XVIII. *On the Laws which regulate the Distribution of the Polarising Force in Plates, Tubes, and Cylinders of Glass, that have received the Polarising Structure.* *By* David Brewster, LL. D. F. R. S. Lond. & Edin.

(*Read June* 17. 1816.)

In the Philosophical Transactions for 1816, I have described at great length the various phenomena which are exhibited by glass and other substances to which the property of double refraction has been communicated by heat, by rapid cooling, by evaporation, or by mechanical compression and dilatation. In pursuing the same subject, I have observed many singular facts respecting the developement of new axes, by a change in the form and condition of the plates; and by submitting the phenomena to accurate measurement, I have succeeded in determining the laws which regulate the distribution of the polarising force. A brief account of these results will form the subject of the following paper.

1. *On Plates of Glass with One Axis of Polarisation.*

If we take a plate of glass perfectly circular, and communicate to it the polarising structure, either transiently, by the transmission of heat from its circumference to its centre, or permanently, by cooling it rapidly, when it has been made red hot, we shall find that it will exhibit, when exposed to polarised light, a system of rings traversed by a black rectangular cross. This system of rings is precisely the same, both in appearance and in the character of its tints, as the system seen

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along the axis of *Ice*, *Quartz*, &c. and other crystals of the *positive* class.

If the circular plate of glass, on the contrary, receive the polarising structure transiently, by heating it uniformly in boiling oil, and allowing it to cool rapidly, it will exhibit a similar system of rings; but this system has a negative polarisation, like the rings formed by *Calcareous spar*, *Beryl*, &c. and other crystals of the *negative* class.

This opposition in the character of the two plates may be finely observed, by combining them together. The resulting system of rings, when two positive or two negative plates are combined, will be the same as that which would have been produced by a plate equal to the sum of their thicknesses; but when the one is positive, and the other negative, the resulting system will be that which would be produced by a plate equal to the difference of their thicknesses. Hence, when the negative system is exactly equal to the positive system, they will destroy each others effects, and the compound plate will have no action whatever upon polarised light.

By comparing the value of the tints with their distances from the centre of the plate, I have found, that they vary as the squares of their distances from the axis. Hence if T is the tint which corresponds to any distance D, the tint t corresponding to any other distance d, will be found by the formula

$$t = \frac{T\,d^2}{D^2}.$$

2. *On Plates of Glass with Two Axes of Polarisation.*

When a plate of glass deviates from the circular form, and is either elliptical or rectangular, it has two axes of polarisation, one of which is perpendicular to the plane of the plate, and the other at right angles to it, and lying in the plane of the plate. When the plate has received the polarising structure transiently, by the transmission of heat, or permanently, by being

being cooled rapidly from a red-heat, the axis perpendicular to the plane of the plate (which is always the principal axis), is *positive*; but when the polarising structure is communicated by heating the plate in boiling oil, and then cooling it rapidly, the principal axis is *negative*.

By measuring carefully the distances of the tints from the centre of the plate, I have found the following formula, deduced from the supposition of two axes, perfectly correct, viz. $t = T - \frac{T d^2}{D^2}$, where D is the distance of either of the black fringes or lines of no polarisation from the centre of the plate. The term $\frac{T d^2}{D^2}$ represents the influence of the principal axis, and would have given us the tint t if that axis had existed alone. But as the axis in the plane of the plate produces an uniform tint T in every part of the plate, which acts in opposition to the other tint; the tint t must be equal to the difference of these tints, or to $T - \frac{T d^2}{D^2}$.

In examining the relative intensities of the two axes in rectangular plates of considerable length, and in elliptical plates, in which the conjugate axis is very small when compared with the transverse axis, I have found that D, or half the distance between the black fringes, is a function of the breadth of the plate, that is, if B is the breadth of the plate $2\,D : B = 10 : 16.02$, and $D = .312\,B^2$. As the excentricity of the elliptical plate diminishes, the value of D diminishes, or the polarising force of the axis in the plane of the plate diminishes; and when the conjugate and transverse axes are equal, D is equal to 0, or the axis in the plane of the plate is destroyed. In elliptical plates, the black fringes which are seen when the transverse axis is inclined 45° to the plane of primitive polarisation, are convex towards the transverse axis, and their curvature is such, that they

they cut perpendicularly every similar ellipse drawn within the plate. Hence, when the ellipse is nearly a circle, the distance of the fringes will be almost nothing, and each of the fringes will form two straight lines at right angles to one another, as in elliptical plates, when the transverse or the conjugate axis is in the plane of primitive polarisation.

