat{\boldsymbol{\omega}}^1 \otimes \hat{\boldsymbol{\omega}}^1 + g_{22}(t)\, \hat{\boldsymbol{\omega}}^2 \otimes \hat{\boldsymbol{\omega}}^2 + g_{33}(t)\, \hat{\boldsymbol{\omega}}^3 \otimes \hat{\boldsymbol{\omega}}^3\,. \tag{2b}$$

[2]Note that the well-posedness of the Einstein equations (for solutions without symmetry) has been questioned in the case $-1/3 < w < 0$, see [32]. The case $w = 1$ is known as the stiff fluid case, for which the speed of sound is equal to the speed of light. The asymptotic dynamics of stiff fluid solutions is simpler than the oscillatory behavior characterizing the models with range $-\frac{1}{3} < w < 1$, and well understood [26, 33]. (In the terminology introduced below, the stiff fluid models are asymptotically self-similar.) We will therefore refrain from discussing the stiff fluid case in this paper.

Bianchi type	$\hat{n}_\alpha$	$\hat{n}_\beta$	$\hat{n}_\gamma$
I	0	0	0
II	0	0	$+$
VI_0	0	$-$	$+$
VII_0	0	$+$	$+$
VIII	$-$	$+$	$+$
IX	$+$	$+$	$+$

Table 1: The class A Bianchi types are characterized by different signs of the structure constants $(\hat{n}_\alpha, \hat{n}_\beta, \hat{n}_\gamma)$, where $(\alpha\beta\gamma)$ is any permutation of (123). In addition to the above representations there exist equivalent representations associated with an overall change of sign of the structure constants; e.g., another type IX representation is $(- - -)$.

Hence, the type IX models naturally belong to the so-called class A Bianchi models, see Table 1. Let

$$n_1(t) := \hat{n}_1 \frac{g_{11}}{\sqrt{\det g}}\,, \quad n_2(t) := \hat{n}_2 \frac{g_{22}}{\sqrt{\det g}}\,, \quad n_3(t) := \hat{n}_3 \frac{g_{33}}{\sqrt{\det g}}\,, \tag{3}$$

where $\det g = g_{11}g_{22}g_{33}$. Furthermore, define

$$\theta = -\operatorname{tr} k \qquad \text{and} \qquad \sigma^\alpha{}_\beta = -k^\alpha{}_\beta + \tfrac{1}{3}\operatorname{tr} k\, \delta^\alpha{}_\beta = \operatorname{diag}(\sigma_1, \sigma_2, \sigma_3) \quad \Big(\Rightarrow \textstyle\sum_\alpha \sigma_\alpha = 0\Big)\,, \tag{4}$$

where $k_{\alpha\beta}$ denotes the second fundamental form associated with (2) of the SH hypersurfaces $t = \text{const}$. The quantities θ and $\sigma_{\alpha\beta}$ can be interpreted as the expansion and the shear, respectively, of the normal congruence of the SH hypersurfaces. In a cosmological context it is customary to replace θ by the Hubble variable $H = \theta/3 = -\operatorname{tr} k/3$; this variable is related to changes of the spatial volume density according to $d\sqrt{\det g}/dt = 3H\sqrt{\det g}$. Evidently, in Bianchi type IX (and type VIII) there is a one-to-one correspondence between the 'orthonormal frame variables' $(H, \sigma_\alpha, n_\alpha)$ (with $\sum_\alpha \sigma_\alpha = 0$) and $(g_{\alpha\beta}, k_{\alpha\beta})$; in particular, the metric $g_{\alpha\beta}$ is obtained from (n_1, n_2, n_3) via (3). (For the lower Bianchi types I–VII_0, some of the variables (n_1, n_2, n_3) are zero, cf. (3); in this case, the other frame variables, i.e., (H, σ_α), are needed as well to reconstruct the metric; see [34] for a group theoretical approach.)

In the Hubble-normalized dynamical systems approach we define dimensionless orthonormal frame variables according to

$$(\Sigma_\alpha, N_\alpha) = (\sigma_\alpha, n_\alpha)/H\,, \qquad \Omega = \rho/(3H^2)\,. \tag{5}$$

In addition we introduce a new dimensionless time variable τ according to $d\tau/dt = H$. Like the cosmological time t, the time τ is directed towards the future; however, to make contact with the well established convention that uses a past-directed 'time' for the discrete Mixmaster map, see Section 5, it will occasionally become necessary to use an inverse time $\tau_- = -\tau$ instead of τ itself.

For all class A models except type IX the Gauss constraint guarantees that H remains positive if it is positive initially. In Bianchi type IX, however, it is known from a theorem by Lin and Wald [35] that all type IX vacuum and orthogonal perfect fluid models with $w \geq 0$ first expand ($H > 0$), reach a point of maximum expansion ($H = 0$), and then recollapse ($H < 0$).[3] The variable transformation (5) breaks down at the point of maximum expansion in the type IX case; however, the variables $(\Sigma_\alpha, N_\alpha)$ correctly describe the dynamics in the expanding phase, which we will focus on henceforth.

[3] In the locally rotationally symmetric case it has been proved that the range of w can be extended to $w > -\frac{1}{3}$, see [36]. There are good reasons to believe that the assumption of local rotational symmetry is superfluous, but this has not been established yet.

When the Einstein field equations are reformulated in terms of H and the Hubble-normalized variables $(\Sigma_\alpha, N_\alpha)$ it follows for dimensional reasons that the equation for the single variable with dimension, H,

$$H' = -(1+q)H \,, \tag{6}$$

decouples from the remaining dimensionless equations [10]; here and henceforth a prime denotes the derivative $d/d\tau$. The equations for $(\Sigma_\alpha, N_\alpha)$ form the following coupled system [10]:

$$\Sigma'_\alpha = -(2-q)\Sigma_\alpha - {}^3\!S_\alpha \,, \tag{7a}$$

$$N'_\alpha = (q + 2\Sigma_\alpha)\, N_\alpha \qquad \text{(no sum over } \alpha) \,, \tag{7b}$$

where

$$q = 2\Sigma^2 + \tfrac{1}{2}(1+3w)\Omega \,, \qquad \Sigma^2 = \tfrac{1}{6}(\Sigma_1^2 + \Sigma_2^2 + \Sigma_3^2) \,, \tag{8a}$$

$$\text{and} \quad {}^3\!S_\alpha = \tfrac{1}{3}\left[N_\alpha(2N_\alpha - N_\beta - N_\gamma) - (N_\beta - N_\gamma)^2\right], \quad (\alpha\beta\gamma) \in \{(123),(231),(312)\} \,. \tag{8b}$$

Note that the 'deceleration parameter' q is non-negative because of the assumption $w > -1/3$.

Apart from the trivial constraint $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$, there exists the Gauss constraint

$$\Sigma^2 + \tfrac{1}{12}\Big[N_1^2 + N_2^2 + N_3^2 - 2\underbrace{(N_1N_2 + N_2N_3 + N_3N_1)}_{\Delta_{\mathrm{II}}}\Big] + \Omega = 1 \,, \tag{9}$$

which is used to globally solve for Ω when $\Omega \neq 0$. Accordingly, the reduced state space is given as the space of all $(\Sigma_1, \Sigma_2, \Sigma_3)$ and (N_1, N_2, N_3) such that $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ and

$$\Sigma^2 + \tfrac{1}{12}\Big[N_1^2 + N_2^2 + N_3^2 - 2\,(N_1N_2 + N_2N_3 + N_3N_1)\Big] \leq 1 \,, \tag{9$'$}$$

which follows from (9) under the assumption that $\Omega \geq 0$ ($\rho \geq 0$). It follows that the dimensionless state space of the Bianchi type IX orthogonal perfect fluid models with a linear equation of state is 5-dimensional,[4] while the state space of the vacuum models (i.e., $\Omega = 0$) is 4-dimensional. The same is true for Bianchi type VIII, while the state spaces of the remaining class A Bianchi models have less degrees of freedom; see Table 2. Once the dynamics in the dimensionless state space is understood, H is obtained from a quadrature by integrating (6), which allows one to reconstruct the metric.

For all class A models except type IX the constraint (9$'$) implies that $\Sigma^2 \leq 1$; in Bianchi type IX, however, $\Sigma^2 > 1$ is possible. For type IX, define

$$\Delta = \tfrac{1}{4}(N_1N_2N_3)^{2/3} \,. \tag{10}$$

Employing (8a) and (9) and using that

$$\tfrac{1}{12}\Big[N_1^2 + N_2^2 + N_3^2 - 2\Delta_{\mathrm{II}}\Big] + \Delta \geq 0 \,, \tag{11}$$

where equality holds iff $N_1 = N_2 = N_3$, we find

$$\Sigma^2 \leq 1 + \Delta \,, \qquad \Omega \leq 1 + \Delta \,, \qquad 2 - q \geq \tfrac{3}{2}(1-w)\Omega - 2\Delta \,. \tag{12}$$

The function Δ is strictly monotonically increasing along orbits of Bianchi type IX. To see this we use (7) and compute

$$\Delta' = 2q\Delta \,, \qquad \Delta''\Big|_{q=0} = 0 \,, \qquad \Delta'''\Big|_{q=0} = \tfrac{4}{3}\left[{}^3\!S_1^2 + {}^3\!S_2^2 + {}^3\!S_3^2\right]\Delta \,, \tag{13}$$

[4] It is common to globally solve $\Sigma_1 + \Sigma_2 + \Sigma_3 = 0$ by introducing new variables according to $\Sigma_1 = -2\Sigma_+$, $\Sigma_2 = \Sigma_+ - \sqrt{3}\Sigma_-$, $\Sigma_3 = \Sigma_+ + \sqrt{3}\Sigma_-$, which yields $\Sigma^2 = \Sigma_+^2 + \Sigma_-^2$. However, since this breaks the permutation symmetry of the three spatial axes (exhibited by type IX models), we choose to retain the variables Σ_1, Σ_2, Σ_3.

Bianchi type	Symbol	Range of $(N_\alpha, N_\beta, N_\gamma)$	State space	D
I	$\boldsymbol{B}_{\mathrm{I}}$	$N_\alpha = 0,\ N_\beta = 0,\ N_\gamma = 0$	$\Sigma^2 \leq 1$	2
II	$\boldsymbol{B}_{\mathrm{II}}$	$N_\alpha = 0,\ N_\beta = 0,\ N_\gamma > 0$	$\Sigma^2 + \frac{1}{12}N_\gamma^2 \leq 1$	3
VI$_0$	$\boldsymbol{B}_{\mathrm{VI}_0}$	$N_\alpha = 0,\ N_\beta < 0,\ N_\gamma > 0$	$\Sigma^2 + \frac{1}{12}[N_\beta - N_\gamma]^2 \leq 1$	4
VII$_0$	$\boldsymbol{B}_{\mathrm{VII}_0}$	$N_\alpha = 0,\ N_\beta > 0,\ N_\gamma > 0$	$\Sigma^2 + \frac{1}{12}[N_\beta - N_\gamma]^2 \leq 1$	4
VIII	$\boldsymbol{B}_{\mathrm{VIII}}$	$N_\alpha < 0,\ N_\beta > 0,\ N_\gamma > 0$	Eq. $(9')$ $(\Rightarrow \Sigma^2 < 1)$	5
IX	$\boldsymbol{B}_{\mathrm{IX}}$	$N_\alpha > 0,\ N_\beta > 0,\ N_\gamma > 0$	Eq. $(9')$ $(\Rightarrow \Sigma^2 \leq 1 + \Delta)$	5

Table 2: The dimensionless state spaces associated with class A Bianchi models; here, $(\alpha\beta\gamma)$ is any permutation of (123). In addition to the above representations there exist equivalent representations associated with an overall change of sign of the variables (N_1, N_2, N_3). The quantity D denotes the dimension of the state space (in the fluid case); the dimensionality of the state space in the vacuum cases is given by $\mathsf{D} - 1$.

where we note that ${}^3S_1^2 + {}^3S_2^2 + {}^3S_3^2 > 0$ because of the constraints. In combination with (12) it follows that Σ^2 and Ω are *bounded towards the past*.

The right hand side of the reduced system (7) consists of polynomials of the state space variables and is thus a regular dynamical system. Solutions of (7) of Bianchi types I–VIII are global in τ, since (9) implies the bounds $\Sigma^2 \leq 1$ and $\Omega \leq 1$ which control the evolution of N_α in (7b). Solutions of (7) of Bianchi type IX are global towards the past, since $q + 2\Sigma_\alpha$ is bounded from below; this follows from (8a), which yields $q \geq 2\Sigma^2$, so that $q + 2\Sigma_\alpha \geq -2 + \frac{1}{2}(\Sigma_\alpha + 2)^2 + \frac{1}{6}(\Sigma_\beta - \Sigma_\gamma)^2$. The decoupled equation for H, cf. (6), yields that $H \to \infty$ as $\tau \to -\infty$, because q is non-negative. Since q is bounded as $\tau \to -\infty$, the asymptotics of H can be bounded by exponential functions from above and below. It follows that the equation $dt/d\tau = H^{-1}$ can be integrated to yield t as a function of τ such that $t \to 0$ as $\tau \to -\infty$.

In addition to (7) it is useful to also consider an auxiliary equation for the matter quantity Ω,

$$\Omega' = [2q - (1 + 3w)]\Omega\,. \tag{14}$$

Making use of (5) and (6) we conclude that for all orthogonal perfect fluid models with a linear equation of state we have $\rho \propto \exp\left(-3[1 + w]\tau\right)$, and hence $\rho \to \infty$ as $\tau \to -\infty$, which yields a past singularity. The divergence of ρ can also be directly read off from the matter equation $\nabla_a T^{ab} = 0$.

3 The Bianchi type IX Lie contraction hierarchy

The Bianchi type IX state space $\boldsymbol{B}_{\mathrm{IX}}$ is characterized by the conditions $N_1 > 0$, $N_2 > 0$, $N_3 > 0$. We write $\boldsymbol{B}_{\mathrm{IX}} = \mathcal{B}_{N_1N_2N_3}$. The notation is such that the subscript denotes the non-zero variables among $\{N_1, N_2, N_3\}$. Setting one or more of these variables to zero (which corresponds to Lie contractions [37]) yields invariant boundary subsets which describe more special Bianchi types. Since the type IX models exhibit discrete symmetries associated with axes permutations, the contractions generate all possible representations of the more special (Lie contracted) Bianchi types (which are associated with such permutations): The Bianchi type VII$_0$ subspace $\boldsymbol{B}_{\mathrm{VII}_0}$ is given by the disjoint union of three equivalent sets, $\boldsymbol{B}_{\mathrm{VII}_0} = \mathcal{B}_{N_1N_2} \cup \mathcal{B}_{N_2N_3} \cup \mathcal{B}_{N_3N_1}$, where, e.g., $\mathcal{B}_{N_1N_2}$ denotes the type VII$_0$ subset with $N_1 > 0$, $N_2 > 0$ and $N_3 = 0$; the Bianchi type II subspace $\boldsymbol{B}_{\mathrm{II}}$ by the union $\boldsymbol{B}_{\mathrm{II}} = \mathcal{B}_{N_1} \cup \mathcal{B}_{N_2} \cup \mathcal{B}_{N_3}$; the Bianchi type I subspace $\boldsymbol{B}_{\mathrm{I}}$ by $\boldsymbol{B}_{\mathrm{I}} = \mathcal{B}_{\emptyset}$. Note that the Bianchi type VI$_0$ subspace does not appear as a boundary subset of $\boldsymbol{B}_{\mathrm{IX}}$. A Bianchi subset contraction diagram for type IX is given in Figure 1.

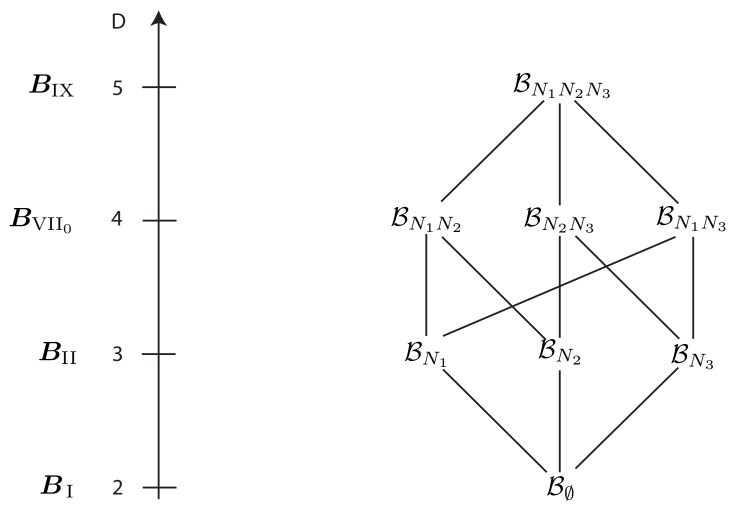

Figure 1: Subset contraction diagram for Bianchi type IX. D denotes the dimension of the dimensionless state space for the various models with an orthogonal perfect fluid with linear equation of state; the associated vacuum subsets have one dimension less; see also Table 2. The notation is such that the subscript of $\mathcal{B}_\star$ denotes the non-zero variables, e.g. $\mathcal{B}_{N_1}$ denotes the type II subset with $N_1 > 0$, $N_2 = 0$ and $N_3 = 0$.

