3. In a recent paper[3] it was shown that the factors in the numerator of expression (1.1) do *not* represent hooks in $[\lambda]$, and, dividing them out, we have

$$f_\lambda = \frac{n!}{H} \tag{3.1}$$

where H is the product of the lengths of *all* the hooks in $[\lambda]$. This formula is of general interest, since it gives a simple interpretation for the quotient $n!/f_\lambda$. Note that expression (2.2) is the special case of expression (3.1) appropriate to the skew diagram $[\lambda_1]\cdot[\lambda_2]\cdot\ldots\cdot[\lambda_n]$.

It should be remarked in conclusion that an operator approach to the reduction of the skew diagram $[\lambda] - [\mu]$ is also available. Indeed, Feit's formula was devised to yield

$$f_{\lambda - \mu} = n!|z_{ij}|, \tag{3.2}$$

where $z_{ij} = 1/(\lambda_j - j - \mu_i + i)!$, with the same conditions as before. Setting $\rho = 0$ when r and $r + 1$ appear in disjoint constituents of a standard skew diagram yields the matrices of the induced representation corresponding to any $[\lambda] - [\mu]$. It is interesting to compare these ideas with the corresponding theory of Schur functions $\{\lambda\}$ as developed by D. E. Littlewood.[4] One might add that this operator approach does yield the enumeration of standard tableaux without the intervention of any other machinery. Moreover, the relation (2.3) provides an immediate proof that the character of a cycle of length n in $[n - r, 1^r]$ is $(-1)^r$; thus the motion of a *hook* derives its significance directly from the raising operator which has proved so important in the modular theory.[5]

[1] W. Feit, *Proc. Am. Math. Soc.*, **4**, 740–744, 1953; F. D. Murnaghan, *The Theory of Group Representations* (Baltimore, 1938), Chaps. V, VII.

[2] G. de B. Robinson and O. E. Taulbee, these PROCEEDINGS, **40**, 723–726, 1954.

[3] J. S. Frame, G. de B. Robinson, and R. M. Thrall, *Can. J. Math.*, **6**, 316–324, 1954.

[4] D. E. Littlewood, *The Theory of Group Characters* (Oxford, 1940), chap. vi.

[5] G. de B. Robinson and O. E. Taulbee, these PROCEEDINGS **41**, 596–598, 1955.

ON A GENERALIZATION OF THE NOTION OF MANIFOLD

BY I. SATAKE

TOKYO UNIVERSITY

Communicated by A. A. Albert, March 7, 1956

In the following, I shall introduce a notion of V-manifold, which is a generalization of the notion of manifold as well as of that of quotient space of a manifold with respect to a properly discontinuous group of transformations. I shall also indicate how de Rham's theorem and Poincaré's duality theorem can be generalized to the case of V-manifolds.

1. Let $\mathfrak{V}$ be a Hausdorff space. A *local uniformizing system* (l.u.s.) $\{\tilde{U}, G, \varphi\}$ for an open set $U \subset \mathfrak{V}$ is by definition a collection of the following objects:
$\tilde{U}$: a connected open subset of R^n.

G: a finite group of linear transformations of $\tilde{U}$ onto itself. We assume that the set of all fixed points of G is of dimension $\leq n - 2$.

φ: a continuous map $\tilde{U} \rightarrow U$ such that $\varphi \circ \sigma = \varphi$ for all $\sigma \in G$. Then φ induces a map from the quotient space $G \backslash \tilde{U}$ onto U, which we assume to be a homeomorphism.

Let $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\}$ be l.u.s. for U, U', respectively, and let $U \subset U'$. By an *injection* λ: $\{\tilde{U}, G, \varphi\} \rightarrow \}\tilde{U}', G', \varphi'\}$ we mean a C^∞-isomorphism λ from $\tilde{U}$ onto a subdomain of $\tilde{U}'$ such that for any $\sigma \in G$ there exists $\sigma' \in G'$ satisfying the relation $\lambda \circ \sigma = \sigma' \circ \lambda$ and such that $\varphi = \varphi' \circ \lambda$. Then σ' is uniquely determined by σ, and the correspondence $\sigma \rightarrow \sigma'$ defines an isomorphism from G into G'. If $U = U'$, then the C^∞-isomorphism λ: $\tilde{U} \rightarrow \tilde{U}'$ and the associated isomorphism $G \rightarrow G'$ become onto, and λ^{-1} is also an injection $\{\tilde{U}', G', \varphi'\} \rightarrow \{\tilde{U}, G, \varphi\}$. In this case $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\}$ are said to be *equivalent*.