The lines of equal tint at the angles of rectangular plates of glass, are highly deserving of attention. They are produced by the external fringes, and the maximum tint at the angles is always less than at the edges, and generally higher than the maximum tint in the middle of the plate. When the external fringes are two in number, viz. $a\,b$, $a'\,b'$, $d\,e$, $d'\,e'$, as shown in Plate VII. Fig. 1.; the outermost, $a\,b$, $a'\,b'$, terminates at c, c', and the other, $d\,e$, $d'\,e'$, branches off at e, e', so as to form the ellipsis $e\,f\,e'\,e$. The tint increases from this elliptical line to its centre o, but decreases from o to p, and from o to q. When the glass plate is a perfect square, the curve $e\,f\,e'\,e$ is a circle *. In a plate whose maximum tint, in the middle, was a *Blue of the 2d order*; the tint at o was a *Pink of the 2d order*; and that at p and q a *White of the 1st order*.

3. *On the Lines of Equal Tint formed by the Transverse Combination of Plates of Glass.*

In the Philosophical Transactions for 1816, I have represented the *Isochromatic lines*, or lines of equal tint, formed by the transverse combination of plates of glass, when the principal axes are both negative, or both positive †; when the principal axis in one is negative, and in the other positive ‡; when the two plates receive their structure by bending ||; and when a bent plate is combined with a plate formed by heat §. In these various cases, the lines of equal tint

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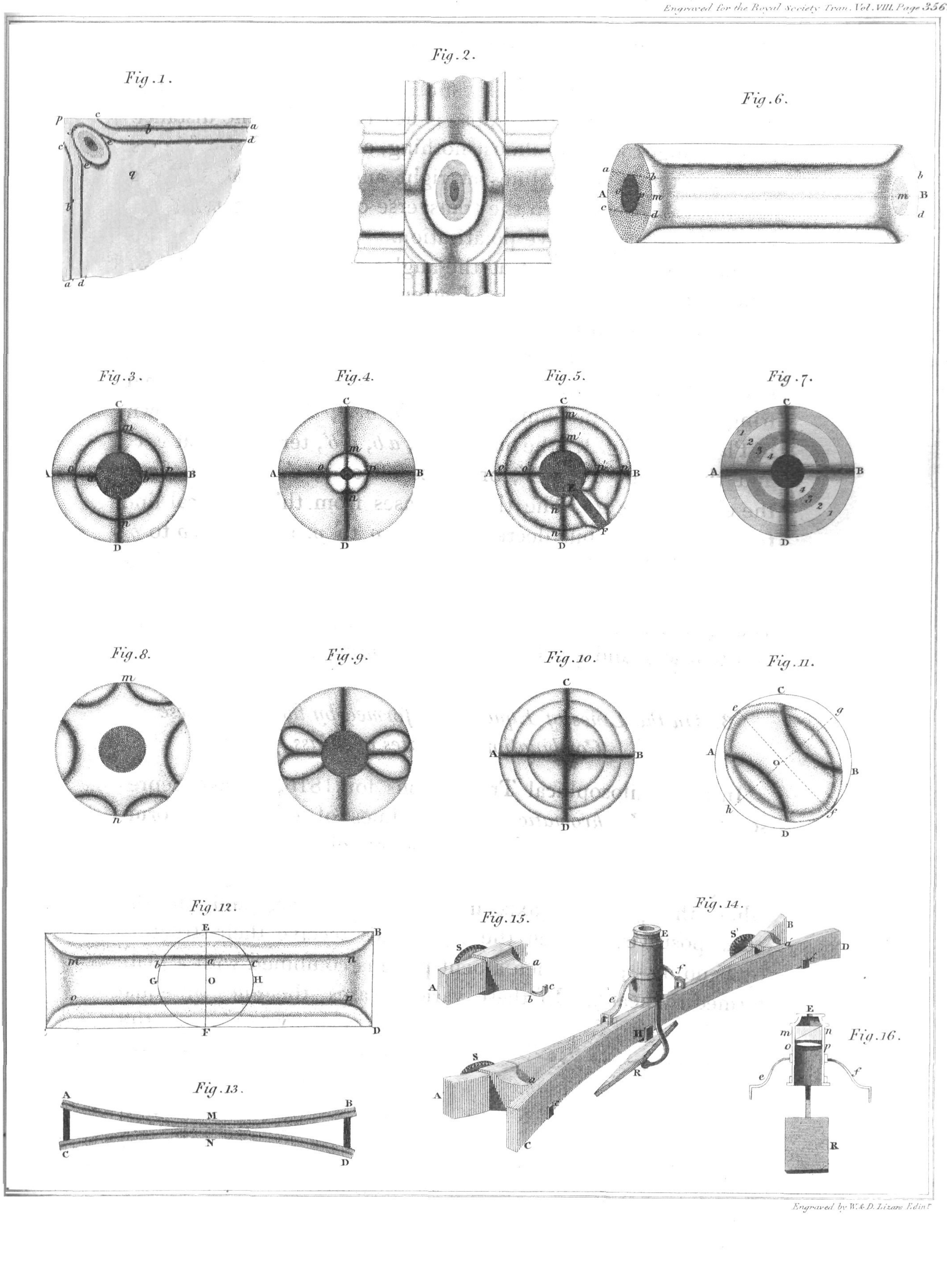
* See *Phil. Trans.* 1816, Plate IV. fig. 29.
† *Id.* Plate II. fig. 3, 4.
‡ *Id.* Plate II. fig. 8.
|| *Id.* Plate IX. fig. 9.
§ *Id.* Plate IX. fig. 10.