Each set of the Lie contraction hierarchy is the union of an invariant vacuum subset, i.e., $\Omega = 0$, and an invariant fluid subset, i.e., $\Omega > 0$. To refer to a vacuum [fluid] subset of a Bianchi set $\mathcal{B}_\star$ we use the notation $\mathcal{B}_\star^{\text{vac.}}$ [$\mathcal{B}_\star^{\text{fl.}}$]. In this spirit, e.g., the type II subset decomposes as $\mathcal{B}_{N_1} = \mathcal{B}_{N_1}^{\text{vac.}} \cup \mathcal{B}_{N_1}^{\text{fl.}}$.

In the following we analyze the boundary subsets to the extent needed in order to understand the asymptotic type IX dynamics.

The Bianchi type I subset

The Bianchi type I subset is given by $N_1 = 0$, $N_2 = 0$, $N_3 = 0$ and $\Omega = 1 - \Sigma^2 \geq 0$; since N_1, N_2, N_3 vanish, we denote this subset by $\mathcal{B}_\emptyset$, cf. Figure 1. The *vacuum subset* consists of a circle of fixed points—*the Kasner circle* $\mathrm{K}^{\bigcirc}$, which is characterized by $\Sigma^2 = 1$. It is common to represent different points on $\mathrm{K}^{\bigcirc}$ in terms of the Kasner exponents p_α,

$$(\Sigma_1, \Sigma_2, \Sigma_3) = (3p_1 - 1, 3p_2 - 1, 3p_3 - 1)\,; \qquad p_1 + p_2 + p_3 = 1\,, \quad p_1^2 + p_2^2 + p_3^2 = 1\,; \quad (15)$$

each fixed point on $\mathrm{K}^{\bigcirc}$ represents a Kasner solution (Kasner metric) with the corresponding exponents. The Kasner circle is divided into six equivalent sectors, denoted by permutations of the triple (123), where sector $(\alpha\beta\gamma)$ is characterized by $p_\alpha < p_\beta < p_\gamma$, see Figure 2. The boundaries of the sectors are six special points that are associated with solutions that are locally rotationally symmetric (LRS): Q_α are given by $(\Sigma_\alpha, \Sigma_\beta, \Sigma_\gamma) = (-2, 1, 1)$ or $(p_\alpha, p_\beta, p_\gamma) = (-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})$ and yield the three equivalent LRS solutions whose intrinsic geometry is non-flat; the *Taub points* T_α are given by $(\Sigma_\alpha, \Sigma_\beta, \Sigma_\gamma) = (2, -1, -1)$ or $(p_\alpha, p_\beta, p_\gamma) = (1, 0, 0)$ and correspond to the flat LRS solutions—the Taub representation of Minkowski spacetime.

The quantity $p_1p_2p_3$ (or, equivalently, $\Sigma_1\Sigma_2\Sigma_3 = 2 + 27p_1p_2p_3$) is invariant under changes of the axes and thus naturally captures the 'physical essence' of a solution independent of the

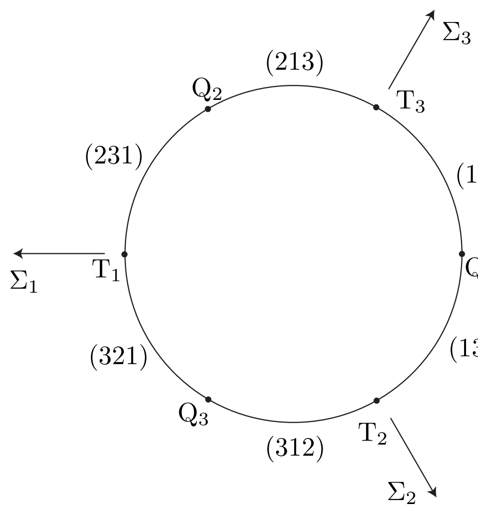

Figure 2: The Kasner circle $K^{\bigcirc}$ of fixed points divided into its six equivalent sectors and the LRS fixed points T_α and Q_α. Sector $(\alpha\beta\gamma)$ is defined by $\Sigma_\alpha < \Sigma_\beta < \Sigma_\gamma$. The sectors are related to each other by permutations of the spatial axes.

chosen frame; however, it is typically replaced by the *Kasner parameter* u through

$$p_1 p_2 p_3 = \frac{-u^2(1+u)^2}{(1+u+u^2)^3}\,, \qquad \text{where} \quad u \in [1,\infty]\,. \tag{16}$$

The Kasner parameter u parameterizes the Kasner exponents uniquely (up to the permutation symmetry); we have

$$p_\alpha = \frac{-u}{1+u+u^2}\,, \qquad p_\beta = \frac{1+u}{1+u+u^2}\,, \qquad p_\gamma = \frac{u(1+u)}{1+u+u^2}\,, \tag{17}$$

for sector $(\alpha\beta\gamma)$ of $K^{\bigcirc}$, where $u \in (1,\infty)$. Therefore, each point on sector $(\alpha\beta\gamma)$ is represented by a unique value of $u \in (1,\infty)$. At the boundary points of sector $(\alpha\beta\gamma)$, which are Q_α and T_γ, the Kasner parameter is $u = 1$ and $u = \infty$, respectively. Permuting $(\alpha\beta\gamma)$ yields a physically equivalent state on a different sector; accordingly, each $u \in (1,\infty)$ represents an equivalence class of six points on $K^{\bigcirc}$. In contrast, $u = 1$ describes the three points $\{Q_1, Q_2, Q_3\}$; $u = \infty$ yields $\{T_1, T_2, T_3\}$.

While the Bianchi type I vacuum subset coincides with the Kasner circle $K^{\bigcirc}$, which is given by $\Sigma^2 = 1$, the Bianchi type I perfect fluid subset is the set $1 - \Omega = \Sigma^2 < 1$. From (14) it is straightforward to deduce that there exists a central fixed point, the *Friedmann fixed point* F, given by $\Sigma_\alpha = 0\ \forall\alpha$, which corresponds to the isotropic Friedmann-Robertson-Walker (FRW) solution. Solutions with $0 < \Sigma^2 < 1$ are given by radial straight lines originating from $K^{\bigcirc}$ and ending at F. These results rely on the assumption $w < 1$, see (1).

The Bianchi type II subset

Let us consider the Bianchi type II subset $\mathcal{B}_{N_\gamma}$ given by $N_\alpha = N_\beta = 0$, $N_\gamma > 0$. In this case, the γ-direction is singled out, while there exists a discrete symmetry associated with the interchange of the α- and β-direction. The LRS subset $\Sigma_\alpha = \Sigma_\beta$ is a subset of codimension one which divides the state space into two equivalent parts, the subsets $\{\Sigma_\alpha > \Sigma_\beta\}$ and $\{\Sigma_\alpha < \Sigma_\beta\}$ (related by permuting the α- and β-axes).

Since the past singularity is of particular interest in our considerations, it is convenient to choose the time direction towards the past, i.e., to use a reversed time variable τ_- according to

$$\tau_- = -\tau \tag{18}$$

which we do in the remainder of this section; accordingly, approach to the past singularity means $\tau_- \to \infty$. (In Section 5 we will see that vacuum type II orbits are the building blocks for the Mixmaster map and the closely related Kasner map (BKL map). It is a well established convention that forward iterations of these maps are directed towards the singularity. To agree with this convention the use of a past-directed time variable in a discussion of Bianchi type II models thus suggests itself.)

On $\mathcal{B}_{N_\gamma}$, the Gauss constraint $\Sigma^2 + \frac{1}{12}N_\gamma^2 + \Omega = 1$ can be used to replace N_γ by Ω as a dependent variable. The system (7) thus becomes

$$\frac{d\Sigma_{\alpha/\beta}}{d\tau_-} = (2-q)\Sigma_{\alpha/\beta} + {}^3S_{\alpha/\beta}\,, \quad \frac{d\Sigma_\gamma}{d\tau_-} = (2-q)\Sigma_\gamma + {}^3S_\gamma\,, \quad \frac{d\Omega}{d\tau_-} = -\Omega\left[2q-(1+3w)\right]\,, \tag{19}$$

where $q = 2\Sigma^2 + \frac{1}{2}(1+3w)\Omega$ and ${}^3S_{\alpha/\beta} = -4(1-\Sigma^2-\Omega)$, ${}^3S_\gamma = 8(1-\Sigma^2-\Omega)$; we have $\Sigma^2+\Omega<1$.

Let us first consider the vacuum subset $\mathcal{B}_{N_\gamma}^{\mathrm{vac.}}$, i.e., $\Omega = 0$. There do not exist any fixed points in the type II vacuum subset $\mathcal{B}_{N_\gamma}^{\mathrm{vac.}}$, but the boundary of the vacuum subset coincides with the Kasner circle $\mathrm{K}^{\bigcirc}$. The orbits of (19) form a family of straight lines in $\mathcal{B}_{N_\gamma}^{\mathrm{vac.}}$, where each orbit connects one fixed point on $\mathrm{K}^{\bigcirc}$ with another fixed point on $\mathrm{K}^{\bigcirc}$; hence each orbit is heteroclinic [10]. Following the nomenclature of [30] we call these heteroclinic orbits Bianchi type II *transitions*, because each orbit can be viewed as representing a transition from one Kasner state to another. We denote these transitions by $\mathcal{T}_{N_\gamma}$, where each $\mathcal{T}_{N_\gamma}$ emanates from $(\gamma\alpha\beta)\cup\mathrm{Q}_\gamma\cup(\gamma\beta\alpha)$. If the initial point is a point of sector $(\gamma\alpha\beta)$, then the final point is a point of $(\alpha\gamma\beta)\cup\{\mathrm{Q}_\alpha\}\cup(\alpha\beta\gamma)$; interchanging α and β yields the transitions emanating from $(\gamma\beta\alpha)$; if the initial point is Q_γ, the final point is T_γ; the points T_α and T_β are not connected with any other fixed point (they are 'fixed points' under the present 'type II map'), see Figure 3.

Let $(\Sigma_\alpha^{\mathrm{i}}, \Sigma_\beta^{\mathrm{i}}, \Sigma_\gamma^{\mathrm{i}}) = (3p_\alpha^{\mathrm{i}}-1, 3p_\beta^{\mathrm{i}}-1, 3p_\gamma^{\mathrm{i}}-1)$ denote the initial fixed point on $\mathrm{K}^{\bigcirc}$; this point can be represented in terms of the Kasner parameter $u = u^{\mathrm{i}}$ by using (17). The orbit (transition) emanating from this fixed point is given in terms of an auxiliary function $\eta = \eta(\tau_-)$ by

$$\Sigma_{\alpha/\beta} = 2[1-\eta] + \eta\,\Sigma_{\alpha/\beta}^{\mathrm{i}}\,, \qquad \Sigma_\gamma = -4[1-\eta] + \eta\,\Sigma_\gamma^{\mathrm{i}}\,, \tag{20a}$$

where η is determined by the equation

$$\frac{d\eta}{d\tau_-} = 2(1-\Sigma^2)\eta \qquad \text{with} \quad (1-\Sigma^2) = \frac{3}{g}(g-\eta)(\eta-1) \quad \text{and} \quad g = \frac{1+u+u^2}{1-u+u^2} \tag{20b}$$

and the conditions that $\lim_{\tau_-\to-\infty}\eta = 1$ and $\lim_{\tau_-\to+\infty}\eta = g$. The quantity g is in the interval $(1,3)$ for $u\in(1,\infty)$; $u=1$ corresponds to $g=3$ and describes the orbit $\mathrm{Q}_\gamma\to\mathrm{T}_\gamma$; $u=\infty$ corresponds to $g=1$ and describes the 'isolated' points T_α, T_β. Since η increases from 1 to g as τ_- goes from $-\infty$ to $+\infty$, g is called the growth factor [30]. In Section 5, the transitions $\mathcal{T}_{N_\gamma}$, as represented by (20), will appear as the building blocks for the Mixmaster/Kasner map.

While there do not exist any fixed points in the vacuum subset of $\mathcal{B}_{N_\gamma}$, there exists one fixed point in $\mathcal{B}_{N_\gamma}$ with $\Omega>0$, the *Collins-Stewart fixed point* CS_γ, which corresponds to one representation of the LRS solutions found by Collins and Stewart [38]. CS_γ is given by $(\Sigma_\alpha,\Sigma_\beta,\Sigma_\gamma) = \frac{1}{8}(1+3w)(1,1,-2)$ and $\Omega = 1-\frac{1}{16}(1+3w)$ (which yields $N_\gamma = \frac{3}{4}\sqrt{1-w}\sqrt{1+3w}$). The fixed point CS_γ is the source (w.r.t. τ_-) for all orbits in $\mathcal{B}_{N_\gamma}$ with $\Omega>0$. In the limit $\tau_-\to\infty$ all solutions in $\mathcal{B}_{N_\gamma}\backslash\{\mathrm{CS}_\gamma\}$ converge to fixed points on the Bianchi type I boundary of $\mathcal{B}_{N_\gamma}$: There

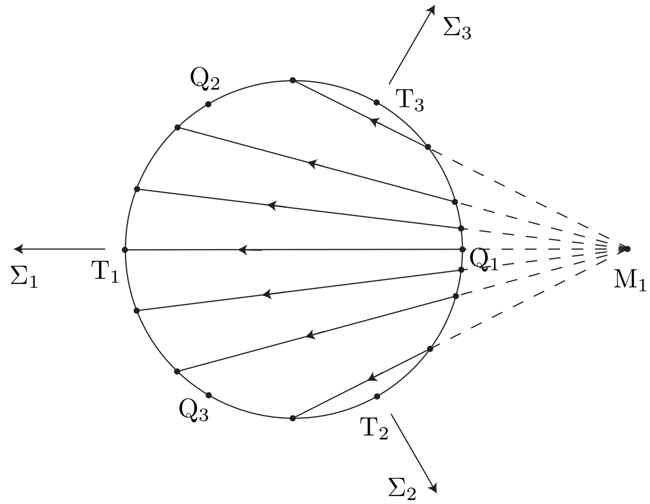

Figure 3: The type II transitions $\mathcal{T}_{N_1}$ on the $\mathcal{B}_{N_1}$ subset; by definition, $N_1 \neq 0$ along $\mathcal{T}_{N_1}$, while $N_2 = N_3 = 0$. The projections of these transition onto $(\Sigma_1, \Sigma_2, \Sigma_3)$-space are straight lines, which possess a common focal point M_1 characterized by $(\Sigma_1, \Sigma_2, \Sigma_3) = (-4, 2, 2)$. The transitions $\mathcal{T}_{N_2}$, $\mathcal{T}_{N_3}$ on the subsets $\mathcal{B}_{N_2}$, $\mathcal{B}_{N_3}$ are obtained by permutations of the axes, see Figure 5. The arrows indicate the direction of time towards the past.

exists one orbit (which corresponds to an LRS solution) that converges to F as $\tau_- \to \infty$; every other orbit converges to a fixed point on $(\alpha\gamma\beta) \cup \{\mathrm{Q}_\alpha\} \cup (\alpha\beta\gamma)$ or $(\beta\gamma\alpha) \cup \{\mathrm{Q}_\beta\} \cup (\beta\alpha\gamma)$ on $\mathrm{K}^{\bigcirc}$, or to T_γ (in the LRS case). For a detailed discussion of these results see [10].

The Bianchi type VII_0 subset

In anticipation of Theorem 6.1 which implies that generic orbits of Bianchi type IX do not have α-limit[5] points (w.r.t. the standard future directed time variable τ) on any of the Bianchi type VII_0 subsets $\mathcal{B}_{N_1N_2}$, $\mathcal{B}_{N_2N_3}$, $\mathcal{B}_{N_3N_1}$, we refrain from giving a detailed discussion of these subspaces here. (However, note that in order to *prove* Theorem 6.1, a detailed understanding of solutions of Bianchi type VII_0 is essential; in fact, in the proof of Theorem 6.1 Bianchi type VII_0 is ubiquitous; we refer to [26] and [27].) In the present context it suffices to note that on each subset $\mathcal{B}_{N_\alpha N_\beta}$ there exists a line of fixed points TL_γ given by $(\Sigma_\alpha, \Sigma_\beta, \Sigma_\gamma) = (-1, -1, 2)$ and $N_\alpha = N_\beta$ (so that $\Omega = 0$). Since TL_γ emanates from the point $\mathrm{T}_\gamma \in \mathrm{K}^{\bigcirc}$ we call it the 'Taub line.' Like T_γ itself, each of fixed points on TL_γ is associated with a representation of Minkowski spacetime (in an LRS type VII_0 symmetry foliation).

