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Nagoya Math. J.
Vol. 51 (1973), 57–77

ON SOME DEGENERATE PARABOLIC EQUATIONS I

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§ 1. Introduction

Let Ω, I be open intervals in $R_x = (-\infty < x < \infty)$, $R_t = (-\infty < t < \infty)$ respectively. For a function $a(x, t) \in C^\infty(\Omega \times I)$, consider the partial differential operator

$$L = \frac{\partial}{\partial t} + a(x, t)\frac{\partial}{\partial x}\,. \tag{1.1}$$

A sufficient condition of Nirenberg and Treves (cf. [7]) for the operator (1.1) to be hypoelliptic[1] in $\Omega \times I$ is expressed as follows:

(1.2) for all $x \in \Omega$, the function $t \to \operatorname{Im} a(x, t)$ has only zeros of even order less than or equal to 2ℓ in the interval I. (ℓ interger, $\geqq 0$).

This is necessary and sufficient condition when $a(x, t)$ is analytic in $\Omega \times I$.

Motivated by this fact, we shall consider the hypoellipticity of a degenerated parabolic operator defined in $\Omega \times I$:

$$P = \frac{\partial}{\partial t} - a(x, t)\frac{\partial^2}{\partial x^2} + b(x, t)\frac{\partial}{\partial x} + c(x, t)\,, \tag{1.3}$$

where $a(x, t)$, $b(x, t)$, $c(x, t) \in C^\infty(\Omega \times I)$ and satisfy the following conditions.

$$\operatorname{Re} a(x, t) \geqq 0 \qquad \text{in } \Omega \times I\,, \tag{1.4}$$

(1.5) for all $x \in \Omega$, the function $t \to \operatorname{Re} a(x, t)$ has only zeros of even order less than or equal to 2ℓ in the interval I,

$$|\operatorname{Im} a(x, t)| \leqq C \operatorname{Re} a(x, t) \qquad \text{in } \Omega \times I\,, \tag{1.6}$$

Received December 11, 1972.

[1] The operator P is said to be hypoelliptic, if any distribution u is infinitely differentiable in every open set where Pu is infintely differentiable.

$$(1.7)\qquad |\mathrm{Im}\, a_x(x, t)| \leqq C[\mathrm{Re}\, a(x, t)]^{1/2} \qquad \text{in } \Omega \times I\,,$$

$$(1.8)\qquad |b(x, t)| \leqq C[\mathrm{Re}\, a(x, t)]^{1/2} \qquad \text{in } \Omega \times I\,,$$

where C denotes a positive constant.

Our aim is to construct the parametrices of the operator P. To do so, we shall mainly rely upon the procedure of [4], [5] and the theory of pseudo-differential operators developped in [1], [2]. Main result is the following.

THEOREM 1.1. *Suppose the operator* (1.3): P *satisfies the conditions* (1.4) $\sim$ (1.8) *then the operator* P *is hypoelliptic in* $\Omega \times I$.

EXAMPLE. The operator

$$\frac{\partial}{\partial t} - (t^{2\ell} + x^{2m})\frac{\partial^2}{\partial x^2} + i(t^{\ell} + x^m)\frac{\partial}{\partial x} + 1, \quad \ell, m \text{ integers} > 0\,,$$

satisfies the conditions (1.4) $\sim$ (1.8) in a neighborhood of the origin.

At the end of § 6, we shall give the other examples of the operators for which our method can be applied.

§ 2. Preliminary lemmas

We prepare the useful lemmas derived from the recent results of Nirenberg and Treves.

LEMMA 2.1. (cf. [7], Lemma 3.1 and [6], part I). *Under the conditions* (1.4) *and* (1.5), *for any compact set* $K \subset \Omega \times I$ *there exists a constant* $C > 0$ *such that*

$$(2.1)\qquad |\mathrm{Re}\, a_x(x, t)| \leqq C(\mathrm{Re}\, a(x, t))^{1/2} \qquad (x, t) \in K\,.$$

LEMMA 2.2. (cf. [7], Lemma C. 1, $\cdots$) *Under the conditions* (1.4) *and* (1.5), *for any compact set* $K \subset \Omega \times I$ *there exists a constant* $\delta > 0$ *such that*

$$(2.2)\qquad \delta(t - t')^{2\ell+1} \leqq \mathrm{Re} \int_{t'}^{t} a(x, \tau)d\tau\,,$$

where (x, t), $(x, t') \in K$ *and* $t' \leqq t$.

§ 3. Formal construction of the parametrices

Our aim is to construct approximate solutions (parametrices) of the equation

$$P_{x,t}E(x, y, t, t') = \delta(x - y, t - t') .$$

For $v(y, t') \in C_0^\infty(R^2_{y,t'})$ we set

$$\hat{v}(\xi, t') = (2\pi)^{-1/2}\int e^{-iy\xi}v(y, t')dy , \qquad \xi \in R_\xi . \tag{3.1}$$

Assume a parametrix of P can be written in the following form

$$[\mathscr{K}v](x, t) = \int_{T_1}^{t}\left(\int_{-\infty}^{\infty} e^{ix\xi}K(x, \xi ; t, t')\hat{v}(\xi, t')d\xi\right) dt' \tag{3.2}$$

for $v(y, t') \in C_0^\infty(\Omega \times I)$ and $(x, t) \in \Omega \times I$. Hereafter we put $I = (T_1 < t < T_2)$. We have, by a formal computation,

$$\begin{aligned} P[\mathscr{K}v](x, t) = \int_{T_1}^{t}\int_{-\infty}^{\infty} e^{ix\xi}\Big[\frac{\partial}{\partial t} - a(x, t)\Big(\frac{\partial}{\partial x} + i\xi\Big)^2 \\ + b(x, t)\Big(\frac{\partial}{\partial x} + i\xi\Big) + c(x, t)\Big]K(x, \xi ; t, t')\hat{v}(\xi, t')d\xi dt' \\ + \int_{-\infty}^{\infty} e^{ix\xi}K(x, \xi ; t, t')\hat{v}(\xi, t')d\xi|_{t'=t} , \end{aligned}$$

from where we arrive at the following Cauchy problem:

$$\begin{aligned} &\Big[\frac{\partial}{\partial t} - a(x, t)\Big(\frac{\partial}{\partial x} + i\xi\Big)^2 + b(x, t)\Big(\frac{\partial}{\partial x} + i\xi\Big) + c(x, t)\Big] \\ &\cdot K(x, \xi ; t, t') = 0 \qquad \text{in } \Omega \times R_\xi \times \Delta , \\ &\Delta \equiv \{(t, t') ; T_1 < t' < t < T_2\} , \end{aligned} \tag{3.3}$$

$$K(x, \xi ; t, t')|_{t=t'} = 1 , \tag{3.4}$$

$$K(x, \xi ; t, t')| = 0 \qquad \text{if } t' > t . \tag{3.5}$$

As a first approximation of the solution of this problem we take $K_0(x, \xi ; t, t')$ as follows:

$$L_1K_0 = \Big[\frac{\partial}{\partial t} + a(x, t)\xi^2\Big]K_0(x, \xi ; t, t') = 0 \qquad \text{in } \Omega \times R_\xi \times \Delta , \tag{3.6}$$

$$K_0(x, \xi ; t, t')|_{t=t'} = 1 , \tag{3.7}$$

$$K_0(x, \xi ; t, t')| = 0 \qquad \text{if } t < t' . \tag{3.8}$$

The solution K_0 can be written explicitely:

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$$(3.9)\qquad K_0(x,\xi;t,t') = \begin{cases} \exp\left(-\int_{t'}^{t} a(x,\tau)\xi^2 d\tau\right) & \text{in } \Omega \times R_\xi \times \bar{\Delta}\,, \\ \qquad \bar{\Delta} \equiv \{(t,t');\, T_1 < t' \leqq t < T_2\}\,, & \\ 0 \quad \text{if } t < t'\,. & \end{cases}$$

Next we set

$$L_2 = -2i\xi a(x,t)\frac{\partial}{\partial x} - a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} + i\xi b(x,t) + c(x,t) \qquad \text{in } \Omega \times R_\xi \times I\,.$$

