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Instantons, Higher-Derivative Terms, and Nonrenormalization Theorems in Supersymmetric Gauge Theories

N. Dorey[1], V.V. Khoze[2], M.P. Mattis[3], M.J. Slater[2], and W.A. Weir[2]

[1] *Physics Department, University of Wales Swansea*
Swansea SA2 8PP UK

[2] *Centre for Particle Theory, University of Durham*
Durham DH1 3LE UK

[3] *Theoretical Division T-8, Los Alamos National Laboratory*
Los Alamos, NM 87545 USA

We discuss the contribution of ADHM multi-instantons to the higher-derivative terms in the gradient expansion along the Coulomb branch of $N = 2$ and $N = 4$ supersymmetric $SU(2)$ gauge theories. In particular, using simple scaling arguments, we confirm the Dine-Seiberg nonperturbative nonrenormalization theorems for the 4-derivative/8-fermion term in the two finite theories ($N = 4$, and $N = 2$ with $N_F = 4$).

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1. Introduction. Thanks largely to the work of Seiberg and Witten [1,2], much is now understood about spontaneously broken 4-dimensional $SU(2)$ gauge theories with extended supersymmetry. One aspect that has received extensive study is the structure of the (suitably defined [3]) Wilsonian effective action along the Coulomb branch in which the models retain an unbroken $U(1)$ gauge symmetry. For energies well below the symmetry breaking scale, the dynamics of the massless $U(1)$ modes may be analyzed in a (supersymmetrized) gradient expansion for the scalar fields, the form of which is constrained by both gauge invariance and $N=2$ supersymmetry. In particular the leading 2-derivative/4-fermion term is expressed in terms of a holomorphic object $\mathcal{F}(\Psi)$ known as the prepotential [4,1]:

$$\mathcal{L}_{\text{2-deriv}} \;=\; \frac{1}{4\pi}\,\mathrm{Im}\int d^4\theta\,\mathcal{F}(\Psi)\;, \tag{1}$$

where Ψ denotes the massless $N=2$ abelian chiral superfield [5]. Equation (1) is an $N=2$ F-term as it involves integration over half the $N=2$ superspace. In contrast, the next-leading term, involving four derivatives and up to eight fermions, is an $N=2$ D-term [6,7]:

$$\mathcal{L}_{\text{4-deriv}} \;=\; \int d^4\theta d^4\bar\theta\,\mathcal{H}(\Psi,\bar\Psi) \tag{2}$$

where $\mathcal{H}$ is a real function of its arguments. This supersymmetric gradient expansion has been systematized by Henningson [6].

By exploiting holomorphicity, together with electric-magnetic duality, Seiberg and Witten were able to produce the exact quantum solution for $\mathcal{F}$ in a variety of $SU(2)$ models [2]. (There are however some interpretational caveats in the case of the two finite models, namely the $N=2$ model with $N_F=4$ flavors of quark hypermultiplets [8], and the $N=4$ model [9].) In contrast, comparatively little is known in general about the function $\mathcal{H}$ [6-7,10-12], since it is real rather than holomorphic (although it does respect duality [6]). But for the two finite models, it turns out that exact statements can nevertheless be made. In particular, Dine and Seiberg have recently argued that in both these cases $\mathcal{L}_{\text{4-deriv}}$ is one-loop exact: the one-loop result receives corrections neither from higher orders in perturbation theory, nor from nonperturbative physics such as instantons [13].

