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Non-Relativistic Gravitation: From Newton to Einstein and Back

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Abstract: We present an improvement to the Classical Effective Theory approach to the non-relativistic or Post-Newtonian approximation of General Relativity. The "potential metric field" is decomposed through a temporal Kaluza-Klein ansatz into three NRG-fields: a scalar identified with the Newtonian potential, a 3-vector corresponding to the gravito-magnetic vector potential and a 3-tensor. The derivation of the Einstein-Infeld-Hoffmann Lagrangian simplifies such that each term corresponds to a single Feynman diagram providing a clear physical interpretation. Spin interactions are dominated by the exchange of the gravito-magnetic field. Leading correction diagrams corresponding to the 3PN correction to the spin-spin interaction and the 2.5PN correction to the spin-orbit interaction are presented.

Contents

1. Introduction

In 2004 Rothstein and Goldberger [1] suggested a novel approach to describe gravity (within Einstein's theory) for extended objects. Their method uses effective field theory and replaces the extended object by a point particle, whose interactions (or effective world-line action) encode all of its physical properties, such as reaction to tidal gravitational forces, ordered by a certain natural order of relevancy. This method has the advantage of applying to General Relativity (GR) tools which are normally associated with Quantum Field Theories including Feynman diagrams, dimensional regularization and effective actions. In particular action methods are more efficient than studying the equations of motion.

Later that method was applied [2] to determine the thermodynamics of "caged black holes", namely small black holes in the presence of compact extra dimensions. Caged black holes were previously analyzed by means of Matched Asymptotic Expansion (MAE) [3, 4, 5, 6] due to their part in the black-hole black-string phase transition [7, 8, 9] associated with the black string instability of Gregory-Laflamme [10]. The effective field theory approach, while formally identical to MAE, typically economizes the computation significantly. Recently an improved Classical Effective Field Theory (ClEFT) approach to caged black hole appeared [11].

The original main application of [1] was the non-relativistic motion of a binary system also known as the Post-Newtonian (PN) approximation. In this paper we apply the improvements of [11] to this case, and especially the stationary decomposition of fields. In section 2 we start by recalling the evolution from Newtonian gravity to Einstein's. In section 3 we consider Non-Relativistic Gravity (NRG), expanding Einstein's theory as Newton's plus corrections.[1] We describe the proposal of [1] for an effective field theory

[1] The limit $v \ll c$ which is *non-relativistic* from the point of view of Einstein's fully relativistic theory, corresponds to *relativistic* corrections from the post-Newtonian perspective, and it is also known by this latter name in the literature.

of NRG and we proceed to suggest an improvement via transforming to NRG-fields. Finally in section 4 we discuss two applications: to the first post-Newtonian correction of the two body problem, known as the Einstein-Infeld-Hoffmann (EIH) Lagrangian, and to spin interactions.

Note added (v2). This published version includes several relatively minor improvements. We detail the value of each individual diagrammatic contribution to the EIH action (4.1). We added details about the diagrams for the spin interactions. Signs were changed in the spin vertices (3.6,3.7) to reflect current conventions. Finally, references were added and updated.

Version 3. The generalization of the EIH Lagrangian to an arbitrary dimension was computed recently in [28] using the EFT approach introduced in [1]. We add a subsection (4.1.1) where the computation is done through the use of our improved version of ClEFT. Our result confirms all terms except for one which is to be corrected in a revision of [28].

2. From Newton to Einstein

Consider the motion of several masses whose sole interaction is through gravity. Without loss of generality we will write the actions for two masses. The Newtonian equations of motion [12] are concisely encoded by the action

$$S = \int dt \left[\sum_{a=1}^{2} \frac{m_a}{2} \dot{\vec{r}}_a^{\,2} + \frac{G\, m_1\, m_2}{r} \right] \tag{2.1}$$

where $\vec{r}_1, \vec{r}_2$ are the locations of the two masses, and $r := |\vec{r}| := |\vec{r}_1 - \vec{r}_2|$.