We define $K_j(x,\xi;t,t')$, $j = 0,1,2,\cdots$, recurcively as the solutions of the following problem:

$$(3.10)\qquad \left[\frac{\partial}{\partial t} + a(x,t)\xi^2\right]K_{j+1}(x,\xi;t,t') = -L_2K_j(x,\xi;t,t') \quad \text{in } \Omega \times R_\xi \times \Delta\,,$$

$$(3.11)\qquad K_{j+1}(x,\xi;t,t')|_{t=t'} = 0$$

$$(3.12)\qquad K_{j+1}(x,\xi;t,t') = 0 \qquad \text{if } t < t'\,.$$

Apparently we have

$$(3.13)\qquad \begin{aligned} K_{j+1}(x,\xi;t,t') &= -\int_{t'}^{t} \exp\left(-\int_{t'}^{t} a(x,\tau)\xi^2 d\tau\right) L_2K_j(x,\xi;s,t')ds \\ &= -\int_{t'}^{t} K_0(x,\xi;t,s)L_2K_j(x,\xi;s,t')ds \\ &\qquad \text{in } \Omega \times R_\xi \times \bar{\Delta}\,. \end{aligned}$$

According to the formula (3.2), we set

$$[\mathscr{K}_j v](x,t) = \int_{T_1}^{t}\left(\int_{-\infty}^{\infty} e^{ix\xi}K_j(x,\xi;t,t')\hat{v}(\xi,t')d\xi\right)dt'\,, \qquad j = 0,1,2,\cdots\,,$$

for $v(y,t') \in C_0^\infty(\Omega \times I)$. A direct computation shows that

$$(3.14)\qquad \begin{aligned} &P[(\mathscr{K}_0 + \mathscr{K}_1 + \cdots + \mathscr{K}_j)v](x,t) \\ &\quad = v(x,t) + \int_{T_1}^{t}\int_{-\infty}^{\infty} e^{ix\xi}L_2K_j(x,\xi;t,t')\hat{v}(\xi,t')d\xi dt'\,. \end{aligned}$$

Symbolically we have

$$(3.14')\qquad P(\mathscr{K}_0 + \mathscr{K}_1 + \cdots + \mathscr{K}_j) = \delta(x-y, t-t') + L_2\mathscr{K}_j\,.$$

This formula suggests us the method of construction of the parametrices. We shall study the properties of $K_j(x, \xi; t, t')$ as a symbol of a pseudo-differential operator with parameter (t, t').

§4. Principal symbol

$K_0(x, \xi; t, t')$.

We recall some notations:

$$\begin{aligned}
&\Omega \text{ an open interval in } R_x = (-\infty < x < \infty)\,,\\
&I = (T_1 < t < T_2)\,,\\
&\varDelta = \{(t, t');\, T_1 < t' < t < T_2\}\,,\\
&\bar{\varDelta} = \{(t, t');\, T_1 < t' \leqq t < T_2\}\,,
\end{aligned}$$

DEFINITION 4.1. (cf. [2], Definition 1.1.1.) Let m, ρ, δ be real numbers with $0 \leqq \rho \leqq 1$, $0 \leqq \delta \leqq 1$. Then we denote by $S^m_{\rho,\delta}(\Omega \times R_\xi)$ the set of all $a = a(x, \xi) \in C^\infty(\Omega \times R_\xi)$ such that for every compact set $K \subset \Omega$ and all integers α, β $(\geqq 0)$ the estimate

$$|D^\beta_x D^\alpha_\xi a(x, \xi)| \leqq C_{\alpha,\beta,K}(1 + |\xi|)^{m - \rho\alpha + \delta\beta}, \quad x \in K, \quad \xi \in R^1_\xi\,, \tag{4.1}$$

is valid for some constant $C_{\alpha,\beta,K}$. The elements of $S^m_{\rho,\delta}$ are called symbols of order m and type ρ, δ. We set

$$S^{-\infty} = S^{-\infty}_{\rho,\delta} = \bigcap_m S^m_{\rho,\delta}\,.$$

Obviously $S^m_{\rho,\delta}(\Omega \times R_\xi)$ is a Fréchet space with the topology defined by taking as seminorms the best constants $C_{\alpha,\beta,K}$ in (4.1).

DEFINITION 4.2. Let $\varLambda$ be a subset of $R_t \times R_{t'}$. We denote by $\mathscr{E}^0(\varLambda; S^m_{\rho,\delta}(\Omega \times R_\xi))$ the set of all $K(x, \xi; t, t')$ such that

$$K(x, \xi; t, t') \in S^m_{\rho,\delta}(\Omega \times R_\xi)$$

for every $(t, t') \in \varLambda$ and continuous with respect to parameter (t, t') in $\varLambda$. For an integer $p \geqq 0$ we say that

$$K(x, \xi; t, t') \in \mathscr{E}^p(\varLambda; S^m_{\rho,\delta}(\Omega \times R_\xi))$$

if $D^j_{t,t'}K(x, \xi; t, t') \in \mathscr{E}^0(\varLambda; S^m_{\rho,\delta}(\Omega \times R_\xi))$ for all j, $0 \leqq j \leqq p$, where $D^j_{t,t'}$, denotes a differential operator of the form

$$\frac{\partial^j}{\partial t^{j_1} \partial t'^{j_2}}, \quad j_1 + j_2 = j\,.$$

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We set

$$\mathscr{E}(\Lambda\,;\,S^m_{\rho,\delta}(\Omega \times R_\xi)) = \bigcap_{p\geqq 0} \mathscr{E}^p(\Lambda\,;\,S^m_{\rho,\delta}(\Omega \times R_\xi))\,.$$

After some shrinking of the sets Ω and I (rewrite them Ω and I) we have

PROPOSITION 4.1. *For any* $\varepsilon > 0$ *we have*

(4.2) $$K_0(x,\xi\,;\,t,t') \in \mathscr{E}(\Delta\,;\,S^{-\infty}(\Omega \times R_\xi)) \cap \bigcap_{p\geqq 0} \mathscr{E}^p(\bar{\Delta}\,;\,S^{\varepsilon+2p}_{1,2\ell/(2\ell+1)}(\Omega \times R_\xi))\,,$$

(4.3) $$|(K_0(x,\xi\,;\,t,t') - 1)(1 + |\xi|)^{-\varepsilon}| \rightrightarrows 0 \textit{ in } \Omega \times R_\xi \textit{ as } t \downarrow t'\,,$$

(4.4) $$|D^p_{t,t'} D^\beta_x D^\alpha_\xi K_0 (1 + |\xi|)^{(-\beta\times 2\ell)/(2\ell+1)-2p+\alpha-\varepsilon}| \rightrightarrows 0$$
in $\Omega \times R_\xi$ *as* $t \downarrow t'$, *if* $2p < \alpha$.
($\rightrightarrows$ *means uniform convergency.*)

Proof. We may assume

(2.2′) $$\delta(t - t')^{2\ell+1} \leqq \operatorname{Re} \int_{t'}^t a(x,\tau)d\tau\,, \qquad (x,t,t') \in \Omega \times \bar{\Delta}$$

for some $\delta > 0$. (See Lemma 2.2.) That is, we have

$$\delta(t - t')^{2\ell+1}\xi^2 \leqq \operatorname{Re} \int_{t'}^t a(x,\tau)\xi^2 d\tau\,, \qquad (x,\xi\,;\,t,t') \in \Omega \times R_\xi \times \bar{\Delta}\,.$$

Remembering that

$$K_0(x,\xi\,;\,t,t') = \begin{cases} \exp\left(-\int_{t'}^t a(x,\tau)\xi^2 d\tau\right) & (x,\xi\,;\,t,t') \in \Omega \times R_\xi \times \bar{\Delta} \\ 0 \qquad t < t'\,, \end{cases}$$

we see

$$K_0(x,\xi\,;\,t,t') \in \mathscr{E}(\Delta\,;\,S^{-\infty}(\Omega \times R_\xi))\,.$$

Next we shall prove

(4.5) $$K_0 \in \bigcap_{p\geqq 0} \mathscr{E}^p(\bar{\Delta}\,;\,S^{\varepsilon+2p}_{1,2\ell/(2\ell+1)}(\Omega \times R_\xi))\,, \qquad \varepsilon > 0$$

in several steps.

