# Spacetime and Fields, a Quantum Texture

Sergio Doplicher Dipartimento di Matematica University of Rome "La Sapienza" 00185 Roma, Italy

October 29, 2018

#### Abstract

We report on joint works, past and in progress, with K.Fredenhagen and with J.E.Roberts, on the quantum structure of spacetime in the small which is dictated by the principles of Quantum Mechanics and of General Relativity; we comment on how these principles point to a deep link between coordinates and fields. This is an expanded version of a lecture delivered at the 37th Karpacz School in Theoretical Physics, February 2001.

# 1 Spacetime Uncertainty Relations

At large scales spacetime is a pseudo Riemanniann manifold locally modeled on Minkowski space. But the concurrence of the principles of Quantum Mechanics and of Classical General Relativity points at difficulties at the small scales, which make that picture untenable. For if we try to locate an event in say a spherically symmetric way around the origin in space with accuracy a, according to Heisenberg principle an uncontrollable energy E of order 1/a has to be transferred, which will generate a gravitational field with Schwarzschild radius  $R \simeq E$  ( $\hbar = c = G = 1$ ). Hence  $a \gtrsim R \simeq 1/a$  and  $a \gtrsim 1$ , i.e. in CGS units

$$a \gtrsim \lambda_P \simeq 1.6 \cdot 10^{-33} cm.$$
 (1)

If however we measure one of the space coordinates of our event with great precision a, but allow large uncertainties L in the knowledge of the other coordinates, the energy 1/a may spread over a thin disk of radius L

and thus generate a gravitational potential that would vanish everywhere as  $L \to \infty$ .

One has therefore to expect Space Time Uncertainty Relations emerging from first principles, already at a semiclassical level. Carrying through such an analysis [1, 2] one finds indeed that, if the smallest and largest space uncertainties of an event are denoted by a,b respectively, and the time uncertainty by  $\tau$ , the gravitational potential generated by the energy  $1/\min(a,\tau)$  localized at some instant with accuracies  $a,b,\tau$ , is at most of the order

$$|V| \simeq \frac{1}{b \cdot \min(a, \tau)} \tag{2}$$

Now our basic requirement is that the localization experiment should not deform spacetime in such a way that no signal from the region we wish to observe can reach infinity in space, otherwise this would put the observed event out of reach for any distant observer; namely

$$g_{00} = 1 + 2V > 0, (3)$$

where V is the potential generated by the energy transferred with the localization measurement itself; hence by (2) a necessary condition is

$$b \cdot \min(a, \tau) \gtrsim 1. \tag{4}$$

The Space Time Uncertainty Relations strongly suggest that spacetime has a Quantum Structure at small scales, expressed, in generic units, by

$$[q_{\mu}, q_{\nu}] = i\lambda_P^2 Q_{\mu\nu} \tag{5}$$

where Q has to be chosen not as a random toy mathematical model, but in such a way that necessary restrictions like (4) follow from (5). Further we want to impose (full) Lorentz invariant conditions on  $Q_{\mu\nu}$ , so that our models are compatible with Special Relativity; since in (5) Q is dimensionless, the commutator will effectively vanish for large distances compared to the Planck scale.

But we do not insist on covariance under general coordinate transformations, which, at a quantum level and at small scales, cannot be supported by conceptual experiments, as the freely falling laboratory, in presence of fields which vary significantly over Planckian distances. Moreover, for the sake of Elementary Particle Physics, an asymptotically flat background is an appropriate idealization, for the distribution of masses in the Universe

should not affect significantly the outcome of collision experiments in our laboratories.

The noncommutativity of the operators  $q_0, ..., q_3$  can be measured by the fundamental invariants

$$Q_{\mu\nu}Q^{\mu\nu};$$

$$[q_0, ..., q_3] := \det \begin{pmatrix} q_0 & \cdots & q_3 \\ \vdots & \ddots & \vdots \\ q_0 & \cdots & q_3 \end{pmatrix} :=$$

$$(6)$$

$$\varepsilon^{\mu\nu\lambda\rho}q_{\mu}q_{\nu}q_{\lambda}q_{\rho}=-(1/2)Q_{\mu\nu}(*Q)^{\mu\nu}$$

but the second is invariant only under the proper Lorentz transformations and only its square is invariant under space and time reflections as well.

