# On the dynamics of subcontinua of a tree

MYKOLA MATVIICHUK

Given a tree map $f : T \to T$, we study the dynamics of subcontinua of $T$ under action of $f$. In particular, we prove that a subcontinuum of $T$ is either asymptotically periodic or asymptotically degenerate. As an application of this result, we show that zero topological entropy of the system $(T, f)$ implies zero topological entropy of its functional envelope (endowed with the Hausdorff metric).

## 1 Introduction

By a *(topological) dynamical system* we mean a pair $(X, f)$ where $X$ is a compact metrizable topological space and $f : X \to X$ is a map, i.e. continuous function. Recall that a *continuum* is a nonempty compact connected metric space. Given a dynamical system $(X, f)$, one can in a natural way extend $f$ to a map $\mathcal{F}$ on the hyperspace $\mathrm{Con}(X)$ of all subcontinua of $X$. We call the system $(\mathrm{Con}(X), \mathcal{F})$ a *connected envelope* (where $\mathrm{Con}(X)$ is endowed with the topology induced by the Hausdorff metric). The natural question arises here: what is the connection between dynamical properties of the base map $f$ and its extension $\mathcal{F}$. For papers related to this topic, see [1], [5], [8], [11].

In the present paper we deal with the case when underlying phase space is a tree. At the end of the paper we will prove (Theorem 4) the equality of topological entropies of a dynamical system on a tree and its connected envelope. As a consequence, we will get a nice result concerning a system which a dynamical system on a tree induces on the hyperspace of all maps on this tree endowed with the Hausdorff metric; following [4] we call it a *functional envelope*. Namely, we prove (Theorem 5) that if a system on a tree has zero topological entropy, then so does its functional envelope (cf. with result due to Glasner and Weiss [9] who proved that zero entropy of any topological dynamical system implies zero entropy of the system induced on the space of all probability Borel measures on the phase space). For the case of interval both these results were done in [12].

In order to prove the mentioned results we study the dynamics of a subcontinuum of a tree under action of a tree map. First, in Section 2, we consider the situation when the subcontinuum contains a periodic point of the map. In [7] it was proved that if a subinterval of an interval contains a periodic point of an interval map, then it is asymptotically periodic with respect to this map. We prove (see Theorem 1) the generalization of this result for tree maps, i.e. we prove that each subcontinuum of a tree containing a periodic point of a tree map is asymptotically periodic with respect to the map. Unfortunately, our method does not provide a good estimate of period of the asymptotically periodic set. For the case of interval such an estimate is known; namely, the period of the set is a divisor of doubled period of each periodic point it contains [7].

Next, in Section 3, we consider in some sense the opposite situation, when only the endpoints of a tree are permitted to be periodic. Recall that, by the fixed point property, it must have at least one of them. It turns out that in this setting there is a unique attracting fixed point which attracts everything which does not eventually glue to a periodic orbit (see Lemmas 4 and 5). As a consequence, we get that any subcontinuum of the tree converges to the attracting fixed point, provided that it does not glue to a periodic orbit; and if it does, then, by previous results, it is asymptotically periodic (see Theorem 2 and the proof).

Finally, in Section 4, we prove that any subcontinuum of a tree when it is iterated under a tree map is either asymptotically periodic or asymptotically degenerate, or both (see Theorem 3). For interval maps such a characterization was known (see for instance [12]) and for transitive graph maps similar result was recently proved in [11]. Still for general graph maps the situation is unclear. We finish the paper with the above-mentioned result that zero entropy of a tree dynamical system implies zero entropy of its functional envelope. We remark that this phenomenon is essentially due to dimension one. There are quite simple examples of zero entropy maps on the square for which the functional envelope has infinite entropy (e.g. $f(x,y) = (x,y^2)$, $(x,y) \in [0,1]^2$ works). So, the following open question seems to be quite natural here.

*Question.* Does Theorem 5 remain true for a) graphs with loops, b) dendrites?

## 2 The dynamics of a subcontinuum of a tree containing a periodic point.

First, let us recall some definitions and fix notations. By an *interval* we mean any space homeomorphic to $[0,1] \subset \mathbb{R}$. A *tree* is a uniquely arcwise connected space that is either a point or a union of finitely many intervals. Remark that any tree is a continuum. Any continuous function from a tree into itself is called a *tree map*. If $T$ is a tree and $x \in T$, we define the *valence* of $x$ to be the number of connected components of $T \setminus \{x\}$. Each point of valence one will be called an *endpoint* of $T$ and the set of such points will be denoted by $\mathrm{En}(T)$. A point of valence greater than one will be called a *cut-point* and the set of cut-points of $T$ will be denoted by $\mathrm{Cut}(T)$. A point of valence different from two will be called a *vertex* of $T$, and the set of vertices of $T$ will be denoted by $V(T)$. The closure of each connected component of $T \setminus V(T)$ will be called an *edge* of $T$.

If $(X, f)$ is a dynamical system and $x \in X$ then the $\omega$*-limit set* of $x$ under $f$ is the set $\omega_f(x)$ of all limit points of the trajectory $x, f(x), f^2(x), \ldots$ regarding it as a sequence. Given a subset $A$ of a topological space, we denote by $\overline{A}$, $\mathrm{Int}(A)$ and $\partial A$ the closure, the interior and the boundary of $A$, respectively. Moreover, for $x \in X$ we will denote by $\mathrm{Comp}\,(A, x)$ the (connected) component of $A$ containing $x$ if $x \in A$, and the singleton $\{x\}$ if $x \notin A$. For a finite set $B$ we will denote its cardinality by $|B|$.

Let us summarize some simple topological facts we will need. Let $T$ be a tree, $M$ be a subcontinuum of $T$ and $A$, $A_n, n \geq 0$ be connected subsets of $T$. Then the following holds.

- $M$ is a tree. Also the factor space $T/_M$ (i.e. we just identify all points within $M$) is a tree.
- The set $\partial A$ is finite.
- Each point in $\overline{A} \setminus A$ is an endpoint of $\overline{A}$.
- The set $\overline{\mathrm{Comp}\,(T \setminus A, x)} \cap \overline{A}$ is a singleton for each $x \in T$.
- If $A_n \cap A_{n+1} \neq \emptyset$ for each $n \geq 0$, then $\cup_{n=0}^{\infty} A_n$ is again a connected set.
- The set $\cap_{n=0}^{\infty} A_n$ is either connected or empty.

