* This work was done at the Carnegie Institution of Washington, Department of Genetics, Cold Spring Harbor, New York, while the author was associated with Dr. A. F. Blakeslee.

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[3] Blakeslee, A. F., Bergner, A. D., and Avery, A. G., *Cytologia*, Fujii Jub.Vol., 1070–1093 (1937).

[4] Numbers representing chromosomes other than those of PT 1 are shown in boldface type.

[5] ⊙ denotes circle.

[6] Parentheses enclose chromosomes within a ⊙.

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[9] Bergner, A. D., *Proc. 7th. Intern. Genet. Congress*, 63–64 (1939).

[10] Parentheses with subscript $_2$ denote that ⊙ consists of two of each kind of chromosome included within the parentheses.

ON MECHANICAL SELF-EXCITED OSCILLATIONS

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1. *Introduction.*—Self-excited oscillations, particularly electrical ones, form the object of numerous studies in recent years. Mechanical oscillations of this kind have been less explored although they are frequently observed in practice.

The object of this note is to describe a typical phenomenon of this kind observed during experimental work on the antirolling stabilization of ships by the activated tanks method. The problem of the antirolling stabilization of ships itself does not form the object of the note and only a few words will be sufficient to explain the principal details of these experiments.

2. *Experimental Arrangement and Principal Data.*—The principal features of ship stabilization by tanks can be conveniently studied both theoretically and experimentally by means of two coupled pendula of which one P is a rigid physical pendulum and the other P' is a pendulum consisting of a liquid (water, oil, etc.) filling a U-tube connected to two tanks and secured to the rigid pendulum P so as to obtain an oscillation coplanar with the latter.

In the experiments described here the activation of the ballast was accomplished by means of an axial variable pitch pump inserted in the Uchannel of the liquid pendulum. The impeller pump was kept running at a constant speed and the activation was accomplished by controlling the blade angle α of the runner by a control equipment responsive to the angular motion of P.

Let θ be the angle of P and φ the relative angle formed by the levels of P' with respect to P. The rate of flow of the ballast is then proportional to $\dot{\varphi}$. The following experimental results will be of importance.

1. It was found that the roll-quenching efficiency is greatest when the following *condition of control* is fulfilled

$$\alpha = \lambda\ddot{\theta} \tag{1}$$

that is, when the blade angle α is made to vary in proportion to the angular acceleration $\ddot{\theta}$ of the rigid pendulum. The coefficient of proportionality λ could be adjusted by changing the amplification factor of an electronic circuit controlling the blade angle α.

2. For a steady harmonic actuation of the ballast by the impeller pump it is observed that there exists the following relation:

$$\dot{\varphi} = \epsilon\ddot{\theta} \tag{2}$$

between the rate of flow $\dot{\varphi}$ and the angular acceleration $\ddot{\theta}$; this means in view of (1) that the rate of flow $\dot{\varphi}$ and the blade angle α are approximately proportional to each other or,

$$\alpha = \lambda/\epsilon = \gamma\dot{\varphi} \tag{3}$$

the coefficient ϵ changes somewhat according to conditions.

3. Moreover, from numerous tests, it has been ascertained that the force imparted by the impeller pump to the liquid column increases more or less linearly with α for small values of α and increases less than in proportion for larger values of α. The moment M of the force created by the impeller and applied to P' can be, therefore, approximated by a relation

$$M = M_1\alpha - M_3\alpha^3 \tag{4}$$

where M_1 and M_3 are independent of α. The form of the curves approximated by this expression varies somewhat for different conditions.

3. Phenomenon of Self Excitation.—It has been ascertained that when the coefficient λ of amplification is made large enough, a self-excited oscillation of relatively high frequency sets in and superimposes itself on the normal stabilizing action. If the coefficient λ continues to increase further, self-excited oscillation exhibits the presence of harmonics. If, however, λ is decreased gradually, at a certain point $\lambda = \lambda_0$ the self-excited oscillation disappears. During the experiments the parasitic fluttering of the blades had a frequency nearly five times that of the oscillatory process which the equipment is intended to quench.