If we (temporarily) assume that the components of Q commute with one another, and let  $\mathbf{e}, \mathbf{m}$  denote the triples of electric respectively magnetic components, we have

$$(-1/2)Q_{\mu\nu}Q^{\mu\nu} = \mathbf{e}^2 - \mathbf{m}^2; \tag{7}$$

since  $\mathbf{e}$  and  $\mathbf{m}$  respectively govern the space-time and space-space uncertainty relations, symmetry and (4) suggest the condition

$$Q_{\mu\nu}Q^{\mu\nu} = 0. (8)$$

Therefore the basic Quantum Condition must read

$$[q_0, ..., q_3]^2 = S, (9)$$

where S is a Lorentz invariant.

We will see later how more general choices for S are important, but (4) suggest a multiple of I. If we also require that the Q commute with the q, we get the Basic Model introduced and discussed in detail in [1] that we will briefly report on in the next Section.

Other approaches to uncertainty relations affected by gravity and related phenomena can be found e.g. in [7],..., [15]. We do not attempt to give a complete list of references related to this subject, which became quite numerous in the last three years; approaches based on the quantum deformations of the Poincaré Algebra received a lot of attention, cf [17] and references therein.

### 2 The Basic Model

In the notation introduced above the quantum conditions of the Basic Model may be rewritten as

$$[q_{\mu}, Q_{\lambda \rho}] = 0, \tag{10}$$

$$\mathbf{e}^2 = \mathbf{m}^2, \mathbf{e} \cdot \mathbf{m} = \pm I. \tag{11}$$

In this model the following weaker form of (4) is implemented, cf [1, 2]:

$$\Delta q_0 \cdot \sum_{j=1}^3 \Delta q_j \gtrsim 1; \sum_{1 \le j < k \le 3} \Delta q_j \Delta q_k \gtrsim 1.$$
 (12)

Relations (11) define an algebraic manifold with two connected components each isomorphic to the coset space of the proper Lorentz group modulo boosts along a fixed direction and rotations around it, i.e. to  $SL(2,\mathbb{C})/\mathbb{C}_*$ , where  $\mathbb{C}_*$  is embedded in  $SL(2,\mathbb{C})$  as the 1,1 component of diagonal matrices. Each pair  $(\mathbf{e}',\mathbf{m}')$  as in (11) can be obtained from a pair such that  $\mathbf{e} = \pm \mathbf{m}$  by a boost with velocity, say  $\mathbf{v}$ , orthogonal to  $\mathbf{e}$ , hence by (11)  $\mathbf{e}$  and  $\mathbf{m}$  are vectors in the unit sphere  $S^2$  in three dimensional space, and  $\mathbf{v}$  is a tangent vector to  $S^2$ ; summarizing

$$\Sigma_{+} \simeq \Sigma_{-} \simeq SL(2, \mathbb{C})/\mathbb{C}_{*} \simeq TS^{2}.$$
 (13)

While classical Spacetime is described by the commutative C\* algebra of continuous functions vanishing at infinity, Quantum Spacetime will be described by a noncommutative C\* algebra  $\mathcal{E}$ , to which the q are affiliated in the sense of [18], cf [1], i.e each representation of  $\mathcal{E}$  determines operators  $q_{\mu}$  fulfilling our Quantum condition, and all "regular" representations appear this way.

We adopted in [1] the following paradigm, which may well apply to more general cases (cf [4]): if we interpret (5) as defining a bundle of Lie algebras, in that case over  $\Sigma$ , the regular representations will be those which are integrable to a representation of the corresponding bundle of simply connected Lie groups; the C\* algebra  $\mathcal{E}$  will then arise as a continuous field of group C\* algebras.

In the basic model the fibers are just Heisenberg groups with a non degenerate  $\mathbb{C}$ -number commutator matrix (the generic point in  $\Sigma$ ), hence we get a continuous field of the algebra of all compact operators (on a

separable infinite Hilbert space) which can be proved to be trivial (see [1]), i.e.

$$\mathcal{E} \simeq \mathcal{C}_0(\Sigma, \mathcal{K}).$$
 (14)

These findings fit very well in the theory of strict deformation quantization [19].