Given a tree $T$, a sequence $\{x_n\}_{n=0}^{\infty} \subseteq T$ is said to be *consistent with* $x \in T$ if $x_m \in \mathrm{Comp}\,(T \setminus \{x_n\}, x)$ whenever $m > n \geq 0$. Of course, a sequence $\{x_n\}_{n=0}^{\infty}$ which is consistent with some $x$ does not need to be convergent; consider the example $T = [-1, 1]$, $x = 0$ and $x_n = (-1)^n \cdot \dfrac{n+1}{2n}$, $n \geq 1$. However, as it is in the example, one can always split the sequence into a finite number of convergent (and, in some sense, monotone) subsequences.

**Lemma 1** *Let $T$ be a tree, $x \in T$ and $\{x_n\}_{n=0}^{\infty} \subseteq T$ be a sequence consistent with $x$. Then there is a finite partition of the set of nonnegative integers into the sets $L_1, L_2, \ldots, L_k$ such that $[x, x_m] \subseteq [x, x_n]$ whenever $m > n$ and $m, n \in L_i$ for some $1 \leq i \leq k$. In particular, each subsequence $\{x_n\}_{n \in L_i}$, $1 \leq i \leq k$ is convergent.*

**Proof** The proof is straightforward. We just express $T$ as the union $\cup_{i=1}^{k}[x,y_i]$ where $y_1,y_2,\ldots,y_k$ is an enumeration of all endpoints of $T$, and then define each $L_i$ to be the set of those indices $n$ for which $x_n \in [x,y_i]$ but $x_n \notin [x,y_j]$ for any $j<i$. $\square$

Given a metric space $X$, we denote by $\mathrm{Con}(X)$ the space of all subcontinua of $X$ endowed with the following topology. For a sequence $\{A_n\}_{n=0}^{\infty} \subset \mathrm{Con}(X)$ we define:

$$\begin{aligned}
\liminf A_n = \{ \quad & x \in X : \text{if } U \text{ is an open subset of } X \text{ with } U \ni x,\\
& \text{then } U \cap A_i \neq \emptyset \text{ for all but finitely many } n\};
\end{aligned}$$

$$\begin{aligned}
\limsup A_n = \{ \quad & x \in X : \text{if } U \text{ is an open subset of } X \text{ with } U \ni x,\\
& \text{then } U \cap A_i \neq \emptyset \text{ for infinitely many } n\}.
\end{aligned}$$

In fact, $\liminf A_n, \limsup A_n \in \mathrm{Con}(X)$ and $\liminf A_n \subseteq \limsup A_n$. If $\liminf A_n = A = \limsup A_n$, then we say that $\{A_n\}_{n=0}^{\infty}$ *converges to* $A$ *as* $n \to \infty$, written $A_n \to A$, $n \to \infty$. It is well known that this convergence defines a topology on $\mathrm{Con}(X)$, and $\mathrm{Con}(X)$ endowed with this topology is a compact metrizable topological space. In fact, this topology is given by the Hausdorff metric, which we will define later when we need it explicitly.

The following easy lemma shows that, given a tree $T$ and $M \in \mathrm{Con}(T)$, convergence in the space $\mathrm{Con}(T)$ is given by convergence in the spaces $\mathrm{Con}(M)$ and $\mathrm{Con}(T/_M)$. Denote by $\pi_M$ the canonical projection $T \to T/_M$.

**Lemma 2** *Let $T$ be a tree and $A_n, n \geq 1, M \in \mathrm{Con}(T)$. Suppose that $M \cap A_n \neq \emptyset$ for each n and both the sequences $\{M \cap A_n\}_{n=0}^{\infty} \subseteq \mathrm{Con}(M)$ and $\{\pi_M(A_n)\}_{n=0}^{\infty} \subseteq \mathrm{Con}(T/_M)$ converge in the corresponding spaces. Then the sequence $\{A_n\}_{n=0}^{\infty}$ converges in* $\mathrm{Con}(T)$.

**Proof** If $x \in T \setminus M$, then one can take an open set $U \ni x$ such that $U \cap M = \emptyset$. So, each $x \in T \setminus M$ belongs to $\liminf A_n$ (resp. $\limsup A_n$) iff $x$ belongs to $\liminf \pi_M(A_n)$ (resp. $\limsup \pi_M(A_n)$). Next, we are going to prove $M \cap \liminf A_n = \liminf(M \cap A_n)$ and $M \cap \limsup A_n = \limsup(M \cap A_n)$. To achieve this, it suffices to show that, given $x \in M$ and a connected open subset $U$ of $T$ with $U \ni x$, if $U$ intersects $A_n$ then it intersects $M \cap A_n$, for each $n$. Let $x$ and $U$ be as above and assume that $U \cap A_n \neq \emptyset$. We take $y \in U \cap A_n$, $z \in M \cap A_n$ and $u \in [z,y] \subseteq A_n$ such that $[z,u] = [z,y] \cap M$. Then $u \in M \cap A_n$, and so it is enough to show $u \in U$. To this end, observe that $u \in \overline{\mathrm{Comp}\,(T \setminus M, y)}$, and hence $\overline{\mathrm{Comp}\,(T \setminus M, y)} \cap M = \{u\}$. Let $v \in [x,y] \subseteq U$

be such that $[x,v] = [x,y] \cap M$. Then $v \in M \cap U$ and $v \in \overline{\mathrm{Comp}\,(T \setminus M, y)}$. Thus $\{v\} = \overline{\mathrm{Comp}\,(T \setminus M, y)} \cap M = \{u\}$ which leads to $u = v \in U$.

To sum it up, we have proved that $(T \setminus M) \cap \liminf A_n = (\liminf \pi_M(A_n)) \setminus \pi_M(M)$ and $M \cap \liminf A_n = \liminf(M \cap A_n)$, and also the same with $\liminf$ replaced by $\limsup$. Therefore, $\liminf A_n = \limsup A_n = \big(A' \setminus \pi_M(M)\big) \cup A''$, where $A'$ and $A''$ denote the limits of the sequences $\{\pi_M(A_n)\}_{n=0}^{\infty}$ and $\{M \cap A_n\}_{n=0}^{\infty}$ respectively. □

Given a dynamical system $(X, f)$, a set $M \subseteq X$ is called *invariant* (resp. *strongly invariant*) if $f(M) \subseteq M$ (resp. $f(M) = M$). For a subset $A \subseteq X$, we denote by $\mathrm{Ls}(f, A)$ the set-theoretical limit superior of the sequence $\{f^n(A)\}_{n=0}^{\infty}$, i.e. $\mathrm{Ls}(f, A) = \cap_{m=0}^{\infty} \cup_{n=m}^{\infty} f^n(A)$.