The interesting feature of the phenomenon lies in the fact that it may originate also when the pendulum P is almost at rest. In fact it is sufficient to disturb the pendulum at rest very little, e.g., by touching it slightly, to be able to release violent surges of water in the tanks with the corresponding blade angle oscillations, which persist indefinitely. The pendulum P also begins to oscillate with the same high frequency but with an amplitude of a fraction of one degree whereas the oscillation of P' is generally very appreciable and with strong amplification may reach the value of about 20 to 25 degrees. In what follows we shall investigate the problem in this particular case which is relatively simple.

4. *Differential Equations.*—The fact that during the self excitation from rest the amplitude of motion in the θ coördinate is very small, makes it possible to neglect the effect of the passive component of coupling between P and P' since this component depends on the existence of a finite oscillation in the θ degree of freedom, and consider only the active component of coupling, i.e., the action of the moment M due to the pump on P' and the reaction of this moment on P.

The simplified differential equations of the system in this case are:

$$j\ddot{\varphi} + k\dot{\varphi} + c\varphi = M; \qquad I\ddot{\theta} + K\dot{\theta} + C\theta = -M \tag{5}$$

where j, k and c for the liquid pendulum P' and I, K and C for the rigid pendulum P are certain constants, the physical significance of which is sufficiently obvious. The left-hand terms of these equations are linearized somewhat which has no further influence on what follows. The neglect of the passive component of the coupling as just mentioned is in the fact that the first equation contains only the relative coördinate φ instead of the absolute one $\varphi - \theta$ since in this particular case θ is very small.

Equations (5) are thus written down with the knowledge of the principal features of the phenomenon and our purpose will be to fit these features into the form of solutions we are going to obtain. Using the equations (4) and (3) we obtain

$$j\ddot{\varphi} + (k - m_1)\dot{\varphi} + m_3\dot{\varphi}^3 + c\varphi = 0 \tag{6}$$

$$I\ddot{\theta} + K\dot{\theta} + C\theta = -m_1\dot{\varphi} + m_3\dot{\varphi}^3 \tag{7}$$

where $$m_1 = M_1\lambda/\epsilon \quad \text{and} \quad m_3 = M_3\lambda^3/\epsilon^3. \tag{8}$$

It is seen that owing to the neglect of the passive component of coupling equations (6) and (7) do not form actually a coupled system since equation (6) contains only φ. Hence, if we succeed in solving the non-linear equation, φ will appear as a known function in (7) which will permit determining θ and through the equation (1) of control we shall be able to establish the condition of coupling.

Thus, the principal interest centers on the equation (6). It is noted that for a sufficiently small value of λ, the coefficient m_1 is also small so that $k - m_1 > 0$. In such a case it is apparent that no self excitation is possible in the φ degree of freedom since equation (6) shows that the liquid pendulum P' has a purely dissipative damping of a combined velocity, and velocity-cube type. The really interesting case arises when the parameter λ is increased so as to make the coefficient $k - m_1 < 0$ which case we shall consider from now on. Physically this means that the energy imparted by the blades outweighs the dissipation of energy caused by the velocity damping $k\dot{\varphi}$.

In order to bring the problem within the scope of non-linear mechanics we shall assume that the damping terms are relatively small in comparison with the inertial $(j\ddot{\varphi})$ and stability $(c\varphi)$ terms which is sufficiently justified by the actual order of magnitude of these quantities. This permits writing

$$m_1 - k = \mu s; \qquad m_3 = \mu q$$

where μ is a small parameter $(\mu > 0)$ and $s > 0$ and $q > 0$ are certain constants of the order of j and c. With these conventions equation (6) can be written

$$j\ddot{\varphi} + c\varphi + \mu(-s\dot{\varphi} + q\dot{\varphi}^3) = 0. \tag{9}$$