This C\* algebra carries a natural action of the (full) Poincaré group  $\mathcal{P}$  as automorphisms, which is actually determined by its extension to the affiliated q's, fulfilling the natural relations

$$\alpha_L(q) = L^{-1}q \quad , \quad L \in \mathcal{P}.$$
 (15)

Thus  $\mathcal{E}$  is a Quantum space but its global symmetries are the classical ones, as expected since at large scales the model turns classical again, and the Poincaré transformations are global motions, acting the same way in the small and in the large. This situation parallels the familiar one in non-relativistic Quantum Mechanics, where the Schroedinger Operators q and p do not commute, but the Galilei invariance is expressed by an action of the classical Galilei Group as automorphisms (representations up to a phase appear only in the unitary implementations). Actually this structure is indeed a special case of our present model, cf below.

The classical concept of points in a space has to be replaced by pure states with minimal uncertainties, i.e. pure states which are optimally localized in the sense that the quantity

$$(\Delta q_0)^2 + \dots + (\Delta q_3)^2 \tag{16}$$

is minimal; this is a frame dependent condition, which picks a point  $\mathbf{e} = \pm \mathbf{m}$  in the spectrum of the Q's, i.e. a point in the base  $S^2 \times \{\pm 1\}$  if we think of  $\Sigma$  as a tangent manifold, so that the q's fulfilling (5) now become the Schroedinger operators q,p for a particle in two dimensions (the four dimensional translations acting as Galilei transformations), and the expression (16) being minimal implies that its value is 2 and that our state is the ground state of the harmonic oscillator for those Schroedinger operators.

Such states ought to have a preferred role in discussing the large scale limit of the Quantum space; since in (14) the Planck length appears only in the exponential in the Weyl relations, which force the fiber to be  $\mathcal{K}$ , and in the large scale limit  $\mathcal{K}$  deforms to  $\mathcal{C}_0(\mathbb{R}^4)$ , we see that the quantum spacetime becomes  $\mathbb{R}^4 \times \Sigma$  in the large scale limit, while, if only optimally localized states are considered, the limit is rather

$$\mathbb{R}^4 \times S^2 \times \{\pm 1\}$$
.

Thus the discrete space  $\{\pm 1\}$  appears because the spectrum of the centre of the algebra generated by the Q's is not connected (a consequence of imposing symmetry under reflections too), and while the continuous factor in the ghost manifold is not compact, only a compact manifold, actually a sphere with radius the square of the Planck length, plays a role if we are testing with optimally localized states.

The paradigm we adopted in attaching a C\* algebra to relations (5) in our model tells us how to calculate functions f(q): as in the von Neumann-Wigner-Moyal calculus, write f as the Fourier transform of its ordinary anti-Fourier transform, and replace the exponentials by the Weyl operators  $\exp i(\alpha q)$ ; the multiplication of these exponentials is precisely governed by the bundle of Lie groups associated to the models; thus this paradigm can, and will be, applied in some more general context. Moreover space integration at time t and spacetime integration can be easily defined and related to the trace in each fiber, so that we can introduce Free Fields on QST, the free Hamiltonian, which turns out to be unchanged by the quantum deformation, and interaction Hamiltonians, i.e. we can lay down the setup to apply the usual perturbation expansion (cf [1]).

While integration over space or spacetime poses no problem in this model, integration over  $\Sigma$  does, since we tacitly assumed that our fields do not depend on the points of  $\Sigma$ ; but we have no bounded invariant measure on  $\Sigma$  so we cannot integrate to get an invariant result.

The way out chosen in [1] was to integrate over the base  $S^2 \times \{\pm 1\}$  of  $\Sigma$ , thus keeping only rotation invariance; but in the end we face a more serious difficulty. Namely the perturbation expansion is found to be exactly that of a non local theory on the classical Minkowski space.

Of course (4) suggests that causality breaks down at short distances: but it should be recovered at large scales with respect to Planck length (say at QCD scales,  $10^{-17}$  cm.), while the acausal effects of ordinary nonlocal theories might cumulate after summing the perturbation expansion in a disruptive way.