(i) We have

(4.6) $$|K_0(x,\xi\,;\,t,t')| \leqq 1 \qquad \text{in } \Omega \times R_\xi \times \bar{\Delta}\,,$$

(4.7) $$|(K_0(x,\xi\,;\,t,t') - 1)(1 + |\xi|)^{-\varepsilon}| \rightrightarrows 0 \text{ in } \Omega \times R_\xi \text{ as } t \downarrow t'\,.$$

In fact, for any compact set $\Xi \subset R_\xi$ we have

$$K_0(x, \xi; t, t') \rightrightarrows 1 \text{ in } \Omega \times \Xi \text{ as } t \downarrow t'.$$

Hence we have (4.7) for any $\varepsilon > 0$.

(ii) Let α be a positive integer. We shall study $D_\xi^\alpha K_0(x, \xi; t, t') = (\partial^\alpha/\partial\xi^\alpha)K_0(x, \xi; t, t')$ which is expressed as a linear combination of terms

$$\left(\prod_j D_\xi^{\alpha(j)}\int_{t'}^t a(x, \tau)\xi^2 d\tau\right)\cdot K_0(x, \xi; t, t'), \tag{4.8}$$
$$0 < \alpha^{(j)} \leqq 2,\ \alpha^{(1)} + \cdots + \alpha^{(j)} + \cdots = \alpha.$$

Rewrite (4.8) as follows:

$$\prod_j \left(D_\xi^{\alpha(j)}\int_{t'}^t a(x, \tau)\xi^2 d\tau\right)\cdot K_0(x, \xi; t, t')^{\alpha(j)/\alpha} \tag{4.9}$$

For a factor with $\alpha^{(j)} = 1$ we see by (1.6)

$$\begin{aligned}&\left|2\int_{t'}^t a(x, \tau)\xi d\tau\cdot \exp\left(-\frac{1}{\alpha}\int_{t'}^t a(x, \tau)\xi^2 d\tau\right)\right|\\ &\quad\leqq 2C|\xi|^{-1}\int_{t'}^t \mathrm{Re}\, a(x, \tau)\xi^2 d\tau\cdot\exp\left(-\frac{1}{\alpha}\int_{t'}^t \mathrm{Re}\, a(x, \tau)\xi^2 d\tau\right)\\ &\quad\leqq C'|\xi|^{-1}\end{aligned}$$

For a factor with $a^{(j)} = 2$ we see

$$\begin{aligned}&\left|2\int_{t'}^t a(x, \tau)d\tau\cdot \exp\left(-\frac{2}{\alpha}\int_{t'}^t a(x, \tau)\xi^2 d\tau\right)\right|\\ &\quad\leqq C|\xi|^{-2}\int_{t'}^t \mathrm{Re}\, a(x, \tau)\xi^2 d\tau\cdot\exp\left(-\frac{2}{\alpha}\int_{t'}^t \mathrm{Re}\, a(x, \tau)\xi^2 d\tau\right)\\ &\quad\leqq C'|\xi|^{-2}.\end{aligned}$$

We use the same symbols $C, C', \cdots$ to express the different constants. Thus we have

$$|D_\xi^\alpha K_0(x, \xi; t, t')| \leqq C_\alpha(1 + |\xi|)^{-\alpha} \qquad \text{in } \Omega \times R_\xi \times \bar{\Delta}. \tag{4.10}$$

Furthermore, as in the step (i), from (4.6), (4.8) and (4.10) we have

$$|D_\xi^\alpha K_0(x, \xi; t, t')(1 + |\xi|)^{\alpha-\varepsilon}| \rightrightarrows 0 \text{ in } \Omega \times R_\xi \text{ as } t \downarrow t' \tag{4.11}$$

for any $\varepsilon > 0$.

(iii) For a positive integer β, $D_x^\beta K_0 = (\partial^\beta/\partial x^\beta)K_0(x, \xi; t, t')$ is expressed as a linear combination of terms

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$$\left(\prod_j D_x^{\beta^{(j)}} \int_{t'}^t a(x,\tau)\xi^2 d\tau\right) \cdot \exp\left(-\int_{t'}^t a(x,\tau)\xi^2 d\tau\right) \tag{4.12}$$
$$0 < \beta^{(j)}, \ \beta^{(1)} + \cdots + \beta^{(j)} + \cdots = \beta .$$

As in the step (ii) rewrite (4.12) as follows:

$$\prod_j \left(D_x^{\beta^{(j)}} \int_{t'}^t a(x,\tau)\xi^2 d\tau\right) \cdot \exp\left(-\frac{\beta^{(j)}}{\beta}\int_{t'}^t a(x,\tau)\xi^2 d\tau\right) . \tag{4.13}$$

First we investigate a factor with $\beta^{(j)} = 1$:

$$\int_{t'}^t a_x(x,\tau)\xi^2 d\tau \cdot \exp\left(-\frac{1}{\beta}\int_{t'}^t a(x,\tau)\xi^2 d\tau\right) . \tag{4.14}$$

By virtue of (1.7) and Lemma 2.1 we have

$$\begin{aligned}\left|\int_{t'}^t a_x(x,\tau)\xi^2 d\tau\right| &\leqq C\xi^2 \int_{t'}^t (\operatorname{Re} a(x,\tau))^{1/2} d\tau \\ &\leqq C|\xi|(t-t')^{1/2}\left(\int_{t'}^t \operatorname{Re} a(x,\tau)\xi^2 d\tau\right)^{1/2} \\ &\leqq C'|\xi|^{2\ell/(2\ell+1)}(\xi^2(t-t')^{2\ell+1})^{1/2(2\ell+1)} \cdot \left(\int_{t'}^t \operatorname{Re} a(x,\tau)\xi^2 d\tau\right)^{1/2}\end{aligned}$$

Hence by using Lemma 2.2 we have

$$|(4.14)| \leqq C''(1+|\xi|)^{2\ell/(2\ell+1)} \tag{4.15}$$

Next consider a factor with $\beta^{(j)} \geqq 2$:

$$\begin{aligned}\left|\int_{t'}^t D_x^{\beta^{(j)}} a(x,\tau)\xi^2 d\tau\right| &\leqq C\xi^2(t-t') \\ &\leqq C|\xi|^{(2\times 2\ell)/(2\ell+1)}(\xi^2(t-t')^{2\ell+1})^{1/(2\ell+1)} .\end{aligned}$$

Hence we have as above

$$\begin{aligned}&\left|\int_{t'}^t D_x^{\beta^{(j)}} a(x,\tau)\xi^2 d\tau \cdot \exp\left(-\frac{\beta^{(j)}}{\beta}\int_{t'}^t a(x,\tau)\xi^2 d\tau\right)\right| \\ &\qquad \leqq C'(1+|\xi|)^{(\beta^{(j)}\times 2\ell)/(2\ell+1)} , \qquad (\beta^{(j)} \geqq 2) .\end{aligned} \tag{4.16}$$

Thus we have for any $\beta > 0$

$$|D_x^\beta K_0(x,\xi;t,t')| \leqq C(1+|\xi|)^{(\beta\times 2\ell)/(2\ell+1)} \qquad \text{in } \Omega \times R_\xi \times \bar{\Delta} . \tag{4.17}$$

Furthermore as in (i), from (4.12) and (4.17) we have

$$|D_x^\beta K_0(x,\xi;t,t')(1+|\xi|)^{(-\beta\times 2\ell)/(2\ell+1)-\varepsilon}| \rightrightarrows 0 \tag{4.18}$$
$$\text{in } \Omega \times R_\xi \text{ as } t \downarrow t'$$

for any $\varepsilon > 0$.