PLATE VII.

Engraved for the Royal Society Tran. Vol. VIII. Page 356.



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tint will be found to be *Hyperbolas*, *Circles*, *Ellipses*, *Straight Lines* and *Parabolas*.

Let us now suppose,

$T\ T'$ = the maximum tints of the two plates.

$B\ B'$ = the breadths of the plates.

x = the distance from the centre of the plate of any point where the resulting tint is required.

y = the distance of the same point from the centre of the other plate.

$t\ t'$ = the tint produced by each plate separately at the distances x and y, and

τ = the resulting tint.

Then, substituting .312 B instead of its equal D, we have

$$t = T - \frac{T\,x^2}{.312\,B^2}, \text{ and } t' = T' - \frac{T'\,y^2}{.312\,B'^2}.$$

But, since the resulting tint τ arising from the combination is equal to the difference of the two tints, we have

$$\tau = T' - T - \frac{T'\,y^2}{.312\,B'^2} + \frac{T\,x^2}{.312\,B^2}, \text{ and}$$

$$y^2 = .312\,B'^2\left(\frac{T' - T - \tau}{T'}\right) + \frac{T\,.312\,B'^2\,x^2}{T'\,.312\,B^2}.$$

Consequently, the lines of equal tint are *Hyperbolas*. When $T = T'$, and $B = B'$, the hyperbolas are equilateral, and

$$y' = .312\,B'^2\left(\frac{-\tau}{T}\right) + x^2.$$

When a plate whose principal axis is negative, is crossed with a plate whose principal axis is positive *, the resulting tint

* *Phil. Trans.* 1816, vol. II. fig. 8.

tint will be the sum of the separate tints. Hence, in this case we have

$$\tau = T + T' - \frac{T\,x^2}{.312\,B^2} - \frac{T'\,y^2}{.312\,B'^2}, \text{ and}$$

$$y^2 = .312\,B'^2\left(\frac{T' + T - \tau}{T'}\right) - \frac{T\,.312\,B'^2 x^2}{T'\,.312\,B^2}.$$

The lines of equal tint will therefore be *Ellipses*, as shewn in Plate VII. Fig. 2. when either the maximum tints, or the breadths of the combined plates are not equal.

But, when $T = T'$ and $B = B'$, we have

$$y^2 = .312\,B'^2\left(\frac{2\,T - \tau}{T}\right) - x^2.$$

Consequently, the lines are in this case *Circles*, as shewn in *Phil. Trans.* 1816, Plate II. Fig. 8.

