* This investigation was supported by a grant from the National Foundation for Infantile Paralysis.

[1] Bertani, G., and Weigle, J., *J. Bact.*, **65**, 113 (1953).

[2] Adams, M. H., in *Methods in Medical Research*, Chicago, 1950.

[3] Dulbecco, R., *J. Bact.*, **59**, 329 (1950).

[4] Jacob, F., personal communication.

[5] We wish to thank Ciba S.A., Basel, Switzerland, who put at our disposal the chlorohydrate of methyl bis-β-chlorethylamine, called Dichlorene, used in these experiments.

[6] See Lwoff, A., *Ann. Inst. Past.*, **84**, 225 (1953).

[7] Hayes, W., *J. Gen. Microb.*, **8**, 72 (1953).

[8] Cavalli, L. L., Lederberg, J., and Lederberg, E. M., *Ibid.*, **8**, 89 (1953).

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*RELATIONS ON ITERATED REDUCED POWERS**

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In this note we present the generalization of the relations on iterated squares[1] to the case of iterated cyclic reduced powers of arbitrary prime period p. As in the case $p = 2$, the new relations are used to solve some particular problems.

Throughout this paper we will use the definitions and notation recently introduced by Steenrod.[2]

1. For any complex K and odd prime p, the cyclic reduced power operations are homomorphism $\mathcal{P}^s$, $(s = 0, 1, \ldots)$,

$$\mathcal{P}^s : H^q(K; Z_p) \rightarrow H^{q+2s(p-1)}(K; Z_p)$$

They satisfy the following properties: $\mathcal{P}^s f^* = f^* \mathcal{P}^s$, where f is a map of one complex into another; $\mathcal{P}^0 =$ identity; if $q = \dim u$ is even, $\mathcal{P}^{q/2}u = u^p$ (in cup-product sense); $\mathcal{P}^s u = 0$ when $s > q/2$.

As in the case of squares, an *iterated* cyclic reduced power is a composition of two or more of the $\mathcal{P}^s$, e.g., $\mathcal{P}^r\mathcal{P}^s\mathcal{P}^t$.

Let δ^* be the coboundary operator associated with the exact coefficient sequence $0 \rightarrow Z \rightarrow Z \rightarrow Z_p \rightarrow 0$. Our main result is the following

THEOREM 1.1 *For all $0 \leq r < sp$ the iterated cyclic reduced powers satisfy the following set of relations*[3]

$$(1.2) \quad \mathcal{P}^r\mathcal{P}^s = \sum_{i=0}^{[r/p]} (-1)^{r+i} \binom{(s-i)(p-1)-1}{r-ip} \mathcal{P}^{r+s-i}\mathcal{P}^i,$$

$$(1.3) \quad \mathcal{P}^r\delta^*\mathcal{P}^s = \sum_{i=0}^{[r/p]} (-1)^{r+i} \binom{(s-i)(p-1)}{r-ip} \delta^*\mathcal{P}^{r+s-i}\mathcal{P}^i + \sum_{i=0}^{[(r-1)/p]} (-1)^{r+i+1} \binom{(s-i)(p-1)-1}{r-ip-1} \mathcal{P}^{r+s-i}\delta^*\mathcal{P}^i, \qquad (\bmod p),$$

where (——) *denotes the binomial coefficient with the usual conventions.*

An induction argument based on (1.2) proves the following

THEOREM 1.4. *The set* $\{\mathcal{P}^{p^k}\}$ *with* $k = 0, 1, \ldots,$ *form a base in the sense that any other* $\mathcal{P}^r$ *can be expressed as a sum of iterated cyclic reduced powers with exponents powers of* p.

For example,

$$\mathcal{P}^r = \frac{1}{r!}(\mathcal{P}^1)^r \qquad \text{for } 0 < r < p.$$

$$\mathcal{P}^{2p} = \frac{1}{2}(\mathcal{P}^p)^2 + \frac{1}{2}(\mathcal{P}^1)^{p-1}\mathcal{P}^p\mathcal{P}^1$$

where $(\mathcal{P}^j)^r$ means $\mathcal{P}^j$ iterated r times.

Again, an induction using formula (1.2) proves

THEOREM 1.5. *The iterated powers of the type* $\mathcal{P}^{i_1} \ldots \mathcal{P}^{i_r}$ *with* $i_1 \geq pi_2, \ldots, i_{r-1} \geq pi_r$ *and* $c = i_1 + \ldots + i_r$, *form an additive base for all iterated powers* $\mathcal{P}^{j_1} \ldots \mathcal{P}^{j_n}$ *where* $j_1 + \ldots + j_n = c$.

