ON THE DERIVATIVES OF FUNCTIONS ANALYTIC IN THE UNIT CIRCLE

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Various results concerning the order of growth of the first and higher derivatives of univalent and of bounded functions analytic in the unit circle are known. Among these we mention Koebe's distortion theorem (Verzerrungssatz) in the univalent case and Schwarz's Lemma and the results of O. Szász[1] in the bounded case. A consequence of these results for a function $f(z)$ analytic in $|z| < 1$ is $|f'(z)| = 0\left(\frac{1}{(1 - |z|)^3}\right)$ in the case that $f(z)$ is univalent and $|f^{(n)}(z)| = 0\left(\frac{1}{(1 - |z|)^n}\right)$ in the case that $f(z)$ is bounded. These relations, however, give no information as to the conditions under which $|f^{(n)}(z_k)|(1 - |z_k|)^n$ tends to zero for a given sequence of points $\{z_k\}$ in the unit circle. In the univalent case an answer to this question is contained in the following result due to J. E. Littlewood and A. J. Macintyre:[2] *Let* f(z) *be analytic and univalent in* $|z| < 1$ *and let it omit there the value* ω. *Then in* $|z| < 1$ *the following inequality is satisfied:*

$$|f'(z)|(1 - |z|^2) \leqq 4\,|\,\omega - f(z)\,|.$$

The purpose of the present note is to summarize the principal results recently obtained by the authors in the study of the relation

$$|f^{(n)}(z_k)|(1 - |z_k|)^n \to 0 \text{ as } k \to \infty, \text{ where } |z_k| < 1,$$

in the case that the analytic function $f(z)$ is univalent, or is bounded, or omits two values in $|z| < 1$. The method involves primarily the systematic and detailed study of the function $\varphi_k(\zeta) = f\left(\frac{z_k + \zeta}{1 + \bar{z}_k\zeta}\right)$. The detailed exposition will be published at a later date.

Let $w = f(z)$ be analytic in $|z| < 1$ and let R denote the Riemann configuration over the w-plane onto which this function maps the region $|z| < 1$. Let w_0 be an arbitrary point of R. Then the radius of the largest smooth circle (boundary not included) with center at w_0 and wholly contained in R is called the *radius of univalence of* R *at* w_0 and will be denoted by $D_1(w_0)$. The following theorem can be easily proved:

Let f(z) *be regular and univalent in* $|z| < 1$, z_0 *any point of* $|z| < 1$, *and* $w_0 = f(z_0)$. *Then*

$$D_1(w_0) \leqq |f'(z_0)|(1 - |z_0|^2) \leqq 4D_1(w_0). \tag{1}$$

Each of these inequalities is sharp and the second one is another form of the Littlewood-Macintyre inequality. An immediate consequence of this theorem is the corollary:

Let f(z) *be regular and univalent in* $|z| < 1$, z_n *any sequence of points in* $|z| < 1$, *and* $w_n = f(z_n)$. *Then a necessary and sufficient condition that*

$$\lim_{n \to \infty} |f'(z_n)|(1 - |z_n|) = 0 \tag{2}$$

is that

$$\lim_{n \to \infty} D_1(w_n) = 0, \tag{3}$$

and a necessary and sufficient condition that $|f'(z_n)|(1 - |z_n|)$ *be bounded is that* $D_1(w_n)$ *be bounded.*

From this theorem it follows for a univalent and bounded function $f(z)$ that $|f'(z)| = o\left(\frac{1}{1 - |z|}\right)$ uniformly in the circle $|z| < 1$ as $|z| \to 1$. In this connection the following theorem is appropriate:

Let f(z) *be regular and univalent in the circle* $|z| < 1$. *Then*

$$\lim_{z \to e^{i\alpha}} |f'(z)|(1 - |z|)^{1/2} = 0$$

for all points $e^{i\alpha}$ *of the circumference* $|z| = 1$ *with the exception of at most a set of Lebesgue measure zero, where* z *in the above limit is taken in any angle less than* π *with vertex in* $e^{i\alpha}$ *and bisected by the corresponding radius.*

For univalent functions the second part of inequality (1) admits of extensions to all higher derivatives, for instance

$$|f^{(n)}(z_0)|(1 - |z_0|^2)^n \leqq 4 \cdot e \cdot n!\,(|z_0| + n)(1 + |z_0|)^{n-2} D_1(w_0).$$

For $n = 2$ or 3 the factor e in the right side may be omitted; the resulting inequalities are sharp. Obviously, $D_1(w_k) \to 0$, $w_k = f(z_k)$, implies

$$|f^{(n)}(z_k)|(1 - |z_k|)^n \to 0. \tag{4}$$

Examples can be constructed to show that the limit in (2) and in (4) can be approached arbitrarily slowly even if $f(z)$ is univalent and continuous in $|z| \leqq 1$.

The situation is entirely analogous for bounded functions:

Let f(z) *be regular and bounded in* $|z| < 1$:

$$|f(z)| \leqq M,$$

let $\{z_n\}$ *be any sequence of points in* $|z| < 1$, *and let* $w_n = f(z_n)$. *Then a necessary and sufficient condition for* (2) *is* (3).

Indeed, for any z_0 $(|z_0| < 1)$ *we have*

$$D_1(w_0) \leqq |f'(z_0)|(1 - |z_0|^2) \leqq \sqrt{8MD_1(w_0)}, \tag{5}$$

where $w_0 = f(z_0)$.