When the two combined plates receive their structure by bending, the tints do not vary, as the squares of their distances from the centre, but simply as their distances. Hence, T being the maximum tint at the edge of the plate, and B B′ the breadths of the plate as formerly, we have $\frac{B}{2} : T = x : t$; and

$\frac{B'}{2} : T' = y : t'$, which give

$t = \frac{2\,T\,x}{B}$, and $t' = \frac{2\,T'\,y}{B'}$. Consequently,

$$\tau = \frac{2\,T\,x}{B} \pm \frac{2\,T'\,y}{B'}, \text{ and}$$

$$y = \frac{B'\,\tau}{2\,T'} - \frac{T\,B'\,x}{T'\,B}.$$

Hence, the lines of equal tint will be *Straight lines.*

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The angle ϕ, which the straight lines of equal tint form with the edges of the plate, will be found by the formula,

$$\text{Tang. } \phi = \frac{T}{T'} = \frac{B}{B'}.$$

When $x : y = B : B'$, and when similar sides of the plates cross each other, we shall have $\tau = 0$, that is, the line of no-polarisation will be the diagonal of the parallelogram formed by the sides of the two plates *.

When $B = B'$, and $T = T$, then

$$y = \frac{B\tau}{2T} - x, \text{ and}$$

the straight lines of equal tint will be inclined 45° to the edges of the plates, for $\frac{T}{T'} = 1$, which is the tangent of 45°.

When a plate of glass with two axes is combined with a plate of bent glass †, we have

$$\tau = \frac{2\,T'y}{B'} + T - \frac{T\,x^2}{.312\,B^2}, \text{ and}$$

$$x^2 = .312\,B^2\left(1 - \frac{\tau}{T} + \frac{2\,T'y}{T\,B'}\right),$$

when the *concave* side of the bent plate crosses a plate with two axes, in which the principal axis is *negative*; or,

$$\tau = \frac{2\,T'y}{B'} - T + \frac{T\,x^2}{.312\,B^2} \text{ and}$$

$$x^2 = .312\,B^2\left(1 + \frac{\tau}{T} - \frac{2\,T'y}{TB}\right),$$

when the *convex* side of the bent plate crosses a plate with two axes, in which the principal axis is *positive*, and *vice versa*. Hence, it follows, that the lines of equal tint are here *Parabolas*.

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* See *Phil. Trans.* 1816, Plate IX. fig. 9. † *Id.* Plate IX. Fig. 10.

When $T = T'$ the line of no-polarisation is a complete parabola, passing through the angles of the parallelogram, and having its vertex at the very edge of the bent plate. The curves within it will also be complete parabolas, but all those without it will be only parabolic segments.

4. *On the Distribution of the Polarising Force in Tubes and Cylinders of Glass.*

We have already seen, that circular plates, or cylinders of glass, have *one negative axis*, like *Quartz;* but when the cylinder has the form of a tube, like AB, Plate VII. Fig. 3. the polarising force is distributed in a very remarkable manner. A black circular fringe *m p n o* forms the line of no-polarisation, and the coloured fringes are placed on each side of this dark ring, and concentric with it. The structure on the outside of *m p n o* is *negative* like *Calcareous spar;* and the structure on the inside *positive*, like *Quartz*, &c.; and the effect is exactly the same as if a plate of glass had been bent into a circular form, and kept in that position by force.

The breadth of the positive annulus *a o* is always less than that of the negative annulus A *o*, the former decreasing, and the latter increasing, as the bore of the tube diminishes; and when the bore becomes extremely small, as in Fig. 4. the positive structure is also extremely small, and sometimes can scarcely be seen without the aid of a microscope.

In comparing the values of the tints with their distances from the line *m p n o,* I have found that they vary as the distances, in the same manner as in bent glass. Hence it is obvious, that the glass is in a state of compression within the black ring *m p n o*, and in a state of dilatation without that ring, and that the particles are held in a state of violent constraint, entirely different from that position of equilibrium in which they are placed in plates of glass with one or two axes.