**Lemma 3** *Let $f : T \to T$ be a tree map and $A \in \mathrm{Con}(T)$ contains a fixed point $x$ of $f$. Then $\mathrm{Ls}(f, A)$ is strongly invariant connected subset of $T$ containing $x$.*

**Proof** Let $\Delta = \mathrm{Ls}(f, A) = \cap_{m=0}^{\infty} \Delta_m$, where $\Delta_m = \cup_{n=m}^{\infty} f^n(A)$ for any $m \geq 0$. First, $\Delta$ is a connected set containing $x$, because each $f^n(A)$ is. Next, since $f(\Delta_m) = \Delta_{m+1}$ for each $m \geq 0$ and $\Delta_m$ decreases on $m$, the set $\Delta$ is invariant as intersection of a family of invariant sets. On the other hand, fix any $x \in \Delta$ and let us show that $x = f(y)$ for some $y \in \Delta$. Whatever the $m \geq 0$, from $x \in \Delta_{m+1} = f(\Delta_m)$ we get $x = f(y_m)$ for some $y_m \in \Delta_m$. Consider the sequence $\{y_m\}_{m=0}^{\infty}$. Let $m_0 = 0$. If $y_m \notin \mathrm{Comp}\,(T \setminus \{y_{m_0}\}, x)$ for infinitely many $m \geq m_0$, then $y_{m_0} \in [x, y_m] \subseteq \Delta_m$ for infinitely many $m \geq m_0$, and so $y_{m_0} \in \Delta$. Otherwise, there is $m_1 > m_0$ such that $y_m \in \mathrm{Comp}\,(T \setminus \{y_{m_0}\}, x)$ for all $m \geq m_1$. On the next step, if $y_m \notin \mathrm{Comp}\,(T \setminus \{y_{m_1}\}, x)$ for infinitely many $m \geq m_1$, then $y_{m_1} \in [x, y_m] \subseteq \Delta_m$ for infinitely many $m \geq m_1$, and so $y_{m_1} \in \Delta$. Otherwise, there is $m_2 > m_1$ such that $y_m \in \mathrm{Comp}\,(T \setminus \{y_{m_1}\}, x)$ for all $m \geq m_2$. Repeating this procedure, we either get that $x = f(y_{m_r})$, $y_{m_r} \in \Delta$ for some $r \geq 0$ or get the subsequence $\{y_{m_r}\}_{r=1}^{\infty}$ which is consistent with $x$ (see the definition of consistent sequence before Lemma 1) and such that $x = f(y_{m_r})$, $y_{m_r} \in \Delta_{m_r}$ for each $r \geq 0$. In the former case we get exactly what we need to complete the proof of strong invariance of $\Delta$. In the latter case, applying Lemma 1, we get a convergent subsequence $\{y_{m_r}\}_{r \in L}$ (here $L$ is an infinite subset of the set of nonnegative integers) such that $[x, y_{m_s}] \subseteq [x, y_{m_r}]$ whenever $s > r$ and $s, r \in L$. Therefore, if $y$ denotes the limit of $\{y_{m_r}\}_{r \in L}$, then $y \in \cap_{r \in L} [x, y_{m_r}] \subseteq \cap_{r \in L} \Delta_{m_r} = \Delta$ and, by continuity, $x = f(y)$. So, we have showed that the set $\Delta$ is strongly invariant. □

Given a dynamical system $(X, f)$ and a nonempty, closed and invariant set $M \subseteq X$, one can consider a *subsystem* $(M, f|_M)$, where $f|_M$ is the restriction of $f$ to $M$. In the same setting, one can define a *factor-system* $(X,f)/_M := (X/_M, f/_M)$, where $X/_M$ is the factor space and $f/_M : X/_M \to X/_M$ is given by $f/_M = \pi_M \circ f \circ \pi_M^{-1}$ where $\pi_M : X \to X/_M$ is the canonical projection.

Let $f : T \to T$ be a tree map. A continuum $A \in \mathrm{Con}(T)$ is called *asymptotically periodic* under $f$ if the sequence $f^{pn}(A), n \geq 0$ converges for some $p \geq 1$.

**Theorem 1** *Let $f : T \to T$ be a tree map and $A \in \mathrm{Con}(T)$ contains a periodic point of $f$. Then $A$ is asymptotically periodic under $f$.*

**Proof** Let $x \in A$ be a periodic point. In the sequel we will freely replace $f$ with $f^k$ and $A$ with $f^m(A)$ for some positive integers $k, m$, because it is enough to prove that the sequence $f^{pkn+m}(A), n \geq 0$ converges for some $p \geq 1$. Thus, at first, it is convenient to assume that $x$ is a just fixed point.

Let $\Delta = \mathrm{Ls}(f, A)$. By Lemma 3, the set $\Delta$ is a connected strongly invariant set containing $x$. If it happens that $\Delta = \{x\}$ then we are done, because we easily get $f^n(A) \to \{x\}, n \to \infty$. Otherwise, we express $\overline{\Delta}$ as the union $\cup_{i=1}^k [x, x_i]$ where $k = |\,\mathrm{En}(\overline{\Delta}) \setminus \{x\}|$ and $\{x_1, x_2, \ldots, x_k\}$ is an enumeration of all endpoints of $\overline{\Delta}$ but possibly $x$ (if $x \in \mathrm{En}(\overline{\Delta})$). Here some of $x_i$'s belong to $\Delta$, while the others belong to $\overline{\Delta} \setminus \Delta$. Next, passing to subsystems and factor-systems, we will decrease the number of $x_i$'s.

First, we consider the case when all $x_i$'s belong to $\Delta$, i.e. $\Delta$ is closed. Since $f(\Delta) = \Delta$ and $\Delta = \cup_{i=1}^k [x, x_i]$, for each $1 \leq i \leq k$ there is $1 \leq j \leq k$ such that $f[x, x_j] \supseteq [x, x_i]$. Hence, there are $1 \leq i, j \leq k$ such that $f^j[x, x_i] \supseteq [x, x_i]$. By definition of $\Delta$, there is a positive integer $m$ such that $x_i \in f^m(A)$. Replacing $A$ with $f^m(A)$ and $f$ with $f^j$, we can assume that $x_i \in A$ and $f[x, x_i] \supseteq [x, x_i]$. Now, we are going to prove that $f^{pn}(A)$ converges as $n \to \infty$, for some $p \geq 1$. To this end, we are going to use Lemma 2 for $M = \cup_{n \geq 0} f^n[x, x_i]$ and for the sequence $A_{np} = f^{np}(A)$, $n \geq 0$. Since clearly $M \cap A_n \to M$, $n \to \infty$, all we need is to prove that $\pi_M(A_{pn})$ converges as $n \to \infty$, for some $p \geq 1$. We remark that $\pi_M(f^n(A)) = g^n(B)$, $n \geq 0$ where $g = f/_M$ and $B = \pi_M(A)$. Thus we consider the factor-system $(T/_M, g)$ and continuum $B = \pi_M(A) \subseteq T/_M$ which contains the fixed point $\pi_M(x)$ of $g$. Therefore, we have reduced the proving of asymptotical periodicity of $A$ under $f$ to the proving of asymptotical periodicity of $B$ under $g$. The set $\mathrm{Ls}(g, B) = \pi_M(\Delta)$ is again, by Lemma 3, a strongly invariant continuum containing the fixed point, but now $\mathrm{En}(\mathrm{Ls}(g, B)) \setminus \{\pi_M(x)\}$ has at most $k-1$ elements, because $\pi_M[x, x_i] = \{\pi_M(x)\}$. By repeating this procedure we will eventually get that $\Delta = \{x\}$, and so the proof is complete for the case of closed $\Delta$.