By the definition above we can prove the following:

LEMMA 1. *Let λ, μ be two injections $\{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$. Then there exists a uniquely determined $\sigma' \in G'$ such that $\mu = \sigma' \circ \lambda$.*

If λ: $\{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$, λ': $\{\tilde{U}', G', \varphi'\} \rightarrow \{\tilde{U}'', G'', \varphi''\}$ are injections, then $\lambda' \circ \lambda$ becomes an injection $\{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}'', G'', \varphi''\}$.

2. DEFINITION. *A C^∞ V-manifold is a composite concept formed of a connected Hausdorff space $\mathfrak{V}$ and a family $\mathfrak{F}$ of l.u.s. for open sets in $\mathfrak{V}$ satisfying the following conditions:*

(1) *Let $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\} \in \mathfrak{F}$ be l.u.s. for U, U', respectively, and let $U \subset U'$. Then there exists an injection λ: $\{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$.*

(2) *The $\mathfrak{F}$-uniformized open sets, i.e., the open sets U for which there exist l.u.s. $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$, form a basis of open sets in $\mathfrak{V}$.*

Two families $\mathfrak{F}$, $\mathfrak{F}'$ of l.u.s. satisfying conditions 1 and 2 are said to be *equivalent* if $\mathfrak{F} \cup \mathfrak{F}'$ satisfies condition 1. Equivalent families are regarded as defining one and the same V-manifold structure on $\mathfrak{V}$.

In a similar way, we can also define real or complex analytic V-manifolds. A V-manifold is called *orientable* if we can assign an orientation of $\tilde{U}$ for each $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$ such that every possible injection λ in condition 1 preserves these orientations. Thus a complex analytic V-manifold is always orientable.

An ordinary manifold is a special case of V-manifold where every group G in $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$ reduces to the unity group. Moreover, it can easily be proved that if $\tilde{\mathfrak{V}}$ is an analytic manifold and $\mathfrak{G}$ is a properly discontinuous group of analytic automorphisms of $\tilde{\mathfrak{V}}$, then the quotient space $\mathfrak{G} \backslash \tilde{\mathfrak{V}}$ possesses canonically an analytic V-manifold structure.

3. Let $\mathfrak{V}$ be a C^∞ V-manifold with a defining family $\mathfrak{F}$ of l.u.s.. For $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$, we denote by $D_{\tilde{U}}^p$ the module of all G-invariant C^∞-forms of degree p on $\tilde{U}$. Let $\{\tilde{U}, G, \varphi\}$, $\{\tilde{U}', G', \varphi'\} \in \mathfrak{F}$ be l.u.s. for U, U', respectively, $U \subset U'$, and let λ be an injection $\{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\}$. Then it is quite clear that for any $\tilde{\omega} \in D_{\tilde{U}'}^p$, we have $\tilde{\omega} \circ \lambda \in D_{\tilde{U}}^p$. Since $\tilde{\omega} \circ \sigma' = \tilde{\omega}$ for $\sigma' \in G'$ and λ is determined up to $\sigma' \in G'$ by Lemma 1, the correspondence $\tilde{\omega} \rightarrow \tilde{\omega} \circ \lambda$ is independent of the choice of λ. Denoting this homomorphism $D_{\tilde{U}'}^p \rightarrow D_{\tilde{U}}^p$ by $\lambda^*_{\tilde{U}\tilde{U}'}$, we have $\lambda^*_{\tilde{U}\tilde{U}} = 1$, and $\lambda^*_{\tilde{U}\tilde{U}'} \circ \lambda^*_{\tilde{U}'\tilde{U}''} = \lambda^*_{\tilde{U}\tilde{U}''}$ for $\{\tilde{U}, G, \varphi\} \rightarrow \{\tilde{U}', G', \varphi'\} \rightarrow \{\tilde{U}'', G'', \varphi''\}$. In particular, if $U = U'$, then $\lambda^*_{\tilde{U}\tilde{U}'}$: $D_{\tilde{U}'}^p \rightarrow D_{\tilde{U}}^p$ is an isomorphism onto.