Introducing the gravitational field ϕ, the familiar equations of motion are encoded by an action which replaces the direct gravitational potential in (2.1) by a coupling of the masses to ϕ, together with a kinetic term for ϕ

$$S = \int dt \sum_{a=1}^{2} \left[\frac{m_a}{2} \dot{\vec{r}}_a^{\,2} - m_a\, \phi(\vec{r}_a) \right] - \frac{1}{8\pi G} \int dt\, d^3x\, \left(\vec{\nabla}\phi \right)^2 \ . \tag{2.2}$$

In Einstein's theory of gravity [13] the gravitational field is promoted to a space-time metric $g_{\mu\nu}$. The two body dynamics is given by the Einstein-Hilbert action together with the relativistic action of point particles

$$S = -\frac{1}{16\pi G} \int \sqrt{-g}\, d^4x\, R[g] - \sum_{a=1}^{2} m_a \int d\tau_a \ . \tag{2.3}$$

where $x^\mu(\tau)$ is the particle's trajectory, the proper time is defined by $\tau^2 := g_{\mu\nu}\, dx^\mu\, dx^\nu$, and for clarity we used $c = 1$ units. Here the two body problem becomes a 4d field dynamics which is fully non-linear, and no closed solution is known or believed to exist.

3. Non-Relativistic Gravity: Stationary decomposition

Consider the two body problem in General Relativity (GR) in the case where both velocities are small with respect to the speed of light. This problem applies to a binary inspiral process at its early stages, which is a conjectured source for the widely sought gravitational waves. The more traditional approach to the limit within GR is known as "the Post-Newtonian (PN) approximation".

In 2004 Goldberger and Rothstein [1] (see also [14] and a pedagogical introduction in [15]) pioneered an Effective Field Theory approach to this problem. Denote a typical separation between the masses by r, and a typical (small) velocity by v. Accordingly the typical variation time for all fields is r/v. In space, however, there are two typical lengths: r and r/v. Accordingly [1] decomposes the metric into

$$g_{\mu\nu} = H_{\mu\nu} + \bar{g}_{\mu\nu} \tag{3.1}$$

where $H_{\mu\nu}$ has length scale r and since the time variation scale is r/v, $H_{\mu\nu}$ is off-shell and it is called the potential component, while $\bar{g}_{\mu\nu}$ is of length scale r/v, it is on-shell and represents the radiation component. This approach is referred to as "Non-Relativistic GR" (NRGR) and can also be called "Non-Relativistic Gravity" (NRG).

In this paper we concentrate on the potential component $H_{\mu\nu}$. According to [1] its propagator includes only the spatial frequencies and not the temporal frequencies, which are subleading in the non-relativistic limit and hence treated as a perturbation. As H is considered static for the purposes of the propagator, it is natural to transform the fields through performing a temporal Kaluza-Klein dimensional reduction as in [11]

$$ds^2 = e^{2\phi}(dt - A_i\, dx^i)^2 - e^{-2\phi}\, \gamma_{ij}\, dx^i dx^j \ . \tag{3.2}$$

This relation defines a change of variables from $g_{\mu\nu}$ to (γ_{ij}, A_i, ϕ), $i, j = 1, 2, 3$ which we call "NRG-fields".

The action, when translated into NRG-fields, and within the static approximation, namely when all fields are t-independent becomes

$$S = -\frac{1}{16\pi G} \int dt\, dx^3 \sqrt{\gamma} \left[R[\gamma] + 2\ (\partial\phi)^2 - \frac{1}{4}\, e^{4\phi} F^2 \right] \ , \tag{3.3}$$

where $(\partial\phi)^2 = \gamma^{ij}\, \partial_i\phi\, \partial_j\phi = (\vec{\nabla}\phi)^2 + \dots$ and the term with the field strength F is conventionally defined by $F^2 = F_{ij}F^{ij}$, $F_{ij} = \partial_i A_j - \partial_j A_i$.