4 Asymptotic self-similarity

In the previous section we have given the fixed points associated with the system (7) on $\overline{\boldsymbol{B}}_{\mathrm{IX}}$. A local dynamical systems analysis of the fixed points shows whether or not these points attract type IX orbits in the limit $\tau \to -\infty$.[6] We merely state the results here and refer to [27] for details.

[5]For a dynamical system on a state space X, the α-limit set $\alpha(x)$ of a point $x \in X$ is defined as the set of all accumulation points towards the past (i.e., as $\tau \to -\infty$) of the orbit $\gamma(\tau)$ through x. The simplest examples of α-limit sets are fixed points and periodic orbits.

[6]In this section we adapt to [10, 26] and use the normal future-directed time variable τ.

$\mathrm{K}^{\bigcirc}$ Each fixed point K on $\mathrm{K}^{\bigcirc}\backslash\{\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3\}$ is a transversally hyperbolic saddle (one stable[7] mode, see Figure 4, and three unstable modes; the stable manifold of K coincides with a vacuum type II transition orbit, see Figure 3). The Taub points $\{\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3\}$ are center saddles with a two-dimensional unstable manifold and a three-dimensional center manifold, where T_α is excluded as an α-limit set on the center manifold, see [27]; consequently there do not exist any type IX solutions that converge to any of the points on $\mathrm{K}^{\bigcirc}$ as $\tau \to -\infty$.

F The fixed point F on $\mathcal{B}^{\text{fl.}}_{\emptyset}$ is a hyperbolic saddle (with $\mathcal{B}^{\text{fl.}}_{\emptyset}$ as a two-dimensional stable manifold and an additional three-dimensional unstable manifold). Accordingly, F attracts a two-parametric family of type IX orbits as $\tau \to -\infty$. These solutions have a so-called isotropic singularity.

CS_α The fixed points CS_α ($\alpha = 1, 2, 3$) on $\mathcal{B}^{\text{fl.}}_{N_\alpha}$ are hyperbolic saddles (with $\mathcal{B}^{\text{fl.}}_{N_\alpha}$ as a three-dimensional stable manifold and an additional two-dimensional unstable manifold). The unstable modes are associated with the equations $N_\beta^{-1} N_\beta' |_{\mathrm{CS}_\alpha} = \frac{3}{4}(1 + 3w)$ (for $\beta \neq \alpha$). Therefore, each of the fixed points CS_α attracts an (equivalent) one-parameter set of type IX orbits in the limit $\tau \to -\infty$.

TL_α Each fixed point on TL_α on $\mathcal{B}^{\text{vac.}}_{N_\beta N_\gamma}$ is a center saddle. On the three-dimensional center manifold (which coincides with $\mathcal{B}^{\text{vac.}}_{N_\beta N_\gamma}$, $\alpha \neq \beta \neq \gamma \neq \alpha$) the point acts as a (non-hyperbolic) sink, see [27]. Since there is a two-dimensional unstable manifold, there exists, for each fixed point on TL_α, a one-parameter family of type IX orbits that converges to it as $\tau \to -\infty$; these orbits correspond to LRS solutions. (Conversely, generic LRS type IX solutions converge to TL_α, see, e.g.,[10].)

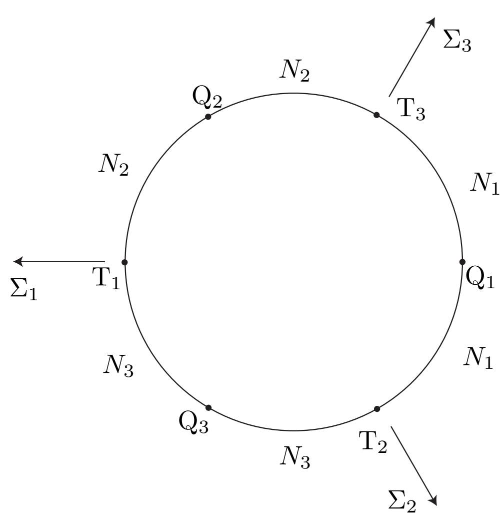

Figure 4: This figure depicts the Kasner circle $\mathrm{K}^{\bigcirc}$ and the stable variables for each sector; N_α is the stable variable in sectors $(\alpha\beta\gamma)$, $(\alpha\gamma\beta)$, and at the point Q_α. Expressed in the time variable τ_-, which is directed towards the past, these variables are the unstable modes. At a given fixed point, the associated unstable manifold orbit is a Bianchi type II transition, see Figure 3.

Collecting the results we see that the solutions whose α-limit is one of the fixed points form a subfamily of measure zero of the (four-parameter) family of Bianchi type IX solutions. Following

[7] When we reverse the direction of time, i.e., when we use τ_- instead of τ, we must replace 'stable' by 'unstable'.

the nomenclature of [26] we thus refer to these solutions as *non-generic* solutions of Bianchi type IX. Alternatively, to capture the asymptotic behavior of these solution, we use the term *past asymptotically self-similar* solutions. (Since a fixed point in the Hubble-normalized dynamical systems formulation corresponds to a self-similar solution, see e.g. [10], solutions that converge to a fixed point are asymptotically self-similar.)

The past asymptotically self-similar solutions comprise the LRS Bianchi type IX solutions. As seen above, generic LRS solutions converge to TL_α towards the past (and each solution that converges to TL_α is LRS), but there exist exceptional LRS solutions that converge to F or CS_α. The remaining orbits whose limit point is either F or CS_α correspond to past asymptotically self-similar solutions that are non-LRS. Clearly, every solution that converges to F or CS_α is a non-vacuum solution, since $\Omega \neq 0$ at F and CS_α.

It is natural to ask how the non-generic orbits are embedded in the state space $\overline{\boldsymbol{B}}_{\mathrm{IX}}$. The LRS orbits form the three LRS subsets $\mathcal{LRS}_\alpha$, which are the hyperplanes given by the conditions $\Sigma_\beta = \Sigma_\gamma$, $N_\beta = N_\gamma$, where $(\alpha\beta\gamma) \in \{(123), (231), (312)\}$. The orbits whose α-limit set is the fixed point CS_α (for some α) form the set $\mathcal{CS}_\alpha$ in $\boldsymbol{B}_{\mathrm{IX}}$; we call $\mathcal{CS}_\alpha$ the Collins-Stewart manifold. The local analysis of the fixed point CS_α and the regularity of the dynamical system (7) imply that the Collins-Stewart manifold $\mathcal{CS}_\alpha$ is a two-dimensional surface; it can be viewed as a two-dimensional manifold with boundary embedded in $\overline{\boldsymbol{B}}_{\mathrm{IX}}$ (where this boundary corresponds to an orbit in $\overline{\boldsymbol{B}}_{\mathrm{VII}_0}$). Analogously, the orbits whose α-limit set is the fixed point F form the set $\mathcal{F}$ in $\boldsymbol{B}_{\mathrm{IX}}$, which we call the isotropic singularity manifold, since solutions converging to F are those with an isotropic singularity. The isotropic singularity manifold $\mathcal{F}$ is a three-dimensional hypersurface; it can be viewed as a three-dimensional manifold with boundary.

Generic Bianchi type IX models are those that are not asymptotic self-similar and thus constitute examples for asymptotic self-similarity breaking; for other such examples, see [39, 40]. The central theme in this paper is the past asymptotic behavior of the generic models.

5 Facts about the Mixmaster and Kasner map

The *Mixmaster attractor* $\mathcal{A}_{\mathrm{IX}}$ (alternatively referred to as the Bianchi type IX attractor) is defined to be the subset of $\overline{\boldsymbol{B}}_{\mathrm{IX}}$ given by the union of the Bianchi type I and II vacuum subsets, i.e., $\mathcal{A}_{\mathrm{IX}} = \boldsymbol{B}_{\mathrm{I}}^{\mathrm{vac.}} \cup \boldsymbol{B}_{\mathrm{II}}^{\mathrm{vac.}}$. Since the type II vacuum subset consists of three equivalent representations we obtain

$$\mathcal{A}_{\mathrm{IX}} = \mathrm{K}^{\bigcirc} \cup \mathcal{B}_{N_1}^{\mathrm{vac.}} \cup \mathcal{B}_{N_2}^{\mathrm{vac.}} \cup \mathcal{B}_{N_3}^{\mathrm{vac.}} \,. \tag{21}$$

In this section we investigate the structures that the flow of dynamical system (7) induces on $\mathcal{A}_{\mathrm{IX}}$. In particular we discuss the Mixmaster map, the Kasner map, and the era map. To agree with the well-established convention for these maps, the direction of time will be taken towards the past.

The Mixmaster, Kasner, and era maps

In Section 3 we have seen that the vacuum type II orbits, i.e., the orbits on $\mathcal{B}_{N_\alpha}^{\mathrm{vac.}}$, $\alpha = 1, 2, 3$, are heteroclinic orbits that emanate from and converge to fixed points on the Kasner circle $\mathrm{K}^{\bigcirc}$, see Figures 3 and 5. In accord with Section 3 we refer to these orbits as *transitions* and denote them by $\mathcal{T}_{N_\alpha}$, $\alpha = 1, 2, 3$.

The type II transitions are the building blocks for the analysis of the Mixmaster attractor. By concatenating transitions we obtain a *sequence of transitions*, also known as a *heteroclinic chain*. Since each fixed point on $\mathrm{K}^{\bigcirc}$ (except for the Taub points) is the initial value for one

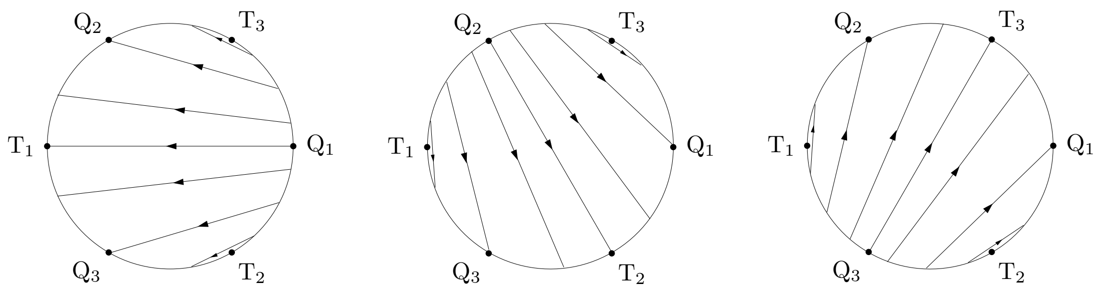

Figure 5: Projection of the type II transitions $\mathcal{T}_{N_1}$ $\mathcal{T}_{N_2}$, $\mathcal{T}_{N_3}$ on the type II subsets $\mathcal{B}^{\text{vac.}}_{N_1}$, $\mathcal{B}^{\text{vac.}}_{N_2}$, $\mathcal{B}^{\text{vac.}}_{N_3}$ onto $(\Sigma_1, \Sigma_2, \Sigma_3)$-space. Concatenation of these transition orbits yields sequences of transitions. The arrows indicate the direction of time towards the past.

single transition, see Figures 4 and 5, the concatenation of transitions is unique: Each fixed point (except for the Taub points) generates a unique sequence of transitions. Note, however, that the 'direction of time' is relevant. For each fixed point P on $\mathrm{K}^{\bigcirc}$, which is not one of the Taub points, there exists one single transition emanating from P at $\tau_- = -\infty$, but there are two transitions converging to P as $\tau_- \to \infty$. Therefore, concatenating transitions in the reversed direction of time leads to ambiguities. (In terms of the standard future-directed time variable τ we have the converse statement: It is possible to make unambiguous retrodictions, but not predictions.)

Let $l = 0, 1, 2, \ldots$ and let $\mathrm{P}_l \in \mathrm{K}^{\bigcirc}$ denote the initial point of the l^{th} transition (P_l is also the end point of the $(l-1)^{\text{th}}$ transition). We refer to the sequence $(\mathrm{P}_l)_{l\in\mathbb{N}}$ of Kasner fixed points, which is induced by the sequence of transitions, as being generated by the *Mixmaster map*. The Mixmaster map can be visualized by a map in $(\Sigma_1, \Sigma_2, \Sigma_3)$-space, obtained by inscribing $\mathrm{K}^{\bigcirc}$ in a triangle with corners at $(\Sigma_1, \Sigma_2, \Sigma_3) = (-4, 2, 2)$ and cyclic permutations, from which the (projections of the) transition orbits 'originate' as straight lines; see Figure 6.

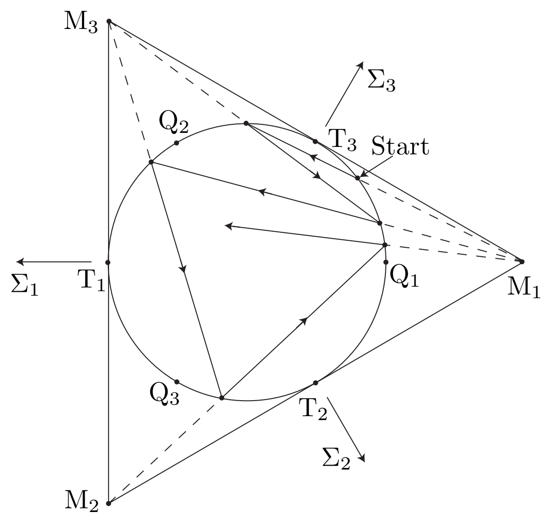

Figure 6: Concatenating type II transition orbits we obtain sequences of transitions—heteroclinic chains. The discrete map governing the associated sequence of fixed points on $\mathrm{K}^{\bigcirc}$ is the Mixmaster map. The arrows indicate the direction of time towards the past.

Let the initial Kasner state of a transition be represented by the Kasner parameter $u = u^{\mathrm{i}}$, where we assume $u^{\mathrm{i}} < \infty$, since neither of the Taub points T_1, T_2, T_3 can be the initial value for a transition. Inserting (15) and (17) into (20) we find that a transition maps the Kasner parameter u^{i} to the parameter u^{f}, where

$$u^{\mathrm{f}} = \begin{cases} u^{\mathrm{i}} - 1 & \text{if} \quad u^{\mathrm{i}} \in [2, \infty)\,, \\ (u^{\mathrm{i}} - 1)^{-1} & \text{if} \quad u^{\mathrm{i}} \in [1, 2]\,. \end{cases} \tag{22}$$

The information contained in this *Kasner map* suffices to represent the collection of all transition orbits (as a whole). However, for each particular mapping $u^{\mathrm{i}} \mapsto u^{\mathrm{f}}$, there exist six (equivalent) associated transitions;[8] this is simply because u^{i} characterizes the initial Kasner fixed point on $\mathrm{K}^{\bigcirc}$ only up to permutations of the axes. Hence, in order to reconstruct a particular transition from (22), we supplement (22) with information about the initial sector of the transition, which determines the position of the axes.

In terms of the Kasner parameter, a sequence of transitions corresponds to an iteration of (22). Let $l = 0, 1, 2, \ldots$ and let u_l denote the initial Kasner state of the l^{th} transition. This transition maps u_l to u_{l+1}, i.e.,

$$u_l \xrightarrow{l^{\mathrm{th}} \text{ transition}} u_{l+1} : \qquad u_{l+1} = \begin{cases} u_l - 1 & \text{if} \quad u_l \in [2, \infty)\,, \\ (u_l - 1)^{-1} & \text{if} \quad u_l \in [1, 2]\,. \end{cases} \tag{23}$$

We refer to this map as the (iterated) *Kasner map* (which is also known as the BKL map [2]). Since each value of the Kasner parameter $u \in (1, \infty)$ represents an equivalence class of six Kasner fixed points, the Kasner map can be regarded as the map induced by the Mixmaster map on these equivalence classes via the equivalence relation.