The situation for the higher derivatives of bounded functions is somewhat more complicated than in the univalent case. We need the following definitions:

Let R be a Riemann surface (configuration) over the w-plane, let w_0 be any point of R not a branch point of order greater than $p - 1$. Suppose that C_p is a simply connected region on R which contains w_0 in its interior, lies over the circle $|w - w_0| < \rho$, and covers this circle precisely p times. We call such a region a *p-sheeted circle of center* w_0 *and radius* ρ; the value $\rho = \infty$ is not excluded. *The radius of* p*-valence* $D_p(w_0)$ *of a Riemann surface* R *at a point* w_0 *belonging to* R *is defined as follows:*

(a) For $p = 1$, $D_p(w_0) = D_1(w_0)$.

(b) If w_0 is a branch point of order greater than $p - 1$ ($p > 1$), then $D_p(w_0) = 0$.

(c) For any other point w_0, the number $D_p(w_0)$ is the radius of the largest p-sheeted circle with center w_0 contained in R if such a circle exists, and is otherwise $D_{p-1}(w_0)$.

With this definition we prove:

Let $\{f_n(z)\}$ *be a sequence of functions analytic in the unit circle* $|z| < 1$ *and converging uniformly in every closed subregion of* $|z| < 1$ *to an analytic function* $f(z)$. *Let* z_0 *be any point in the circle* $|z| < 1$ *and set* $w_n = f_n(z_0)$, $w_0 = f(z_0)$. *Let* $D_p(w_n)$ *and* $D_p(w_0)$ *pertain to the images of* $|z| < 1$ *by the maps* $w = f_n(z)$ *and* $w = f(z)$, *respectively. Then*

$$\lim_{n \to \infty} D_p(w_n) = D_p(w_0).$$

As an analogue of the inequalities (5) we obtain:

Let $f(z)$ *be regular and bounded in* $|z| < 1$: $|f(z)| \leqq M$, *let* z_0 *be any point of* $|z| < 1$ *and* $w_0 = f(z_0)$. *Let* p *be a positive integer. Then there exist two positive constants* λ_p *depending only on* p *and* Λ_p *depending on* p *and* M *such that we have*

$$\lambda_p \cdot D_p(w_0) \leqq \sum_{k=1}^{p} \frac{1}{k!} \left| \sum_{\nu=0}^{k-1} (-1)^{\nu} \cdot \nu! \binom{k}{\nu} \binom{k-1}{\nu} \bar{z}_0^{\nu} \cdot (1 - |z_0|^2)^{k-\nu} \cdot f^{(k-\nu)} z_0 \right| \leqq \Lambda_p [D_p(w_0)]^{2^{-p}}. \quad (6)$$

Consequently, for any sequence $\{z_n\}$ ($|z_n| < 1$), *a necessary and sufficient condition for*

$$\lim_{n \to \infty} f^{(k)}(z_n)|(1 - |z_n|)^k = 0 \qquad (k = 1, 2, \ldots, p)$$

with $w_n = f(z_n)$, *is that*

$$\lim_{n \to \infty} D_p(w_n) = 0.$$

The last part of this theorem may also be inferred from the preceding theorem. Constants λ_p and Λ_p for which (6) holds can be explicitly determined.

Schottky's theorem enables us to extend the above results for bounded functions to the case of functions $f(z)$ omitting two values provided the sequence $w_n = f(z_n)$ is bounded. The case $|w_n| \to \infty$ can be treated by other methods.

[1] O. Szász, *Math Zeitschrift*, **8**, 303–309 (1920).

[2] J. E. Littlewood, *Proc. London Math. Soc.*, **23**, 507 (1924); A. J. Macintyre, *Jour. London Math. Soc.*, **11**, 7–11 (1936).

DIFFERENTIAL CALCULUS IN LINEAR TOPOLOGICAL SPACES[1]

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1. *Introduction.*—The most valuable definitions of differentials of functions in the classical differential calculi of finite as well as of infinite dimensional spaces are those that give the differential as a "first order approximation" to the difference. In this paper we give a definition of such a differential for functions whose arguments are in a linear topological space T_1 and whose values are in a linear topological space T_2, not necessarily the same[2] as T_1. Some of the fundamental properties of this differential are given as well as the properties of other related topological differentials.

We wish to emphasize here the fact that the spaces T_1 and T_2 are not necessarily metric—not even metrizable—and that the differential calculus in linear topological spaces has important applications to general differential geometry, general dynamics and general continuous group theory.

2. *Topological* M-*Differential.*—By a linear topological space we shall mean an abstract linear space with a Hausdorff topology in which the functions $x + y$ and αx are respectively continuous functions of both variables.

Let T_1 and T_2 be any two linear topological spaces. A function $l(x)$ on T_1 to T_2 is termed *linear* if it is additive and continuous—hence homogeneous of degree one.

DEFINITION OF M-DIFFERENTIAL.[3] *Let* $\mathrm{f(x)}$ *be a function with values in* $\mathrm{T_2}$ *and defined on a Hausdorff neighborhood* $\mathrm{S_{x_0}}$ *of* $\mathrm{x_0} \in \mathrm{T_1}$. *The function* $\mathrm{f(x)}$ *will be said to be* M-*differentiable at* $\mathrm{x = x_0}$ *and* $\mathrm{f(x_0;\ \delta x)}$ *will be called an* M-*differential of* $\mathrm{f(x)}$ *at* $\mathrm{x = x_0}$ *with increment* $\delta\mathrm{x}$ *if*

(1) *there exists a linear function* $\mathrm{f}(x_0;\ \delta\mathrm{x})$ *of* $\delta\mathrm{x}$ *with arguments in* $\mathrm{T_1}$ *and values in* $\mathrm{T_2}$