In this note, we discuss the contribution of Atiyah-Drinfeld-Hitchin-Manin (ADHM) multi-instantons [14-17] to these higher terms in the gradient expansion. Our principal result (Eq. (25) below) is a formal expression for the leading semiclassical contribution of the pure n-instanton (or pure n-antiinstanton) sector to $\mathcal{H}$, expressed as a finite-dimensional integral over the bosonic and fermionic collective coordinates of the supersymmetrized ADHM multi-instanton. As a simple illustration, we calculate the 1-instanton contribution to $\mathcal{H}$ in the case of pure $N=2$ SYM theory, and reproduce an earlier result of Yung's [12]. When $n>2$ the expression (25) is truly a formal one only, since the measure for this integration is not currently known [16,17]. Nevertheless, for the finite model with $N=2$ and $N_F=4$, we can verify, using a simple scaling argument, the vanishing of these multi-instanton contributions to $\mathcal{H}$, for all values of the topological charge n. A slightly modified scaling argument extends this null result to the $N=4$ model as well. Thus the Dine-Seiberg nonperturbative nonrenormalization theorems are built into the ADHM instanton calculus.

2. Instanton basics. The massless $U(1)$ modes of $N=2$ supersymmetric $SU(2)$ gauge theory along the Coulomb branch consist of a photon field v_m, two Weyl spinors λ and ψ, and a complex scalar A; these assemble into a single neutral massless $N=2$ chiral superfield Ψ. We start by reviewing the instanton representation of the prepotential $\mathcal{F}$ [18]; a straightforward extension of these methods will then yield an analogous formula for $\mathcal{H}$. Let us expand $\mathcal{L}_{\text{2-deriv}}$ in component fields, and focus (as in Sec. 5 of [17]) on the following three effective vertices:

$$\mathcal{L}_{\text{2-deriv}} \;\supset\; \frac{1}{4\pi}\sum_{k=1,2,3}\mathcal{V}^k\circ\mathcal{F}(\mathrm{v}) \;+\; \text{H.c.}\;, \tag{3}$$

where v denotes the VEV of the Higgs field A, and

$$\mathcal{V}^1 \;=\; \frac{i}{4}\left(v_{mn}^{\text{SD}}\right)^2\frac{\partial^2}{\partial\mathrm{v}^2}\;, \tag{4a}$$

$$\mathcal{V}^2 \;=\; \frac{i}{2\sqrt{2}}\,\psi\sigma^{mn}\lambda v_{mn}^{\text{SD}}\frac{\partial^3}{\partial\mathrm{v}^3}\;, \tag{4b}$$

$$\mathcal{V}^3 \;=\; -\frac{i}{8}\,\psi^2\lambda^2\frac{\partial^4}{\partial\mathrm{v}^4}\;. \tag{4c}$$

The superscript SD indicates the self-dual part of the gauge field strength v_{mn}.[1] We will call such vertices "holomorphic" as the fields ψ, λ and v_{mn}^{SD} live in the chiral superfield Ψ rather than in $\bar\Psi$. To extract the (multi-)instanton contribution to these three holomorphic vertices, one analyzes, respectively, the three *anti*holomorphic Green's functions $\langle\bar{\mathcal{O}}^k(x_1,\ldots,x_{k+1})\rangle$, $k=1,2,3$, with

$$\bar{\mathcal{O}}^1(x_1,x_2) \;=\; v_{mn}^{\text{ASD}}(x_1)\,v_{pq}^{\text{ASD}}(x_2)\;, \tag{5a}$$

$$\bar{\mathcal{O}}^2(x_1,x_2,x_3) \;=\; \bar\psi_{\dot\alpha}(x_1)\,v_{mn}^{\text{ASD}}(x_2)\,\bar\lambda_{\dot\beta}(x_3)\;, \tag{5b}$$

$$\bar{\mathcal{O}}^3(x_1,x_2,x_3,x_4) \;=\; \bar\psi_{\dot\alpha}(x_1)\,\bar\psi_{\dot\beta}(x_2)\,\bar\lambda_{\dot\gamma}(x_3)\,\bar\lambda_{\dot\delta}(x_4)\;. \tag{5c}$$