In a sequence $(u_l)_{l=0,1,2,\ldots}$ that is generated by the Kasner map (23), each Kasner state u_l is called an *epoch*. Every sequence $(u_l)_{l=0,1,2,\ldots}$ possess a natural partition into pieces (which contain a finite number of epochs each) where the Kasner parameter is monotonically decreasing according to the simple rule $u_l \mapsto u_{l+1} = u_l - 1$; these pieces are called *eras* [2]. An era begins with a maximal value $u_{l_{\mathrm{in}}}$ of the Kasner parameter (where $u_{l_{\mathrm{in}}}$ is generated from $u_{l_{\mathrm{in}}-1}$ by $u_{l_{\mathrm{in}}} = [u_{l_{\mathrm{in}}-1} - 1]^{-1}$), continues with a sequence of Kasner parameters obtained via $u_l \mapsto u_{l+1} = u_l - 1$, and ends with a minimal value $u_{l_{\mathrm{out}}}$ that satisfies $1 < u_{l_{\mathrm{out}}} < 2$, so that $u_{l_{\mathrm{out}}+1} = [u_{l_{\mathrm{out}}} - 1]^{-1}$ begins a new era.

$$\underbrace{6.29 \to 5.29 \to 4.29 \to 3.29 \to 2.29 \to 1.29}_{\text{era}} \to \underbrace{3.45 \to 2.45 \to 1.45}_{\text{era}} \to \underbrace{2.23 \to 1.23}_{\text{era}} \to \underbrace{4.33 \to \ldots}_{\text{era}} \tag{24}$$

Let us denote the initial (= maximal) value of the Kasner parameter u in era number s (where $s = 0, 1, 2, \ldots$) by u_s. Following [2, 7] we decompose u_s into its integer part $k_s = [\mathsf{u}_s]$ and its fractional part $x_s = \{\mathsf{u}_s\}$, i.e.,

$$\mathsf{u}_s = k_s + x_s\,, \qquad \text{where} \quad k_s = [\mathsf{u}_s]\,, \quad x_s = \{\mathsf{u}_s\}\,. \tag{25}$$

The number k_s represents the (discrete) length of era s, which is simply the number of Kasner epochs it contains. The final (= minimal) value of the Kasner parameter in era s is given by $1 + x_s$, which implies that era number $(s + 1)$ begins with

$$\mathsf{u}_{s+1} = \frac{1}{x_s} = \frac{1}{\{\mathsf{u}_s\}}\,.$$

The map $\mathsf{u}_s \mapsto \mathsf{u}_{s+1}$ is (a variant of) the so-called *era map*; starting from $\mathsf{u}_0 = u_0$ it recursively determines u_s, $s = 0, 1, 2, \ldots$, and thereby the complete Kasner sequence $(u_l)_{l=0,1,\ldots}$.

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[8] In the exceptional case $u^{\mathrm{i}} = 1$ there exist only three (equivalent) associated transitions: $\mathrm{Q}_1 \to \mathrm{T}_1$, $\mathrm{Q}_2 \to \mathrm{T}_2$, $\mathrm{Q}_3 \to \mathrm{T}_3$.

The era map admits a straightforward interpretation in terms of continued fractions. Consider the continued fraction representation of the initial value, i.e.,

$$\mathsf{u}_0 = k_0 + \cfrac{1}{k_1 + \cfrac{1}{k_2 + \cdots}} = [k_0; k_1, k_2, k_3, \dots] . \tag{26}$$

The fractional part of u_0 is $x_0 = [0; k_1, k_2, k_3, \dots]$; since u_1 is the reciprocal of x_0 we find

$$\mathsf{u}_1 = [k_1; k_2, k_3, k_4, \dots] . \tag{27}$$

Therefore, the era map is simply a shift to the left in the continued fraction expansion,

$$\mathsf{u}_s = [k_s; k_{s+1}, k_{s+2}, \dots] \;\mapsto\; \mathsf{u}_{s+1} = [k_{s+1}; k_{s+2}, k_{s+3}, \dots] . \tag{28}$$

The properties of the Kasner sequence depend on the initial value $\mathsf{u}_0 = u_0$.

(i) If and only if the initial Kasner parameter u_0 is a rational number, i.e., if and only if $u_0 \in \mathbb{Q}$, then its continued fraction representation is finite, i.e.,

$$u_0 = [k_0; k_1, k_2, \dots, k_n] , \tag{29i}$$

where $k_n > 1$. Therefore, there exists only a finite number of eras (where the last one begins with $\mathsf{u}_n = k_n$), and the Kasner sequence is finite. At the end of era number n, the Kasner parameter reaches $u = 1$, which subsequently terminates the recursion (23). Since $\mathbb{Q}$ is a set of measure zero in $\mathbb{R}$, this case is non-generic.

(ii) A quadratic irrational (quadratic surd) is an algebraic number of degree 2, i.e., an irrational solution of a quadratic equation with integer coefficients. If and only if the initial Kasner parameter u_0 is a quadratic irrational, i.e., if and only if $u_0 = q_1 + \sqrt{q_2}$, where $q_1 \in \mathbb{Q}$ and $q_2 \in \mathbb{Q}$ is not a perfect square (i.e., $\sqrt{q_2} \notin \mathbb{Q}$), then its continued fraction representation is periodic, i.e.,

$$u_0 = \big[k_0; k_1, \dots, k_n, (\bar{k}_1, \dots, \bar{k}_{\bar{n}})\big] ; \tag{29ii}$$

the notation is such that the part in parenthesis, i.e., $(\bar{k}_1, \dots, \bar{k}_{\bar{n}})$, is repeated ad infinitum.[9] Consequently, the era map becomes periodic (after the n^{th} era), and we thus obtain a periodic sequence of eras and a periodic Kasner sequence $(u_l)_{l=0,1,2,\dots}$. It is straightforward to see that while the period of the era sequence is $\bar{n}$, the period of the Kasner sequence is $(\bar{k}_1 + \cdots + \bar{k}_{\bar{n}})$; see the examples below. Since the set of algebraic numbers of degree two (or equivalently the set of equations with integer coefficients—it is a subset of $\mathbb{N}^3$) is a countable set, case (ii) is also non-generic.

(iii) An irrational number is called badly approximable if its Markov constant[10] is finite. If and only if u_0 is badly approximable, then the coefficients (partial quotients) in its continued fraction representation are bounded, i.e.,

$$u_0 = \big[k_0; k_1, k_2, k_3, \dots\big] \qquad \text{with} \quad k_i \leq K \ \forall i \tag{29iii}$$

for some positive constant K. Consequently, the sequence of eras and the Kasner sequence $(u_l)_{l=0,1,2,\dots}$ are bounded, i.e., $u_l \leq K \ \forall l$. Obviously, case (ii) is a subcase of case (iii).

[9] If $u_0 = q_1 + \sqrt{q_2} > 1$ is a quadratic irrational such that $q_1 - \sqrt{q_2} \in (-1, 0)$, then $u_0 = \big[(\bar{k}_1, \dots, \bar{k}_{\bar{n}})\big]$, i.e., the continued fraction is purely periodic without any preperiod.

[10] For $x \in \mathbb{R}$, let $\|x\|$ denote the distance from x to the nearest integer, i.e., $\|x\| = \min_{n \in \mathbb{Z}} |x - n|$. The Markov constant $M(x)$ of a number $x \in \mathbb{R}\backslash\mathbb{Q}$ is defined as $M(x)^{-1} = \liminf_{\mathbb{N} \ni n \to \infty} n \, \|nx\|$, see [41]. It is known that $M(x) \geq \sqrt{5}$ for all $x \in \mathbb{R}\backslash\mathbb{Q}$.

(Note, however, that there is probably no relationship between case (iii) and algebraic numbers of degree greater than 2, since it is expected that these numbers are well approximable.) The set of badly approximable numbers are a set of Lebesgue measure zero, hence this case is non-generic.

(iv) If and only if the initial Kasner parameter u_0 is a well approximable irrational number, then the partial quotients k_i in the continued fraction representation

$$u_0 = \big[k_0; k_1, k_2, k_3, \dots\big] \tag{29iv}$$

are unbounded (and we can construct a diverging subsequence from the sequence of partial quotients $(k_i)_{i\in\mathbb{N}}$). This is the generic case, and hence generically the Kasner sequence $(u_l)_{l=0,1,2,\dots}$ is infinite and unbounded.

In terms of continued fractions, the Kasner sequence $(u_l)_{l\in\mathbb{R}}$ generated by $u_0 = \big[k_0; k_1, k_2, \dots\big]$ is

$$\begin{aligned} u_0 = \mathsf{u}_0 &= \big[k_0; k_1, k_2, \dots\big] \to \big[k_0-1; k_1, k_2, \dots\big] \to \big[k_0-2; k_1, k_2, \dots\big] \to \dots \to \big[1; k_1, k_2, \dots\big] \\ \to \mathsf{u}_1 &= \big[k_1; k_2, k_3, \dots\big] \to \big[k_1-1; k_2, k_3, \dots\big] \to \big[k_1-2; k_2, k_3, \dots\big] \to \dots \to \big[1; k_2, k_3, \dots\big] \\ \to \mathsf{u}_2 &= \big[k_2; k_3, k_4, \dots\big] \to \big[k_2-1; k_3, k_4, \dots\big] \to \big[k_2-2; k_3, k_4, \dots\big] \to \dots \end{aligned}$$

Let us give some examples for periodic *era sequences* and *Kasner sequences*. If $u_0 = [(1)] = (1+\sqrt{5})/2$, which is the golden ratio, then $\mathsf{u}_s = (1+\sqrt{5})/2\ \forall s$. It follows that the Kasner sequence is also a sequence with period 1,

$$(u_l)_{l\in\mathbb{N}} : \tfrac{1}{2}\big(1+\sqrt{5}\big) \to \tfrac{1}{2}\big(1+\sqrt{5}\big) \to \tfrac{1}{2}\big(1+\sqrt{5}\big) \to \tfrac{1}{2}\big(1+\sqrt{5}\big) \to \tfrac{1}{2}\big(1+\sqrt{5}\big) \to \dots$$

If $u_0 = [(2)] = 1+\sqrt{2}$, then $\mathsf{u}_s = 1+\sqrt{2}\ \forall s$; hence the era sequence is a sequence of period 1. However, the associated Kasner sequence has period 2,

$$(u_l)_{l\in\mathbb{N}} : \underbrace{(1+\sqrt{2}) \to \sqrt{2}}_{\text{era}} \to \underbrace{(1+\sqrt{2}) \to \sqrt{2}}_{\text{era}} \to \underbrace{(1+\sqrt{2}) \to \sqrt{2}}_{\text{era}} \to \underbrace{(1+\sqrt{2}) \to \sqrt{2}}_{\text{era}} \to \dots$$

Analogously, the initial value $u_0 = [(3)] = (3+\sqrt{13})/2$ generates an era sequence of period 1 and an associated Kasner sequence of period 3. Finally let $u_0 = [(2,4)] = 1+\sqrt{3/2} \simeq 2.2247$. Then the era sequence has period 2 with $\mathsf{u}_n = [(2,4)] \simeq 2.2247$ for even n and $\mathsf{u}_n = [(4,2)] \simeq 4.4495$ for odd n. The associated Kasner sequence $(u_l)_{l\in\mathbb{N}}$ has period 6,

$$\underbrace{2.22 \to 1.22}_{\text{era}} \to \underbrace{4.45 \to 3.45 \to 2.45 \to 1.45}_{\text{era}} \to \underbrace{2.22 \to 1.22}_{\text{era}} \to \underbrace{4.45 \to 3.45 \to 2.44 \to 1.45}_{\text{era}} \to \dots$$

In the state space description of sequences (in terms of the Mixmaster map), an *epoch* is simply a point P_l on the Kasner circle. (It is one of the six points in the equivalence class associated with the Kasner parameter u_l.) Transitions connect epochs and thus generate the Mixmaster map.

The Kasner parameter u can be employed to measure the (angular) distance of a point P on $\mathrm{K}^{\bigcirc}$ from the Taub points or from the non-flat LRS points: If $u \gg 1$, then P is at an angular distance of approximately u^{-1} from one of the Taub points. On the other hand, in the vicinity of the non-flat LRS points (where $u-1 \ll 1$), $u-1$ is a linear measure for the angular distance of P from the closest of the non-flat LRS points. If $u < 2$, then P is closer to one of the non-flat LRS points Q_α than to any of the Taub points T_α. Therefore, in the state space picture, an *era* can be described as a (finite) sequence of points $\mathrm{K}^{\bigcirc} \ni \mathrm{P}_l$, $l_{\text{in}} \le l \le l_{\text{out}}$, obtained from the Mixmaster map, whose angular distance from the Taub points is monotonically increasing. The era begins with a fixed point $\mathrm{P}_{l_{\mathrm{in}}}$ that is close to one of the Taub points T_α (the preceding point $\mathrm{P}_{l_{\mathrm{in}}-1}$ was closer to one of the other two Taub points T_β, $\beta \neq \alpha$, than to T_α); then the distance from the Taub point T_α monotonically increases until, for $\mathrm{P}_{l_{\mathrm{out}}}$, it exceeds the distance to the non-flat LRS points; this is the terminal point for the era; it is connected by the following transition with the initial point of the next era; a good illustration is Figure 7(d), where each era consists of three epochs. Note that due to the equivalence of Kasner points, there exist several realizations of one and the same Kasner sequence as Mixmaster sequences in the state space picture; this is exemplified by the heteroclinic cycles in Figure 7(a) and 7(b).

The state space description of the cases (i)–(iv), which are characterized by the initial Kasner parameter $\mathsf{u}_0 = u_0$, is the following:

(i) Iff $u_0 \in \mathbb{Q}$, see (29i), the Mixmaster sequence of Kasner fixed points $(\mathrm{P}_l)_{l=0,1,\dots}$ is finite. After a finite number of transitions, at the end of era n, the sequence reaches one of the LRS points Q_α (where $u = 1$); a last transition follows, namely the transition $\mathrm{Q}_\alpha \to \mathrm{T}_\alpha$, and the sequence terminates in one of the Taub points.

(ii) Iff u_0 is a quadratic surd, i.e., $u_0 = q_1 + \sqrt{q_2}$ for some $q_1, q_2 \in \mathbb{Q}$ where q_2 is not a perfect square, see (29ii), then the Mixmaster sequence of Kasner points $(\mathrm{P}_l)_{l\in\mathbb{N}}$ is eventually periodic, where the period is a multiple of $(\bar{k}_1 + \cdots + \bar{k}_{\bar{n}})$; see Figure 7. Viewed as a periodic sequence of transitions (which are heteroclinic orbits) we obtain a heteroclinic cycle. In Figure 7 we give some of the heteroclinic cycles associated with Kasner sequences with periods 1, 2, and 3 in their projection onto $(\Sigma_1, \Sigma_2, \Sigma_3)$-space. Note that due to permutation symmetry there are several cycles associated with a given periodic Kasner sequence.

(iii) Iff u_0 is a badly approximable irrational number, see (29iii), there exists a neighborhood of the Taub points T_α such that the Mixmaster sequence of Kasner points $(\mathrm{P}_l)_{l\in\mathbb{N}}$ does not enter this neighborhood. This is simply because there exists a maximal value of the sequence $(u_l)_{l\in\mathbb{N}}$.

(iv) Iff u_0 is a well approximable irrational number, see (29iv), then the Mixmaster sequence $(\mathrm{P}_l)_{l\in\mathbb{N}}$ comes arbitrarily close to the Taub points. This is the generic case.