(iv) We shall study $D_x^\beta D_\xi^\alpha K_0(x, \xi\,; t, t')$ $(\alpha+\beta>0)$ which is expressed as a linear combination of terms

$$(4.19)\quad \left(\prod_{i,j} D_x^{\beta^{(j)}} D_\xi^{\alpha^{(i)}} \int_{t'}^t a(x,\tau)\xi^2 d\tau\right)\cdot K_0(x,\xi\,; t, t')$$
$$= \prod_{i,j}\left(D_x^{\beta^{(j)}} D_\xi^{\alpha^{(i)}} \int_{t'}^t a(x,\tau)\xi^2 d\tau\cdot K_0(x,\xi\,; t, t')^{(\beta^{(j)}+\alpha^{(i)})/(\beta+\alpha)}\right),$$
$$0 \leqq \alpha^{(i)} \leqq 2, \quad 0<\beta^{(j)}+\alpha^{(i)}, \quad \sum_{i,j}(\beta^{(j)}+\alpha^{(i)}) = \beta+\alpha\,.$$

There are three cases in the factors in (4.19): $\alpha^{(i)}=0$, $\alpha^{(i)}=1$ and $\alpha^{(i)}=2$.

(a) For the factors with $\alpha^{(i)}=0$, we have examined in the step (iii).

(b) For the factors with $\alpha^{(i)}=1$ we have to examine the cases $\beta^{(j)}=1$ and $\beta^{(j)}\geqq 2$.

(b.1) Case $\alpha^{(i)}=1$, $\beta^{(j)}=1$:

$$\left|\xi\int_{t'}^t a_x(x,\tau)d\xi\right| \leqq |\xi|(t-t')^{1/2}\left(\int_{t'}^t \operatorname{Re} a(x,\tau)d\tau\right)^{1/2}$$
$$= |\xi|^{2\ell/(2\ell+1)-1}(\xi^2(t-t')^{2\ell+1})^{1/2(2\ell+1)}\left(\int_{t'}^t \xi^2 \operatorname{Re} a(x,\tau)d\tau\right)^{1/2}.$$

By using this inequality we have as before

$$(4.20)\quad \left|D_x D_\xi \int_{t'}^t a(x,\tau)\xi^2 d\tau\cdot K_0^{2/(\alpha+\beta)}\right| \leqq C(1+|\xi|)^{2\ell/(2\ell+1)-1}$$
$$\text{in } \Omega\times R_\xi\times\bar{\Delta}\,.$$

(b.2) Case $\alpha^{(i)}=1$, $\beta^{(j)}\geqq 2$:

$$\left|\xi\int_{t'}^t D_x^{\beta^{(j)}} a(x,\tau)d\tau\right| \leqq C|\xi|(t-t')$$
$$= C|\xi|^{(2\times 2\ell)/(2\ell+1)-1}(\xi^2(t-t')^{2\ell+1})^{1/(2\ell+1)}\,.$$

By using this inequality we have

$$(4.21)\quad \left|D_x^{\beta^{(j)}} D_\xi \int_{t'}^t a(x,\tau)\xi^2 d\tau\cdot K_0^{(1+\beta^{(j)})/(\alpha+\beta)}\right| \leqq C(1+|\xi|)^{(\beta^{(j)}\times 2\ell)/(2\ell+1)-1}$$
$$\text{in } \Omega\times R_\xi\times\bar{\Delta}\,.$$

(c) For the factors with $\alpha^{(i)}=2$ we also have to examine the cases $\beta^{(j)}=1$ and $\beta^{(j)}\geqq 2$.

(c.1) Case $\alpha^{(i)}=2$, $\beta^{(j)}=1$:

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$$\left|\int_{t'}^{t} a_x(x,\tau)d\tau\right| \leqq C(t-t')^{1/2}\left(\int_{t'}^{t} \operatorname{Re} a(x,\tau)d\tau\right)^{1/2}$$
$$= C|\xi|^{2\ell/(2\ell+1)-2}(\xi^2(t-t')^{2\ell+1})^{1/2(2\ell+1)}\left(\int_{t'}^{t} \operatorname{Re} a(x,\tau)\xi^2 d\tau\right)^{1/2}.$$

Thus we have

$$\text{(4.22)} \qquad \left|D_x D_\xi^2 \int_{t'}^{t} a(x,\tau)\xi^2 d\tau \cdot K_0^{3/(\alpha+\beta)}\right| \leqq C(1+|\xi|)^{2\ell/(2\ell+1)-2}$$
$$\text{in } \Omega \times R_\xi \times \bar{\Delta}.$$

(c.2) Case $\alpha^{(i)} = 2$, $\beta^{(j)} \geqq 2$:

$$\left|D_x^{\beta^{(j)}} \int_{t'}^{t} a(x,\tau)d\tau\right| \leqq C(t-t')$$
$$= C|\xi|^{(2\times 2\ell)/(2\ell+1)-2}(\xi^2(t-t')^{2\ell+1})^{1/(2\ell+1)},$$

from where we have

$$\text{(4.23)} \qquad \left|D_x^{\beta^{(j)}} D_\xi^2 \int_{t'}^{t} a(x,\tau)\xi^2 d\tau \cdot K_0^{(2+\beta^{(j)}/(\alpha+\beta)}\right| \leqq C(1+|\xi|)^{(\beta^{(j)}\times 2\ell)/(2\ell+1)-2}$$
$$\text{in } \Omega \times R_\xi \times \bar{\Delta}.$$

Combining all the above investigation we have finally

$$\text{(4.24)} \quad |D_x^\beta D_\xi^\alpha K_0(x,\xi;t,t')| \leqq C_{\alpha,\beta}(1+|\xi|)^{(\beta\times 2\ell)/(2\ell+1)-\alpha} \qquad \text{in } \Omega \times R_\xi \times \bar{\Delta}$$

for all integers $\alpha, \beta \geqq 0$.
Furthermore from (4.19) and (4.24) we have

$$\text{(4.25)} \qquad |D_x^\beta D_\xi^\alpha K_0(x,\xi;t,t')(1+|\xi|)^{(-\beta\times 2\ell)/(2\ell+1)+\alpha-\varepsilon}| \rightrightarrows 0$$
$$\text{in } \Omega \times R_\xi \text{ as } t \downarrow t'$$

for any $\varepsilon > 0$, $(\beta+\alpha > 0)$.

(d) From (4.19), (4.24) we can easily see

$$\text{(4.26)} \qquad |D_{t,t'}^p D_x^\beta D_\xi^\alpha K_0| \leqq C_{\alpha,\beta,p}(1+|\xi|)^{(\beta\times 2\ell)/(2\ell+1)+2p-\alpha}$$
$$\text{in } \Omega \times R_\xi \times \bar{\Delta}.$$

Furthermore, since $0 \leqq \alpha^{(i)} \leqq 2$ in (4.19) and by the proof of (4.25), we have for any $\varepsilon > 0$

$$\text{(4.27)} \qquad |D_{t,t'}^p D_x^\beta D_\xi^\alpha K_0(1+|\xi|)^{(-\beta\times 2\ell)/(2\ell+1)-2p+\alpha-\varepsilon}| \rightrightarrows 0$$
$$\text{in } \Omega \times R_\xi \text{ as } t \downarrow t'$$

if $2p < \alpha$.

Thus the proof of Proposition 4.1 is completed.

PROPOSITION 4.2. *The oscillatory integral* (cf. [2]):

$$\mathscr{K}_0(x, y, t, t') = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} e^{i(x-y)\xi} K_0(x, \xi; t, t') d\xi \tag{4.28}$$

defines a function in $C^\infty(W)$, $W = \{(x, y, t, t') \in \Omega \times R_y \times I \times I; |x-y| + |t-t'| > 0\}$.

Proof. By definition $\mathscr{K}_0(x, y, t, t') = 0$ for $t' > t$. By (4.12) we can see

$$\begin{aligned} &|D^p_{t,t'} D^\beta_x D^\alpha_y (e^{i(x-y)\xi} K_0(x, \xi; t, t'))| \\ &\qquad \leqq C_{\alpha,\beta,p}(1 + |\xi|)^{\alpha+2\beta+2p} \exp(-\delta\xi^2(t-t')^{2\ell+1}) \end{aligned} \tag{4.29}$$

if $t' < t$.

Hence $\mathscr{K}_0(x, y, t, t')$ is infinitely differentiable in $\{(x, y, t, t') \in \Omega \times R_y \times I \times I; |t-t'| > 0\}$.

