AUTOMORPHISMS OF THE DIHEDRAL GROUPS

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The group of inner automorphisms of a dihedral group whose order is twice an odd number is obviously the group itself while the group of inner automorphisms of a dihedral group whose order is divisible by 4 is the quotient group of this dihedral group with respect to its invariant subgroup of order 2 when the dihedral group is not the four group. It is well known that the four group is the only abelian dihedral group and that its group of automorphisms is the symmetric group of order 6. All of these automorphisms except the identity are outer automorphisms but this symmetric group admits no outer automorphisms. In fact, we shall prove that it is the only dihedral group which does not admit any outer automorphisms. To emphasize this fact it may be noted here that on page 152 of the *Survey of Modern Algebra* by Birkhoff and MacLane (1941) it is stated that the group of symmetries of the square, which is also a dihedral group, admits no outer automorphisms. This is obviously not in agreement with the theorem noted above.

To prove that the symmetric group of order 6, which is also the group of movements of the equilateral triangle, is the only dihedral group which does not admit any outer automorphisms it may first be noted that if a dihedral group whose order is twice an odd number admits no outer automorphisms its cyclic subgroup of index 2 cannot have an order which exceeds 3 since in the group of automorphisms of a cyclic group every operator of highest order corresponds to every other such operator and every operator may correspond to its inverse. If a dihedral group whose order is divisible by 4 admits no outer automorphisms it cannot be the four group and hence it contains a cyclic subgroup which involves an invariant subgroup of order 2. Its non-invariant operators correspond to themselves multiplied by every operator of this subgroup in some automorphism of the group. That is, *the dihedral group of order 6 is the only dihedral group which admits no outer automorphisms.*

Since the dihedral group of order 6 involves no invariant operator besides the identity and admits no outer automorphisms it is its own group of automorphisms. To prove that no other dihedral group whose order is twice an odd number is its own group of automorphisms it may be noted that if such a group has this property its cyclic subgroup of index 2 cannot admit any automorphisms besides the one in which each of its operators corresponds to its inverse and hence this cyclic subgroup of odd order must be of order 3 and hence the group must be the dihedral group of order 6.

If a dihedral group whose order is divisible by 4 is its own group of automorphisms its cyclic subgroup of index 2 may be assumed to have an order which exceeds 2 and to admit also no automorphism besides the identity and the one in which each of its operators corresponds to its inverse. Hence this cyclic group is of order 4 and it results that *the octic group and the symmetric group of order 6 are the only two dihedral groups which are their own groups of automorphisms.*

If a dihedral group whose order is twice an odd number admits the same number of outer automorphisms as inner automorphisms its cyclic subgroup of index 2 must have the cyclic group of order 4 for its group of automorphisms and hence it is the metacyclic group of order 20. This results from the fact that the group of automorphisms of the cyclic group of prime order p is the cyclic group of order $p-1$. If a dihedral group whose order is divisible by 4 admits the same number of outer automorphisms as inner automorphisms and is not the four group its cyclic subgroup of index 2 cannot have a larger order than 4 since its non-invariant operators of order 2 are transformed into themselves multiplied by the operators of the invariant subgroup of order 2 under its group of inner automorphisms, and hence they are transformed into themselves multiplied by the remaining operators of this cyclic subgroup by the outer automorphisms of the group. Hence it results that *there are two and only two dihedral groups which have the property that they admit just as many outer automorphisms as inner automorphisms.* One of these is the octic group while the other is the metacyclic group of order 20.

It was noted at the opening of this article that the group of inner automorphisms of a non-abelian dihedral group whose order is divisible by 4 is its quotient group with respect to its invariant subgroup of order 2. Hence every such dihedral group involves exactly two subgroups which are separately simply isomorphic with their groups of inner automorphisms while every other dihedral group except the four group is simply isomorphic with its group of inner automorphisms. On the other hand, the group of automorphisms of every non-abelian dihedral group is known[1] to be the holomorph of its cyclic subgroup of index 2. There is therefore no upper limit to the ratio of the orders of the group of automorphisms of the general dihedral group and the order of this dihedral group, and the study of the outer automorphisms of a dihedral group is practically reduced to the study of the holomorphs of cyclic groups.