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With the view of confirming these results, I took a file with a very sharp edge, and cut the tube entirely through by a notch EF, Fig. 5. By this operation the particles were freed from the state of violence in which they were held, and assumed the very same arrangement which they never fail to take in rectangular plates of glass. By exposing the tube, thus divided, to polarised light, it exhibited the appearance shewn in Fig. 5. where $m\,p\,n\,o$, $m'\,p'\,n'\,o'$, are two dark fringes having a negative structure on the outer side of each, and a positive structure between them, as in plates of glass with two axes. I obtained the same result with a tube of a very large bore, having its exterior diameter 0.875 of an inch, and its interior diameter 0.816; so that the thickness of the glass was only 0.059 of an inch. When the tube is cut into two or more pieces, each piece has the same structure as a plate of glass with two axes. By a number of measurements, I have found that the diameter $o\,p$, Fig. 3. of the black fringe or circle of no-polarisation, is a geometrical mean between the interior and exterior diameters of the tube, that is, $o\,p = \sqrt{\mathrm{AB} \times a\,b}$.

If a tube of glass is brought to a red heat, and then cooled by inserting in its bore a cylinder of iron, or any other conducting body, the structure will then be the same as is represented in Fig. 5.

If a solid cylinder of glass which has only one structure, is perforated in its centre, it will exhibit the appearance in Fig. 3., and if it is divided by a notch, it will acquire the structure shewn in Fig. 5.

When polarised light is transmitted through a long cylinder of unannealed glass, in a line perpendicular to its axis, after it is immersed in a fluid of the same refractive power, it exhibits the same phenomena as rectangular plates of glass, having a positive axis perpendicular to its length, and a negative axis perpendicular to the positive one; but in the cylinder, the ex-

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ternal tints do not rise so high, from the diminution in the thickness of the glass.

The same phenomena are exhibited by a glass-tube AB, Fig. 6. similarly placed. In this case, however, the maximum tint does not appear along the line *m m*, the axis of the tube; but it is seen both in the lines *b b* and *d d*, equidistant from *m m*. This effect is obviously occasioned by the greater thickness of the glass in these directions; for *a b* and *c d* are each greater than $A\,o + p\,m$, and the difference is sufficiently great to overbalance the diminution of the tints at a greater distance from the principal axis.

In examining very carefully the structure of glass tubes, when exposed to polarised light, it will be found that they are as it were divided into different elementary concentric tubes; and that in some cases there is an actual separation between them. Hence, there arises a remarkable singularity in the progression of the tints. Instead of shading into one another by imperceptible gradations, each elementary tube has an uniform tint of its own, as is represented in Fig. 7. where the tube AB is divided into four tubes 1, 1; 2, 2; 3, 3; 4, 4; the tube 4, 4 is in every part of it a white of the first order; the tube 3, 3 is every where equally dark, being the black circular fringe *m p n o* of Fig. 3.; the tubes 2, 2 and 4, 4 are a white of the first order; and the tube 1, 1 is a yellow of the first order. Hence it follows, that in tubes which possess this peculiarity of structure, each elementary tube is uniformly dense throughout, and that the variation of density takes place by leaps.

If a portion of a glass tube perfectly annealed, is exposed to pressure, it exhibits the tints shown in Fig. 8. when the line *m n*, joining the points of pressure, is parallel or perpendicular to the plane of primitive polarisation; and the tints shown in Fig. 9. when the same line is inclined 45° to that plane.

5. *On*

5. *On the Conversion of Plates of Glass with One Axis into Plates with Two Axes.*

From the different experiments which I have described, both in this and in other papers, it appears, that in almost every case where polarising forces are developed, two negative structures are separated by a positive structure, or two positive structures by a negative structure, both of which are simultaneously produced, like the two opposite polarities in electricity and magnetism. When the two structures are disposed in a different manner, or when only one structure is developed, the regular arrangement of the polarising forces may be effected by a slight change in the form or in the mechanical condition of the plate.

If we take a piece of unannealed glass perfectly circular, we have only one axis, or one structure, as shewn in Fig. 10., where AB, CD is the black cross, which preserves the same appearance by turning the plate round its centre. But if we grind the smallest quantity from any two opposite sides CB, AD, so as to induce a slight degree of ellipticity, a new axis is created; the tint which it produces appears at the centre of the plate, and the system of rings has the form shewn in Fig. 8., where the internal structure within the black fringes AD, CB, is negative, and the two external structures without AD, CB, positive.