Now, assume that $\Delta$ is not closed. Since $\Delta$ is strongly invariant, $\overline{\Delta}\setminus\Delta$ is also strongly invariant. Moreover, $\overline{\Delta}\setminus\Delta$ is finite as it is contained in the boundary of connected subset of a tree. Replacing $f$ with $f^{|\overline{\Delta}\setminus\Delta|!}$, we can assume that all points of $\overline{\Delta}\setminus\Delta$ are fixed under $f$. Let $x_i\in\overline{\Delta}\setminus\Delta$, $f(x_i)=x_i$. Let us show that $\overline{\Delta_m}$ contains only one preimage of $x_i$ for each $m$ which is large enough, where $\Delta_m=\cup_{n=m}^{\infty}f^n(A)$. In order to see this, note that $\{x\in\overline{\Delta}:f(x)=x_i\}$ is just a singleton $\{x_i\}$, for the set $\Delta$ is invariant and each point in $\overline{\Delta}\setminus\Delta$ is fixed. Choose $m'$ such that $x_i\notin\Delta_{m'}$. Then $x_i\in\overline{\Delta_{m'}}\setminus\Delta_{m'}$, in particular, $x_i$ is endpoint of $\overline{\Delta_{m'}}$. Consider the closed set $\{x\in\overline{\Delta_{m'}}:f(x)=x_i\}$. As we remarked above, it intersects $\overline{\Delta}$ at exactly one point $x_i$. Moreover, $x_i$ is isolated in $\overline{\Delta_{m'}}$, for $x_i$ is an endpoint for both $\overline{\Delta_{m'}}$ and $\overline{\Delta}$. Therefore, $\{x\in\overline{\Delta_m}:f(x)=x_i\}=\{x_i\}$ for each $m\geq m'$ which is large enough. So, replacing $(T,f)$ with the subsystem $(\overline{\Delta_m},f|_{\overline{\Delta_m}})$, we can assume that $x_i$ is an endpoint of $T$ and $f^{-1}(x_i)=\{x_i\}$.

Since $x_i\notin\Delta_m\supseteq f^m(A)$, there is a small enough neighbourhood $[x_i,y)$ of $x_i$ such that $T\setminus[x_i,y)\supseteq f^m(A)$. It follows that if a closed invariant set contains $T\setminus[x_i,y)$, it must coincide with whole $\overline{\Delta_m}=T$. Since $f^{-1}(x_i)=\{x_i\}$, we can take a neighbourhood $[x_i,z)\subseteq[x_i,y)$ of $x_i$ such that $T\setminus[x_i,z)\supseteq f(T\setminus[x_i,y))$. Then $f[x_i,z]\supset[x_i,z]$, for otherwise the set $T\setminus[x_i,z)=(T\setminus[x_i,y))\cup[x,z]$ would be proper closed invariant subset of $T$ which contains $T\setminus[x_i,y)$. Similarly we get $\overline{\cup_{n\geq0}f^n[x,z]}\ni x_i$, for otherwise the set $(T\setminus[x_i,y))\cup\left(\overline{\cup_{n\geq0}f^n[x,z]}\right)$ would be proper closed invariant subset of $T$ which contains $T\setminus[x_i,y)$. Now, we using Lemma 2 pass to the factor-system $(T/_M,f/_M)$, where $M=\overline{\cup_{n\geq0}f^n[x,z]}$. Putting $A_n=f^n(A)$, $n\geq m$ we get $M\cap A_n\to M$, $n\to\infty$, so we need only to show that $\pi_M(A_{pn})$ converges as $n\to\infty$, for some $p\geq1$. Since $\pi_M(A_{n+m})=g^n(B)$ where $g=f/_m$ and $B=\pi_M(f^m(A))$, we need only to prove that the continuum $B$, which contains the fixed point $\pi_M(x)$, is asymptotically periodic under the tree map $g$. We remark that $\mathrm{Ls}(g,B)=\pi_M(\Delta)$, and $\overline{\mathrm{Ls}(g,B)}\setminus\mathrm{Ls}(g,B)=\pi_M(\overline{\Delta})\setminus\pi_M(\Delta)$ consist of at most $|\overline{\Delta}\setminus\Delta|-1$ points, for $\pi_M(x_i)=\pi_M(x)\in\pi_M(\Delta)$. Thus step by step we reduce the general case to the case when $\overline{\Delta}\setminus\Delta$ is empty, i.e. $\Delta$ is closed (this case was considered earlier). $\square$

## 3 The dynamics of a tree system without periodic cut-points.

Recall that, given a map $f:X\to X$, a point $x\in X$ is called an *attracting fixed point* (AFP, for short) if for any open set $U\ni x$ there is an open set $U\supseteq V\ni x$ such that $f(\overline{V})\subseteq V$. Let $f:T\to T$ be a tree map such that no cut-point of $T$ is fixed under $f$. Then one can easily see that if $x\in\mathrm{En}\,T$ such that $f(y)\in[x,y)$ for some $y$ within the edge of $T$ containing $x$, then $x$ is AFP. On the other hand, if $x \in \operatorname{En} T$ is AFP, then $f(y) \in [x, y)$ for each $y$ within the edge of $T$ containing $x$.

**Lemma 4** *Let $f : T \to T$ be a tree map such that no cut-point of $T$ is fixed. Then there is unique AFP of $f$ in $T$ (which is, of course, an endpoint of $T$).*

**Proof** *Existence*. We will say that a point $y \in \mathrm{Cut}(T)$ *moves towards* $x \in \mathrm{En}(T)$ if $f(y) \in \mathrm{Comp}\,(T \setminus \{y\}, x)$ (equivalent condition is $y \notin [x, f(y)]$).