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[1] In Minkowski space the self-dual and anti-self-dual components of v_{mn} are projected out using $v_{mn}^{\text{SD}} = \frac14\big(\eta_{mk}\eta_{nl}-\eta_{ml}\eta_{nk}+i\epsilon_{mnkl}\big)v^{kl}$ and $v_{mn}^{\text{ASD}}=(v_{mn}^{\text{SD}})^*$, where $\epsilon_{0123}=-\epsilon^{0123}=-1$. Also, since $\sigma^{mn}=\frac14\,\sigma^{[\,m}\bar\sigma^{n\,]}$ and $\bar\sigma^{mn}=\frac14\,\bar\sigma^{[\,m}\sigma^{n\,]}$ are self-dual and anti-self-dual, respectively, it follows that $\sigma^{mn}{}_\alpha{}^\beta v_{mn}=\sigma^{mn}{}_\alpha{}^\beta v_{mn}^{\text{SD}}$ and $\bar\sigma^{mn\dot\alpha}{}_{\dot\beta}v_{mn}=\bar\sigma^{mn\dot\alpha}{}_{\dot\beta}v_{mn}^{\text{ASD}}$. Here σ_m and $\bar\sigma_m$ are spin matrices in Wess and Bagger conventions; see [17] for a complete set of our SUSY and ADHM conventions.

In the semiclassical approximation these field insertions are simply replaced by their values in the classical (multi-)instanton background, projected onto the unbroken $U(1)$ direction in color space, then integrated over all bosonic and fermionic instanton collective coordinates, which we now briefly review.

In the ADHM formulation, the general multi-instanton solution of topological charge n in $SU(2)$ gauge theory may be parametrized by an $(n+1)\times n$ quaternion-valued[2] collective coordinate matrix $a_{\alpha\dot\alpha}$, while the adjoint fermionic zero modes for the gaugino λ^γ and Higgsino ψ^γ are expressed in terms of $(n+1)\times n$ Weyl-valued collective coordinate matrices $\mathcal{M}^\gamma$ and $\mathcal{N}^\gamma$, respectively [14-17]:

$$a_{\alpha\dot\alpha} = \begin{pmatrix} w_{1\alpha\dot\alpha} & \cdots & w_{n\alpha\dot\alpha} \\ & & \\ & a'_{\alpha\dot\alpha} & \\ & & \end{pmatrix}\ ,\quad \mathcal{M}^\gamma = \begin{pmatrix} \mu_1^\gamma & \cdots & \mu_n^\gamma \\ & & \\ & \mathcal{M}'^\gamma & \\ & & \end{pmatrix}\ ,\quad \mathcal{N}^\gamma = \begin{pmatrix} \nu_1^\gamma & \cdots & \nu_n^\gamma \\ & & \\ & \mathcal{N}'^\gamma & \\ & & \end{pmatrix} \tag{6}$$

with $a'_{\alpha\dot\alpha} = a'^T_{\alpha\dot\alpha}$, $\mathcal{M}'_\gamma = \mathcal{M}'^T_\gamma$, and $\mathcal{N}'_\gamma = \mathcal{N}'^T_\gamma$. Furthermore the matrices a, $\mathcal{M}$ and $\mathcal{N}$ are subject to a set of algebraic constraints which may be used, for example, to eliminate the off-diagonal elements of the $n\times n$ submatrices a', $\mathcal{M}'$ and $\mathcal{N}'$. This leaves $4\times 2n$ independent scalar degrees of freedom in a (i.e., the w_k and the diagonal elements of a'), and likewise $2\times 2n$ independent Grassmann degrees of freedom in each of $\mathcal{M}$ and $\mathcal{N}$. Of these, the 'trace' components of a', $\mathcal{M}'$, and $\mathcal{N}'$, respectively, play a special role: that of the position $(x_{0\alpha\dot\alpha}\,,\,\xi^1_\alpha\,,\,\xi^2_\alpha)$ of the multi-instanton in $N=2$ superspace (see Eq. (8.1) of Ref. [17]).