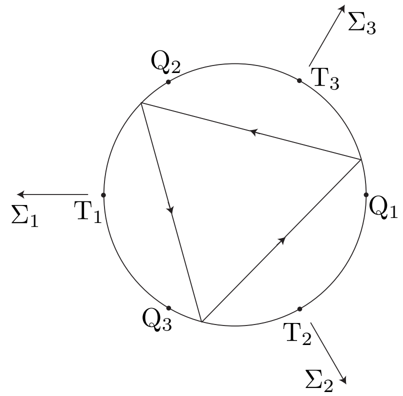
In the following we analyze the generic case (iv) in more detail. Let u_0 be a well approximable irrational number, i.e., a number whose continued fraction expansion

$$u_0 = \big[k_0; k_1, k_2, k_3, \dots\big] \tag{30}$$

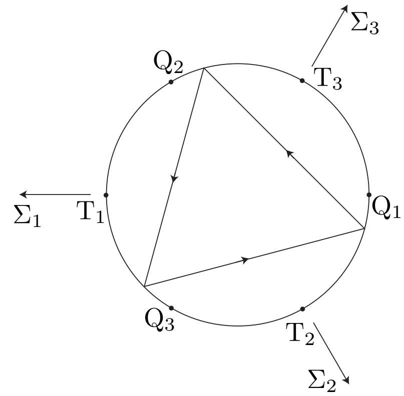
defines an unbounded sequence $(k_i)_{i\in\mathbb{N}}$. By construction, era number i contains k_i epochs (which we call its length). A natural question to ask concerns the distribution of the partial quotients k_i. For a 'typical' well approximable irrational number, how often does the number 1 appear in the sequence $(k_i)_{i\in\mathbb{N}}$? How often the number 2? And what about the number 1000? The answer is given by Khinchin's law [42]. Let $P_n(k{=}m)$ denote the probability that a randomly chosen partial quotient among $(k_1, \dots, k_n)$ equals $m \in \mathbb{N}$. In the asymptotic limit, i.e., for $P(k{=}m) = \lim_{n\to\infty} P_n(k{=}m)$ we have

$$P(k{=}m) \,=\, {}^2\!\log\big(\frac{m+1}{m+2}\big) - {}^2\!\log\big(\frac{m}{m+1}\big)\,, \tag{31}$$

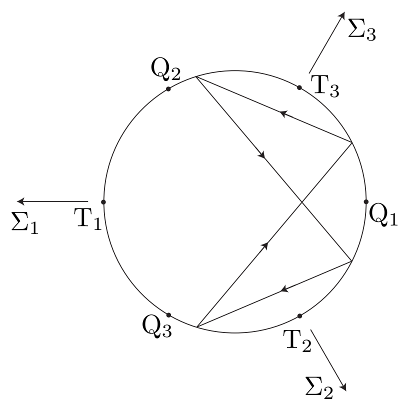
i.e., the partial quotients of the continued fraction representation of (30) are distributed like a random variable whose probability distribution is given by (31). (By (31), the number 1 appears in 42% of the slots, the number 2 in 17%, and the number 1000 in $1.4 * 10^{-6}$% of the slots.) Khinchin's law applies for *almost all* numbers u_0.



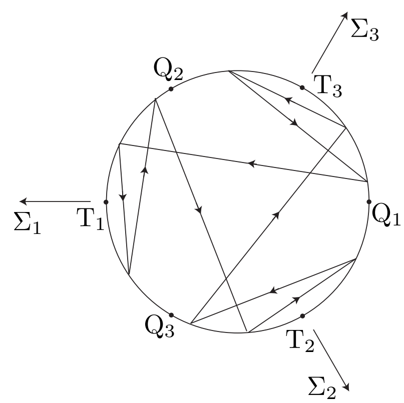
(a) One of the two heteroclinic cycles associated with the Kasner sequence generated by $u_0 = [(1)]$.



(b) The second of the two heteroclinic cycles associated with the Kasner sequence generated by $u_0 = [(1)]$.



(c) One of the heteroclinic cycles associated with the Kasner sequence generated by $u_0 = [(2)]$.



(d) One of the heteroclinic cycles associated with the Kasner sequence generated by $u_0 = [(3)]$.

Figure 7: Examples of heteroclinic cycles associated with era sequences with period 1. The Kasner sequences have period 1, 2, and 3, respectively; the period of the heteroclinic cycles is a multiple of that period. Note that the direction of time is towards the past.

For a (generic) Kasner sequence $(u_l)_{l\in\mathbb{N}}$ with initial parameter $u_0 = [k_0; k_1, k_2, \ldots]$ and its associated era sequence $(\mathsf{u}_s)_{s\in\mathbb{N}}$, where $\mathsf{u}_s = [k_s; k_{s+1}, k_{s+2}, \ldots]$, the expression $P(k{=}m)$ of (31) represents the probability that a randomly chosen era of $(\mathsf{u}_s)_{s\in\mathbb{N}}$ has length m; this corresponds to the probability that the initial value of an era is contained in the interval $[m, m+1)$. In this manner, the probability distribution (31) makes possible a stochastic interpretation of generic Kasner sequences.

The probability distribution (31) results in extraordinary properties of the Mixmaster/Kasner map, which will be of crucial importance in the considerations of Section 7. We do not discuss details here but refer to future work; however, we cannot refrain from giving a teaser: For a generic Kasner sequence $(u_l)_{l\in\mathbb{N}}$ and its associated era sequence $(\mathsf{u}_s)_{s\in\mathbb{N}}$ there exist infinitely many eras such that the length (i.e., the number of epochs) of the n^{th} era is larger than $n \log n$; however, for sufficiently large n, the length is guaranteed to be bounded by $n \log^2 n$. (For a proof of this result by Borel and Bernstein see [43]; see also [44].) Properties of this kind underline the remarkable intricacies of the heteroclinic structures on the Mixmaster attractor.

6 Mixmaster facts

In this section we turn to what is known about the past asymptotic dynamics of generic type IX solutions. The main Mixmaster fact is Ringström's 'Bianchi type IX attractor theorem'.

The main Mixmaster fact

Consider a solution of Bianchi type IX that is either vacuum or associated with a perfect fluid satisfying $-\frac{1}{3} < w < 1$. Recall that such a solution is called *generic* if it is not past asymptotically self-similar, i.e., if its α-limit set is neither the point F, nor any of the points CS_α, nor a point on TL_α; in other words, a generic solution corresponds to an orbit in $\boldsymbol{B}_{\mathrm{IX}}$ that is neither contained in $\mathcal{F}$, nor in $\mathcal{CS}_\alpha$, nor in $\mathcal{LRS}_\alpha$. Therefore, the set of generic Bianchi type IX states is an open set in $\boldsymbol{B}_{\mathrm{IX}}$. (To conform with [10, 26] we use the future directed time variable τ.)

The main results concerning generic Bianchi type IX models are due to Ringström [26]; these results rest on earlier work that is reviewed and derived in [10], and on [24, 25]. In the following we state the main theorem in a version adapted to our purposes.

Theorem 6.1 ([26])**.** *Let $(\Sigma_1, \Sigma_2, \Sigma_3, N_1, N_2, N_3)(\tau)$ be a generic solution of Bianchi type IX, i.e., a generic solution of* (7) *in $\boldsymbol{B}_{\mathrm{IX}}$. Then*

$$\Delta_{\mathrm{II}} = N_1N_2 + N_2N_3 + N_3N_1 \ \to\ 0 \qquad \textit{and} \qquad \Omega \ \to\ 0 \tag{32}$$

as $\tau \to -\infty$.

Note that this Theorem applies to both the fluid and the vacuum case; in the latter case (32) becomes $\Delta_{\mathrm{II}} \to 0$ and $\Omega \equiv 0$.

The proof of Theorem 6.1 given in [26] is delicate. In the first part it is proved that the α-limit set of a generic Bianchi type IX solution is non-empty and must contain a point on the Kasner circle. The main part of the proof deals with the fact that the function $\Delta_{\mathrm{II}}(\tau)$ is in general not monotone. There exist times where $\Delta_{\mathrm{II}}(\tau)$ increases (as $\tau \to -\infty$); the associated growth must therefore be controlled and shown to be negligible compared to the overall decrease in Δ_{II}. This is done by a careful analysis of the equations and (approximate) solutions. In [27] we give an alternative and relatively short and succinct proof of Theorem 6.1 which is based on an in-depth understanding of the hierarchical structure of the dynamical system (7) (as represented by Figure 1). It is important to note, however, that either of the proofs fail in the other Bianchi types that are conjectured to exhibit an oscillatory approach towards the singularity, i.e., the proofs fail for types $\mathrm{VI}_{-1/9}$ and VIII. This is unfortunate, since there are reasons to believe that these models are more relevant than type IX as regards the dynamics of generic (inhomogeneous) cosmologies, see [27].

Using the concept of the Mixmaster attractor, cf. (21), we obtain an equivalent formulation of Theorem 6.1: Let $X(\tau) = (\Sigma_1, \Sigma_2, \Sigma_3, N_1, N_2, N_3)(\tau)$ be a generic solution of Bianchi type IX. Then

$$\|X(\tau) - \mathcal{A}_{\mathrm{IX}}\| \to 0 \qquad (\tau \to -\infty)\,, \tag{33}$$

where the distance $\|X - \mathcal{A}_{\mathrm{IX}}\|$ is given as $\min_{Y\in\mathcal{A}_{\mathrm{IX}}} \|X - Y\|$.

Theorem 6.1 thus states that the attractor of generic type IX solutions resides on $\mathcal{A}_{\mathrm{IX}}$; however, whether the past attractor is in fact $\mathcal{A}_{\mathrm{IX}}$ or merely a subset thereof remains open. (The terminology 'Mixmaster attractor' is seductive but might turn out to be quite misleading.) Likewise, the theorem does not provide any direct information about the details of the asymptotic behavior of solutions.

Consequences

The prerequisite for a deeper understanding of the asymptotic behavior and the oscillatory dynamics of generic type IX solutions is an understanding of the Mixmaster attractor. In Section 5 we have identified the structures on the Mixmaster attractor $\mathcal{A}_{\mathrm{IX}}$ that are induced by the flow of the dynamical system: heteroclinic cycles [case (ii)] and finite [case (i)] and infinite heteroclinic sequences [cases (iii) and (iv)]. All these structures qualify (a priori) as possible α-limit sets of generic type IX orbits. In conjunction with these results, the main theorem 6.1 implies a number of further facts about the attractor—'Mixmaster attractor facts', which we give as a list of **corollaries**. (For proofs see [27], and also [26].)

1. A generic, i.e. not past asymptotically self-similar, type IX orbit possesses an α-limit point on $\mathcal{A}_{\mathrm{IX}}$.
2. If $\mathrm{P} \in \mathcal{A}_{\mathrm{IX}}$ is an α-limit point of a type IX orbit, then the entire heteroclinic cycle/sequence (Mixmaster sequence) through P must be contained in the α-limit set.
3. If one of the Taub points $\{\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3\}$ is an α-limit point of a type IX orbit, then the α-limit set contains Kasner fixed points associated with arbitrarily large values of the Kasner parameter u.
4. For generic solutions of Bianchi type IX the Weyl curvature scalar $C_{abcd}C^{abcd}$ (and therefore also the Kretschmann scalar) becomes unbounded towards the past.
5. Taking into account both the expanding and contracting phases of Bianchi type IX solutions, generic Bianchi type IX initial data generate an inextendible maximally globally hyperbolic development associated with past and future singularities where the curvature becomes unbounded.
6. Convergence to the Mixmaster attractor is uniform on compact sets of generic initial data: Let $\mathcal{X}$ be a compact set in $\boldsymbol{B}_{\mathrm{IX}}$ that does not intersect any of the manifolds $\mathcal{F}$, $\mathcal{CS}_\alpha$, $\mathcal{LRS}_\alpha$, so that each initial data $\mathring{x} \in \mathcal{X}$ generates a generic type IX solution. Let $X(\mathring{x};\tau)$ denote the type IX solution with $X(\mathring{x},0) = \mathring{x}$. Then
$$\|X(\mathring{x};\tau) - \mathcal{A}_{\mathrm{IX}}\| \to 0 \qquad (\tau \to -\infty) \tag{34}$$
uniformly in $\mathring{x} \in \mathcal{X}$.

Corollary 3 implies that the α-limit set contains an infinite set of Kasner fixed points in a neighborhood of the Taub point(s), but this set is not necessarily a continuum of fixed points; cf. the previous discussion about possible α-limit sets. If a Kasner point with $u \in \mathbb{Q}$ is contained in the α-limit of an orbit, so is a Taub point. This is an immediate consequences of the results of Section 5, case (i). On the other hand, if the α-limit set of an orbit is a heteroclinic cycle or a heteroclinic sequence (possibly in combination with a cycle) associated with cases (ii) and (iii) of Section 5, then there exists a neighborhood of the Taub points whose intersection with the α-limit set is empty.

Note that the oscillatory behavior of asymptotic type IX dynamics, which we unfortunately know no details about, constitutes an example of asymptotic self-similarity breaking [39]. In order to make progress as regards the details of the asymptotic oscillatory behavior, it is natural to first establish the connection between the Mixmaster/Kasner map and dynamics for a finite time interval $\Delta\tau$, discussed next.

Finite Mixmaster shadowing

To make contact between the Mixmaster map and Bianchi type IX asymptotic dynamics we introduce the concept of *finite Mixmaster shadowing* which formalizes the following basic idea: Given a sequence of transitions we can choose type IX initial data sufficiently close to the initial data of the sequence so that the type IX solution generated by this data remains close to the sequence for some 'time'.

Let P_0 be a Kasner fixed point (but $\mathrm{P}_0 \not\in \{\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3\}$) and let u_0 be the associated value of the Kasner parameter. There exists a unique sequence of transitions $(\mathcal{T}_l)_{l\in\mathbb{N}}$ and an associated Mixmaster sequence $(\mathrm{P}_l)_{l\in\mathbb{N}}$ with P_0 as initial data; the associated Kasner sequence is $(u_l)_{l\in\mathbb{N}}$. (If $u_0 \in \mathbb{Q}$, the sequence terminates at one of the Taub points after a finite number of transitions.) We shall make the definition that a type IX solution shadows a finite piece $(\mathcal{T}_l)_{l=0,1,\ldots,L}$ of the sequence of transitions if it is contained in a prescribed (small) tubular neighborhood of $(\mathcal{T}_l)_{l=0,1,\ldots,L}$. However, a standard ϵ-neighborhood of the sequence fails to be a reasonable measure of closeness, because in the vicinity of a Taub point the transitions lie so dense that consecutive transitions are not separated from each other by their respective ϵ-neighborhoods. Therefore, the introduction of *adapted* tubular neighborhoods is necessary to take into account the sensitivity of the flow at the Taub points and to capture more accurately the intuitive idea of shadowing.

Let $\epsilon > 0$ be small. A '*Taub-adapted neighborhood*' of the Mixmaster sequence $(\mathrm{P}_l)_{l\in\mathbb{N}}$ is a sequence $(\mathcal{U}_l)_{l\in\mathbb{N}}$ of open balls, where $\mathcal{U}_l$ is centered at the point P_l and has radius ϵu_l^{-2}, i.e., $\mathcal{U}_l = \{X \in \overline{\boldsymbol{B}}_{\mathrm{IX}} : \|X - \mathrm{P}_l\| < \epsilon u_l^{-2}\}$. (The radius ϵu_l^{-2} is chosen to ensure that the intersection of the ball $\mathcal{U}_l$ with the Kasner circle induces more or less a standard ϵ-neighborhood $(u_l - \epsilon, u_l + \epsilon)$ of the Kasner parameter u_l; recall from Section 5 that u^{-1} measures the angular distance of a fixed point from a Taub point.) A Taub-adapted tubular neighborhood of the sequence of transitions $(\mathcal{T}_l)_{l\in\mathbb{N}}$ is the sequence of tubes $(\mathcal{V}_l)_{l\in\mathbb{N}}$ that linearly interpolate between $\mathcal{U}_l$ and $\mathcal{U}_{l+1}$. Based on this definition we say that a type IX solution $X(\tau)$ *shadows* a finite piece $(\mathcal{T}_l)_{l=0,1,\ldots,L}$ of the sequence of transitions if it moves in a prescribed Taub-adapted tubular neighborhood $(\mathcal{T}_l)_{l=0,1,\ldots,L}$, i.e., if there exists a sequence of times $(\tau_l)_{l=0,1,\ldots,L+1}$ such that $X(\tau)$ is contained in $\mathcal{V}_l$ for all $\tau \in (\tau_{l+1}, \tau_l]$ for all $0 \leq l \leq L$.

Making use of these concepts, a formulation of finite Mixmaster shadowing is the following: Let $\epsilon > 0$ and $L \in \mathbb{N}$. Consider the sequence of transitions $(\mathcal{T}_l)_{l\in\mathbb{N}}$ emanating from an initial Kasner point P_0 and its Taub-adapted tubular neighborhood (associated with ϵ). Then there exists $\delta_\epsilon > 0$ such that each type IX orbit $X(\tau)$ that is generated by initial data X_0 with $\|X_0 - \mathrm{P}_0\| < \delta_\epsilon$ shadows the finite piece $(\mathcal{T}_l)_{l=0,1,\ldots,L}$ of the sequence of transitions. (A proof of this statement—in a slightly different form—has been given by Rendall [24]. Alternatively, one can invoke the regularity of the dynamical system, the center manifold reduction theorem and continuous dependence on initial data.) Evidently, δ_ϵ depends on the choice of ϵ and L. More importantly, however, δ_ϵ depends on the position of P_0—shadowing is not uniform; in particular, if we consider a series of initial points that approach one of the Taub points, then δ_ϵ necessarily converges to zero along this series—shadowing is more delicate in the vicinity of a Taub point. We will return to this issue in some detail in the next Section.