Strangely enough, these lessons of [1] have been largely neglected; well after the appearance of [1] we assisted to a flow of papers on QFT models on a QST which is characterized by (5) with a fixed C-number tensor on the right, disregarding the physical meaning of noncommutativity and the need of Lorentz invariance, but extensively applying the calculation aspect of what we exposed, summarized by the use of the "star product".

The negative conclusions referred to above might lead us to reanalyze the concept of interaction over QST; in particular the ordinary Wick product should not be allowed: if e.g. we are to define the Wick square of the field A, we can evaluate A(q)A(q') on distinct variables q, q', but then we cannot set q-q'=0, since these operators obey similar relations to (4); we can however evaluate a conditional expectation defined by an optimally localized state on q-q'; as this is the ground state of an harmonic oscillator, it introduces a Gaussian nonlocality factor which violates again causality, but might fully regularize the theory, and in fact might give rise to a Gaussian fall off of cross sections at large energies [5].

The problems with causality lead us [5] to enquire about light propagation, i.e. classical ED on QST; while the local gauge group of Classical ED on Minkowski space is the unitary group of  $C_0(\mathbb{R}^4)+\mathbb{C}\cdot I$ , it is that of  $\mathcal{E}+\mathbb{C}\cdot I$  in the case of QST. Treating it the usual way we found that ED is characterized by nonlinearly selfinteracting equations, for which a plane wave is a stationary solution, but superpositions of two different plane waves are not, with a propagation into massive modes; in principle, this effect ought to be detectable splitting a monochromatic laser beam into a superposition of states with different momenta with the help of a partially reflecting mirror (or detecting the light of a distant galaxy split by a gravitational lens); but the fraction of energy density which would go into these massive modes, calculated to the lowest order in the Planck length, turns out to be of order lower than  $10^{-130}$ .

More seriously, such a theory has a huge gauge group, so it is difficult to propose testable effects, no matter how tenuous, in terms of gauge invariant quantities.

For recent discussions of possible testable effects of Quantum Gravity cf e.g. [21, 22].

Another drastic consequence is the nonvanishing of the current divergence, due to quantum gravitational anomalies.

But the Gauge Principle expresses the point nature of interactions, and is the basic principle lying behind locality in ordinary QFT, so it might well by itself provide a rigid substitute to causality in QFT on QST. This hope motivates a long standing attempt to a general formulation of gauge theories on a noncommutative manifolds using the absolute differential calculus ([5, 6]; this calculus emerged also in other papers appeared meanwhile, cf e.g. [16]). One might expect that a proper noncommutative approach might lead to a new picture of interactions at Planck scale, which avoids the unpleasant features met when we just replace products with \*-products.

An approach to gauge theories on noncommutative spaces based on the

notion of covariant coordinates has been proposed in [24] and references therein.

### 3 Deformed Models

The remaining sections are based on work in progress with K. Fredenhagen, D. Bahns and G. Piacitelli.

The basic model discussed in the previous section has many virtues including simplicity, even if it does not implement in full relations (4). However these relations appear, as relations (12), only necessary to guarantee (at least at a semiclassical level) the gravitational stability of localization of events, but not a priori sufficient to that purpose; furthermore it might well turn out to be impossible to formulate a necessary and sufficient condition which involves solely the background geometry, without a dynamical description of spacetime, cf next section. However, if we relax the drastic simplification (10) that the Q's are central, and instead we only assume that they commute with one another, keeping the other Quantum Conditions, there is room for deformed models where the Spacetime Uncertainty relations are implemented in forms stronger than (12).

One such model ([5]; partly announced in [4]) can be formulated introducing self adjoint central operators, which form two antisymmetric 2-tensors H, T and a four vector C, and adding to the q's two scalar commuting generators R, S, and imposing

$$[q_{\mu}, q_{\nu}] = i(H_{\mu\nu} + T_{\mu\nu}R),$$
  
 $[q_{\mu}, R] = iC_{\mu}S,$   
 $[q_{\mu}, S] = -iC_{\mu}R,$   
 $[R, S] = 0,$ 
(17)

where  $S^2 + R^2$  is central and can be set equal to I, and, by Jakobi identity,

$$C_{\mu}T_{\nu\lambda} + C_{\nu}T_{\lambda\mu} + C_{\lambda}T_{\mu\nu} = 0, \quad for \quad all \quad \mu, \nu, \lambda.$$
 (18)

Furthermore, the contraction of T and of its Hodge dual \*T with itself and with H should vanish, while H fulfills the same conditions as Q in the basic model.