This equation is of the Rayleigh type. By differentiating it becomes of the Van der Pol type in $\dot{\varphi}$ and thus may be shown[1] to have the approximate solution

$$\varphi = \frac{1}{\omega_0}\sqrt{\frac{4(m_1 - k)}{3m_3}}\cos(\omega_0 t + \psi_0)$$

$$\dot{\varphi} = -\sqrt{\frac{4(m_1 - k)}{3m_3}}\sin(\omega_0 t + \psi_0). \tag{10}$$

Substituting this value for $\dot{\varphi}$ into equation (7) one easily obtains θ since this equation is linear. Since the term $m_3\dot{\varphi}^3$ contains the third harmonic it is apparent that the oscillation of P will also contain that harmonic and in view of equations (1) and (3) this harmonic will also appear in the φ oscillation. The amplitudes of these harmonics are small if the ratio m_3/m_1 is small as we have assumed. We shall limit this study to the first approximation only assuming that $m_3 = 0$ in equation (7). Since the phase ψ_0 is clearly of no interest here we can write

$$I\ddot{\theta} + K\dot{\theta} + C\theta = -m_1\dot{\varphi} = m_1\sqrt{\frac{4(m_1 - k)}{3m_3}}e^{i\omega_0 t} = Ae^{i\omega_0 t}$$

where $A = m_1\sqrt{\frac{4(m_1 - k)}{3m_3}}$; whence the approximate expression for the amplitude θ_0 is

$$\theta_0 = \frac{m_1}{I\omega_0^2}\sqrt{\frac{4(m_1 - k)}{3m_3}}$$

and the amplitude of the angular acceleration is then

$$\ddot{\theta}_0 = \frac{m_1}{I}\sqrt{\frac{4(m_1 - k)}{3m_3}}. \tag{11}$$

The blade angle is given by equation (1)

$$\alpha_0 = \lambda\ddot{\theta}_0.$$

It is seen that the blade angle generally increases with the amplification factor but not exactly in proportion to λ since by equations (8) both m_1 and m_3 are functions of λ. It is generally observed that the blade angle α increases with the amplification factor but the observation is handicapped owing to the presence of harmonics which are left out in this study.

6. *Critical Value of Parameter.*—We shall now attempt to establish analytically the principal feature of the phenomenon mentioned at the beginning of Section 3, namely, a rather sudden appearance of self excitation at a certain value $\lambda = \lambda_0$ of the parameter when it is increased from small values as well as its sudden disappearance at the same value $\lambda = \lambda_0$, when λ is gradually decreased from larger values.

Although this fact has been discussed to some extent in Section 4 we shall analyze it now from a more general standpoint resulting from the theory of Poincaré-Liapounoff.[2] Only equation (6) will interest us here because equation (7) does not present any difficulty.

Putting $m_1 = \lambda a_1$ and $m_3 = \lambda^3 a_3$ where a_1 and a_3 are certain positive constants at least within the range of the experiment as follows from (8) we write equation (6) in the form

$$j\ddot{\varphi} + (k - \lambda a_1)\dot{\varphi} + m_3\dot{\varphi}^3 + c\varphi = 0. \tag{6a}$$

This equation is equivalent to the following system of differential equations

$$\begin{aligned} \dot{\varphi} &= y \\ \dot{y} &= -g(\lambda)y - \omega_0^2\varphi - m_3'y^3 \end{aligned} \tag{12}$$

where

$$g(\lambda) = (k - \lambda a_1)/j \text{ and } m_3' = m_3/j.$$

We shall investigate the conditions of stability of the liquid pendulum when at rest. Liapounoff has shown that for this purpose it is legitimate to neglect the non-linear term in the second equation (12) which gives

$$\begin{aligned}\dot{\varphi} &= y \\ \dot{y} &= -g(\lambda)y - \omega_0^2\varphi.\end{aligned} \tag{13}$$

We can now apply the classical method of Poincaré which consists in investigating the singularities of the system (13) as a function of the parameter. In this case there is clearly one singular point of interest, viz., $\varphi = y = 0$. The characteristic equation is

$$S^2 + g(\lambda)S + \omega_0^2 = 0 \tag{14}$$

and its roots are

$$S_{1,2} = -g(\lambda)/2 \pm \sqrt{\frac{g^2(\lambda)}{4} - \omega_0^2}. \tag{15}$$

There are four possibilities according to the sign of $g(\lambda)/2$ and of the quantity under the square root.