If, on the contrary, we now grind a small quantity from the sides AC, BD, so as to reconvert the elliptical plate into a circular plate, we shall find that the new axis which was generated by the change of form, has entirely disappeared, so that the plate exhibits the figure shewn in Fig. 10.

This singular experiment, in which one of the axes may be extinguished and revived at pleasure, is worthy of the most attentive

attentive consideration. If the polarising forces depend solely on the mechanical condition of the particles of the glass, then it necessarily follows, that the central parts of the glass, which were in a state of variable expansion when it had a circular shape, are in a state of variable compression when the glass has received an elliptical form, and we are presented with a new law relative to the equilibrium of the cohesive forces in solids of variable density. But if the variation of density is merely the means of developing a new agent in the same manner as heat excites electricity in the tourmaline, or as pressure excites it in calcareous spar, then we cannot avoid regarding this agent in the same light as the electrical and magnetical fluid which are decomposed by certain mechanical operations, and distribute themselves according to regular laws.

But whatever be the origin of the polarising forces, it becomes a matter of great importance to discover the law of their distribution, when they are not controuled by opposite actions, and to apply this law to the explanation of the phenomena which they develope when they are either modified or extinguished by the external form of the body in which they reside.

Those who have studied the papers to which I have already referred, cannot have failed to remark, that when the polarising forces are unconstrained in their developement, a negative structure is generated in the middle of the plate, when a positive structure is generated at one or both of its edges; and that the intensity of the negative, is to the intensity of either of the positive structures, as 10 to 16.02.

In order to apply this principle to a circular plate, let EHFG, Fig. 12. be a plate of this description, whose centre is O. Then if this plate were a part of the rectangular plate ABDC, *mn*, *op* would be the lines of no-polarisation which separate the internal negative

negative structure from the two external positive structures. The tint at a, or any part of the line $b\,c$, would be $T - \frac{T\,d^2}{.312\,B^2}$; but if the circular plate were part of a plate similar to, and at right angles to ABCD, the tint at a, or any part of the line EOF, would be equal to T; and as this tint is rectangular to the other tint at a, the resulting tint will be equal to the difference of these tints, or to

$$T - T - \frac{T\,d^2}{.312\,B^2} = \frac{T\,d^2}{.312\,B^2}.$$

In like manner, it may be shewn, that in every point of the circular plate, the tint is represented by $\frac{T\,d^2}{.312\,B^2}$, which is the experimental expression for it already found. In plates, therefore, that have only a positive structure, the negative structure still exists, but is overpowered by opposite actions.

We are now prepared to understand how the negative structure re-appears, as shewn in Fig. 11., by giving an elliptical form to the plate. For, the maximum negative tint produced at O, in the direction $g\,h$, is no longer counterbalanced by the tint in the direction $e\,f$; and therefore the difference of these tints appears at O, with a negative character. As the points e, f remove from O, or as the ellipticity increases, the tint at O gradually rises till it becomes equal to T, or $\frac{10}{16.02}$ times the tint at g, when the action of the edges at e and f has no longer any influence at O. The very same results are obtained by the conversion of a sphere into a spheroid, and they are explicable upon the same principles.

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The distribution of the polarising force at the angles of square or rectangular plates, characterises a very remarkable state of equilibrium among the cohesive forces; and this state of equilibrium is not an accidental result of the mode in which the heat either enters or leaves the angles of the glass plate, for the very same distribution takes place when a new angle is formed, by cutting the plate into two parts *. In all these cases, the lines of equal tint are the lines of equal density.

As the two kinds of polarising forces seem to be co-existent in every part of a glass plate, and as each of them, when it appears to exist alone, is merely the resultant of two opposite forces, it is easy to assign a reason for the singular phenomena which are exhibited, by dividing a plate of glass into two parts. The two forces which reside in every part of the glass cannot be in equilibrio, unless a negative structure is placed between two positive structures; and therefore each half of the glass plate, or each portion of it of the same shape as the whole plate, must acquire the same property as the plate itself, or have the forces distributed in the same manner. This view of the distribution of the polarising forces is analogous to COULOMB's theory of the construction of the magnet. Every elementary portion of the magnet has a north and south pole, and therefore wherever it is broken, the fragment must have a north and south pole, like the magnet of which it formed a part.

