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GRAVITY AND SIGNATURE CHANGE

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ABSTRACT

The use of proper "time" to describe classical "spacetimes" which contain both Euclidean and Lorentzian regions permits the introduction of smooth (generalized) orthonormal frames. This remarkable fact permits one to describe both a variational treatment of Einstein's equations and distribution theory using straightforward generalizations of the standard treatments for constant signature.

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1. INTRODUCTION

A signature-changing spacetime is a manifold which contains both Euclidean ³ and Lorentzian regions. Signature-changing metrics must be either degenerate (vanishing determinant) or discontinuous, but Einstein's equations implicitly assume that the metric is nondegenerate and at least continuous. ⁴ Thus, in the presence of signature change, it is not obvious what "the" field equations should be.

For discontinuous signature-changing metrics, one can derive such equations from a suitable variational principle [2]. This turns out to follow from the existence in this case of a natural generalization of the notion of orthonormal frame. The standard theory of tensor distributions, as well as the usual variation of the Einstein-Hilbert action, can both be expressed in terms of orthonormal frames, and thus generalize in a straightforward manner to these models. No such derivation is known for continuous signature-changing metrics. Our key point is that although signature change requires the metric to exhibit some sort of degeneracy, there is in the discontinuous case a more fundamental field, namely the (generalized) orthonormal frame, which remains smooth.

We introduce here two simple examples in order to establish our terminology. A typical *continuous* signature-changing metric is

$$ds^2 = t \, dt^2 + a(t)^2 \, dx^2 \tag{1}$$

whereas a typical *discontinuous* signature-changing metric is

$$ds^{2} = \operatorname{sgn}(\tau) \, d\tau^{2} + a(\tau)^{2} \, dx^{2} \tag{2}$$

Away from the surface of signature change at $\Sigma = \{t = 0\} = \{\tau = 0\}$, these metrics are related by a smooth coordinate transformation, with τ denoting proper "time" away from Σ . However, since $d\tau = \sqrt{|t|} dt$, the notions of smooth tensors associated with these coordinates are different at Σ , corresponding to different differentiable structures.

We argue here in favor of the discontinuous metric approach, both physically and mathematically. Physically, because of the fundamental role played by proper time. Mathematically, because of the geometric invariance of the unit normal to the surface of signature change. The resulting (generalized) orthonormal frames provide a clear path leading to a straightforward generalization of both Einstein's equations and the theory of tensor distributions.

2. PHYSICS

A standard tool in the description of physical processes is the introduction of an orthonormal frame. Physical quantities can be expressed in terms of tensor components in

³ Due to the frequent misuse of the word *Riemannian* to describe manifolds with metrics of any signature, we instead use *Euclidean* to describe manifolds with a positive-definite metric and *Lorentzian* for the usual signature of relativity. This is not meant to imply flatness in the former case, nor curvature in the latter.

⁴ This can be weakened [1] to allow locally integrable metrics admitting a squareintegrable weak derivative. Discontinuous metrics do not satisfy this condition.

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an orthonormal frame, corresponding to measurements using proper distance and proper time.

For example, when studying a scalar field on signature-changing backgrounds such as (1) or (2), it is important to know the value of the canonical momentum at the boundary, which is essentially the derivative of the field with respect to proper time. Furthermore, the well-posedness of the initial-value problem in the Lorentzian region tells us that the canonical momentum will be well-behaved at Σ if it is well-behaved at early times.

Continuous signature-changing metrics necessarily have vanishing determinant at the surface of signature change, which prevents one from defining an orthonormal frame there. The situation is different for signature-changing metrics such that proper "time" τ is an admissible coordinate. Although the metric is necessarily discontinuous, 1-sided orthonormal frames can be smoothly joined at Σ . Remarkably, the resulting *generalized orthonormal frame* is smooth, and is as orthonormal as possible. In fact, requiring not only that the 1-sided induced metrics on Σ , but also the 1-sided orthonormal frames, should agree at Σ implies that either the full metric is continuous (and nondegenerate) or that the signature changes.

Such frames can be used to derive Einstein's field equations from the Einstein-Hilbert action, obtained for constant signature by integrating the Lagrangian density

$$\mathcal{L} = g_{ac} R^c{}_b \wedge * \left(e^a \wedge e^b \right) \tag{3}$$

where

$$R^a{}_b = d\omega^a{}_b + \omega^a{}_c \wedge \omega^c{}_b \tag{4}$$

are the curvature 2-forms and * denotes the Hodge dual. Varying this action with respect to the metric-compatible connection ω leads to the further condition that ω be torsion-free, while varying with respect to the (arbitrary) frame *e* leads to Einstein's equations. In the presence of a boundary, one obtains ⁵ (in vacuum) the Darmois junction condition [3], namely that the extrinsic curvature of the boundary must be the same as seen from each side. (In general, one obtains the usual Lanczos equation [4] relating the stress tensor of the boundary to the discontinuity in the extrinsic curvature.)