If $\left|\frac{g(\lambda)}{2}\right| > \omega_0$ the roots are real and of the same sign which corresponds to a nodal point of the system. This case is to be discarded because the experiment always shows that a self-excited oscillation builds itself up in a series of swings and never reaches a steady state aperiodically which would require a pump of an enormous power. Hence, we have to consider the case when $\left|\frac{g(\lambda)}{2}\right| < \omega_0$ in which case the roots are conjugate complex and the singularity $\varphi = y = 0$ is a focal point, stable if $g(\lambda) > 0$ and unstable if $g(\lambda) < 0$. The value $\lambda = \lambda_0$ for which $g(\lambda_0) = 0$ and which separates the stable focal points from the unstable ones is thus the critical or *bifurcation* value of the parameter λ (Poincaré).

Since $g(\lambda)$ is monotone decreasing with λ increasing and is positive for small λ, it follows that by gradually increasing the value of the parameter λ the original stable focal point for $\lambda \leqslant \lambda_0$ becomes unstable for $\lambda \geqslant \lambda_0$.

From the theory of Poincaré it follows that for the critical value $\lambda = \lambda_0$ for which a stable singularity becomes unstable there may appear at least one stable *limit cycle* which means a stationary oscillation. The proof of the existence of a limit cycle is rather laborious and requires the investigation of a series expansion satisfying the non-linear system (12). This need not be done here since we know that the self-excited oscillation exists for $\lambda \geqslant \lambda_0$ and the theory of the first approximation gives the answer in a simple manner.

7. *Conclusion.*—It is seen that the principal features of the phenomenon in question are consistently explained on the basis of non-linear mechanics. In fact, if one had to follow purely linear methods it would be impossible even to predict the existence of a phenomenon of this nature.

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FIRST PROOF THAT THE MERSENNE NUMBER M_{157} *IS COMPOSITE*

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The form of a Mersenne number is $M_p = 2^p - 1$ where p is prime. Just three hundred years ago Mersenne published, in effect, the statement that the only values of p not greater than 257 which make M_p prime are 2, 3, 5, 7, 13, 17, 19, 31, 67, 127 and 257. The prime or composite character of all of the 55 numbers included under this conjecture, except the six corresponding, respectively, to p = 157, 167, 193, 199, 227 and 229, had been investigated prior to the year 1935. Contrary to Mersenne's surmise it was found that for p = 67 and 257 M_p is composite, and that M_p is prime for p = 61, 89 and 107. The data presented above have been derived from a comprehensive paper by R. C. Archibald.[1]

The present contribution marks the first fruits of a friendly suggestion made last year by Professor Archibald and seconded by Professor D. H. Lehmer that the author turn his attention to problems of factorization in general, beginning with the special problem of the character of the six heretofore uninvestigated Mersenne numbers.

The modern technique of this problem is based explicitly upon the following theorem, discovered by E. Lucas and clarified by D. H. Lehmer,[2] namely: "*The number* $N = 2^n - 1$, *where n is an odd prime, is a prime if, and only if, N divides the* $(n - 1)$-*st term of the series,*

$$S_1 = 4, \qquad S_2 = 14, \qquad S_3 = 194, \ldots, S_k, \ldots,$$

where $S_k = S_{k-1}^2 - 2$."

Since the number M_{157}, which is the subject of the present study, equals 182 68770 46663 62864 77546 06040 89535 37745 69915 67871 and since the above theorem requires that this 48-digit number be used as a divisor 149 times (8 to 156, inclusive) it should be obvious that the prospective investigator would focus attention upon this aspect of the computation. The procedure followed by R. E. Powers[3] in showing that M_{241} is composite ap-