The full Lorentz group  $\mathcal{L}$  will act transitively on the spectrum of the centre, so that here, at large scales, the classical limit of our QST will be

$$\mathbb{R}^4 \times \mathcal{L} \tag{19}$$

a manifold with 10 dimensions, which would effectively reduce here too if we restricted attention to optimally localized states. A more detailed account of this model will be discussed elsewhere [27].

### 4 A Dynamical Picture of Quantum Spacetime

The models of QST outlined above try to implement in the noncommutative nature of the underlying geometry some of the minimal limitations on the localization of an event which are imposed by our present knowledge of the principles of Physics. Developing QFT in the appropriate way on this underlying geometry rather than on Minkowski space should avoid some of the contradictions we would be otherwise bound to meet. But we might expect that the very structure of spacetime in the small, hence the algebraic structure of the underlying model, should depend on the dynamics, and thus on the quantum state. We propose a general scenario where this would be the case.

Let us first note that if we develop QFT over a fixed geometric background described by a (noncommutative  $C^*$ ) algebra  $\mathcal{E}$ , carrying an action of the Poincaré group by automorphisms, the (bounded functions of the) field operators (or more generally [23] local observables) should take values in a quasilocal  $C^*$  algebra  $\mathfrak{A}$ , and fields would be (functions from QST to  $\mathfrak{A}$ ) described by elements of (or affiliated to) the tensor product

$$\mathcal{E} \otimes \mathfrak{A}$$
 (20)

But a more realistic picture of QST might well involve operators in (20) which cannot be easily split in the two factors; the commutators of the q's would then appear as functions of the fields, more specifically of the metric  $g_{\mu\nu}$ , coupled to all fields by Einstein Equation, the fields themselves being at the same time functions of the q's. Thus the commutation relations between

the q's should appear as part of the equations of motion:

$$[q_{\mu}, q_{\nu}] = iQ_{\mu\nu}(g)$$
 
$$R_{\mu\nu} - (1/2)Rg_{\mu\nu} = 8\pi T_{\mu\nu}(\psi)$$
 (21) 
$$F(\psi) = 0,$$

where  $T_{\mu\nu}(\psi)$  is the energy momentum tensor of the fields involved, except gravitation, and the last line is symbolic for their equation of motion, where g enters too through the covariant derivatives. Of course the action of translations on the q's will no longer be just the addition of multiples of the identity, since the q's depend on the metric g on which translations act as well.

As an attempt to investigate the form of (21,a), suppose we adopt a semiclassical approach, replacing the right hand side of (21,b) by its expectation value in a given state and let g be a classical solution; if we perform a measurement to localize an event in this state, we should repeat the considerations of [1], cf Section 1 above, in the background g; in the approximation of linearized gravity, with V as in equation (3), we should now impose

$$q_{00} + 2V > 0; (22)$$

hence

$$q_{00} \cdot b \cdot \min(a, \tau) \ge 1. \tag{23}$$

If we forget for a moment not only general covariance, according to which g should have no intrinsic meaning, but even Lorentz covariance, we could fulfill (23) requiring

$$[q_{\mu}, q_{\nu}] = iQ_{\mu\nu}g_{00}^{-1} \tag{24}$$

where  $Q_{\mu\nu}$  does not depend of g, and is defined as above in this report. According to General Relativity the Ricci tensor  $R_{\mu\nu}$  is physically significant but the metric tensor g is not; yet it has been proposed [15] that Quantum Mechanics might alter this view, a possibility to be kept in mind while trying to rewrite a more convincing covariant extrapolation of (24).