The above derivation of Einstein's equations requires that the connection 1-forms admit (1-sided) limits to the boundary. For continuous signature-changing metrics, the connection 1-forms typically blow up at the boundary, but for discontinuous signature-changing metrics in a (1-sided) orthonormal frame they don't. ⁶ It thus seems reasonable to propose that "Einstein's equations" for signature-changing manifolds should be obtained by varying the (piecewise extension of) the above action ⁷ with respect to the (generalized)

⁵ A surface term (the trace of the extrinsic curvature) must be added to the Einstein-Hilbert action in the presence of boundaries; this has nothing to do with signature change.

⁶ This will be the case if each 1-sided manifold-with-boundary has a well-defined connection, as for instance when glueing manifolds together or, on the Lorentzian side, when starting from well-posed initial data.

⁷ There are a number of relative sign ambiguities between regions of different signature, so that the relative sign in the action — and hence in the boundary conditions — can be chosen arbitrarily.

orthonormal frame. As expected, one obtains Einstein's equations separately in the two regions together with the Darmois junction conditions at the boundary [2].⁸

3. MATHEMATICS

Theories involving internal boundaries are typically formulated using distribution theory. The standard theory of hypersurface distributions is based on a nondegenerate volume element, which is usually taken to be the metric volume element if available. It is a remarkable property of signature-changing spacetimes for which τ is an admissible coordinate that, even though the metric is discontinuous, the (continuous extension of) the metric volume element is smooth. This of course follows immediately from the smoothness of the generalized orthonormal frame, from which the volume element can be constructed. Thus, standard distribution theory can be used with no further ado [6].

Smooth signature-changing metrics, on the other hand, have metric volume elements which vanish at Σ . In fact, the combined requirements that the metric volume element be used where possible and that smooth tensors be distributions result in this case in a theory [6] in which the Dirac delta distribution is identically zero!

To illustrate these results, consider the following informal example. Consider first the discontinuous signature-changing metric (2) with metric volume element

$$\omega = d\tau \wedge dx \tag{5}$$

defined initially away from $\tau = 0$, then continuously extended. Let $V = V^{\tau} \partial_{\tau}$ be a smooth vector field, and let

$$\boldsymbol{\delta} = d\boldsymbol{\Theta} = \delta(\tau) \, d\tau \tag{6}$$

be the standard hypersurface distribution associated with $\tau = 0$, namely the derivative of the Heaviside distribution Θ . Then

$$\langle \boldsymbol{\delta}, V \rangle = \int_{M} V^{\tau} \, \delta(\tau) \, \omega = \int_{t=0} V^{\tau} dx \tag{7}$$

Now repeat the above construction for the smooth signature-changing metric (1) with metric volume element

$$\hat{\omega} = \sqrt{|t|} \, dt \wedge dx \tag{8}$$

again defined initially away from t = 0, then continuously extended. The hypersurface distribution associated with t = 0 is now

$$\boldsymbol{\delta} = d\boldsymbol{\Theta} = \delta(t) \, dt \tag{9}$$

so that if $\hat{V} = \hat{V}^t \partial_t$ is a smooth vector field then

$$\langle \boldsymbol{\delta}, \hat{V} \rangle = \int_{M} \hat{V}^{t} \,\delta(t) \,\hat{\omega} = 0 \tag{10}$$

⁸ Embacher [5] has derived field equations from a number of different versions of the Einstein-Hilbert action, including the one given here.

since $\hat{\omega} = 0$ at t = 0. The essential difference is not a change in δ , nor in the volume element, but rather fundamentally different notions of what it means for the vector fields V and \hat{V} to be smooth. For further details, see [6].

This problem can of course be avoided for smooth signature-changing metrics by using a nonmetric volume element. For the above example, choosing the volume element

$$\Omega = dt \wedge dx \tag{11}$$

in the definition of distributions leads to (10) being replaced by

$$\langle \boldsymbol{\delta}, \hat{V} \rangle = \int_{M} \hat{V}^{t} \,\delta(t) \,\Omega = \int_{t=0} V^{t} dx \tag{12}$$

This theory is perfectly viable, and has been used to study the scalar field on signature changing backgrounds. However, the resulting distributions — foremost among them the Heaviside distribution — differ from the distributions one would naturally define on the Lorentzian region alone. While this does not limit the usefulness of this approach, we find it attractive that for discontinuous signature-changing metrics no such problem arises.

4. DISCUSSION

We have given both mathematical and physical examples of calculations which are greatly simplified by working with generalized orthonormal frames when the signature changes, and hence with proper "time" τ . Choosing a manifold structure such that τ is a coordinate seems most likely to lead one correctly through the minefield of choices one must make when dealing with a degenerate metric.

Even in the constant signature case, while there is no need to use orthonormal frames, many calculations become simpler if one does so. One well-known example is classical relativity itself, where the use of orthonormal tetrads rather than, say, coordinate basis vectors, causes a vast reduction in the number of independent components of the curvature tensor [7]. This fact formed the basis for the early work on the classification of solutions of Einstein's equations using computer algebra; the coordinate-based computations would have been too unwieldy.

The results described here for gravity are completely analogous to the work of Dray *et al.* for the scalar field [8], in which it was proposed that the field and its canonical momentum be continuous at the surface of signature change. Ellis and coworkers proposed similar boundary conditions for both the scalar field and for gravity [9]. Some of the implications of these boundary conditions for gravity have been further explored by Hellaby and Dray [10].

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