Finite Mixmaster shadowing concerns any generic type IX orbit $X(\tau)$. Let $\mathrm{P} \in \mathrm{K}^{\bigcirc}$ be an α-limit point of the type IX orbit $X(\tau)$; without loss of generality we may assume that P is not one of the Taub points. (The existence of such a point is ensured by Corollaries 1–3 of Section 6.) For simplicity we assume that P is associated with an irrational value of the Kasner parameter, which guarantees that the sequence $(\mathrm{P}_l)_{l\in\mathbb{N}}$ emanating from $\mathrm{P} = \mathrm{P}_0$ is an infinite sequence. Since P is an α-limit point of $X(\tau)$, there exists a sequence of times $(\tau_n)_{n\in\mathbb{N}}$, $\tau_n \to -\infty$ $(n \to \infty)$, such that $X(\tau_n) \to \mathrm{P}$ $(n \to \infty)$. Therefore, we observe a recurrence of phases, where the orbit $X(\tau)$ shadows $(\mathrm{P}_l)_{l\in\mathbb{N}}$ with an increasing degree of accuracy, i.e., shadowing takes place for increasingly longer pieces of the sequence or in ever smaller neighborhoods. (If the Kasner parameter of P is rational, then the sequence $(\mathrm{P}_l)_{l\in\mathbb{N}}$ is finite and terminates at a Taub point. $X(\tau)$ will shadow this finite sequence recurrently with an increasing degree of precision.) We will return to this issue in some more detail in the subsection 'Stochastic Mixmaster beliefs' of Section 7.

The concept of shadowing leads directly to the concept of *approximate sequences* which we introduce next. Consider a generic type IX orbit $X(\tau) = (\Sigma_1, \Sigma_2, \Sigma_3, N_1, N_2, N_3)(\tau)$ and the function $\|X(\tau) - \mathrm{K}^{\bigcirc}\|$, where the distance $\|X - \mathrm{K}^{\bigcirc}\|$ is given as $\min_{Y\in\mathrm{K}^{\bigcirc}} \|X - Y\|$. When the orbit $X(\tau)$ traverses a (sufficiently small) neighborhood of a fixed point on $\mathrm{K}^{\bigcirc}\backslash\{\mathrm{T}_1, \mathrm{T}_2, \mathrm{T}_3\}$, the function $\|X(\tau) - \mathrm{K}^{\bigcirc}\|$ exhibits a unique local minimum. This is immediate from the transversal hyperbolic saddle structure of the fixed point. (However, the flow in the vicinity of the Taub points is more intricate, since these points are not transversally hyperbolic.) It follows that the function $\|X(\tau) - \mathrm{K}^{\bigcirc}\|$ can be used to partition $X(\tau)$ into a sequence of segments in a straightforward manner:[11] The local minima of $\|X(\tau) - \mathrm{K}^{\bigcirc}\|$ form an infinite sequence $(\tau_l)_{l\in\mathbb{N}}$ such that $\tau_l \to -\infty$ as $l \to \infty$. (This follows directly from Corollaries 1 and 2 because $X(\tau)$ has α-limit point(s) on the Kasner circle and α-limit points on the type II subset.) A *segment* of $X(\tau)$ is defined to be the solution curve between two consecutive minima, i.e., the image of the interval $(\tau_{l+1}, \tau_l]$. (Note that $\tau \to -\infty$ in the approach to the singularity, while the discrete 'time' l is past-directed, i.e., $l \to \infty$ towards the singularity. This convention is chosen to agree with the standard convention for the Mixmaster and the Kasner map.) In the asymptotic regime, i.e., in the approach to the Mixmaster attractor, finite shadowing entails that a finite sequence of segments will resemble a finite sequence of type II transitions; this assumes, however, that the type IX orbit does not come too close to any of the Taub points. (In the neighborhood of a Taub point the flow of the dynamical system is much more intricate. This might yield recurring interruptions of the 'standard behavior'.) We call the type IX orbit in its segmented form an *approximate sequence of transitions*.

In addition, we define a sequence of 'check points' that is associated with an approximate sequence of transitions. Each minimum τ_l of $\|X(\tau) - \mathrm{K}^{\bigcirc}\|$ is associated with a Kasner fixed point $\check{\mathrm{P}}_l$ (a 'check point') that is defined as the minimizer on $\mathrm{K}^{\bigcirc}$ of the distance between $X(\tau_l)$ and $\mathrm{K}^{\bigcirc}$. (Note that the check points $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ do not lie on the type IX orbit $X(\tau)$.) Since each check point $\check{\mathrm{P}}_l$ is associated with a value $\check{u}_l$ of the Kasner parameter, the sequence $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ induces a sequence $(\check{u}_l)_{l\in\mathbb{N}}$. The value $\check{u}_{l+1}$ is in general not generated from $\check{u}_l$ by the exact Kasner map (23), but differs from that value by an error of $\delta\check{u}_l$. In our terminology, the sequence $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ is an approximate Mixmaster sequence; $(\check{u}_l)_{l\in\mathbb{N}}$ is an approximate Kasner sequence, see Figure 8.

The approximate Kasner sequence $\check{u}_l$ associated with a type IX orbit $X(\tau)$ does not follow the Kasner map (23) exactly. A natural question to ask, however, is whether the errors $\delta\check{u}_l$ of the approximate Kasner sequence converge to zero as $l \to \infty$ or not. If $\delta\check{u}_l \to 0$ as $l \to \infty$ for a type IX orbit, this means that its dynamics is completely described by an 'asymptotic Mixmaster/Kasner map', i.e., by a map that converges to the Mixmaster/Kasner map towards the singularity. However, if $\delta\check{u}_l \not\to 0$ as $l \to \infty$, then the evolution is interrupted repeatedly—infinitely many times—by phases where the dynamics is completely different from the Mixmaster dynamics, e.g., 'eras of small oscillations' [2]. (In the present state space description an era of small oscillations is associated with type VII_0 behavior in the vicinity of one of the Taub lines TL_α, where N_β and N_γ are small and of the same order; for details we refer to the discussion of type VII_0 dynamics in [27].) In the subsection 'Stochastic Mixmaster beliefs' of Section 7 we

[11] The definition of a partition of $X(\tau)$ into segments seems natural; it is important to note, however, that any definition depends on the formulation of the problem and is to a certain extent arbitrary. For instance, instead of using the minima of $\|X(\tau) - \mathrm{K}^{\bigcirc}\|$ one might prefer to analyze the projection of the orbit onto $(\Sigma_1, \Sigma_2, \Sigma_3)$-space and use the extrema of $\Sigma^2(\tau)$. However, the conclusions drawn from any construction of segments are quite insensitive to the details of the definition.

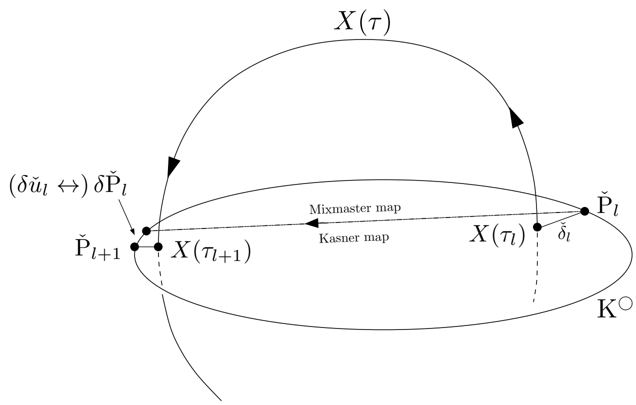


Figure 8: A type IX orbit decomposes into segments $X(\tau_l) \to X(\tau_{l+1})$. The points $X(\tau_l)$, $l \in \mathbb{N}$, are the local minima of the distance between $X(\tau)$ and $\mathrm{K}^{\bigcirc}$. The 'check point' $\check{\mathrm{P}}_l$ is the point on $\mathrm{K}^{\bigcirc}$ that is closest to $X(\tau)$ (for τ in a neighborhood of τ_l). Since each check point $\check{\mathrm{P}}_l$ is associated with a value $\check{u}_l$ of the Kasner parameter, the 'approximate Mixmaster sequence' $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ induces an 'approximate Kasner sequence' $(\check{u}_l)_{l\in\mathbb{N}}$. In general, $\check{\mathrm{P}}_{l+1}/\check{u}_{l+1}$ are not generated from $\check{\mathrm{P}}_l/\check{u}_l$ by the exact Mixmaster/Kasner map (23), but differ by an error $\delta\check{\mathrm{P}}_l/\delta\check{u}_l$.

will investigate the behavior of $\delta\check{u}_l$ along type IX orbits in detail.

7 Mixmaster beliefs

Attractor beliefs

There remain several important open problems. In the following we will address the most pressing questions; in particular we will transform vague beliefs into refutable conjectures, and give arguments in their favor.

An immediate question is the following: What are the actual α-limit sets of type IX orbits? Consider a type IX orbit γ. The α-limit set $\alpha(\gamma)$ contains a number of Kasner fixed points (and the associated type II transitions), each of which is characterized by a particular value of the Kasner parameter u. The set of Kasner parameters obtained in this way from $\alpha(\gamma)$ we denote by $U(\gamma)$. The question of which form $U(\gamma)$ can take for different orbits γ is open, the most significant issues being the following:

(i) Is it possible that there exists γ whose limit set $U(\gamma)$ consists of rational numbers only? It is straightforward to exclude that $U(\gamma)$ coincides with $\mathbb{Q}$ itself (or a dense subset thereof), the reason being that α-limit sets are necessarily closed. Another a priori constraint is that $U(\gamma)$ must contain $u = \infty$ (which characterizes the Taub points) if $U(\gamma)$ contains a rational number; see Section 5. Corollary 3 of Section 6 then implies that $U(\gamma)$ is unbounded, i.e., $U(\gamma)$ contains arbitrarily large values of the Kasner parameter. A set that is compatible with these basic requirements is, e.g., $U(\gamma) = \mathbb{N} \cup \{\infty\}$. Whether there exist orbits γ such that $U(\gamma)$ takes this (or a related) form remains open.

(ii) Is it conceivable that there exist orbits γ such that $\alpha(\gamma)$ is a heteroclinic cycle, see Figure 7? In this case the set $U(\gamma)$ is generated by a quadratic surd $u = [(\bar{k}_1, \bar{k}_2, \ldots, \bar{k}_n)]$ via the Kasner map; see Section 5; in particular, $U(\gamma)$ is finite. However, whether orbits γ with this particular past asymptotic behavior really exist is an open problem.

(iii) Can there exist orbits γ such that $U(\gamma)$ is bounded but contains infinitely many u-values? Is the Kasner sequence generated by a badly approximable number a candidate? The Kasner sequence $(u_l)_{l\in\mathbb{N}}$ generated by a badly approximable number is an infinite sequence that is bounded. However, there must be at least one accumulation point of this sequence; if $U(\gamma)$ contains $(u_l)_{l\in\mathbb{N}}$, then $U(\gamma)$ must contain the accumulation point of the sequence as well (since α-limit sets are closed). If this accumulation point is a well approximable number, $U(\gamma)$ cannot be bounded; however, if the accumulation point is a quadratic surd, no inconsistencies arise. Hence, a priori, there might exist orbits γ such that $U(\gamma)$ is not finite but still bounded. Whether this is indeed the case is doubtful, but hard to exclude a priori.

An open problem that might be quite separate from the questions raised above concerns the behavior of all type IX orbits save a set of measure zero.

Definition. *The past attractor of a dynamical system given on a state space X is defined as the smallest closed invariant set $\mathcal{A}^- \subseteq \overline{X}$ such that $\alpha(p) \subseteq \mathcal{A}^-$ for all $p \in X$ apart from a set of measure zero [45].*

Conjecture (Mixmaster attractor conjecture)**.** *The past attractor of the type IX dynamical system coincides with the Mixmaster attractor $\mathcal{A}_{\mathrm{IX}}$ (rather than being a subset thereof).*

Why is this a belief and not a fact? Theorem 6.1 implies that $\mathcal{A}^- \subseteq \mathcal{A}_{\mathrm{IX}}$; however, it is believed that $\mathcal{A}^- = \mathcal{A}_{\mathrm{IX}} = \mathrm{K}^{\bigcirc} \cup \mathcal{B}_{N_1}^{\mathrm{vac.}} \cup \mathcal{B}_{N_2}^{\mathrm{vac.}} \cup \mathcal{B}_{N_3}^{\mathrm{vac.}}$. (The usage of the terminology 'Mixmaster attractor' for the set $\mathcal{A}_{\mathrm{IX}}$ reflects the strong belief in the Mixmaster conjecture.) It is difficult to imagine how the Mixmaster attractor conjecture could possibly be violated. For instance, it seems rather absurd that the past attractor consists only of (a subset of) heteroclinic cycles—but there exist no proofs. A closely related belief is the following stronger statement.

Conjecture. *For almost all Bianchi type IX solutions γ the α-limit set $\alpha(\gamma)$ coincides with the Mixmaster attractor $\mathcal{A}_{\mathrm{IX}}$.*

We use the term 'almost all' in a noncommittal way without specifying the measure; recall that the word 'generic' already has the well-defined meaning of 'not past asymptotically self-similar'. (The usage of the word 'generic' is in accord with [26].)

Stochastic beliefs

This subsection is concerned with the (open) question of which role the Mixmaster/Kasner map plays in the asymptotic evolution of type IX solutions. The basis for our considerations are the results of Section 5 where we discussed the Mixmaster/Kasner map and the stochastic aspects of (generic) Kasner sequences. The *Mixmaster stochasticity conjecture* supposes that these stochastic properties carry over to almost every type IX orbit when represented as an approximate Kasner sequence.

Conjecture (Mixmaster stochasticity)**.** *The approximate Kasner sequence $(\breve{u}_l)_{l\in\mathbb{N}}$ associated with a generic type IX orbit admits a stochastic interpretation in terms of the probability distribution associated with the Kasner map, cf. Section 5. (This holds with probability one, i.e., for almost every generic type IX orbit.)*

The Mixmaster stochasticity conjecture is based on a rather suggestive simple idea: Type IX evolution is like trying to follow a path of an (infinite) network of paths while the ground is shaking randomly, and where the shaking subsides with time but never stops. A type IX orbit tries to follow a sequence of transitions on the Mixmaster attractor while random errors cause the orbit to lose track. Due to the errors, the orbit is incapable of following one particular sequence forever; after a finite time, the type IX orbit has deviated too much and it leaves the vicinity of the sequence. Although, temporarily, the orbit is contained in a neighborhood of a different sequence, it is bound to lose track of that sequence as well eventually. Accordingly the type IX orbit is thrown around in the space of Mixmaster sequence with the effect that the type IX orbit inherits the stochastic properties of generic Mixmaster sequences.

In the following we paint a heuristic picture that makes this idea a little more concrete. Consider a type IX orbit (approximate sequence of transitions) and the associated approximate Mixmaster sequence $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$. Let $(\mathrm{P}_l^{(0)})_{l\in\mathbb{N}}$ be the exact Mixmaster sequence with $\mathrm{P}_0^{(0)} = \check{\mathrm{P}}_0$ and consider the Taub-adapted neighborhood of this sequence associated with some prescribed value $\epsilon > 0$. Finite Mixmaster shadowing entails that there exists a finite piece $(\check{\mathrm{P}}_l)_{l=0,1,\ldots,L_1-1}$ of the approximate sequence that is contained in the prescribed Taub-adapted neighborhood of the exact sequence $(\mathrm{P}_l^{(0)})_{l\in\mathbb{N}}$. However, at $l = L_1$, the approximate Mixmaster sequence $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ leaves the prescribed tolerance interval due to the accumulation of errors. Hence, at $l = L_1$ we reset the system and consider the exact Mixmaster sequence $(\mathrm{P}_l^{(1)})_{l\geq L_1}$ with initial data $\mathrm{P}_{L_1}^{(1)} = \check{\mathrm{P}}_{L_1}$. The approximate Mixmaster sequence $(\check{\mathrm{P}}_l)_{l\geq L_1}$ is contained in the Taub-adapted neighborhood of the exact sequence $(\mathrm{P}_l^{(1)})_{l\geq L_1}$ up to $l = L_2 - 1$. At $l = L_2$ another readjustment becomes necessary. Iterating this procedure and concatenating the finite pieces $(\mathrm{P}_l^{(i)})_{l=L_i,\ldots,L_{i+1}-1}$ we are able to construct a sequence $(\bar{\mathrm{P}}_l)_{l\in\mathbb{N}}$ with the property that the approximate sequence $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ is contained within the Taub-adapted neighborhood (associated with ϵ) of $(\bar{\mathrm{P}}_l)_{l\in\mathbb{N}}$ for all $l \in \mathbb{N}$. The sequence $(\bar{\mathrm{P}}_l)_{l\in\mathbb{N}}$ is a piecewise exact Mixmaster sequence; it is exact in intervals $[L_i, L_{i+1})$. The length $\delta L_i = L_{i+1} - L_i$ of these intervals grows beyond all bounds as $i \to \infty$, because shadowing takes place with an increasing degree of accuracy. The error (of the order ϵ) between the approximate sequence $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ and the piecewise exact sequence $(\bar{\mathrm{P}}_l)_{l\in\mathbb{N}}$ results from the accumulation of numerous small errors. This obliterates the deterministic origin of the problem and generates a 'randomness' that leads to stochastic properties. Accordingly, we expect that the exact sequences $(\mathrm{P}_l^{(i)})_{l\geq L_i}$ from which $(\bar{\mathrm{P}}_l)_{l\in\mathbb{N}}$ is built constitute a random sample of Mixmaster sequences and thus truly reflect the stochastic properties of the Mixmaster map. As a consequence, although the sequence $(\bar{\mathrm{P}}_l)_{l\in\mathbb{N}}$ is only a piecewise exact Mixmaster sequence, it possesses the same stochastic properties as a generic Mixmaster sequence. Extrapolating this line of reasoning we are able to complete the argument and find that the approximate sequence $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ itself reflects the stochastic properties of the Mixmaster map. To emphasize this aspect of stochasticity of $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ we use the term 'randomized approximate sequence'. Some comments are in order.