On the other hand, if $x \neq y$, $t \geqq t'$, we have (in the oscillatory sense)

$$\begin{aligned} &(x-y)^j \mathscr{K}_0(x, y, t, t') \\ &\quad = \int_{-\infty}^{\infty} \left(\frac{1}{i} \frac{\partial}{\partial \xi}\right)^j e^{i(x-y)\xi} \cdot K_0(x, \xi; t, t') d\xi \\ &\quad = (-1)^\alpha \int_{-\infty}^{\infty} e^{i(x-y)\xi} \left(\frac{1}{i} \frac{\partial}{\partial \xi}\right)^j K_0(x, \xi; t, t') d\xi \,. \end{aligned}$$

By Proposition 4.1 it is obviously verified that

$$\lim_{t \downarrow t'} D^p_{t,t'} D^\beta_x D^\alpha_y (x-y)^j \mathscr{K}_0(x, y, t, t') = 0$$

if $2p + (\beta \times 2\ell)/(2\ell + 1) + \alpha < j - 1$. This completes the proof since j is arbitrary.

PROPOSITION 4.3. *Let* $\mathscr{K}_0(x, y, t, t')$ *be as in Proposition* 4.2. *Then* $\mathscr{K}_0(x, y, t, t')$ *is regular in* (y, t') *as well as in* (x, t).

Proof. (i) For $v = v(y, t') \in C_0^\infty(R_y \times I)$ consider the integral

$$\begin{aligned} &\int_{R_y \times I} \mathscr{K}_0(x, y, t, t') v(y, t') dy dt' \\ &\qquad = \int_{T'}^{t} \left(\int_{-\infty}^{\infty} e^{ix\xi} K_0(x, \xi; t, t') \hat{v}(\xi, t') d\xi \right) dt' \,. \end{aligned}$$

By using the fact that

$$|\hat{v}(\xi, t')| \leqq C_N(1 + |\xi|)^{-N} \qquad N \geqq 0$$

and by Proposition 4.1 we can see this integral defines a function in $C^\infty(\Omega_x \times I_t)$.

(ii) To prove the regularity in (y, t') we need the following lemma.

LEMMA 4.4 (cf. [1], Lemma 2.3.) *Let Ω be an open set in R^n_x. Let $a(x, \xi) \in S^m_{\rho,\delta}(\Omega \times R^n_\xi)$ and $v(x) \in C_0^\infty(\Omega)$. Then we have*

$$\left|\int e^{ix\xi} a(x, \xi) v(x) dx\right| \leqq C_N(1 + |\xi|)^{m+\delta N - N}, \qquad \xi \in R^n_\xi, \tag{4.30}$$

where N is an arbitrary positive integer.

Now let $\psi(x, t) \in C_0^\infty(\Omega \times I)$ and consider the integral

$$\begin{aligned}
&\int_\Omega \mathscr{K}_0(x, y, t, t')\psi(x, t) dx \\
&\quad = \int_\Omega \left(\int_{-\infty}^{\infty} e^{i(x-y)\xi} K_0(x, \xi; t, t') d\xi\right) \psi(x, t) dx \\
&\quad = \int_{-\infty}^{\infty} e^{-iy\xi} \left(\int_\Omega e^{ix\xi} K_0(x, \xi; t, t')\psi(x, t) dx\right) d\xi .
\end{aligned}$$

We set

$$F_0(\xi; t, t') = \int e^{ix\xi} K_0(x, \xi; t, t')\psi(x, t) dx .$$

Then by Lemma 4.4 and Proposition 4.1 we have $F_0(\xi; t, t') \in \mathscr{E}(\bar{\varDelta}; S^{-\infty})$. Whence we have

$$\begin{aligned}
&\int_{\Omega \times I} \mathscr{K}_0(x, y, t, t')\psi(x, t) dx dt \\
&\quad = \int_{t'}^{T_2} \int_{-\infty}^{\infty} e^{-iy\xi} F_0(\xi; t, t') d\xi dt \in C^\infty(R_y \times I_{t'}) .
\end{aligned}$$

This completes the proof of Proposition 4.3. (cf. Lemma 5.2.)

§ 5. Symbols

$K_j(x, \xi; t, t')$, $j = 0, 1, 2, \cdots$. We recall some notations:

$$K_0(x, \xi; t, t') = \begin{cases} \exp\left(-\int_{t'}^t a(x, \tau)\xi^2 d\tau\right) & t \geqq t' \\ 0 \qquad t < t' \end{cases}$$

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$$L_1 = \frac{\partial}{\partial x} + a(x,t)\xi^2 ,$$

$$L_2 = -2ia(x,t)\xi\frac{\partial}{\partial x} - a(x,t)\frac{\partial^2}{\partial x^2} + b(x,t)\frac{\partial}{\partial x} + i\xi b(x,t) + c(x,t) ,$$

For $j = 0, 1, 2, \cdots$

$$K_{j+1}(x,\xi; t,t') = \begin{cases} \int_{t'}^t \exp\left(-\int_s^t a(x,\tau)\xi^2 d\tau\right) \cdot L_2 K_j(x,\xi; s,t')ds , & t \geqq t' \\ 0 \quad t \leqq t' . \end{cases}$$

PROPOSITION 5.1. *For any* $\varepsilon > 0$ *it holds that*

$$(5.1) \quad K_j(x,\xi; t,t') \in \bigcap_{p\geqq 0} \mathscr{E}^p(\bar{\varDelta}; S^{\varepsilon+2p-(j/2(2\ell+1))}_{1,2\ell/(2\ell+1)}(\Omega \times R_\xi)), \quad j = 0,1,2,\cdots ,$$

$$(5.2) \quad |D^p_{t,t'} D^\beta_x D^\alpha_\xi K_j (1 + |\xi|)^{(-\beta\times 2\ell)/(2\ell+1) - 2p + \alpha + (j/2(2\ell+1)) - \varepsilon}| \rightrightarrows 0$$
$$\text{in } \Omega \times R_\xi \text{ as } t \downarrow t', \text{ if } 0 \leqq p < j .$$

Proof. As in the proof of Proposition 4.1 we shall prove in several steps. We shall start by showing that

$$K_j(x,\xi; t,t') \in \mathscr{E}^0(\bar{\varDelta}; S^{\varepsilon-(j/2(2\ell+1))}_{1,2\ell/(2\ell+1)}(\Omega \times R_\xi)) , \qquad j = 0,1,2,\cdots .$$

Assuming the case j, we shall show the case $j+1$. The reasoning is very similar to the proof of Proposition 4.1 so we shall only give a sketch.

(i) We have

$$(5.3) \quad |K_{j+1}(x,\xi; t,t')| \leqq C(1 + |\xi|)^{(-(j+1))/(2(2\ell+1))} \qquad \text{in } \Omega \times R_\xi \times \bar{\varDelta} ,$$

and

$$(5.4) \quad |K_{j+1}(x,\xi; t,t')(1 + |\xi|)^{(j+1)/(2(2\ell+1)) - \varepsilon}| \rightrightarrows 0$$
$$\text{in } \Omega \times R_\xi \text{ as } t \downarrow t'$$

for any $\varepsilon > 0$.