Let us discuss the physical meaning of the new fields $\phi, A_i,\ \gamma_{ij}$. Comparing the kinetic term for ϕ in the action (3.3), with the Newtonian field action (2.2) we find it natural to identify ϕ with the Newtonian potential (this is obvious when the 3-metric is flat $\gamma_{ij} = \delta_{ij}$). In this sense we are back to Newton (plus correction terms). The constant pre-factor 2 in this kinetic term is related to the polarization dependence of the g propagator (the original graviton): if we were to compute the Newtonian potential in the original action, prior to dimensional reduction, this same factor would have emerged from the graviton propagator in the standard Feynman gauge.

The vector potential A_i has an action which resembles the Maxwell action in 3d, and accordingly it is natural to call F the gravito-magnetic field. This name originates in a certain similarity between gravity and electro-magnetism. The strong similarity between Newton's gravitational force and Coulomb's static electrical force, together with the observation that the transition from electro-statics to electro-dynamics requires to supplement the scalar electric potential by a vector potential, promoted already in the 19th century suggestions to add a vector potential to the gravitational degrees of freedom. It is in fact known how to obtain such a vector potential in the weak gravity/ Post-Newtonian approximation to GR, a point of view known as "Gravito-Electro-Magnetism (GEM)" (see [16] and references within). We note the reversed sign of the kinetic term for F. This is directly related to the fact that the spin-spin force in gravity has an opposite sign relative to electro-dynamics, namely "north poles attract" [17].

Finally the 3-metric tensor γ_{ij} comes with a standard Einstein-Hilbert action in 3d (this is achieved through the Weyl rescaling factor in front of γ_{ij} in the ansatz).

Once time-dependence is permitted, the action contains time derivatives. Of particular interest are terms quadratic in the fields (including time derivatives) which are considered as vertices rather than being part of the propagator as mentioned above. The only such term which will be required here is

$$S \supset \frac{1}{16\pi G} \int dt\, dx^3\, 2\dot{\phi}^2 \tag{3.4}$$

rendering the ϕ^2 sector Lorentz invariant. It is obtained after decoupling ϕ and A at the quadratic level through the use of a Lorentz invariant gauge-fixing term. In addition we note that the definition of $\phi, A_i,\ \gamma_{ij}$ (3.2) is given here in the t-independent case, and when time dependence is incorporated it is conceivable that it should be supplemented by terms with time derivatives.

In the new variables (3.2) the point-particle action which appears in (2.3) becomes

$$\begin{aligned} S_{pp} \equiv -m_0 \int d\tau &= -m_0 \int dt\, e^{\phi} \sqrt{(1-\vec{A}\cdot\vec{v})^2 - e^{-4\phi}\,\gamma_{ij}\,v^i\,v^j} = \\ &= -m_0 \int dt \left(1 - \frac{1}{2}v^2 + \phi - \vec{A}\cdot\vec{v} + \frac{3}{2}\,\phi\, v^2 + \dots\right) \end{aligned} \tag{3.5}$$

where $\vec{v} \equiv \dot{\vec{r}}$ is the velocity vector. The change of variables has the advantage that the propagator is diagonal with respect to the field ϕ which couples to the world-line at lowest order (through the interaction $(-)m_0\,\phi$).

For a spinning object the lowest order interaction, found in [18] and translated to NRG-fields in [11] reads (in the conventions of [11])

$$S \supset \frac{1}{4} \int dt\, J_0^{ij}\, F_{ij} = \frac{1}{2}\,\vec{J}\cdot\vec{B} \tag{3.6}$$

where we denoted the angular momentum vector $J_i = \epsilon_{ijk} J^{jk}/2$ and the gravito-magnetic field strength $B_i = \epsilon_{ijk} F^{jk}/2$. Additional terms come with additional fields

$$S \supset \int dt\, J_0^{ij} \left(F_{ij}\,\phi - \frac{1}{2}\,A_i\,\partial_j\phi - \frac{1}{4}\delta\gamma_j{}^k\,F_{ik}\right) \tag{3.7}$$

Yet other terms come with powers of v (by departing from stationarity).