The first natural guess would be to replace  $g_{00}^{-1}$  in (24) by a scalar depending only on the local variations of g, as the scalar curvature R; hence, using Einstein Equation (21,b) we would write

$$[q_{\mu}, q_{\nu}] = -8\pi\alpha i Q_{\mu\nu} g^{\lambda\rho} T_{\lambda\rho}(\psi), \qquad (25)$$

where a further constant factor  $\alpha$  has been allowed; or, even more generally, we could replace the Quantum Conditions by

$$Q_{\mu\nu}Q^{\mu\nu} = 0,$$
  
 $[q_0, ..., q_3]^2 = (\alpha R)^4.$  (26)

Equations (25) and (26) do not reduce to our background model where R = 0, so we are tempted to replace  $\alpha R$  by  $I + \alpha R$ ; we limit ourself here to support the scenario expressed in (21) without committing ourself to a choice, but point out (maybe only as a curiosity) that if our state is strictly localized in a tiny region, expectations of observables which are spacelike to that tiny region will be the same as in the vacuum and there we will find the following semiclassical approximation to (25)

$$[q_{\mu}, q_{\nu}] = -8\pi\alpha i Q_{\mu\nu} g^{\lambda\rho} \langle T_{\lambda\rho} \rangle_0; \tag{27}$$

we might here insert the empiric evidence that  $\langle T_{00} \rangle_0$  is not zero but equal to the cosmological constant  $\Lambda$ ; in a relativistic vacuum

$$\langle T_{\lambda\rho}\rangle_0 = \Lambda \cdot diag(1, -1, -1, -1);$$

now if g is a spherically symmetric stationary solution with  $g_{j0} = g_{0j} = 0$ , j = 1, 2, 3, and  $-\mathbf{g}$  is its space part, (27) takes the form

$$[q_{\mu}, q_{\nu}] = -8\pi\alpha\Lambda i Q_{\mu\nu}(g_{00}^{-1} + tr(\mathbf{g}^{-1}));$$

for the Schwarzschild solution, for instance, the last term in brackets would be equal to  $g_{00}^{-1} + g_{00} + 2$ ; but if there is a preferred frame (that of the Cosmic Background Radiation) where  $\langle T_{00} \rangle_0 = \Lambda$ ,  $\langle T_{jj} \rangle_0 = 0$ , j = 1, 2, 3, we would get exactly (24).

These comments do not pretend to be neither satisfactory nor in a final shape (we used in our heuristic argument strict locality, which is bound to fail at Planck distances); yet it might well turn out that the quantum nature of spacetime does say something on the problem of the cosmological constant; for the presence of T in the right hand side of our spacetime commutation relations should imply an effective repulsion at short distances, and since quantum spacetime links aspects in the small (ultraviolet) to aspects in the large (infrared), this short range repulsion might well give rise to long range effects.

# 5 Hints of Relations to String Theory

With the notation of Section 1, our Space Time Uncertainty Relations read

$$a \cdot b \gtrsim 1,$$
 (28) 
$$\tau \cdot b \gtrsim 1;$$

the second one had been actually proposed earlier on the basis of a qualitative argument in String Theory [7], and derived later in the context of D-branes [11, 8]. Other recent findings in that domain lead to relations similar to our first relation too [16].

Other superficial coincidences can be noted: U(1) gauge theory on QST described by the basic model is actually a  $U(\infty)$  gauge theory (more precisely, the gauge group will be the unitary group of  $\mathcal{E} + \mathbb{C} \cdot I$ , namely the product of the torus  $\mathbb{T}$  with the group of continuous functions equal to I at infinity from  $\Sigma$  to the group of unitaries which are perturbations of I by a compact operator), while U(N) gauge theory in the limit  $N \to \infty$  is believed to merge with String Theory; the QST version of Wick product leads to ultraviolet finite theories with a Gaussian like falloff of the transition matrix elements above Planckian energy-momentum values [26].

These facts might be no more than fortuitous coincidences, but suggest that the physical principles underlying the proposal of Quantum Spacetime might even turn out to provide the fundamental physical motivations which are still lacking in String Theory.