For example, we shall examine a term in the integral of K_{j+1}:

$$I \equiv \left|\int_{t'}^t \exp\left(-\int_s^t a(x,\tau)\xi^2 d\tau\right) i\xi b(x,s) K_j(x,\xi; s,t')ds\right|$$
$$\leqq C_1(1 + |\xi|)^{(-j)/(2(2\ell+1))} \int_{t'}^t \exp\left(-\int_s^t \mathrm{Re}\, a(x,\tau)\xi^2 d\tau\right) \cdot |b(x,s)\xi| ds$$
$$\leqq C_1'(1 + |\xi|)^{(-j)/(2(2\ell+1))} \int_{t'}^t \exp\left(-\int_s^t \mathrm{Re}\, a(x,\tau)\xi^2 d\tau\right) (\mathrm{Re}\, a(x,s)\xi^2)^{1/2} ds ,$$

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$$\int_{t'}^{t}\exp\left(-\int_{s}^{t}\mathrm{Re}\,a(x,\tau)\xi^2 d\tau\right)\cdot(\mathrm{Re}\,a(x,s)\xi^2)^{1/2}ds$$
$$\leqq\left(\int_{t'}^{t}\exp\left(-\int_{s}^{t}\mathrm{Re}\,a(x,\tau)\xi^2 d\tau\right)\mathrm{Re}\,a(x,s)\xi^2 ds\right)^{1/2}$$
$$\cdot\left(\int_{t'}^{t}\exp\left(-\int_{s}^{t}\mathrm{Re}\,a(x,\tau)\xi^2 d\tau\right)ds\right)^{1/2}=II\times III\,,$$

$$II^2=1-\exp\left(-\int_{t'}^{t}\mathrm{Re}\,a(x,\tau)\xi^2 d\tau\right)\leqq 1\,,$$

$$III^2=\int_{t'}^{t}(t-s)^{(-1)/2}(t-s)^{1/2}\exp\left(-\int_{s}^{t}\mathrm{Re}\,a(x,\tau)\xi^2 d\tau\right)ds$$
$$\leqq\int_{t'}^{t}(t-s)^{(-1)/2}ds\sup_{t'\leqq s\leqq t}(t-s)^{1/2}\exp\left(-\int_{s}^{t}\mathrm{Re}\,a(x,\tau)\xi^2 d\tau\right)$$
$$\leqq C_2(1+|\xi|)^{(-1)/(2\ell+1)}\,.$$

(cf. § 4)

Hence we have

$$I\leqq C_3(1+|\xi|)^{(-(j+1))/(2(2\ell+1))}\,.$$

By the similar calculations we have (5.3).
Furthermore as in § 4, by definition of K_{j+1} and by (5.3), we have (5.4).

(ii) We have

$$|D_\xi^\alpha K_{j+1}(x,\xi;t,t')|\leqq C(1+|\xi|)^{(-(j+1))/(2(2\ell+1))-\alpha}\quad\text{in } \Omega\times R_\xi\times\bar{\Delta}\,,\tag{5.5}$$

and

$$|D_\xi^\alpha K_{j+1}(1+|\xi|)^{(j+1)/(2(2\ell+1))+\alpha-\varepsilon}|\rightrightarrows 0\quad\text{in } \Omega\times R_\xi \text{ as } t\downarrow t'\tag{5.6}$$

for any $\varepsilon>0$.
To obtain (5.5) and (5.6), we have to estimate each term of the expression

$$D_\xi^\alpha K_{j+1}(x,\xi;t,t')=D_\xi^\alpha\int_{t'}^{t}\exp\left(-\int_{s}^{t}a(x,\tau)\xi^2 d\tau\right)L_2K_j(x,\xi;s,t')ds\tag{5.7}$$
$$=\sum_{\alpha_1+\alpha_2=\alpha}C_{\alpha_1,\alpha_2}\int_{t'}^{t}D_\xi^{\alpha_1}\exp\left(-\int_{s}^{t}a(x,\tau)\xi^2 d\tau\right)D_\xi^{\alpha_2}L_2K_j(x,\xi;s,t')ds$$
$$=\sum_{\alpha_1+\alpha_2=\alpha}C_{\alpha_1,\alpha_2}\cdot I_{\alpha_1,\alpha_2}\,.$$

By the similar calculation as in the step (i) we have

$$|I_{\alpha_1,\alpha_2}|\leqq C(1+|\xi|)^{-\alpha_1}\int_{s}^{t}\exp\left(-\gamma\int_{s}^{t}\mathrm{Re}\,a(x,\tau)\xi^2 d\tau\right)\cdot|D_\xi^{\alpha_2}L_2K_j(x,\xi;s,t')|ds$$
$$=C(1+|\xi|)^{-\alpha_1}\cdot I_{\alpha_2}$$

for some constant $\gamma > 0$ and we have

$$I_{\alpha_2} \leqq C'(1 + |\xi|)^{-\alpha_2 - (j+1)/(2(2\ell+1))} .$$

Hence we have (5.5) and by the above expression of $D_\xi^\alpha K_{j+1}$ and by (5.5) we obtain (5.6).

(iii) We have

$$|D_x^\beta K_{j+1}(x, \xi ; t, t')| \leqq C(1 + |\xi|)^{(\beta \times 2\ell)/(2\ell+1) - (j+1)(2(2\ell+1))} \quad \text{in } \Omega \times R_\xi \times \bar{\Delta}, \tag{5.8}$$

and

$$|D_x^\beta K_{j+1}(1 + |\xi|)^{-(\beta \times 2\ell)/(2\ell+1) + (j+1)/2(2(2\ell+1)) - \varepsilon}| \rightrightarrows 0 \quad \text{in } \Omega \times R_\xi \text{ as } t \downarrow t' \tag{5.9}$$

for any $\varepsilon > 0$.

To obtain (5.8) and (5.9), we have to examine each term in the expression

$$\begin{aligned} D_x^\beta K_{j+1} &= D_x^\beta \int_{t'}^t \exp\left(-\int_s^t a(x, \tau)\xi^2 d\tau\right) L_2 K_j(x, \xi ; s, t') ds \\ &= \sum_{\beta_1 + \beta_2 = \beta} C_{\beta_1, \beta_2} \int_{t'}^t D_x^{\beta_1} \exp\left(-\int_s^t a(x, \tau)\xi^2 d\tau\right) \cdot D_x^{\beta_2} L_2 K_j(x, \xi ; s, t') ds . \end{aligned} \tag{5.10}$$

Obviously there are two constants C, $\gamma > 0$ such that

$$\begin{aligned} &\left| \int_{t'}^t D_x^{\beta_1} \exp\left(-\int_s^t a(x, \tau)\xi^2 d\tau\right) D_x^{\beta_2} L_2 K_j(x, \xi ; s, t') ds \right| \\ &\quad \leqq C(1 + |\xi|)^{(\beta_1 \times 2\ell)/(2\ell+1)} \int_{t'}^t \exp\left(-\gamma \int_s^t a(x, \tau)\xi^2 d\tau\right) |{}_x^{\beta_2} L_2 K_j(x, \xi ; s, t')| ds \\ &\quad = C(1 + |\xi|)^{(\beta_1 \times 2\ell)/(2\ell+1)} \cdot I . \end{aligned}$$

Therefore we have to show

$$I \leqq C_1(1 + |\xi|)^{(\beta_2 \times 2\ell)/(2\ell+1) - (j+1)(2(2\ell+1))} \qquad \text{in } \Omega \times R_\xi \times \bar{\Delta} . \tag{5.11}$$

We shall study some terms of the integral I:

$$\begin{aligned} &\left| \int_{t'}^t \exp\left(-\gamma \int_s^t \operatorname{Re} a(x, \tau)\xi^2 d\tau\right) \cdot D_x^{\beta_2}(\xi a(x, s) D_x K_j(x, \xi ; s, t')) ds \right| \\ &\quad \leqq C_2 \left(\left| \int_{t'}^t \exp\left(-\gamma \int_s^t \operatorname{Re} a(x, \tau)\xi^2 d\tau\right) \cdot \xi a(x, s) D_x^{\beta_2+1} K_j(x, \xi ; s, t') ds \right| \right. \\ &\qquad + \left| \int_{t'}^t \exp\left(-\gamma \int_s^t \operatorname{Re} a(x, \tau)\xi^2 d\tau\right) \cdot \xi a_x(x, s) D_x^{\beta_2} K_j(x, \xi ; s, t') ds \right| \\ &\qquad \left. + \left| \int_{t'}^t \exp\left(-\gamma \int_s^t \operatorname{Re} a(x, \tau)\xi^2 d\tau\right) \cdot \xi a_{xx}(x, s) D_x^{\beta_2-1} K_j(x, \xi ; s, t') ds \right| + \cdots \right) \\ &\quad = C_2(I' + II' + III') , \end{aligned}$$