**Claim.** *For each* $1 \leq k \leq |\,\mathrm{En}(T)| - 1$ *there is a cut-point $y$ and an endpoint $x$ such that $y$ moves towards $x$ and* $\mathrm{Comp}\,(T \setminus \{y\}, x)$ *contains at most $k$ endpoints of $T$.*

For $k = |\,\mathrm{En}(T)| - 1$ our claim is clear, because we can take arbitrary $y \in \mathrm{Cut}(T)$ and then any endpoint $x$ from $\mathrm{Comp}\,(T \setminus \{y\}, f(y))$, so one can see that our claim holds for the chosen $x$ and $y$. By induction, assume we have proved the claim for some $k$ and let us prove it for $k - 1$.

So, suppose that a cut-point $y$ moves towards an endpoint $x$ and $\mathrm{Comp}\,(T \setminus \{y\}, x)$ contains at most $k$ endpoints of $T$. We take $y_1$ close enough to $x$ so that $y_1$ belongs to the edge of $T$ containing $x$ and $[x, y_1] \subseteq [x, y)$. If $x$ is AFP, then we are done, otherwise we get $f(y_1) \notin [x, y_1)$. The latter is equivalent to $y_1 \in [x, f(y_1))$. Let us define a continuous map $g : [y_1, y] \to [y_1, y]$ by $g = Pr_{[y_1,y]} \circ f|_{[y_1,y]}$, where $Pr_{[y_1,y]}$ denotes the "projection" onto the set $[y_1, y]$, i.e. $Pr_{[y_1,y]}(z)$ is the unique point in $\overline{\mathrm{Comp}\,(T \setminus [y_1, y], z)} \cap [y_1, y]$. By the fixed point property, there is $y_2 \in [y_1, y]$ such that $g(y_2) = y_2$. Therefore, $y_2 = Pr_{[y_1,y]}(f(y_2))$, and so $[y_2, f(y_2)] \cap [y_1, y] = \{y_2\}$. Moreover $y_2 \in [y_1, y)$, because $y_1 \in [x, y)$ and $y$ moves towards $x$. So, $y \notin [x, y_2]$ which leads to $y_2 \in \mathrm{Comp}\,(T \setminus \{y\}, x)$. This means that all the components of $T \setminus \{y_2\}$ but $\mathrm{Comp}\,(T \setminus \{y_2\}, y)$ are subsets of $\mathrm{Comp}\,(T \setminus \{y\}, x)$. On the other hand, $y_2 \in [y, f(y_2)]$, which means that $\mathrm{Comp}\,(T \setminus \{y_2\}, f(y_2))$ is subset of $\mathrm{Comp}\,(T \setminus \{y\}, x)$, while it does not contain all the endpoints of $T$ which are within $\mathrm{Comp}\,(T \setminus \{y\}, x)$ (namely, it does not contain $x$). Thus, $\mathrm{Comp}\,(T \setminus \{y_2\}, f(y_2))$ contains at most $k - 1$ endpoints of $T$, and the claim follows.

In particular, if $k = 1$ in the claim above, we get that there is a cut-point $y$ moving towards an endpoint $x$ and such that $\mathrm{Comp}\,(T \setminus \{y\}, x)$ is just the semi-open interval $[x, y)$. So, $x$ is AFP.

*Uniqueness*. On the contrary, suppose that there are two distinct AFP's $x$ and $x'$. Consider the set

$$W = \{y \in (x, x') : y \text{ moves towards } x\}$$

By continuity, both the sets $W$ and $(x, x') \setminus W$ are nonempty and open in $(x, x')$, which contradicts connectedness of $(x, x')$. $\square$

**Remark 1** *If a tree map $f : T \to T$ is free of periodic cut-points, then for each iterate $f^n$ the unique AFP is well defined and coincides with that of $f$. The reason for that is the following. If $s \in \mathrm{En}(T)$ is the AFP of $f$, then for each neighbourhood of the form $[s,y)$ we have $f[s,y) \subset [s,y)$. Thus $f^n[s,y) \subset [s,y)$ for each $n \geq 0$, and so $s$ is the AFP of each $f^n$.*

**Remark 2** *One can show, in the same way as in the proof of uniqueness above, that each cut-point of $T$ moves towards the AFP. Thus, taking into account Remark 1, we see that if $x \in T$, then either $f^n(x) \in \mathrm{En}(T)$ for some $n \geq 0$ or the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is consistent with the AFP.*

Next, we describe the dynamics of points and subcontinua in the system on a tree without periodic cut-points.

**Lemma 5** *Let $f : T \to T$ be a tree map such that no cut-point of $T$ is periodic. Let $s \in \mathrm{En}(T)$ be its unique AFP. Then for each $x \in T$ either*

*(a) $f^n(x)$ is a periodic cut-point for some $n \geq 0$, or*

*(b) $f^n(x) \to s, n \to \infty$.*

**Proof** Let us suppose that no iterate $f^n(x), n \geq 0$ is periodic and prove that $f^n(x) \to s, n \to \infty$. According to Remark 2 after Lemma 4, the sequence $\{f^n(x)\}_{n=0}^{\infty}$ is consistent with $s$. Therefore, by Lemma 1, the $\omega$-limit set of $x$ is a finite subset of $\mathrm{Cut}(T) \cup \{s\}$. Once $\Omega$ is finite, it must contain a periodic point, for $\Omega$ is an invariant set. Once $\Omega \subset \mathrm{Cut}(T) \cup \{s\}$, the only periodic point it may contain is $s$. So $s \in \Omega$. Then we immediately get $f^n(x) \to s, n \to \infty$, because $s$ is an AFP. $\square$

Let $f : T \to T$ be a tree map which is free of periodic cut-point, $s \in \mathrm{En}(T)$ be its unique AFP and $[s,y]$ be the edge of $T$ containing $s$. By the *immediate basin of attraction* of $s$ we mean the open set $\mathrm{IB}(s) = \mathrm{Comp}\,(\cup_{n=0}^{\infty} f^{-n}[s,y), s)$. It is not hard to see that both $\mathrm{IB}(s)$ and $\partial\,\mathrm{IB}(s)$ are invariant sets. Clearly, if $A \in \mathrm{Con}(T)$ is a subset of the immediate basin of attraction of $s$, then $f^n(A) \to \{s\}, n \to \infty$. Of course, the immediate basin of attraction of $s$ does not need to contain all cut-point of $T$, in other words, $f^n(A)$ does not need to converge to $\{s\}$ even if $A \subset \mathrm{Cut}(T)$. However, as we will see, the only way to escape converging to $s$ is to 'cling' to some of other periodic end-points of $T$.