4. Applications

Here we apply the new NRG-fields to obtain valuable insight into the Einstein-Infeld-Hoffmann Lagrangian and into spin interactions within NRG.

4.1 Einstein-Infeld-Hoffmann

When passing from electro-statics to electro-dynamics we may integrate out the electromagnetic field and obtain an action for two interacting charged particles which depends on their velocities as well as their locations thus including the effects of both electric and magnetic forces. The analogous 1PN correction to the Newtonian gravitational action (2.1) is called Einstein-Infeld-Hoffmann (EIH) [19]

$$\mathcal{L}_{EIH} = \frac{1}{8}\sum_{a=1}^{2} m_a\, \vec{v}_a^4 + \frac{Gm_1m_2}{2r}\left[3(\vec{v_1}^2+\vec{v_2}^2) - 8\vec{v_1}\cdot\vec{v_2} + \vec{v}_{1\perp}\cdot\vec{v}_{2\perp}\right] - \frac{G^2\, m_1m_2(m_1+m_2)}{2\, r^2} \tag{4.1}$$

where $\vec{v}_{i\perp} := \vec{v_i} - \vec{r}\,(\vec{v_i}\cdot\vec{r})/r^2$. From an intuitive Newtonian perspective, this correction represents the contribution to the gravitational interaction from both kinetic and potential energies, as well as a correction accounting for the finite speed of light.

In [1] the calculation of this action, whose associated equations of motion required some 36 pages of traditional GR methods in 1938, was reduced to the computation of 5 Feynman diagrams over less than 2 pages.[2] After translation into NRG-fields we obtain the Feynman diagrams shown in figure 1. The first pay-off is that the triple vertex diagram (5(a) of [1]) gets eliminated, since there is no cubic vertex for ϕ in our action (3.3). Noting that both figures 5(a) and 5(b) of [1] are proportional to the last term in the EIH action (4.1), it is not surprising that they can be replaced by the single diagram in fig. 1(c). This is especially fortunate since the eliminated diagram was the only one to use the awkward 3-graviton vertex and the only one including a loop. As a result of this economization, each diagram in figure 1 is responsible for precisely one term in (4.1): the v^4 term comes from the kinetic part of (3.5); the next 3 terms all proportional to v^2 come from the diagrams in figure 1(b): the $v_1^2+v_2^2$ term comes from the top diagram, the $\vec{v_1}\cdot\vec{v_2}$ term comes from the middle diagram and the $\vec{v}_{1\perp}\cdot\vec{v}_{2\perp}$ comes from the bottom diagram where the vertex (3.4) is used; finally the last term of (4.1) comes from the top diagram of figure 1(c).

Additional pay-off comes in terms of insight into the fields which propagate in the diagram. Almost all of them are the gravitational potential ϕ. An exception is the second from top diagram in fig. 1(b) , where the vector potential propagates and it is responsible for the $v_1\, v_2$ factor coming from both vertices.

4.1.1 In higher dimensions

Following the recent generalization of the EIH Lagrangian to an arbitrary space-time dimension d [28] using the EFT approach introduced in [1], we perform the computation within the improved version of ClEFT discussed above, and compare the results.

The d-dimensional analogs of (3.2, 3.3, 3.4) and (3.5) are

$$ds^2 = e^{2\phi}(dt - A_i\, dx^i)^2 - e^{-2\phi/(d-3)}\,\gamma_{ij}\, dx^i dx^j\ , \tag{4.2}$$

[2] Obviously quite a number of works touched upon the EIH result since 1938 and we shall not attempt to provide a comprehensive list. At the suggestion of a referee we would like to mention the "ADM Hamiltonian approach" [20] which yielded the two body Hamiltonian up to 2PN through "tedious calculations". For field theoretic approaches which predate [1] see references therein.