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$$I' \leqq C'(1+|\xi|)^{((\beta_2+1)\times 2\ell)/(2\ell+1)-j/(2(2\ell+1))-1}\int_{t'}^{t}\exp\left(-\gamma\int_s^t \operatorname{Re} a(x,\tau)\xi^2 d\tau\right)$$
$$\cdot \operatorname{Re} a(x,s)\xi^2 ds$$
$$\leqq C'(1+|\xi|)^{(\beta_2\times 2\ell)/(2\ell+1)-(j+1)/(2(2\ell+1))},$$

$$II' \leqq C''(1+|\xi|)^{(\beta_2\times 2\ell)/(2\ell+1)-j/(2(2\ell+1))}\left|\int_{t'}^{t}\exp\left(-\gamma\int_s^t a(x,\tau)\xi^2 d\tau\right)\operatorname{Re} a_x(x,s)\xi ds\right|$$
$$\leqq C_1''(1+|\xi|)^{(\beta_2\times 2\ell)/(2\ell+1)-j/(2(2\ell+1))}\int_{t'}^{t}\exp\left(-\gamma\int_s^t a(x,\tau)\xi^2 d\tau\right)(\operatorname{Re} a(x,s)\xi^2)^{1/2} ds$$
$$\leqq C_2''(1+|\xi|)^{(\beta_2\times 2\ell)/(2\ell+1)-j/(2(2\ell+1))}(1+|\xi|)^{(-1)/(2(2\ell+1))},$$
(cf. (i))

$$III' \leqq C'''(1+|\xi|)^{((\beta_2-1)\times 2\ell)/(2\ell+1)-j/(2(2\ell+1))+1}\int_{t'}^{t}\exp\left(-\gamma\int_s^t a(x,\tau)\xi^2 d\tau\right)ds,$$
$$\int_{t'}^{t}\exp\left(-\gamma\int_s^t a(x,\tau)\xi^2 d\tau\right)ds \leqq C_1'''(1+|\xi|)^{(-3)/(2(2\ell+1))}.$$
(cf. (i))

For other terms we can treat in the similar manner and we obtain (5.8), then by (5.8) and (5.10) we have (5.9).

(iv) We can treat $D_x^\beta D_\xi^\alpha K_{j+1}(x,\xi;t,t')$ by the similar way as in the step (iv) of the proof of Proposition 4.1, that is, by the same calculation as above, we can estimate each term in the expression:

$$\begin{aligned}(5.12)\qquad & D_x^\beta D_\xi^\alpha K_{j+1}(x,\xi;t,t') \\ &= \sum_{\substack{0\leqq k\leqq\beta\\0\leqq j\leqq\alpha}} C_{j,k}\int_{t'}^{t} D_x^{\beta-k}D_\xi^{\alpha-j}\exp\left(-\int_s^t a(x,\tau)\xi^2 d\tau\right) \\ &\quad\cdot D_x^k D_\xi^j L_2 K_j(x,\xi;s,t')ds.\end{aligned}$$

We obtain the following properties:

$$(5.13)\qquad |D_x^\beta D_\xi^\alpha K_{j+1}(x,\xi;t,t')| \leqq C(1+|\xi|)^{(\beta\times 2\ell)/(2\ell+1)-\alpha-(j+1)/(2(2\ell+1))}$$
$$\text{in } \Omega\times R_\xi\times\bar{\Delta},$$

and

$$(5.14)\qquad |D_x^\beta D_\xi^\alpha K_{j+1}(1+|\xi|)^{(-\beta\times 2\ell)/(2\ell+1)+(j+1)/(2(2\ell+1))-\varepsilon}| \rightrightarrows 0$$
$$\text{in } \Omega\times R_\xi \text{ as } t\downarrow t' \text{ for any } \varepsilon>0.$$

Thus we have proved that

$$K_j(x,\xi;t,t')\in\mathscr{E}^0(\bar{\Delta};S^{\varepsilon-j/(2(2\ell+1))}_{1,2\ell/(2\ell+1)}(\Omega\times R_\xi)),\qquad j=0,1,2,\cdots.$$

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(v) Next we shall study $D^p_{t,t'} D^\beta_x D^\alpha_\xi K_j(x,\xi\,;\,t,t')$. We need the following lemma.

LEMMA 5.2. *Let $f(t,t',s)$ be infinitely differentiable function in the set $\{(t,t',s)\,;\,t'\leqq s\leqq t\}$. Then we have*

$$D^p_t\int_{t'}^t f(t,t',s)ds=\int_{t'}^t D^p_t f(t,t',s)ds + \sum_{j=1}^{p-1}\frac{p(p-1)\cdots(p-j+1)}{j\,!}D^{p-j}_t D^{j-1}_s f(t,t',s)|_{s=t}\,, \tag{5.15}$$

$$D^q_{t'}\int_{t'}^t f(t,t',s)ds=\int_{t'}^t D^q_{t'} f(t,t',s)ds - \sum_{k=1}^{q}\frac{q(q-1)\cdots(q-k+1)}{k\,!}D^{q-k}_{t'} D^{k-1}_s f(t,t',s)|_{s=s'}\,, \tag{5.16}$$

$$\begin{aligned} D^q_{t'}D^p_t\int_{t'}^t f(t,t',s)ds&=\int_{t'}^t D^q_{t'}D^p_t f(t,t',s)ds\\ &\quad + D^q_{t'}\Big(\sum_{j=1}^{p-1}\frac{p(p-1)\cdots(p-j+1)}{j\,!}D^{p-j}_t D^{j-1}_s f(t,t',s)|_{s=t}\Big)\\ &\quad - \sum_{k=1}^{q-1}\frac{q(q-1)\cdots(q-k+1)}{h\,!}D^{q-k}_{t'} D^{q-1}_s (D^p_t f(t,t',s)|_{s=t'})\,. \end{aligned} \tag{5.17}$$

By an induction in j, using Lemma 5.2 and (5.12), we obtain the following estimates:

$$\begin{aligned} &|D^p_{t,t'}D^\beta_x D^\alpha_\xi K_j(x,\xi\,;\,t,t')|\\ &\qquad\leqq C_{\alpha,\beta,p}(1+|\xi|)^{(\beta\times 2\ell)/(2\ell+1)-\alpha-j/(2(2\ell+1))+2p}\,. \end{aligned} \tag{5.18}$$

The method of calculation is very similar to that used in the step (i) ~ (iv) so we omit the detail.

It remains to prove (5.2). By definition of K_j it holds that

$$K_j(x,\xi\,;\,t,t')\in C^\infty(\Omega\times R_\xi\times\bar{\Delta})\,,\qquad j=0,1,2,\cdots$$

and

$$D^p_{t,t'}D^\beta_x D^\alpha_\xi K_j(x,\xi\,;\,t,t')|_{t=t'}=0$$

if $p<j$. Thus we have (5.2) by virtue of (5.18). Q.E.D.

§ 6. Parametrices

As in § 4 we consider the oscillatory integrals:

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$$(6.1)\quad \mathscr{K}_j(x,y,t,t') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(x-y)\xi} K_j(x,\xi;t,t')d\xi\,, \qquad j=0,1,2,\cdots,$$

$$(6.2)\quad F_j(x,y,t,t') = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} e^{i(x-y)\xi} L_2K_j(x,\xi;t,t')d\xi\,, \qquad j=0,1,2,\cdots,$$

PROPOSITION 6.1. *$\mathscr{K}_j(x,y,t,t')$ and $F_j(x,y,t,t')$ define functions in $C^\infty(W)$, $W=\{(x,y,t,t')\in\Omega\times R_y\times I\times I;|x-y|+|t-t'|>0\}$. Furthermore we have*

$$(6.3)\qquad D^p_{t,t'}D^\beta_x D^\alpha_y \mathscr{K}_j(x,y,t,t')\in C^0(\Omega_x\times R_y\times I\times I)$$

if $(\beta\times 2\ell)/(2\ell+1)+\alpha+2p<j/(2(2\ell+1))-1$ *and*

$$(6.4)\qquad D^p_{t,t'}D^\beta_x D^\alpha_y F_j(x,y,t,t')\in C^0(\Omega_x\times\Omega_y\times I\times I)$$

if $(\beta\times 2\ell)/(2\ell+1)+\alpha+2p+2<j/(2(2\ell+1))-1.$

Proof. By induction in j, we easily obtain the following estimates:

$$(6.5)\quad |D^p_{t,t'}D^\beta_x K_j(x,\xi;t,t')|\leqq C_{p,\beta,j}(1+|\xi|)^{2p+2\beta+4j}\cdot\exp(-\delta(t-t')^{2\ell+1}\xi^2)$$
$$\text{in } \Omega\times R_\xi\times\bar{\Delta}\,.$$

By using (6.5), we have the first assertion as in the proof of Proposition 4.2. By (5.2) and (5.18) we have (6.3) and (6.4).