In our discussion we have assumed implicitly that the approximate sequence $(\check{\mathrm{P}}_l)_{l\in\mathbb{N}}$ we consider can shadow any exact Mixmaster sequence for a finite number of transitions only. This is not necessarily the case. A type IX orbit whose α-limit set is one of the heteroclinic cycles (if such an orbit exists!) is an obvious counterexample: For every $\epsilon > 0$ there exists $L \in \mathbb{N}$ such that $\check{\mathrm{P}}_l$ is contained in the Taub-adapted ϵ-neighborhood of the Mixmaster sequence associated with the cycle. However, we expect that this type of 'infinite shadowing' holds at most for orbits of a set of measure zero.

A more serious limitation of the intuitive picture that we have sketched is illustrated by the following related example. Consider a type IX orbit whose α-limit set is the heteroclinic cycle depicted in Figure 7(a) (where we note again that the existence of such an orbit is not proven). For the associated approximate Kasner sequence $(\check{u}_l)_{l\in\mathbb{N}}$ we have $\check{u}_l \to (1+\sqrt{5})/2$ as $l \to \infty$. Consider the piecewise exact Kasner sequence $(\bar{u}_l)_{l\in\mathbb{N}}$ that is associated with the piecewise exact Mixmaster sequence. Each piece $(u_l^{(i)})_{l=L_i,\ldots,L_{i+1}-1}$ is generated from a value $u_{L_i}^{(i)} = [k_0; k_1, k_2, \ldots] = [1; 1, 1, 1, 1, \ldots, 1, k_n^{(i)}, k_{n+1}^{(i)}, \ldots]$; we have $n \to \infty$ as $i \to \infty$. It is evident that these pieces do *not* form a random sample of Kasner sequences. At best one could conjecture (and it is probably safe to do so) that the collection of the remainders $[k_n^{(i)}; k_{n+1}^{(i)}, \ldots]$ is such a random sample. Even so, the stochastic aspect does not carry over to the approximate sequence $(\check{u}_l)_{l\in\mathbb{N}}$, since the approximate sequence leaves the neighborhood of each sequence $(u_l^{(i)})_{l=L_i,\ldots}$ before that sequence has entered its stochastic regime (which is characterized by the remainder $[k_n^{(i)}; k_{n+1}^{(i)}, \ldots]$). Again we invoke 'genericity' to save the day: We conjecture that, for almost all approximate sequences, the sequences $(u_l^{(i)})_{l\geq L_i}$ are a true random sample of Kasner sequences and that the associated stochastic properties indeed carry over to the approximate sequence $(\check{u}_l)_{l\in\mathbb{N}}$ itself.

The piecewise exact sequence $(\bar{\mathrm{P}}_l)_{l\in\mathbb{N}}$ consists of pieces of length $\delta L_i = L_{i+1} - L_i$, where L_i grows beyond all bounds as $i \to \infty$, because shadowing takes place with an increasing degree of accuracy. However, this does not necessarily imply that $\delta L_i \to \infty$ as $i \to \infty$. The latter property is directly connected with the *Kasner map convergence conjecture*.

Conjecture (Kasner map convergence)**.** *Almost every generic type IX orbit is associated with an approximate Kasner sequence* $(\check{u}_l)_{l\in\mathbb{N}}$ *such that* $\delta\check{u}_l \to 0$ *as* $l \to \infty$*, where* $\delta\check{u}_l$ *describes the error between* $\check{u}_{l+1}$ *and the value of the Kasner parameter generated from* $\check{u}_l$ *by the exact Kasner map* (23)*.*

The relevance of the Kasner map for the asymptotic evolution of type IX solutions rests on the validity of the Kasner map convergence conjecture. Let us thus give a more detailed discussion and present the line of arguments that leads to this conjecture.

An orbit $X(\tau)$ (with segments $X(\tau_l) \to X(\tau_{l+1})$, $l \in \mathbb{N}$) generates a sequence of check points $\cdots \mapsto \check{\mathrm{P}}_l \mapsto \check{\mathrm{P}}_{l+1} \mapsto \cdots$, which in turn yields a map $\cdots \mapsto \check{u}_l \mapsto \check{u}_{l+1} \mapsto \cdots$. The parameter $\check{u}_{l+1}$ (associated with $\check{P}_{l+1}$) is generated from $\check{u}_l$ (associated with $\check{P}_l$) by the Kasner map plus an error $\delta\check{u}_l$, see Figure 8. The magnitude of the error $\delta\check{u}_l$ depends on the initial data of the segment, i.e., on $X(\tau_l)$; equivalently, we may view $\delta\check{u}_l$ as a function depending on (i) the position of $\check{\mathrm{P}}_l$ on $\mathrm{K}^{\bigcirc}$, and (ii) the vector $X(\tau_l) - \check{\mathrm{P}}_l$, which is orthogonal to $\mathrm{K}^{\bigcirc}$ at $\check{\mathrm{P}}_l$.

To obtain an estimate for $\delta\check{u}_l$ we introduce the *order of magnitude* of the error, which we denote by $\Delta\check{u}_l$. We define $\Delta\check{u}_l$ to be the average of $|\delta\check{u}_l|$ over all vectors $X(\tau_l) - \check{\mathrm{P}}_l$ of equal length that are orthogonal to $\mathrm{K}^{\bigcirc}$; alternatively, we use the somewhat 'safer' definition of $\Delta\check{u}_l$ as the maximum of $|\delta\check{u}_l|$. By design, the order of magnitude $\Delta\check{u}_l$ of the error is a function of two variables: (i) the position of $\check{\mathrm{P}}_l$ on $\mathrm{K}^{\bigcirc}$, which is invariantly represented by $\check{u}_l$, and, instead of $X(\tau_l) - \check{\mathrm{P}}_l$ itself, (ii) the (orthogonal) distance of $X(\tau_l)$ from $\check{\mathrm{P}}_l$ (or, equivalently, from $\mathrm{K}^{\bigcirc}$), i.e., $\|X(\tau_l) - \check{\mathrm{P}}_l\|$ $(= \|X(\tau_l) - \mathrm{K}^{\bigcirc}\|)$, which we denote by $\check{\delta}_l$, see Figure 8; in brief, $\Delta\check{u}_l = (\Delta\check{u})(\check{u}_l, \check{\delta}_l)$. In the following we investigate the behavior of the function $(\Delta\check{u})(\check{u}, \check{\delta})$ under variations of the two arguments.

(i) $(\Delta\check{u})(\check{u}, \cdot)$. Keep $\check{u}$ fixed (and assume that $\check{u}$ lies in the interval $\check{u} \in (1, \infty)$ so that its image under the Kasner map is finite). Then $\Delta\check{u}$ is a function of the distance $\check{\delta}$ such that $\Delta\check{u} \to 0$ as $\check{\delta} \to 0$ (which is a simple consequence of the regularity of the dynamical system and continuous dependence on initial data). It is reasonable to conjecture that the relation is in fact monotone.

(ii) $(\Delta\check{u})(\cdot, \check{\delta})$. Keep the distance $\check{\delta}$ fixed, so that $\Delta\check{u}$ is a function of $\check{u}$. The fundamental observation is that $\Delta\check{u}$ becomes unbounded as (a) $\check{u} \to \infty$, and (b) $\check{u} \to 1$. Case (a) is due to the intricacies of the flow in the vicinity of the non-transversally hyperbolic Taub points (where $u = \infty$); recall that $\check{u}^{-1}$ measures the angular distance of the check point from the nearest Taub point. For (b) we consider the orbit $\mathrm{Q}_\alpha \to \mathrm{T}_\alpha$ (for some α) which corresponds to $u = 1 \mapsto u = \infty$. Suppose that $\check{\mathrm{P}}_l$ coincides with Q_α (i.e., $\check{u}_l = 1$). In general, $\check{\mathrm{P}}_{l+1}$ will not coincide with T_α (independently of the choice of $\check{\delta} = \check{\delta}_l$); therefore $\Delta\check{u}_l = (\Delta\check{u})(\check{u}_l, \check{\delta}_l) = \infty$. If $\check{\mathrm{P}}_l$ is very close to Q_α, which means that $\check{u}_l$ is close to 1, a small deviation of $X(\tau)$ from the type II transition emanating from $\check{\mathrm{P}}_l$ will still be small at the end point $X(\tau_{l+1})$; however, in the vicinity of the Taub point T_α even a small deviation can translate to a large error $\delta\check{u}_l$ between $\check{u}_{l+1}$ and $(\check{u}_l - 1)^{-1}$, which in turn results in the asserted blow-up of the order of magnitude of the error. The qualitative properties of $\Delta\check{u}$ as a function of $\check{\delta}$ and $\check{u}$ are depicted in Figure 9.

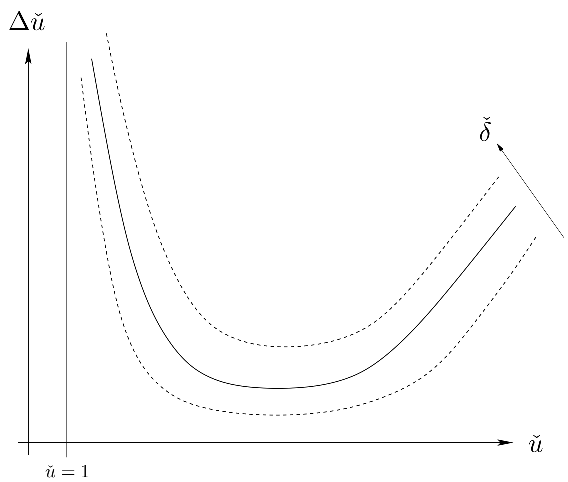


Figure 9: We consider a segment of an orbit $X(\tau)$ with initial check point $\check{\mathrm{P}}$ associated with the Kasner value $\check{u}$. (In accordance with the standard convention for the Mixmaster/Kasner map, the terms 'initial' and 'final' refer to a past-directed time.) The quantity $\check{\delta}$ denotes the initial distance of the orbit from the Kasner circle, which coincides with its distance from $\check{\mathrm{P}}$. The final check point is not generated from $\check{\mathrm{P}}$ via the Mixmaster map, but with an error represented by $\delta\check{u}$. The figure gives a schematic depiction of the order of magnitude $\Delta\check{u}$ of the error in dependence on $\check{u}$ and $\check{\delta}$. Each curve represents $(\Delta\check{u})(\cdot,\check{\delta})$, i.e., $\Delta\check{u}$ as a function of $\check{u}$ for a constant value of the initial distance $\check{\delta}$; the lower curve is associated with a smaller value of $\check{\delta}$, the top curve with a larger value.

Consider a generic type IX orbit $X(\tau)$ and its representation as an approximate sequence of transitions. By Theorem 6.1 each orbit converges to the Mixmaster attractor. As a consequence the distance $\check{\delta}_l = \|X(\tau_l) - \mathrm{K}^{\bigcirc}\|$ $(= \|X(\tau_l) - \check{\mathrm{P}}_l\|)$ converges to zero as $l \rightarrow \infty$. Therefore, as $\check{\delta}_l \rightarrow 0$ $(l \rightarrow \infty)$, the sequence of functions $(\Delta\check{u})(\cdot,\check{\delta}_l)$ of Figure 9 converges to zero. However, this convergence is merely pointwise and not uniform; this behavior is crucial to understand the behavior of $\delta\check{u}_l$ as $l \rightarrow \infty$.

If a type IX orbit $X(\tau)$ converges to one of the heteroclinic cycles on the Mixmaster attractor, then there exists ε such that $\check{u}_l \in (1+\varepsilon, \varepsilon^{-1})$ for all sufficiently large l. On this interval the function $\Delta\check{u}$ of Figure 9 converges to zero uniformly. We therefore obtain that $\Delta\check{u}_l = (\Delta\check{u})(\check{u}_l,\check{\delta}_l) \rightarrow 0$ and thus $\delta\check{u}_l \rightarrow 0$ as $l \rightarrow \infty$ for this type IX orbit. However, the Mixmaster attractor conjecture suggests that almost every type IX orbit $X(\tau)$ has an associated approximate Kasner sequence $(\check{u}_l)_{l\in\mathbb{N}}$ that is unbounded, hence the general case is not so clear. Since $(\check{u}_l)_{l\in\mathbb{N}}$ enters the intervals $[1, 1+\varepsilon)$ and $(\varepsilon^{-1}, \infty)$ for any ε, $\Delta\check{u}_l = (\Delta\check{u})(\check{u}_l,\check{\delta}_l)$ (and thus $\delta\check{u}_l$) need not necessarily converge to zero. Let us elaborate.

Let $\kappa > 0$ be fixed. By the implicit function theorem, the inequality $(\Delta\check{u})(\check{u},\check{\delta}) \geq \kappa$ is satisfied if and only if $\check{u} \leq 1 + f_\kappa(\check{\delta})$ or $\check{u} \geq g_\kappa(\check{\delta})$ for some functions f_κ, g_κ, which satisfy $f_\kappa(\check{\delta}) \rightarrow 0$ and $g_\kappa(\check{\delta}) \rightarrow \infty$ as $\check{\delta} \rightarrow 0$. Accordingly, for a given type IX orbit (and its associated approximated Kasner sequence $\check{u}_l$), we obtain

$$\Delta\check{u}_l = (\Delta\check{u})(\check{u}_l,\check{\delta}_l) \geq \kappa \quad \Leftrightarrow \quad \check{u}_l \leq 1 + f_\kappa(\check{\delta}_l) \tag{35}$$

or $\check{u}_l \geq g_\kappa(\check{\delta}_l)$. We call the union of the two intervals $\big(1, 1+f_\kappa(\check{\delta}_l)\big)$ and $\big(g_\kappa(\check{\delta}_l), \infty\big)$ the 'hazard zone' (associated with epoch number l); obviously, the hazard zone is decreasing with l. Hence, the order of magnitude of the error, $\Delta\check{u}_l$, is larger than κ at epoch number l if and only if the approximate Kasner parameter $\check{u}_l$ falls into the hazard zone (associated with l).

In connection with the Kasner map convergence conjecture the question is how often $(\check{u}_l)_{l\in\mathbb{N}}$ enters the hazard zone, i.e., how often (35) occurs: finitely many times or infinitely many times?

Suppose that the Mixmaster stochasticity conjecture is correct. Then almost every sequence $(\check{u}_l)_{l\in\mathbb{N}}$ admits a probabilistic interpretation in terms of the probability distribution (31). Accordingly, we expect the question to turn into an example of a '0–1 law': There exists two alternatives: (i) Type IX orbits (i.e., sequences $(\check{u}_l)_{l\in\mathbb{N}}$) that satisfy (35) infinitely many times are generic (i.e., of measure 1 in the state space); type IX orbits that do not are non-generic (i.e., of measure 0); (ii) Type IX orbits that satisfy (35) infinitely many times are non-generic; orbits that do not are generic. (In both cases, the two sets, the 'generic set' and the 'non-generic set', are probably non-empty.) Which of these alternatives is actually realized depends on the rate of decay of $(\check{\delta}_l)_{l\in\mathbb{N}}$ and the decay and growth of f_κ and g_κ, respectively, as $\check{\delta} \to 0$. Case (i) is associated with small (subcritical) rates of decay; case (ii) with large (overcritical) decay. We conjecture that case (ii) applies, i.e., (35) is satisfied only finitely many times for generic type IX orbits. The reason for this conjecture is the fast rate of convergence of the orbit to the Mixmaster attractor which follows from the decay of the functions $N_1N_2N_3$, cf. (10), and $N_1N_2 + N_2N_3 + N_3N_1$, cf. [26]; this leads to the expectation that the r.h. side of (35) represents a rapidly decaying function.[12] To conclude our line of arguments in favor of the Kasner map convergence conjecture, we note that the actual error $\delta\check{u}_l$ can be estimated by $\Delta\check{u}_l$. Therefore, if $\Delta\check{u}_l < \kappa$ for all l except for a finite set, then also $\delta\check{u}_l < \kappa$ for all l except for a finite set. Since κ is an arbitrary positive number, the statement of the Kasner map convergence conjecture ensues.