**Theorem 2** *Let $f : T \to T$ be a tree map such that no cut-point of $T$ is periodic. Then each $A \in \mathrm{Con}(T)$ is asymptotically periodic under $f$.*

**Proof** Fix any $A \in \mathrm{Con}(T)$. Then, by Lemma 5, either $f^m(A)$ contains a periodic point for some $m$, or $f^m(A)$ intersects the immediate basin of attraction $\mathrm{IB}(s)$ for some $m$. In the former case $A$ is asymptotically periodic in view of Theorem 1. In the latter one we consider two subcases: $f^m(A) \subseteq \mathrm{IB}(s)$ and $f^m(A) \nsubseteq \mathrm{IB}(s)$, but $f^m(A) \cap IB(s) \neq \emptyset$. If $f^m(A) \subseteq \mathrm{IB}(s)$, then we get $f^n(A) \to \{s\}, n \to \infty$. If $f^m(A) \nsubseteq \mathrm{IB}(s)$, but $f^m(A) \cap IB(s) \neq \emptyset$, then $f^m(A)$ intersects $\partial\,\mathrm{IB}(s)$. As we remarked above, $\partial\,\mathrm{IB}(s)$ is an invariant set. Moreover, it is finite as boundary of a connected subset of a tree. So $f^{m+k}(A)$ contains a periodic point for some $k$ and we, using again Theorem 1, deduce asymptotical periodicity of $A$. $\square$

## 4 Entropy of induced systems for tree maps

In this section, using our previous results, we will compute the topological entropy of connected envelope and functional envelope of a dynamical system on a tree. Throughout the section we will regard a tree as a metric, rather than topological, space.

First, we give the following description of the dynamics of subcontinua of a tree (cf. Proposition in [12]). The proof just mixes Theorems 1 and 2. Given a tree map $f : T \to T$, an element $A \in \mathrm{Con}(T)$ is called *asymptotically degenerate* under $f$ if $\mathit{diam}\, f^n(A) \to 0, n \to \infty$, where *diam* stands for diameter of the set.

**Theorem 3** *Let $f : T \to T$ be a tree map. Then each $A \in \mathrm{Con}(T)$ is either asymptotically periodic or asymptotically degenerate under $f$ (or both).*

**Proof** Fix $A \in \mathrm{Con}(T)$. If all iterates $f^n(A)$ are pairwise disjoint, then obviously $A$ is asymptotically degenerate. So, we assume that $f^k(A) \cap f^m(A) \neq \emptyset$ for some $m > k \geq 0$. Replacing $A$ with $f^k(A)$ and $f$ with $f^{m-k}$ we can assume that $A \cap f(A) \neq \emptyset$. Then the set $\cup_{n\geq 0} f^n(A)$ is an invariant connected subset of $T$. Passing to the subspace we can assume that $T = \overline{\cup_{n\geq 0} f^n(A)}$. Now, if there is a periodic cut-point in $T$, then it belongs to some $f^n(A)$, and thus by Theorem 1 $A$ is asymptotically periodic. On the other hand, if no cut-point of $T$ is periodic, then by Theorem 2 $A$ is asymptotically periodic, too. $\square$

The notion of *topological entropy* of a system on a compact topological space was introduced by Adler, Konheim and McAndrew in [2] as a measure of chaotic character of a dynamical system. In this paper we will use the Bowen-Dinaburg's definitions of the topological entropy (see e.g. [6]) for systems on compact metric spaces, which agree with Adler-Konheim-McAndrew's one for systems on topological metrizable spaces. Let $(X,\rho)$ be a compact metric space and let $f: X \to X$ be a map. Fix $n \geq 1$ and $\varepsilon > 0$. Consider another metric $\rho^{(n)}$ which takes into account the distance between the respective $n$ initial iterates of points, namely put $\rho^{(n)}(x,y) = \max_{0 \leq j < n} \rho(f^j(x), f^j(y))$. A subset $E$ of $X$ is called $(n,f,\varepsilon)$*–separated* if for every two different points $x, y \in E$ it holds $\rho^{(n)}(x,y) > \varepsilon$. We say that a subset $F \subset X$ $(n,f,\varepsilon)$*–spans* $X$, if for every $x \in X$ there is $y \in F$ for which $\rho^{(n)}(x,y) \leq \varepsilon$.

We by *sep* $(n,f,\varepsilon)$ denote the maximal possible cardinality of an $(n,f,\varepsilon)$-separated set in $X$, and by *span* $(n,f,\varepsilon)$ the minimal possible cardinality of a set which $(n,f,\varepsilon)$-spans $X$.

Then the topological entropy of $f$ is defined by

$$h(f) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log sep\,(n,f,\varepsilon) = \lim_{\varepsilon \to 0} \limsup_{n \to \infty} \frac{1}{n} \log span\,(n,f,\varepsilon)$$

The following well-known lemma (see for example [3]) shows a way of computation of entropy when a system can be divided into the smaller subsystems.

**Lemma 6** *If* $X = \bigcup_{\alpha \in A} X_\alpha$ *where each* $X_\alpha$ *is closed and invariant set then* $h(f) = \sup_{\alpha \in A} h(f|_{X_\alpha})$.

Recall that $\mathrm{Con}(X)$ denotes the space of all subcontinua of $X$ endowed with the Hausdorff metric. Given a dynamical system $(X,f)$, by its *connected envelope* we mean the system $(\mathrm{Con}(X), \mathcal{F})$, where $\mathcal{F} : \mathrm{Con}(X) \to \mathrm{Con}(X)$ is given by $\mathcal{F}(A) = f(A)$, where, as usual, $f(A)$ denotes the set of all $f(x)$, $x \in A$. Clearly, the system $(\mathrm{Con}(X), \mathcal{F})$ contains a copy of the original system $(X,f)$ (consider the subspace of all singletons $\{x\}$, $x \in X$). In [12] it was proved that topological entropy of an interval dynamical system is equal to that of its connected envelope. In [11] the same was proved for transitive systems on graphs. Our next theorem establishes this equality for any dynamical system on a tree.

**Theorem 4** *Let* $(T,f)$ *be dynamical system on a tree and* $(\mathrm{Con}(T), \mathcal{F})$ *be its connected envelope. Then* $h(\mathcal{F}) = h(f)$.

