SUPPORT PROJECTIONS ON BANACH SPACES

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Each bounded linear operator a on a Hilbert space K has a hermitian left-support projection p such that $pK = \overline{aK} = \overline{aa^*K}$ and $(1-p)K = \ker a^* = \ker aa^*$. I demonstrate here that certain operators on Banach spaces also have left supports.

Throughout this paper X will be a complex Banach space with norm-dual X', and $L(X)$ will be the Banach algebra of bounded linear operators on X. Two linear subspaces Y and Z of X are orthogonal (in the sense of G. Birkhoff) if $\|y\| \leqq \|y+z\| \, (y \in Y, z \in Z)$; this orthogonality relation is not, in general, symmetric. It is easy to see that pX is orthogonal to $(1-p)X$ if and only if the norm of p is 0 or 1, when p is a projection on X.

An element h of a complex unital Banach algebra A is *hermitian* if $\|\exp(ith)\| = 1 (t \in \mathbf{R})$; equivalently, h is hermitian if its numerical range, $\{f(h) : f \in A', f(1) = \|f\| = 1\}$, is real.

PROPOSITION. *Let X be a reflexive Banach space and let h be a hermitian operator on X (that is, a hermitian element of $L(X)$). Then there is a projection p of norm 0 or 1 such that*

$$pX = \ker h \quad \textit{and} \quad (1-p)X = \overline{hX};$$

so $\ker h$ *is orthogonal to* hX.

Proof. This proposition is an immediate consequence of [**4** : VII. 7.5] and the inequality $\|\alpha(\alpha-h)^{-1}\| \leqq 1$ which holds for hermitian h and purely imaginary α. To prove this inequality, put $k = (\alpha-h)^{-1}$ and choose f in A' with $\|f\| = 1, f(k) = \|k\|$; define g in A' by $g(a) = \|k\|^{-1} f(ak)$; then $g(1) = \|g\| = 1$; so, $g(h)$ being real, $|\alpha| \leqq |\alpha - g(h)| = |g(\alpha-h)| = \|k\|^{-1} |f(1)| \leqq \|k\|^{-1}$.

Alternatively, the result can be derived from [**6**] where it is shown that, first, $\ker t$ is orthogonal to $\overline{tX}$ for any operator t the boundary of whose numerical range contains 0, and, second, that $X = \overline{tX} \oplus \ker t$ if X is reflexive.

The Vidav–Palmer theorem [**2**, §6] characterises unital C^*-algebras among unital Banach algebras; a unital Banach algebra A is a C^*-algebra if and only if $A = H + iH$, where H is the set of hermitian elements of A. I say that A is a *V^*-algebra* (*on X*) if A contains the identity operator on X and $A = H + iH$ (so that A is, abstractly, a C^*-algebra).

Suppose that A is a V^*-algebra on X and that its closed unit ball A_1 is relatively compact in the weak operator topology; this will happen if X is reflexive [**8**] or if X is weakly sequentially complete and A is commutative [**7**, Theorem 2 and Corollary 2]. Let $\tilde{A}$ be the linear span of the closure $(A_1)^w$ of A_1 in the weak operator topology: $\tilde{A} = \cup \{k(A_1)^w : k \in \mathbf{N}\}$. Then $\tilde{A}$ is a V^*-algebra; indeed, $\tilde{A}$ is a W^*-algebra (an abstract von Neumann algebra) [**8**].

I say that an element n of a unital Banach algebra A is *normal* if n can be expressed as $h + ik$ where h and k commute and $h^r k^s$ is hermitian $(r, s = 0, 1, 2, \ldots)$; so n is normal if and only if there is a commutative subalgebra of A which contains n and the identity of A

and, further, is a C^*-algebra. In particular, n is a normal operator on X (that is, normal in $L(X)$) if and only if n belongs to a commutative V^*-algebra on X. (This definition of normality is narrower than that in [**2**].)

THEOREM. *Let X be a complex Banach space, let A be a V^*-algebra on X with weakly relatively compact unit ball and let $a \in A$. Then a has a hermitian left-support projection p such that*

$$pX = \overline{aX} = \overline{aa^*X} \quad \text{and} \quad (1-p)X = \ker a^* = \ker aa^*.$$

Moreover, pX and $(1-p)X$ are mutually orthogonal. Further, $\ker a = \ker a^*$ *if a is normal.*

Proof. Let B be the subalgebra generated by aa^*. Write C for the closure of B in the topology σ induced on A by its predual. Let p be the identity of the W^*-algebra C. By Kaplansky's density theorem [**5**, 1.9] there exists a bounded net P (in B) which σ-converges to p. But then P converges to p in the weak operator topology (for on A_1 these two topologies are comparable, compact and Hausdorff: hence identical). Thus there exists a bounded net Q (of convex combinations of P) which converges to p in the strong operator topology. So $pX \subseteq \overline{aa^*X} \subseteq \overline{aX}$ and $(1-p)X \subseteq \ker a^*$. Now $0 = (1-p)(aa^*) = (a-pa)(a-pa)^*$; so $a = pa$; from which $\overline{aa^*X} \subseteq \overline{aX} \subseteq pX$ and $\ker aa^* \subseteq (1-p)X$. Therefore $pX = \overline{aX} = \overline{aa^*X}$ and $(1-p)X = \ker a^* = \ker aa^*$. The norms of p and $1-p$ are 0 or 1, because p is hermitian; so pX and $(1-p)X$ are mutually orthogonal. Finally, if a is normal, then

$$\ker a = \ker a^*a = \ker aa^* = \ker a^*.$$

This theorem generalises Lemma 3.1 of [**1**].

COROLLARY. *Let s be a scalar-type spectral operator on a Banach space X. Then* $X = \overline{sX} \oplus \ker s$.

Proof. There is a spectral measure e supported by the spectrum of s with $s = \int z\, e(dz)$ [**4**, XV]. Now X can be given a new norm (equivalent to the original norm) with respect to which all the values of e are hermitian. (This is a result of E. Berkson: see [**3**, §33].) Thus s is normal (for the new norm). Also, by Theorem 2 of [**7**], the norm-closed algebra generated by e has weakly relatively compact unit ball. The theorem may therefore be applied to give $\ker s = \ker s^*$ and $X = \overline{sX} \oplus \ker s$.

H. R. Dowson has remarked to me that this corollary can be derived from results of S. R. Foguel; see [**4**, XV. 8.2 and 8.3].

The corollary extends neither to spectral operators in general (consider $\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}$ acting on $\mathbf{C}^2$) nor to all scalar-type prespectral operators (the operator s on l^∞ defined by $s(x_n) = (n^{-1}x_n)$ has zero kernel and separable range).

The proposition and theorem suggest the question: must $X = \overline{hX} \oplus \ker h$ whenever X is weakly complete and h is a hermitian operator on X?

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