The following proposition is obtained just like as Proposition 4.3.

PROPOSITION 6.2. *$\mathscr{K}_j(x,y,t,t')$ is regular in (y,t') as well as in (x,t).*

Now we consider the parametrices. By definition of $K_j(x,\xi;t,t')$ and $\mathscr{K}_j(x,y,t,t')$ we have

$$P_{x,t}\Big(\sum_{j=0}^{\mu}\mathscr{K}_j(x,y,t,t')\Big) = \delta(x-y,t-t')+F_\mu(x,y,t,t')\,, \qquad \mu=0,1,2,\cdots$$

By Proposition 6.1 and 6.2, we have

(i) $\sum_{j=0}^{\mu}\mathscr{K}_j(x,y,t,t')\in C^\infty(W)$, $\mu=0,1,2,\cdots$,

(ii) $\sum_{j=0}^{\mu}\mathscr{K}_j(x,y,t,t')$ is very regular, as a distribution, in the sense of Schwartz [8], $\mu=0,1,2,\cdots$,

(iii) $F_\mu(x,y,t,t')$ becomes smoother in $\Omega\times R_y\times I\times I$ according as μ becomes larger.

Thus we obtain that the operator ${}^tP$ defined by

$$\iint {}^tP\varphi\cdot\psi dxdt = \iint\varphi\cdot P\psi dxdt,\ \ \varphi,\ \psi\in C_0(\Omega_x\times I_t)$$

is hypoelliptic in $\Omega \times I$ (cf. [9]). We can also prove the hypoellipticity of the operator P since a translation of the variable t for the operator ${}^tP$ satisfies the conditions given in § 1.

By the above investigation we have obtained Theorem 1.1.

Remark 1. The case of many variables: It is easily verified that our method can be applied for the following operator:

$$\frac{\partial}{\partial t} - a(x,t)\sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial}{\partial x_j} + c(x,t) \tag{6.6}$$

where $a(x,t)$, $a_{ij}(x,t)$, $b_j(x,t)$, $c(x,t)$ are infinitely differentiable functions in a open set $U = \Omega_x \times I_t$ of $R_x^n \times R_t^1$ and we have

$$\mathrm{Re}\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geqq \delta|\xi|^2 \qquad (x,t) \in U, \xi \in R^n$$

for a positive constant δ. The functions $a(x,t)$ and $b_j(x,t)$ satisfy the analogous conditions (1.4) $\sim$ (1.8) in U.

Remark 2. Another example for the case of many variables: Our method can be applied for the following operator.

$$\frac{\partial}{\partial t} - \sum_{i,j=1}^{n} a_{ij}(x,t)\frac{\partial^2}{\partial x_i \partial x_j} + \sum_{j=1}^{n} b_j(x,t)\frac{\partial}{\partial x_j} + c(x,t)\,, \tag{6.7}$$

where $a_{ij}(x,t)$, $b_j(x,t)$, $c(x,t)$ are infinitely differentiable functions in a open set $U = \Omega_n \times I_t$ of $R_x^n \times R_t^1$ and we suppose

$$\mathrm{Re}\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \geqq 0\,, \qquad (x,t) \in U,\ \ \xi \in R_u\,, \tag{6.8}$$

(6.9) for all $x \in \Omega$ and all $\xi \in R^n$, $\xi \neq 0$, the function $t \mapsto \mathrm{Re}\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j$ has only zeros of even order less than or equal to 2ℓ in the interval I,

$$|\mathrm{Im}\, a_{ij}(x,t)\xi_i\xi_j| \leqq C\,\mathrm{Re}\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j\,, \qquad (x,t) \in U\,, \quad \xi \in R^n\,, \tag{6.10}$$

$$\left|D_x^\beta \sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j\right| \leqq C_\beta\,\mathrm{Re}\sum_{i,j=1}^{n} a_{ij}(x,t)\xi_i\xi_j \quad \text{for all } \beta,\ (x,t) \in U \text{ and } \xi \in R^n\,, \tag{6.11}$$

$$\sum_{i=1}^{n}\left|\sum_{j=1}^{n}\frac{\partial}{\partial x_j}a_{ij}(x,t)\xi_i\right| \leqq C(\mathrm{Re}\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j)^{1/2} \qquad (x,t)\in U\,,\quad \xi\in R^n\,, \tag{6.12}$$

$$\sum_{j=1}^{n}|b_j(x,t)\xi_j| \leqq C(\mathrm{Re}\sum_{i,j=1}^{n}a_{ij}(x,t)\xi_i\xi_j)^{1/2} \qquad (x,t)\in U\,,\quad \xi\in R^n\,. \tag{6.13}$$

Then the operator (6.7) is hypoelliptic in U. In fact we can construct the symbols $K_j(x,\xi;t,t')$, $j=0,1,2,\cdots$, just as in §3 and we have

$$K_j\in\mathscr{E}(\varDelta;S^{-\infty}(\Omega\times R_\xi))\cap\bigcap_{p\geqq 0}\mathscr{E}^p(\varDelta;S^{\varepsilon+2p-j/(2(2\ell+1))}_{1/(2\ell+1),0}(\Omega\times R^n_\xi))$$
$$j=0,1,2,\cdots,$$

and so on.

EXAMPLE: The operator

$$\frac{\partial}{\partial t}-\sum_{j=1}^{n}t^{2\ell_j}\frac{\partial^2}{\partial x_j^2}+\sum_{j=1}^{n}t^{\ell_j}\frac{\partial}{\partial x_j}+1\,,\qquad \ell_j \text{ integers, } \geqq 0\,, \tag{6.14}$$

satisfies the above condition in a neighbourhood of the origin of $R^n_x\times R^1_t$.

Remark 3. The case of infinite order degeneracy: As an example, we consider the operator:

$$\frac{\partial}{\partial t}-a(t)\frac{\partial^2}{\partial x^2} \tag{6.15}$$

where $a(t)$ is infinitely differentiable function in the interval $I=(-1<t<1)$ and we suppose $a(t)>0$ for $t\neq 0$ and $a(0)=0$. Take $\Omega=(-\infty<x<\infty)$ and set

$$K_0(x,\xi;t,t')=\begin{cases}\exp\left(-\int_{t'}^{t}a(\tau)\xi^2d\tau\right) & -1<t'\leqq t<1\\ 0 & -1<t<t'<1\,.\end{cases}$$

Then for any $\varepsilon>0$, we have easily

(a) $K_0(x,\xi;t,t')\in\varepsilon(\varDelta;S^{-\infty}(\Omega\times R_\xi))\cap\bigcap_{p=0}\mathscr{E}^p(\bar{\varDelta};S^{\varepsilon+2p}_{1,0}(\Omega\times R_\xi))$,

(b) $|D^p_{t,t'}D^\alpha_\xi K_0(1+|\xi|)^{-2p+\alpha-\varepsilon}|\rightrightarrows 0$ in $\Omega\times R_\xi$ as $t\downarrow t'$ if $2p<\alpha$.

Thus we have the hypoellipticity of the operator (6.15) in $\Omega\times I$ and we have the fundamental solution defined by an oscillatory integral:

$$\mathscr{K}_0(x, y, t, t') = (2\pi)^{(-1)/2}\int_{-\infty}^{\infty} e^{i(x-y)\xi} K_0(x, \xi; t, t')d\xi .$$
$$= \frac{1}{2\sqrt{\pi}\, A(t, t')^{1/2}} \exp\left[-\frac{(x-y)^2}{4A(t, t')}\right],$$
$$A(t, t') = \int_{t'}^{t} a(\tau)d\tau , \quad t \geqq t' .$$

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