**Proof** The proof is based on Theorem 3 and Lemma 6. Consider the family of closed invariant sets $\{N_A\}_{A \in \mathrm{Con}(T)}$ where $N_A = \overline{\{f^n(A) : n \geq 0\}}$. Since each $A \in N_A$, $\mathrm{Con}(T)$ is the union of all $N_A$. Now, we can apply Lemma 6:

$$h(\mathcal{F}) = \sup_{A \in \mathrm{Con}(T)} h(\mathcal{F}|_{N_A}).$$

Let $A \in \mathrm{Con}(T)$ is given. If $A$ is asymptotically periodic, then it can be derived directly from the definition of the topological entropy that $h(\mathcal{F}|_{N_A}) = 0$. Otherwise, by Theorem 3, $A$ is asymptotically degenerate. So, the $\omega$-limit set $\omega_{\mathcal{F}}(A)$ is a subset of $T_{sing} := \{\{x\} : x \in T\}$. Thus $h(\mathcal{F}|_{N_A}) = h(f|_{\omega_{\mathcal{F}}(A)}) \leq h(\mathcal{F}|_{T_{sing}}) = h(f)$.

We see that $h(\mathcal{F}|_{N_A}) \leq h(f)$ for every $A \in \mathrm{Con}(T)$. In view of Lemma 6 this implies inequality $h(\mathcal{F}) \leq h(f)$. The converse inequality holds, because $(T, f)$ is a subsystem of $(\mathrm{Con}(T), \mathcal{F})$. □

Recall that the *Hausdorff distance* between two sets $A_1$ and $A_2$ in a metric space $X$ is given by $d_H(A_1, A_2) = \inf\{\varepsilon > 0 : A_1 \subseteq \overline{B}(A_2, \varepsilon) \text{ and } A_2 \subseteq \overline{B}(A_1, \varepsilon)\}$ where $\overline{B}(A, \varepsilon)$ denotes the union of all closed balls of radius $\varepsilon > 0$ whose centres run over $A$. This is a metric on the family of all bounded, nonempty closed subsets of $X$. As we remarked above, the Hausdorff metric generates the same topology on $\mathrm{Con}(X)$ as that given by $\liminf$ and $\limsup$.

Recall the definition of a functional envelope of a dynamical system (see [4]). For the general references see [10, 13, 14, 15]. Given a metric space $(X, \rho)$, denote the set of all continuous maps $X \to X$ by $S(X)$. We endow the space $S(X)$ with the Hausdorff metric $\rho_H$ (derived from the metric $\rho_{max}((x_1, y_1), (x_2, y_2)) = \max\{\rho(x_1, x_2),\ \rho(y_1, y_2)\}$ in $X \times X$) applied to the graphs of maps. Denote the corresponding metric space by $S_H(X)$. Given a dynamical system $(X, f)$, consider the uniformly continuous map $F : S_H(X) \to S_H(X)$ defined by $F(\varphi) = f \circ \varphi$ (first apply $\varphi$) for any $\varphi \in S_H(X)$. The space $S_H(X)$ is *not* compact (because it is not complete). However, if we view $S_H(X)$ as a subset of the space of all closed subsets of $X \times X$ endowed with the Hausdorff metric, then the closure $\overline{S_H(X)}$ will be compact. The uniformly continuous map $F$ can be uniquely extended to a continuous selfmap of a compact metric space $\overline{S_H(X)}$. We will denote this map by the same letter $F$ as well; that is $F : \overline{S_H(X)} \to \overline{S_H(X)}$. The system $(\overline{S_H(X)}, F)$ is called a *functional envelope* of $(X, f)$. Again, as in the case of connected envelope, the system $(\overline{S_H(X)}, F)$ contains a copy of the original system $(X, f)$ (consider the subspace of all constant maps).

If $X = T$ is a tree, then the extension $F : \overline{S_H(T)} \to \overline{S_H(T)}$ can be described precisely in the following way. Recall that a set-valued map $M : T \to T$ is *upper semicontinuous* if for every point $x \in T$ and every open subset $V$ of $T$ such that $V \supseteq M(x)$ the set $\{y \in T : M(y) \subseteq V\}$ contains a neighbourhood of $x$. One can prove that $\overline{S_H(T)}$ consists of graphs of all set-valued maps $T \to T$ which have nonempty connected, compact values and are upper semicontinuous, and the extension $F : \overline{S_H(T)} \to \overline{S_H(T)}$ is given by $F(\varphi) = \mathcal{F} \circ \varphi$ for any $\varphi \in \overline{S_H(T)}$.

In [12] it was proved that if an interval dynamical system has zero topological entropy, then so does its functional envelope. Now, we are going to prove the generalization of this result for dynamical systems on trees. To do this, we need the following estimates on the numbers used in the definitions of topological entropy.

**Lemma 7** *Let* $(T,f)$ *be dynamical system on a tree,* $(\mathrm{Con}(T),\mathcal{F})$ *be its connected envelope and* $(\overline{S_H(T)},F)$ *be its functional envelope. Then for any* $\varepsilon>0$*,* $n\geq 1$ *it holds*

$$sep\,(n,f,\varepsilon)^{N_1(\varepsilon)}\leq sep\,(n,F,\varepsilon)\leq span\,\left(n,\mathcal{F},\varepsilon/2\right)^{N_2(\varepsilon)},$$

*for some numbers* $N_{1,2}(\varepsilon)$ *which do not depend on* $n$ *and* $N_{1,2}(\varepsilon)\to+\infty,\varepsilon\to 0+$*.*

**Proof** Fix $\varepsilon>0$ and $n\geq 1$. First, let us prove the right-hand inequality. Let $\{T_k\}_{k=1}^N$ be a cover of $T$ with continua of diameter less than $\varepsilon$. Then for each pair $\varphi,\psi\in\overline{S_H(T)}$ the inequality $\rho_H(\varphi,\psi)>\varepsilon$ implies $d_H(\varphi(T_k),\psi(T_k))>\varepsilon$ for some $k$ (here $d_H$ denotes the Hausdorff metric on the space $\mathrm{Con}(T)$ and $\rho_H$ denotes the Hausdorff metric on the space $\overline{S_H(T)}$). Moreover, $\rho_H^{(n)}(\varphi,\psi)>\varepsilon$ implies $d_H^{(n)}(\varphi(T_k),\psi(T_k))>\varepsilon$ for some $k$. Now, suppose that there is an $(n,F,\varepsilon)$-separated set $E_0$ of cardinality $M^N+1$ where $M$ is minimal possible cardinality of a set in $\mathrm{Con}(T)$ which $(n,\mathcal{F},\varepsilon/2)$-spans $\mathrm{Con}(T)$. Consecutively, for each $1\leq k\leq N$, by Dirichlet's box principle, we take a subset $E_k\subset E_{k-1}$ of cardinality $M^{N-k}+1$ such that $d_H^{(n)}(\varphi(T_k),\psi(T_k))\leq\varepsilon$. On the last step we get a set $E_N\subset E_0$ which contains two different elements $\varphi,\psi$ such that $d_H^{(n)}(\varphi(T_k),\psi(T_k))\leq\varepsilon$ for each $1\leq k\leq N$. This implies $\rho_H^{(n)}(\varphi,\psi)\leq\varepsilon$, a contradiction to the fact that $E_0$ is $(n,F,\varepsilon)$-separated set. Thus the maximal possible cardinality of an $(n,F,\varepsilon)$-separated set is less than or equal to $M^N$. We put $N_2(\varepsilon)=N$.