## References

- [1] S.Doplicher, K.Fredenhagen, J.E.Roberts: The Quantum Structure of Spacetime at the Planck Scale and Quantum Fields, Commun. Math. Phys. 172, 187 220 (1995);
- [2] S.Doplicher, K.Fredenhagen, J.E.Roberts: Spacetime Quantization Induced by Classical Gravity, Phys. Letters B 331, 39 44 (1994);
- [3] S.Doplicher: Quantum Physics, Classical Gravity, and Noncommutative Spacetime, Proceedings of the XIth International Conference of Mathematical Physics, D.Iagolnitzer ed, 324 -329, World Sci. 1995;

- [4] S.Doplicher: Quantum Spacetime, Annales Inst. Henri Poincare' vol.64, 543 553, 1996;
- [5] work in progress with D.Bahns, K.Fredenhagen and G.Piacitelli;
- [6] K.Fredenhagen: address to the Goslar Meeting, 1998; Quantum fields and noncommutative space-time, Proceedings of the Hesselberg Meeting March 1999, F.Scheck, W.Werner and H.Upmeier eds., Springer L.N.P. 596 (2002);
- [7] T.Yoneya: Duality and Indeterminacy Principle in String Theory, in Wandering in the Fields, K.Kawarabayashi, A.Uwaka eds., World Sc. (1987);
- [8] T.Yoneya: String Theory and Space-Time Uncertainty Principle, hep-th/0004074;
- [9] C.A.Mead: Possible Connection between Gravitation and Fundamental Length, Phys. Rev. 135B, 849-862 (1964);
- [10] D.Amati, M.Ciafaloni, G.Veneziano: Can Spacetime be probed below the String Size? Phys. Lett. B, 216 41 (1989);
- [11] G.Amelino-Camelia, J.Ellis, N.E.Mavromatos, D.V.Nanopoulos: On the Spacetime Uncertainty Relations of Liouville Strings and D-Branes, hep-th/9701144;
- [12] J.Lukierski, A.Nowicki, H.Ruegg: New Quantum Poincare' Algebra and k-deformed field theory, Phys. Lett.B 293, 344-352 (1992);
- [13] A.Kempf: Uncertainty relations in quantum Mechanics with quantum group symmetries, J. Math. Phys. 35, 4483-4496 (1994);
- [14] M.Maggiore: Quantum groups, gravity and the generalized uncertainty principle, Phys. Rev. D49, 5182-5187 (1994);
- [15] Chong-Sun-Chu, Pei-Ming Ho, Yeong-Chuan Kao: Worldvolume uncertainty relations for D-branes, hep-th/9904133;
- [16] S.Cho, R.Hinterding, J.Madore, H.Steinacker: Finite Field Theory on Noncommutative Geometry, Int. J. Mod. Phys. D9, 161-199 (2000);

- [17] P.Kosinski, J.Lukierski, P.Maslanka: Noncommutative parameters of quantum symmetries and Star Products hep-th/0012056;
- [18] S.L.Woronowicz: Unbounded Elements Affiliated with C\* Algebras and non compact Quantum Groups, Commun. Math. Phys. 136, 399-432 (1991);
- [19] M.A.Rieffel: On the Operator Algebra for the Spacetime Uncertainty Relations, in Operator Algebras and Quantum Field Theory, S.Doplicher, R.Longo, J.E.Roberts and L.Zsido eds, I.P. 1997;
- [20] D.V.Ahluwalia, Principle of equivalence and wave-particle duality in quantum gravity, gr-qc/0009033;
- [21] G.Amelino-Camelia: Gravity mediated interferometers as probes of a low-energy effective quantum gravity gr-qc/9903080;
- [22] J.Ellis, N.E.Mavromatos, C.V.Nanopoulos: *Probing Models of Quantum Space-Time Foam*, gr-qc/9909085;
- [23] R.Haag: Local Quantum Physics, Texts and Monographs in Physics, Springer 1994;
- [24] J.Madore, S.Schraml, P.Schupp, J.Wess: Gauge theory on noncommutative spaces, hep-th/0001203; B.Jurco, L.Mueller, S.Schraml, P.Schupp, J.Wess: Construction of non-abelian gauge theories on noncommutative spaces, hep-th/0104153.
- [25] D.Bahns, S.Doplicher, K.Fredenhagen, G. Piacitelli: On the unitarity problem in space/time noncommutative theories, Phys. Lett. B553 (2002) 178 - 181, hep-th/0201222.
- [26] D.Bahns, S.Doplicher, K.Fredenhagen, G. Piacitelli: *Ultraviolet finite quantum field theory on quantum spacetime*, hep-th/0301100.
- [27] D.Bahns, S.Doplicher, K.Fredenhagen, G. Piacitelli: work in progress.