We round off this section with some further remarks on the conjectures. First, we note that one might be led to suppose that there could in fact exist type IX solutions for which the statement of the Kasner map convergence conjecture is violated, i.e., $\delta\check{u}_l \not\to 0$ as $l \to \infty$. In any case, the class of these special solutions is at most of measure zero. Second, we note that the statement of the Kasner map convergence conjecture, i.e., $\delta\check{u}_l \to 0$ $(l \to \infty)$, and the statement $\delta L_i \to \infty$ $(i \to \infty)$, cf. the discussion on piecewise exact sequences, are expressions of one and the same stochastic property. This emphasizes that the two 'stochastic' conjectures tightly intertwine. Third, we remark that it is conceivable that 'almost every' in the two conjectures might be in fact 'every'. In that case, the non-generic examples of Kasner/Mixmaster sequences, see Section 5, would not have any counterparts among the type IX solutions. Of particular interest in this context would be to have an answer to question (ii) of the second subsection of Section 7. If there do not exist type IX orbits whose α-limit set is one of the heteroclinic cycles, this would be a strong indication in favor of 'every' and against 'almost every'. If there exist type IX orbits that converge to a cycle, 'almost every' is the best one can aim for in the Mixmaster stochasticity conjecture.

Finally, let us briefly comment on claims of stochastic and chaotic properties of Bianchi type IX asymptotic dynamics. Chaotic aspects of the Kasner map and related maps have been studied under various aspects [7, 8, 19, 17, 20]. However, the relevance of these maps for type IX asymptotics rests on the two conjectures in this section. If the conjectures turn out to be wrong (e.g., if there exists a generic set of solutions that converge to heteroclinic cycles), then none of the results on the Kasner map carry over to full type IX dynamics. Numerical investigations, see [13] and references therein, reflect Theorem 6.1 and finite shadowing, but it is implausible that numerics can possibly shed light on the actual asymptotic limit of type IX solutions. Numerical errors are unavoidable and random in nature; these errors generate precisely the type of stochasticity the simulation is looking for. Accordingly, numerical studies will necessarily

[12] Additional support for the conjecture comes from toy models that reflect the instability of the Kasner map and its consequences. We will come back to this issue in future work.

find type IX solutions to exhibit the same stochastic behavior as generic Kasner sequences and are thus not suited to explore the validity of the conjectures.

8 Billiards and billiard beliefs

In this section we briefly review the Hamiltonian billiard approach to type IX dynamics in the spirit of [6, 31] and make contact with the dynamical systems approach. The metric (2b) can be written as

$$ {}^4\mathbf{g} = -(\det g)\tilde{N}^2 dx^0 \otimes dx^0 + e^{-2b^1}\,\hat{\boldsymbol{\omega}}^1 \otimes \hat{\boldsymbol{\omega}}^1 + e^{-2b^2}\,\hat{\boldsymbol{\omega}}^2 \otimes \hat{\boldsymbol{\omega}}^2 + e^{-2b^3}\,\hat{\boldsymbol{\omega}}^3 \otimes \hat{\boldsymbol{\omega}}^3\,, \tag{36} $$

where x^0 is an arbitrary time variable that is related to proper time t by a densitized lapse function $\tilde{N}$ according to $dt = \sqrt{\det g}\,\tilde{N}\,dx^0$; evidently, $\det g = \exp[-2(b^1 + b^2 + b^3)]$. The Hamiltonian for the orthogonal Bianchi type IX perfect fluid models is given by

$$ \mathcal{H} = \tilde{N}\left(\tfrac{1}{4}\sum_{\alpha,\beta}\mathcal{G}^{\alpha\beta}\pi_\alpha\pi_\beta - {}^3R\,\det g + 2\rho\,\det g\right) = 0\,, \tag{37} $$

where ${}^3R$ is three-curvature and $\rho = \rho_0(\det g)^{-(1+w)/2}$, cf. the remark following (14) (where we recall that $d\log(\det g) = 6d\tau$). $\mathcal{G}^{\alpha\beta}$ is the inverse of the so-called minisuperspace metric $\mathcal{G}_{\alpha\beta}$,

$$ \sum_{\alpha,\beta}\mathcal{G}_{\alpha\beta}v^\alpha w^\beta := -\sum_{\gamma\neq\delta}v^\gamma w^\delta = \sum_\alpha v^\alpha w^\alpha - \Big(\sum_\alpha v^\alpha\Big)\Big(\sum_\beta w^\beta\Big)\,, \tag{38} $$

i.e., $\mathcal{G}_{\alpha\beta}$ is a $2+1$-dimensional Lorentzian metric. The gravitational potential $U_G = -{}^3R\,\det g$ is given by

$$ U_G = \tfrac{1}{2}\left(e^{-4b^1} + e^{-4b^2} + e^{-4b^3} - 2e^{-2(b^1+b^2)} - 2e^{-2(b^2+b^3)} - 2e^{-2(b^3+b^1)}\right)\;; \tag{39} $$

where we have set the structure constants $\hat{n}^\alpha$ to one; the potential for the fluid is given by $U_F = 2\rho\,\det g = 2\rho_0\exp[-(1-w)(b^1+b^2+b^3)]$. For further details see, e.g., [37] or [10, Chapter 10]; note that $b^\alpha = -\beta^\alpha$, which is used in these references; see also [30] and [31].

As argued in [4, 31], b^α is expected to be timelike in the asymptotic regime, i.e., $\sum_\alpha \mathcal{G}_{\alpha\beta}\,b^\alpha b^\beta < 0$. Assuming that b^α is timelike allows to introduce new metric variables instead of $\{b^1, b^2, b^3\}$. Defining $\bar{\rho}^2 = -\sum_\alpha \mathcal{G}_{\alpha\beta}\,b^\alpha b^\beta$ and orthogonal angular metric variables, collectively denoted by γ, leads to

$$ \sum_{\alpha,\beta}\mathcal{G}_{\alpha\beta}\,db^\alpha db^\beta = -d\bar{\rho}^2 + \bar{\rho}^2 d\Omega_h^2\,, \tag{40} $$

where $d\Omega_h^2$ is the standard metric on hyperbolic space. Making a further change of variables,

$$ \lambda = \log\bar{\rho} = \tfrac{1}{2}\log\Big(-\sum_{\alpha,\beta}\mathcal{G}_{\alpha\beta}\,b^\alpha b^\beta\Big)\,, \tag{41} $$

yields

$$ \sum_{\alpha,\beta}\mathcal{G}^{\alpha\beta}\,\pi_\alpha\pi_\beta = -\pi_{\bar{\rho}}^2 + (\bar{\rho})^{-2}\pi_\gamma^2 = (\bar{\rho})^{-2}\left[-\pi_\lambda^2 + \pi_\gamma^2\right]\,. \tag{42} $$

Choosing the lapse according to $\tilde{N} = \bar{\rho}^2$ leads to a Hamiltonian of the form

$$ \mathcal{H} = \tfrac{1}{4}\left[-\pi_\lambda^2 + \pi_\gamma^2\right] + \bar{\rho}^2\sum_A c_A\,\exp(-2\bar{\rho}\,w_A(\gamma))\,, \tag{43} $$

where $w_A(\gamma)$ denotes certain linear forms of the variables γ^α, i.e., $w_A(\gamma) = \sum_\beta w_{A\,\beta}\,\gamma^\beta$; see [31] for details.

The essential point is that one expects that $\bar{\rho} \to \infty$ towards the singularity and hence that each term $\bar{\rho}^2 \exp[-2\bar{\rho} w_A(\gamma)]$ becomes an infinitely high sharp wall described by an infinite step function $\Theta_\infty(x)$ that vanishes for $x < 0$ and is infinite for $x \geq 0$. Accordingly, only 'dominant' terms in the potential are assumed to be of importance for the generic asymptotic dynamics, while 'subdominant' terms, i.e., terms whose exponential functions can be obtained by multiplying dominant wall terms, are neglected. In the present case there are three dominant terms in U_G (which is the minimal set of terms required to define the billiard table), the three exponentials $\exp(-4b^\alpha)$. Dropping the subdominant terms in the limit $\bar{\rho} \to \infty$ leads to an asymptotic Hamiltonian of the form

$$\mathcal{H}_\infty = \tfrac{1}{4}\left[-\pi_\lambda^2 + \pi_\gamma^2\right] + \sum_{A=1}^{3} \Theta_\infty(-2w_A(\gamma))\,, \tag{44}$$

where only the three dominant terms appear in the sum. The correspondence between the dynamical systems picture and the Hamiltonian picture is easily obtained by noting that the Hamiltonian constraint (37) is proportional to the Gauss constraint (9). The dominant terms correspond to the terms N_α^2, $\alpha = 1, 2, 3$, in (9); the subdominant terms are collected in Δ_{II}.

The non-trivial dynamics described by (44) resides in the variables γ, i.e., in the hyperbolic space. It can be described asymptotically as geodesic motion in hyperbolic space constrained by the existence of sharp reflective walls, i.e., the asymptotic dynamics is determined by the type IX 'billiard' given in Figure 10. Based on the heuristic considerations the limiting Hamiltonian (44) is believed to describe generic asymptotic dynamics.

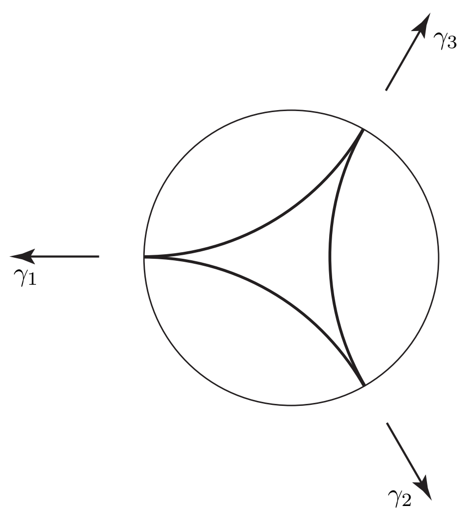


Figure 10: The Bianchi type IX billiard: The disc represents hyperbolic space. The asymptotic description of a solution is given by geodesic billiard motion inside the triangle, which acts as a stationary infinite potential wall, yielding a 'configuration space picture' of the asymptotic dynamics.

The Hamiltonian picture (as represented by the billiard) and the dynamical systems picture (as represented by the Mixmaster attractor) are 'dual' to each other. In the Hamiltonian picture the asymptotic dynamics is described as the evolution of a point in the billiard. Straight lines (geodesics) in hyperbolic space correspond to Kasner states. Wall bounces correspond to Bianchi type II solutions; the bounces change the Kasner states according to the Kasner map.

Since the billiard picture emphasizes the dynamics of the configuration space variables, one may say that the Hamiltonian billiard approach yields a 'configuration space' representations of the essential asymptotic dynamics.

In the dynamical systems state space picture the motion in the Hamiltonian billiard picture becomes a 'wall' of fixed points—the Kasner circle $\mathrm{K}^{\bigcirc}$. The walls in the Hamiltonian billiard are translated to motion in the dynamical systems picture—Bianchi type II heteroclinic orbits; these type II transitions yield exactly the same rule for changing Kasner states as the wall bounces: the Kasner map. Since the variables Σ_α are proportional to π_α, it is natural to refer to the projected dynamical systems picture as a 'momentum space' representation of the asymptotic dynamics.

Summing up, 'walls' and 'straight line motion' switch places between the Hamiltonian formulation and dynamical systems description of asymptotic dynamics, and the two pictures give equivalent complementary asymptotic pictures.

The above heuristic derivation of the limiting Hamiltonian rests on two basic assumptions: It is assumed that b^α is timelike in the asymptotic regime and that one can drop the subdominant terms. These assumptions correspond to assuming that Ω and Δ_{II} can be set to zero asymptotically, i.e., the procedure precisely assumes Theorem 6.1. (The Hamiltonian analysis in [46, Chapter 2] uses the same assumptions, and hence the present discussion is of direct relevance for that work as well.) That these assumptions are highly non-trivial is indicated by the difficulties that the proof of Theorem 6.1 has presented; cf. [26, 27]. An alarming example is Bianchi type VIII; in this case the heuristic procedure in this respect leads to exactly the same asymptotic results, but so far there exist no proof that Ω and Δ_{II} tend to zero towards the singularity for generic solutions. We elaborate on the differences between type VIII and type IX in [27]. Moreover, like in the state space picture there exists no proof that all of the possible billiard trajectories are of relevance for the asymptotic dynamics of type IX solutions. For example, there exists a correspondence between periodic orbits in the dynamical systems picture and the Hamiltonian billiard picture, and it is not excluded a priori that solutions are forced to one, or several, of these. We emphasize again that all proposed measures of chaos that take billiards as the starting point, see [46] and references therein, rely the conjectured connection between the Mixmaster map and asymptotic type IX dynamics.

The above discussion shows that there are non-trivial assumptions and subtle phenomena that are being glossed over in heuristic billiard 'derivations.' Nevertheless, we do believe that the billiard procedure elegantly uncovers the main generic asymptotic features, and it might be fruitful to attempt to combine the dynamical systems and Hamiltonian picture in order to prove the Mixmaster conjectures. A tantalizing hint is that in the billiard picture π_λ becomes an asymptotic constant of the motion. Rewriting this dimensional constant of the motion in terms of the dynamical systems variables yields

$$\pi_\lambda = 2H\sqrt{\det g}\left[-(\tau-\tau_0) + \tfrac{1}{12}\log\left(N_1^{\Sigma_1}N_2^{\Sigma_2}N_3^{\Sigma_3}\right)\right]\,, \tag{45}$$

where $\log\sqrt{\det g} = 3(\tau-\tau_0)$ and τ_0 is a constant.

9 Concluding remarks

The purpose of this paper is two-fold. On the one hand, we analyze the main known results on the asymptotic dynamics of Bianchi type IX vacuum and orthogonal perfect fluid models towards the initial singularity. The setting for our discussion is the Hubble-normalized dynamical systems approach, since this is essentially the set-up which has led to the first rigorous statements on Bianchi type IX asymptotics [26]. (We choose slightly different variables to emphasize the permutation symmetry that underlies the problem.) The main result ('Mixmaster fact') is Theorem 6.1 which is due to Ringström [26]; for an alternative proof see [27]. This theorem, in conjunction with an analysis of the Mixmaster attractor, leads to a number of further rigorous results which we list as consequences [27].

On the other hand, we draw a clear line between rigorous results ('facts') and heuristic considerations ('beliefs'). We make explicit that the implications of Theorem 6.1 and its consequences are rather limited, in particular, the rigorous results do not give any information on the details of the oscillatory nature of Mixmaster asymptotics. The mathematical methods required to obtain proofs about the actual asymptotic Mixmaster oscillations are yet to be developed; it is likely that radically new ideas are needed. This paper provides the infrastructure that might yield the basis for further developments. Our framework enables us to transform vague beliefs to a number of specific conjectures that describe the expected 'complete picture' of Bianchi type IX asymptotics. The arguments we give in their favor are based on an in-depth analysis of the Mixmaster map and its stochastic aspects in combination with the dynamical systems concept of shadowing.

We conclude with a few pertinent comments. First, as elaborated in [27] there exists no corresponding theorem to Theorem 6.1 in other oscillatory Bianchi models such as Bianchi type $\mathrm{VI}_{-1/9}$ or VIII; this suggests that the situation in the general inhomogeneous case is even more complicated than expected. Furthermore, numerical studies are incapable of shedding light on the asymptotic limit. This is mainly due to the accumulation of inevitable random numerical errors that make it a priori impossible to track a particular type IX orbit. Finally, although the Hamiltonian methods are a formidable heuristic tool, so far this approach has not yielded any proofs about asymptotics. Nevertheless, it might prove to be beneficial to explore the possible synergies between dynamical systems and Hamiltonian methods.

In this paper and in [27, 30] we have encountered remarkable subtleties as regards the asymptotic dynamics of oscillatory singularities; this emphasizes the importance of a clear distinction between facts and beliefs.

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