2. For the particular values $r = 1$, $s = 2^k - 1$, formula (1.2) becomes

$$\mathcal{P}^1\mathcal{P}^{2^k-1} = 2^k\mathcal{P}^{2^k},$$

therefore, if $\dim u = 2^{k+1}$ we have

$$u^p = \frac{1}{2^k}\mathcal{P}^1\mathcal{P}^{2^k-1}u \qquad (\mathrm{mod}\ p), \tag{2.1}$$

where u^p is the p-power of u in the cup-product sense.

Let $H(K)$ denote the integral cohomology ring of a complex K. We say that $H(K)$ is a truncated polynomial ring on u if $H(K)$ is generated by the cup-product powers of u and each power is of infinite order. The height of u is the minimal integer n such that $u^n = 0$.

THEOREM 2.2. *If* $H(K)$ *is a non-trivial truncated polynomial ring on* u *then* $\dim u = 2^{k+1}$. *Moreover, if* $\dim u \geq 8$ *then the height of* u *is at most 3.*

We will show how this theorem is implied by our relations on reduced powers. Let $q = \dim u$. First, that q cannot be odd follows from the commutative law for cup-products. Now, if q is not a power of 2, then, because $\{\mathrm{Sq}^{2^k}\}$ is a base for squares,[1] we have $u \smile u = \mathrm{Sq}^q u = 0 \pmod 2$, and this is a contradiction. Finally, suppose $q = 2^{k+1}$. Using (2.1) for $p = 3$, we have

$$u \smile u \smile u = (-1)^k\mathcal{P}^1\mathcal{P}^{2^k-1}u \qquad (\mathrm{mod}\ 3),$$

and $\dim \mathcal{P}^{2^k-1}u = 3\cdot 2^{k+1} - 4$. Therefore $u \smile u \smile u = 0 \bmod 3$, unless $3\cdot 2^{k+1} - 4$ is a multiple of 2^{k+1}. That is the case only if $k = 0, 1$.

Let S^{r-1} be a sphere bundle with S^{s-1} as fiber. Examples are known for the following forms of r and s: $r = s$; all r, $s = 1$; $r = 2n$, $s = 2$; $r = 4n$, $s = 4$; $r = 16$, $s = 8$.

COROLLARY 2.3. *The other possible values of r and s for which S^{r-1} can be a sphere bundle with fiber S^{s-1} are of the form $r = 2^{k+1}$ and $s = 2^k$, $(k \geq 4)$.*

Proof: If S^{r-1} is fibered by S^{s-1}, it follows from Gysin's sequence for sphere bundles that the integral cohomology ring of the base space B is a truncated polynomial ring, generated by the characteristic class u of dimension s. Then $r = ns$ for some integer n and $\dim B = s(n - 1)$; therefore the height of u is n. If $n > 3$, then 2.3 follows from 2.2. If $n = 3$ and $f:S^{r-1} \rightarrow B$ is the projection, adjoin an r cell E^r to B by means of f, so that $M = B \cup e^r$ is a manifold. By duality $H(M)$ is a truncated polynomial ring generated by u with height 4. This contradicts 2.2.

3. Our proof of relations (1.2), (1.3) is purely algebraic. The relations are obtained as homology relations on the symmetric group S_{p^2} of degree p^2, and makes full use of the general definition for reduced power operations found recently by Steenrod.[2] We will indicate briefly this method. Let G be a p-sylow group of S_{p^2} and $\theta:G \rightarrow S_{p^2}$ the inclusion homomorphism. For each $C \in H_i(G; Z_p)$ we have a reduced power operation. If $u \in H^q(K; Z_p)$ then $u^{p^2}/C \in H^{p^2q-i}(K; Z_p)$.

To obtain the relations we first identify the operations induced by some cycles of $H_i(G; Z_p)$ with sums of cyclic reduced powers. The relations are then obtained, according to the general principle of Steenrod,[2] as elements on the kernel of $\theta_*:H_i(G; Z_p) \rightarrow H_i(S_{p^2}; Z_p)$, i.e., if $\theta_*(C_1 - C_2) = 0$, then $u^{p^2}/C_1 = u^{p^2}/C_2$.

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[1] Adem, J., these PROCEEDINGS, 38, 720–726 (1952).

[2] Steenrod, N. E., *Ibid.*, 39, 213–223 (1953).

[3] I have heard that H. Cartan has obtained relations of the same type, using methods quite different from mine.