6. *On*

* See *Phil. Trans.* 1816, p. 82.

6. *On the Intensity and Distribution of the Polarising Force in Plates of Bent Glass.*

In order to ascertain the intensity of the polarising force, as produced by different degrees of flexure, I placed together two similar plates A B, C D, Fig. 13, and separated their extremities by two pieces of glass A C, B D, of equal thickness. After pressing them into contact at M and N, I observed the maximum tints which they exhibited at these points, as given in the following Table.

Length of Plates, or A B.	Distance of Extremities, or A C.	Thickness of the Glass.	Breadth of the Plate.	Maximum Tint.
16 inches	0.16	0.133	0.35	4
13	0.16	0.133	0.35	9
12	0.048	0.207	0.967	10.6
6	0.054	0.133	0.35	10.5

With the view of ascertaining the tints which they yielded before they broke, I took plates, whose thickness was 0.1333 of an inch, and breadth 0.33. One of these broke when the tint was 10.4, another when the tint was 12, and a third when it had reached 13.55. In the phenomena of bent glass, the polarising force is distributed in such a manner, that the lines of equal tint are the lines of equal compression or dilatation, and the tint at the edge is every where inversely proportional to the radius of curvature.

In order to compare the effects produced by the application of the same force to plates of different thicknesses, I placed them as in Fig. 13. The thickness of A B was .133, and its tint 3.4.

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The thickness of C D was .230, and its tint 10.2. But $133^2 : 230^2 = 3.4 : 10.2$; so that the tints and the elasticities were, in this case, inversely as the squares of the thicknesses.

The preceding experiment furnishes us with the principle of a new instrument, which may be called a *Teinometer*, for ascertaining the elasticities of bodies. The tints of AB and CD are obviously measures of the elasticities of the two plates; so that the elasticities of plates of glass of different dimensions, or formed of different materials, may be readily determined. The power of the instrument however, is not limited to glass; for, by using a glass plate AB as a standard, the other plate CD may be a plate of metal, or any other opaque substance, whose elasticity it is required to ascertain. The tint of AB, when opposed by a plate of equal elasticity, is known by experiment; and therefore, the tint which it affords, when opposed by a similar plate, of a substance possessing a greater or a less elasticity than itself, is a measure of this elasticity. Although I consider the variation of the tint as an excellent means of determining the degree of curvature, yet the principle o 'exhibiting the relative elasticities of two plates, by the application of the same force, may be employed in an instrument entirely mechanical, in which the *sagitta* of the inflected plate is actually measured, as in the ingenious machine invented by S'GRAVESANDE.

The chromatic *Teinometer* is represented in Figs. 14, 15. and 16. where AB is the standard plate, of well annealed glass, having its edges highly polished. Along this plate, there are moved two brass pieces, S *a b c*, S' *a' b' c'*, which can be fixed in any position, by means of the screws S, S. The plate CD, where elasticity is to be measured, rests with its lower edge upon the projection *b c*, Fig. 15. and with one of its faces against *a b*. It is then pressed into contact with the plate AB,

and

and kept in this position by the wooden holdfast H; the brass pieces S *a b c*, S′ *a′ b′ c′*, having been previously placed at such a distance from each other, that the two plates will meet at H, without breaking, or without any permanent change of form. The apparatus ER, for observing the tint, is shewn separately in Fig. 16. It consists of an eye-piece E, to which is attached a reflector R, made of several plates of the thinnest glass, about 1½ long and an inch broad, and placed close to each other. The eye-piece E consists of two tubes, one of which is moveable within the other. The moveable tube contains an achromatic prism, *m n*, of calcareous-spar, with a convex lens, *o p*, about an inch in focal length, placed either above or below it. When this apparatus is set upon the edge of AB, by means of the forked arms *e*, *f*, the reflector R is turned round, till the plane of reflection is cut at an angle of 45°, by the plane of the plate AB, and is placed at such an angle, that the light which it reflects through the edge of the plate AB, and up the tube, is completely polarised. The moveable tube is then turned round, till the tints appear on the edge of one of the images of the glass plate. In order to avoid the confusion arising from two images, the achromatic prism may be constructed in such a manner that only one of the images is visible.