Hence the system $\{D_{\tilde{U}}^p, \lambda^*_{\tilde{U}\tilde{U}'}\}$ ($\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$) defines a sheaf D^p on $\mathfrak{V}$ which we call the sheaf of germs of C^∞-forms of degree p on $\mathfrak{V}$. For any open set $U \subset \mathfrak{V}$ an element ω of the module $D_U{}^p$ of the sections of D^p on U is called a *C^∞-form* of degree p on U. A C^∞-form f of degree 0 on U can be regarded as a (real-valued) function on U, which we call a *C^∞-function*. If U is uniformized by $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$, then $D_U{}^p$ is isomorphic to $D_{\tilde{U}}{}^p$ in a canonical way. We can define the operations d, $\wedge$ on the graded module $D_U = \sum_p D_U{}^p$ in a natural manner.

4. Let $\mathfrak{W}$ be another C^∞ V-manifold defined by a family $\mathfrak{K}$ of l.u.s.

DEFINITION. *A map f from $\mathfrak{V}$ into $\mathfrak{W}$ is called a C^∞-map if the following conditions are satisfied: Let $x \in \mathfrak{V}$, and let V be a neighborhood of $f(x)$ which is uniformized by $\{\tilde{V}, H, \psi\} \in \mathfrak{K}$; then there exists a neighborhood U of x uniformized by $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$ and a C^∞-map $\tilde{f}$ from $\tilde{U}$ into $\tilde{V}$ such that $f \circ \varphi = \psi \circ \tilde{f}$.*

Clearly this definition does not depend on the choice of the families $\mathfrak{F}$, $\mathfrak{K}$. Moreover we can prove the following lemma.

LEMMA 2. *The notations being as above, let $\tilde{\omega} \in D_{\tilde{V}}{}^p$. Then $\tilde{\omega} \circ \tilde{f}$ is independent of the choice of $\tilde{f}$ satisfying the above condition.*

It follows, in particular, that $\tilde{\omega} \circ \tilde{f} \in D_{\tilde{U}}{}^p$. Furthermore, suppose that $\{\tilde{U}', G', \varphi'\} \in \mathfrak{F}$, $\{\tilde{V}', H', \psi'\} \in \mathfrak{K}$, λ: $\{\tilde{U}, G, \varphi\} \to \{\tilde{U}', G', \varphi'\}$, μ: $\{\tilde{V}, H, \psi\} \to \{\tilde{V}', H', \psi'\}$ and that there exists a C^∞-map $\tilde{f}'$: $\tilde{U}' \to \tilde{V}'$ such that $f \circ \varphi' = \psi' \circ \tilde{f}'$. Then, for $\tilde{\omega} \in D_{\tilde{V}'}{}^p$, we have $\tilde{\omega} \circ \tilde{f}' \circ \lambda = \tilde{\omega} \circ \mu \circ \tilde{f}$.

Hence, if $U \subset \mathfrak{V}$, $V \subset \mathfrak{W}$ are any open sets such that $f(U) \subset V$ and $\omega \in D_v{}^p$ we can define $\omega \circ f \in D_U{}^p$. It is clear that $(d\omega) \circ f = d(\omega \circ f)$, $(\omega \wedge \eta) \circ f = (\omega \circ f) \wedge (\eta \circ f)$.

5. A C^∞ singular simplex $s = [f; a_0, \ldots, a_p]$ of dimension p in $\mathfrak{V}$ is defined as usual by a C^∞-map f from a neighborhood of a Euclidean simplex $[a_0, \ldots, a_p]$ into $\mathfrak{V}$. Then we define the integral $\int_s \omega$ of a C^∞-form ω of degree p on s by the formula

$$\int_s \omega = \int_{[a_0, \ldots, a_p]} \omega \circ f.$$

If the carrier of s is contained in U uniformized by $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$ and $\tilde{f}$ is a C^∞-map from a neighborhood of $[a_0, \ldots, a_p]$ into $\tilde{U}$ such that $f = \varphi \circ \tilde{f}$, then

$$\int_s \omega = \int_{\tilde{s}} \tilde{\omega},$$

where $\tilde{s} = [\tilde{f}; a_0, \ldots, a_p]$ is a C^∞ singular simplex in $\tilde{U}$ and $\tilde{\omega} = \omega \circ \varphi \in D_{\tilde{U}}{}^p$. We can prove Stoke's formula.