Now, we are going to prove the left-hand inequality. Let $I\subseteq T$ be an edge in $T$. For convenience, we assume that $I=[0,1]$. Let $x_k=\frac{k}{K},0\leq k\leq K$ where $K=[\frac{1}{2\varepsilon}]-1$. (It suffices to prove the inequality for small enough $\varepsilon$, so we can assume that $K\geq 1$.) Let $F$ be an $(n,f,\varepsilon)$-separated set in $T$ of the maximal possible cardinality. For any $K$-tuple $\overline{y}=(y_1,y_2,\ldots,y_K)$ of elements of $F$ we define the (multivalued) map $\varphi_{\overline{y}}\in\overline{S_H(T)}$ by

- $\varphi_{\overline{y}}(x)=\{y_j\}$, if $x\in(x_{j-1},x_j)\subset I$, for some $1\leq j\leq K$,
- $\varphi_{\overline{y}}(x)=T$, if $x=x_j$ for some $0\leq j\leq K$, or $x\in T\setminus I$.

One can see that collection $\{\varphi_{\overline{y}}\}_{\overline{y}\in F^K}$ forms an $(n,F,\varepsilon)$-separated set in $\overline{S_H(T)}$. Thus $sep\,(n,F,\varepsilon)\geq|F|^K=sep\,(n,f,\varepsilon)^K$. We put $N_1(\varepsilon)=K$. □

**Theorem 5** *Let* $(T,f)$ *be dynamical system on a tree and* $(S_H(T),F)$ *be its functional envelope.*

(1) *If* $h(f) = 0$*, then* $h(F) = 0$*.*

(2) *If* $h(f) > 0$*, then* $h(F) = +\infty$*.*

**Proof** Let $h(f) = 0$. Let $(\mathrm{Con}(T), \mathcal{F})$ be connected envelope of $(T, f)$. Then, by Theorem 4, $h(\mathcal{F}) = 0$. By right-hand inequality in Lemma 7 we get

$$\limsup_{n\to\infty} \frac{1}{n} \log sep\,(n, F, \varepsilon) \le N_2(\varepsilon) \limsup_{n\to\infty} \frac{1}{n} \log span\,\left(n, \mathcal{F}, \varepsilon/2\right),$$

for every $\varepsilon > 0$. Since $h(\mathcal{F}) = 0$, the right-hand side of the last inequality equals $0$ for any $\varepsilon > 0$. So, $h(F) = 0$.

Let $h(f) > 0$. Then, by left-hand inequality in Lemma 7, we get

$$\limsup_{n\to\infty} \frac{1}{n} \log sep\,(n, F, \varepsilon) \ge N_1(\varepsilon) \limsup_{n\to\infty} \frac{1}{n} \log sep\,(n, f, \varepsilon),$$

for every $\varepsilon > 0$. Since $N_1(\varepsilon) \to +\infty, \varepsilon \to 0+$, we see that $h(F) \ge Ch(f)$ for any positive $C$. So, $h(F) = +\infty$. □

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## References

[1] G.Acosta, A.Illanes, H.Mendez-Lango, *The transitivity of induced maps*, Topology and its Applications, *156*, (2009), 1013–1033.

[2] R.Adler, A.Konheim and J.McAndrew, *Topological entropy*, Trans. Amer. Math. Soc. *114* (1965), 309–319.

[3] Ll.Alseda, J.Llibre and M.Misiurewicz, *Combinatorial dynamics and entropy in dimension one*, Advanced Series in Nonlinear Dynamics *5* (1993). World Scientific Publishing Co., Inc., River Edge, NJ, xiv+329 pp.

[4] J.Auslander, S.Kolyada and L.Snoha, *Functional envelope of a dynamical system*, Nonlinearity *20* (2007), no. 9, 2245–2269.

[5] J.Banks, *Chaos for induced hyperspace maps*, Chaos, Solitons, Fractals **25** (2005), 681-685.

[6] R.Bowen, *Entropy for group endomorphisms and homogeneous spaces*, Trans. Amer. Math. Soc. *153* (1971), 401–414.

[7] V.V.Fedorenko, *Topological limit of trajectories of intervals of one-dimensional dymaical systems*, Grazer Math. Ber., Bericht Nr. *346* (2002), 107–111. In: J. Sousa Ramos et al (Eds.) Proc. ECIT'02.

[8] V.V.Fedorenko, E.Yu.Romanenko and A.N.Sharkovsky, *Trajectories of intervals in one-dimensional dynamical systems*, Difference Eqns. Appl. *13* (2007), no. 8-9, 821–828.

[9] E.Glasner and B.Weiss, *Quasi-factors of zero-entropy systems*, J Amer Math Soc *8(3)* 1995, 665-86.

[10] S.F.Kolyada, *Topological entropy of a dynamical system on the space of one-dimensional maps* (Ukrainian), Neliniĭni Koliv. *7* (2004), no. 2, 180-187; translation: Nonlin Oscillations *7* (2004), no. 1, 83–89.

[11] D.Kwietniak and P.Oprocha, *Topological entropy and chaos for maps induced on hyperspaces*, Chaos Solitons Fractals *33* (2007), 76–86.

[12] M.Matviichuk, *Entropy of induced maps for one-dimensional dynamics*, Grazer Math. Ber., Bericht „ 354 (2009). In: A.N. Sharkovky et al. (Eds.) Proc. ECIT '08, 180-185

[13] E.Yu.Romanenko, *Dynamical systems induced by continuous time difference equations and long-time behavior of solutions*, Difference Eqns Appl. *9* (2003), no. 3-4, 263–280.

[14] A.N.Sharkovsky, Yu.L.Maystrenko and E.Yu.Romanenko, *Difference equations and their applications*, Mathematics and its Applications *250* (1993), Kluwer Academic Publishers Group, Dordrecht, xii+358 pp.

[15] A.N.Sharkovsky and E.Yu. Romanenko, *Difference equations and dynamical systems generated by certain classes of boundary value problems*, Proc. Steklov Inst. Math. *244* (2004), 264–279.

*National University of Kyiv,*
*Academician Glushkov ave., 2, build.7, 03127, Kyiv, Ukraine, and*
*Institute of Mathematics of NASU,*
*Tereschenkivs'ka 3, 01601 Kyiv, Ukraine*

mykola.matviichuk@gmail.com