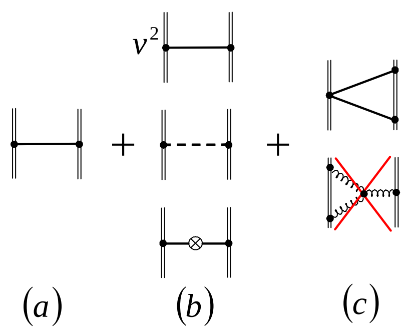

Figure 1: Feynman diagrams contributing to the Newtonian two body Lagrangian (2.1) and the 1PN Einstein-Infeld-Hoffmann Lagrangian (4.1). Double lines represent the masses and do not carry propagators. Solid lines represent ϕ, the Newtonian potential, dashed lines represent A_i, the vector potential, while spring-like lines are non-discriminating notation for all the polarizations of the original graviton g. A power of v near a vertex denotes that the vertex was expanded in v, while a circled cross denoted v-dependent correction to the (static) propagator. A dot vertex for a dashed line represent the $\vec{v} \cdot \vec{A}$ vertex of the world-line action (3.5). Diagram (a) is the Newtonian potential. Each diagram in (b,c) matches a specific terms in the EIH Lagrangian (4.1) as described in the text. Diagrams in (b) represent the v dependent EIH terms, while (c) are v-independent. Note that the relatively complicated triple vertex diagram, fig. 5(a) of [1], disappeared.

$$S = -\frac{1}{16\pi G} \int dt\, d^{d-1}x \sqrt{\gamma} \left[R[\gamma] + \frac{d-2}{d-3}\, (\partial\phi)^2 - \frac{1}{4}\, e^{2(d-2)\phi/(d-3)}\, F^2 \right] \tag{4.3}$$

$$S \supset \frac{1}{16\pi G} \int dt\, d^{d-1}x\, \frac{d-2}{d-3}\, \dot{\phi}^2 \ . \tag{4.4}$$

and

$$\begin{aligned} S_{pp} \equiv -m_0 \int d\tau &= -m_0 \int dt\, e^{\phi} \sqrt{(1 - \vec{A}\cdot\vec{v})^2 - e^{-2(d-2)\phi/(d-3)}\, \gamma_{ij}\, v^i\, v^j} = \\ &= -m_0 \int dt \left(1 - \frac{1}{2} v^2 + \phi - \vec{A}\cdot\vec{v} + \frac{d-1}{2(d-3)}\, \phi\, v^2 + \dots \right) \end{aligned} \tag{4.5}$$

The Feynman diagrams contributing to the 1PN correction to the Newtonian gravitational action are the same diagrams as in 4d, namely those shown in figure 1. The triple vertex diagram gets eliminated as previously. As a result the d-dimensional generalization of EIH Lagrangian is given by

$$\begin{aligned}\mathcal{L}_{EIH} &= \frac{1}{8}\sum_{a=1}^{2} m_a\,\vec{v}_a^{\,4} \\ &+ \frac{1}{2}U_N(r)\left[\frac{d-1}{(d-3)}(\vec{v_1}^{\,2}+\vec{v_2}^{\,2}) - \frac{4(d-2)}{d-3}\vec{v_1}\cdot\vec{v_2} + (\vec{v_1}\cdot\vec{v_2} - (d-3)(\vec{v_1}\cdot\hat{r})(\vec{v_2}\cdot\hat{r}))\right] \\ &- \frac{m_1+m_2}{2m_1m_2}\,U_N(r)^2 \end{aligned} \tag{4.6}$$

where $U_N(r)$ is the Newtonian potential energy corresponding to figure 1(a)

$$U_N(r) = 2\,\frac{d-3}{d-2}\,\frac{\Gamma(\frac{d-3}{2})}{\pi^{(d-3)/2}}\,\frac{Gm_1m_2}{r^{d-3}} = \frac{8\pi}{(d-2)\Omega_{d-2}}\,\frac{Gm_1m_2}{r^{d-3}}\ , \tag{4.7}$$

and Ω_{d-2} denotes the volume of the $d-2$ dimensional sphere. The v^4 term originates from the correction to the kinetic energy and comes from the expansion of (4.5); the terms on second line are of order v^2 and come from the diagrams in figure 1(b): the first term proportional to $v_1^2+v_2^2$ comes from the top diagram, the second term proportional to $\vec{v_1}\cdot\vec{v_2}$ comes from the middle diagram while the last terms come from the bottom diagram; finally the term on the third line comes from the top diagram of figure figure 1(c).