We denote by $S = \sum_p S_p$ the graded module (with boundary operator b) of locally finite C^∞ singular chains in $\mathfrak{V}$.

6. Now let $\mathfrak{V}$ be a paracompact C^∞ V-manifold. Then, for any (open) covering $\mathfrak{U} = \{U_\iota\}_{\iota \in I}$ of $\mathfrak{V}$, we can construct a locally finite C^∞-partition of unity $\{f_\iota\}_{\iota \in I}$ such that the support of f_ι is contained in U_ι.

We call a C^∞-map Φ: $U \times R \to U$ a C^∞-retraction if $\Phi(x, t) = x$ for $t \geqq 1$, and x_0 for $t \leqq 0$. If $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$ is such that $\tilde{U}$ is a Euclidean ball, then $\tilde{U}$ has a C^∞-retraction $\tilde{\Phi}$ such that $\tilde{\Phi}(\sigma(\tilde{x}), t) = \sigma(\tilde{\Phi}(\tilde{x}, t))$ for $\sigma \in G$, and so $U = \varphi(\tilde{U})$ has a C^∞-retraction Φ defined by $\Phi \circ (\varphi \times 1) = \varphi \circ \tilde{\Phi}$. Hence open sets in $\mathfrak{V}$ having a C^∞-retraction form a basis of open sets in $\mathfrak{V}$. We can prove easily

LEMMA 3. *For any covering* $\mathfrak{U} = \{U_i\}_{i \in I}$ *of* $\mathfrak{V}$, *there exists a covering* $\mathfrak{U}' = \{U_j'\}_{j \in J}$ *such that*

(1) *Each* U_j' *has a* C^∞*-retraction.*

(2) *There exists a map* $\tau: J \rightarrow I$ *such that if* $U_{j_0}' \cap \ldots \cap U_{j_p}' \neq \emptyset$, *then* $U_{j_0}' \subset U_{\tau j_0} \cap \ldots \cap U_{\tau j_p}$.

7. Let $H^p(\mathfrak{U}, R)$ and $H_p(\mathfrak{U}, R)$ be the Čech cohomology and homology group, respectively, of a covering $\mathfrak{U}$ of $\mathfrak{V}$ with coefficients in R. We can prove by the method of Weil[1] that for any covering $\mathfrak{U}$ there exists a canonical homomorphism

$$H^p(\mathfrak{U}, R) \rightarrow H^p(D_{\mathfrak{V}}),$$

which is an isomorphism onto if $\mathfrak{U}$ is a simple covering, i.e., if every $U_{i_0} \cap \ldots \cap U_{i_p}$, $U_{i_\nu} \in \mathfrak{U}$, has a C^∞-retraction.

Let $H^p(\mathfrak{V}, R)$ be the inductive limit group of $H^p(\mathfrak{U}, R)$. Then, by Lemma 3, we can prove

THEOREM 1. $H^p(D_{\mathfrak{V}})$ *is isomorphic to* $H^p(\mathfrak{V}, R)$ *canonically.*

Next, let $H_p(\mathfrak{V}, R)$ be the projective limit group of $H_p(\mathfrak{U}, R)$; then we have, similarly,

THEOREM 2. $H_p(S)$ *is isomorphic to* $H_p(\mathfrak{V}, R)$ *canonically.*

We have quite analogous results, restricting cochains and chains to finite ones in the construction of $H^p(\mathfrak{V}, R)$ and $H_p(\mathfrak{V}, R)$, respectively, and also restricting C^∞-forms and C^∞-chains to those with compact supports in defining $H^p(D_{\mathfrak{V}})$ and $H_p(S)$, respectively.

Since $H^p(\mathfrak{V}, R)$ and $H_p(\mathfrak{V}, R)$, one of which is constructed in the restricted sense, are mutually dual, the same relation holds between $H^p(D_{\mathfrak{V}})$ and $H_p(S)$. Writing this duality explicitly, we can see that the inner product is given by the integral $\int_t \omega$, ω being a closed C^∞-form of degree p and t a C^∞-cycle of dimension p. We have thus the first and second theorems of de Rham.