Every abelian group H can be extended by operators which transform every operator of H into its inverse so as to obtain a group G of twice the order of H known as the generalized dihedral group of H. A necessary and sufficient condition that G is abelian is that all of its operators besides the identity are of order 2. As the group of inner automorphisms of G is its quotient group with respect to its subgroup generated by its invariant operators it results that a necessary and sufficient condition that it is simply isomorphic with G is that H is of odd order. It therefore results that a necessary and sufficient condition that the group of inner automorphisms of the generalized dihedral group is simply isomorphic with itself is the same for both the dihedral group and the generalized dihedral group.

To prove the fact that it is not possible for a generalized dihedral group which is not also a dihedral group to be its own group of automorphisms it may first be noted that the order of such a generalized dihedral group could clearly not be twice an odd number since in that case it would be simply isomorphic with its group of inner automorphisms but would clearly also admit outer automorphisms since in every such inner automorphism an operator of odd order corresponds either to itself or to its inverse. Since every non-cyclic abelian group involves at least one non-cyclic Sylow subgroup and every non-cyclic Sylow group involves a non-identity automorphism in which not every operator corresponds to its inverse, it results that *there is no generalized dihedral group which is not also a dihedral group but which is its own group of automorphisms.* It was noted above that such a group may be its own group of inner automorphisms.

To prove that the group of automorphisms of every dihedral group whose order is twice an odd number as well as the group of automorphisms of every non-abelian generalized dihedral group whose order is twice an odd number is a complete group, it may first be noted that each of these groups of automorphisms is known to be the holomorph of its invariant abelian subgroup. Moreover, this holomorph involves no invariant operator besides the identity. If this group of automorphisms is transformed into itself by an operator which is not contained in it the co-set with respect to the given group of automorphisms which contains this transforming operator contains also an operator which is commutative with every operator of the given dihedral group or of the given generalized dihedral group, respectively.

The latter operator can therefore be so selected that its first power which appears in the given group of automorphisms is the identity and that it is commutative with every operator of this group since no two of the operators of this group transform the operators of the given invariant dihedral group in the same manner. It may be emphasized here that while a group may be its own group of automorphisms *there is no group whose order exceeds 2 which is its own holomorph.* This results from the fact that the holomorph of a group necessarily includes the group and if a group is non-abelian its holomorph contains a subgroup which is simply isomorphic with it and is composed of operators which are separately commutative with every operator of the given group. The operation of forming successive holomorphs beginning with a group whose order exceeds 2 therefore leads to an infinite system of groups of increasing orders.

Since the group of automorphisms of a dihedral group whose order is divisible by 4 contains an invariant operator of order 2 it cannot be a complete group. Moreover, it does not follow in this case that if an operator is commutative with every operator of a dihedral group it is also commutative with every operator of its group of automorphisms, since two distinct operators of this group do not necessarily transform the operators of the given invariant dihedral group in a different manner when the order of this dihedral group is divisible by 4. The automorphisms of the dihedral groups are unusually well adapted for the study of the general properties of the group of automorphisms of a given group in view of the properties noted above.

[1] Miller, Blichfeldt, Dickson, *Finite Groups*, p. 169 (1916).

SOME ASYMPTOTIC RELATIONS

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1. Let $(1 - x + tx)^m = \sum_{n=0}^{\infty} a_n t^n$, $(1 - t)^{-r-1} = \sum_{n=0}^{\infty} A_n^{(r)} t^n \quad |t| < |$

$$A_n^{(r)} S_n^{(r)} = A_n^{(r)} a_0 + A_{n-1}^{(r)} a_1 + \ldots + a_n,$$

then the equation

$$(1 - x + tx)^m (1 - t)^{-r-1} = \sum_{n=0}^{\infty} A_n^{(r)} F(-m, -r; -n-r; x) t^n \tag{1}$$

mentioned in a previous paper,[1] indicates that

$$S_n^{(r)} = F(-m, -r; -n-r; x).$$

Hence

$$\lim_{n \to \infty} S_n^{(r)} = 1.$$

This means that the Cesàro sum (C, r) of the series $\sum_{n=0}^{\infty} a_m$ is 1. Equation (1) still holds for $x = 1$ if m is a positive integer and we write

$$F(-m, -r; -n-r; 1) = n!\Gamma(n + r - m + 1)/(n - m)!\Gamma(n + r + 1)$$