A comparison with [28] reveals that our computation confirms theirs for all diagrams but one: the numerical coefficient in the term corresponding to figure 1(c) is different. We are told that it will be corrected in a revision of [28].

4.2 Spin interactions

NRG-fields will significantly simplify the computation of spin interactions, and stress the role of the gravito-magnetic field. Here we limit ourselves to a discussion of the relevant diagrams in terms of NRG-fields, leaving for the future the detailed calculation which involves issues such as the spin supplementary condition (the relation between J_0^{ij} and $\vec{v}$) whose current presentation in the literature leaves room for improved understanding, in our opinion.

The leading spin-spin interaction

$$S_{ss} = G\int dt\,\frac{\vec{J_1}\cdot\vec{J_2} - 3(\vec{J_1}\cdot\hat{r})(\vec{J_2}\cdot\hat{r})}{r^3} \tag{4.8}$$

where $\hat{r} := \vec{r}/r$, is given by the Feynman diagram in figure 2(a) and is seen to consist of an exchange of the gravito-magnetic field A_i at order 2PN.

The next to leading spin-spin interaction was computed in [21] (see also [22, 23, 24]) in terms of 5 diagrams (fig. 1 and fig.2 of [21]) which are quite analogous to the 5 diagrams for EIH in [1]. In terms of NRG-fields we obtain the diagrams of figure 3. The pay-off is that the "voluminous" 3-graviton vertex in fig. 2(a) of [21] is replaced by the compact $\phi\,F^2$ cubic vertex which can be read from the dimensionally reduced action (3.3) avoiding the need for a symbolic manipulation program. Actually a similar diagram appeared in the calculation of the vanishing renormalization of the angular momentum in [11]. Following the appearance of the first arXiv version of this paper, the next to leading spin-spin

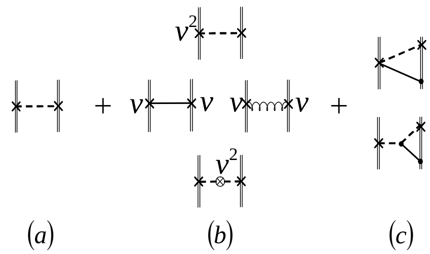

Figure 2: Diagrams representing spin-spin interactions. The cross vertex represents the $\vec{J} \cdot \vec{B}$ vertex (3.6). Unlike the previous figure, here the spring-like line represents the 3-tensor γ. Other notation remains the same. Diagram (a) represents the leading order at 2PN, in terms of the gravito-magnetic field. The other diagrams represent the next to leading contributions at 3PN: (b) represent v^2 corrections, while (c) represent Gm/r corrections.

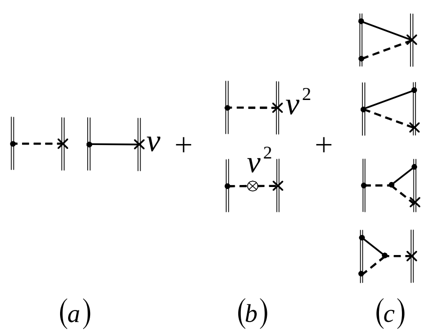

Figure 3: Diagrams representing contributions to the spin-orbit interaction. Diagrams (a) represent the leading order at 1.5PN. The diagrams in (b) and (c) are a sample of those representing the first correction at 2.5PN. The notation is the same as in the previous figures.

interaction was computed with NRG-fields [25] and was found to considerably simplify the calculations.

The situation for the spin-orbit interaction seems to be quite similar. The leading contribution is given now by the two diagrams in figure 3(a), which are related through Galilei invariance. The next to leading terms at order 2.5PN were computed in [26] in terms of equations of motion and in [27] in terms of a Hamiltonian. Here we limit ourselves to pointing out some of the diagrams which would appear at this order in figures 3(b,c).

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