8. Let $\mathfrak{V}$ be orientable. Then, for $\omega \in D_{\mathfrak{V}}^n$ with a compact carrier, we can define the integral $\int_{\mathfrak{V}} \omega$ as follows: if the carrier of ω is contained in U uniformized by $\{\tilde{U}, G, \varphi\} \in \mathfrak{F}$, we put

$$\int_{\mathfrak{V}} \omega = \frac{1}{N_G} \int_{\tilde{U}} \tilde{\omega},$$

where $\tilde{\omega} = \omega \circ \varphi$ and N_G is the order of G. Let $\{f_i\}_{i \in I}$ be a locally finite C^∞-partition of unity such that the support of f_i is contained in an $\mathfrak{F}$-uniformized open set U_i. For general ω, we put

$$\int_{\mathfrak{V}} \omega = \sum_{i \in I} \int_{\mathfrak{V}} f_i \omega.$$

Then we can prove easily that this definition does not depend on the choice of $\{f_i\}$.

From what has been proved above, it follows that $\mathfrak{V}$ is a "homological manifold" for real coefficients. This means that every $x \in \mathfrak{V}$ has arbitrarily small open neighborhoods, each one of which has the same cohomology with compact carriers (over real coefficients) as the space R^n. Then, assuming for simplicity that V is compact, we can prove (again by Weil's method) the following theorem:

THEOREM 3. *$H^{n-p}(D_{\mathfrak{B}})$ is isomorphic to $H_p(\mathfrak{B}, R)$ canonically.*

Hence $H^p(D_{\mathfrak{B}})$ and $H^{n-p}(D_{\mathfrak{B}})$ are mutually dual. We can see that the inner product is given by the integral $\pm\int_{\mathfrak{B}} \omega \wedge \eta$, ω and η being closed C^∞-forms of degree p and $n - p$, respectively. We have thus the duality theorem of Poincaré on Betti groups.

[1] A. Weil, "Sur les théorèmes de de Rham," *Comm. Math. Helv.*, **26**, 119–145, 1952.

THE KINETIC ENERGY OF RELATIVE MOTION*

BY JOSEPH O. HIRSCHFELDER AND JOHN S. DAHLER†

UNIVERSITY OF WISCONSIN NAVAL RESEARCH LABORATORY, MADISON, WISCONSIN

Communicated March 26, 1956

The kinetic energy for a system of N particles is

$$T = \frac{1}{2} \sum_i m_i r_i^2, \tag{1}$$

where m_i is the mass of the ith particle and $\vec{r}_i$ is its position. Very often it is desirable to separate off the kinetic energy of the center of mass and express the relative kinetic energy in terms of a set of relative co-ordinates. There are many different ways of defining the relative co-ordinates, each having characteristic advantages for a particular type of physical problem. For the motion of an N-particle system in configuration space, it is most convenient to define the relative co-ordinates $\vec{Q}_2, \vec{Q}_3, \ldots, \vec{Q}_N$, so that the kinetic energy becomes

$$T = \frac{1}{2} [\dot{Q}_1^2 + \dot{Q}_2^2 + \dot{Q}_3^2 + \ldots + \dot{Q}_N^2]. \tag{2}$$

Here $^1/_2\dot{Q}_1^2$ is the kinetic energy of the center of mass, and $\vec{Q}_1$ itself is the square root of the mass of the system times the co-ordinate of the center of mass. If we let

$$M_k = m_1 + m_2 + m_3 + \ldots + m_k, \tag{3}$$

then

$$\vec{Q}_1 = (M_N)^{-1/2} \sum_i m_i \vec{\ }_i. \tag{4}$$

In order to convert the kinetic energy from the form of equation (1) to that of equation (2), it is necessary to make the transformation

$$(m_i)^{1/2} \vec{\ }_i = \sum_k S_{ki} \vec{Q}_k, \tag{5}$$

where the coefficients S_{ki} form a unitary matrix, so that

$$\sum S_{ki} S_{ji} = \delta_{jk} \tag{6}$$