The above describes the collective coordinate space for pure $N=2$ $SU(2)$ gauge theory. When N_F (massive) flavors of $N=2$ "quark hypermultiplets" in the fundamental representation of the gauge group are coupled in, one needs $2nN_F$ additional Grassmann collective coordinates for the fundamental fermion zero modes [15]; following [18], we label these $\mathcal{K}_{ki}$ and $\tilde{\mathcal{K}}_{ki}$ with $k=1,\cdots,n$ and $i=1,\cdots,N_F$. Alternatively, if a single (massive) adjoint hypermultiplet is coupled in, there are new adjoint fermion zero modes requiring $8n$ Grassmann degrees of freedom; following [9], these may be taken to live in the $(n+1)\times n$ Weyl-valued collective coordinate matrices

$$\mathcal{R}^\gamma = \begin{pmatrix} \rho_1^\gamma & \cdots & \rho_n^\gamma \\ & & \\ & \mathcal{R}'^\gamma & \\ & & \end{pmatrix}\ ,\quad \tilde{\mathcal{R}}^\gamma = \begin{pmatrix} \tilde\rho_1^\gamma & \cdots & \tilde\rho_n^\gamma \\ & & \\ & \tilde{\mathcal{R}}'^\gamma & \\ & & \end{pmatrix}\ , \tag{7}$$

which are subject to the same algebraic constraints as $\mathcal{M}$ and $\mathcal{N}$.

[2] We use quaternionic notation, e.g., $x = x_{\alpha\dot\alpha} = x_m\sigma^m_{\alpha\dot\alpha}$ and $\bar x = \bar x^{\dot\alpha\alpha} = x^m\bar\sigma_m^{\dot\alpha\alpha}$.

In previous work we have derived explicit formulae for the instanton action $S^n_{\rm inst}$ for arbitrary topological charge n, as functions of these collective coordinates, as well as of the VEVs v and $\bar{\rm v}$ and of the hypermultiplet masses m_i :

$$S^n_{\rm inst} \;=\; S^n_{\rm inst}\big(\,\{a\}\,,\,\{\mathcal{M}\,,\mathcal{N}\}\,;\,\{\mathcal{K}\,,\tilde{\mathcal{K}}\}\ \text{or}\ \{\mathcal{R}\,,\tilde{\mathcal{R}}\}\,;\,{\rm v},\bar{\rm v}\,;\,\{m_i\}\,\big) \tag{8}$$

Here we will not actually need these formulae[3]; it will suffice to note some general features of $S^n_{\rm inst}$. To begin with (save for one special case discussed below, that of exact $N=4$ SYM theory), $S^n_{\rm inst}$ explicitly depends on all the Grassmann collective coordinates in the problem *except* for the four exact $N=2$ SUSY modes ξ^1_α and ξ^2_α, $\alpha=1,2$, described above. Recall the rules of Grassmann integration: $\int d^2\xi^i\,(\xi^i)^2 \;=\; 1$ while $\int d^2\xi^i \;=\; 0$. Thus, in order to saturate the $d^2\xi^1\,d^2\xi^2$ integration, one requires the explicit insertion of ξ^i-dependent component fields, e.g., the $\bar{\mathcal{O}}^k$ of Eq. (5). The remaining Grassmann integrations are saturated by pulling down the appropriate power of $S^n_{\rm inst}$ from the exponent.

We illustrate these comments by focusing, first, on the n-instanton contribution to the 4-antifermion Green's function (5c):

$$\begin{aligned}&\langle\bar\psi_{\dot\alpha}(x_1)\,\bar\psi_{\dot\beta}(x_2)\,\bar\lambda_{\dot\gamma}(x_3)\,\bar\lambda_{\dot\delta}(x_4)\rangle\Big|_{n\text{-inst}} \;\cong\\ &\int d^4x_0\,d^2\xi^1\,d^2\xi^2\int d\tilde\mu\,\bar\psi^{\rm LD}_{\dot\alpha}(x_1)\,\bar\psi^{\rm LD}_{\dot\beta}(x_2)\,\bar\lambda^{\rm LD}_{\dot\gamma}(x_3)\,\bar\lambda^{\rm LD}_{\dot\delta}(x_4)\,\exp(-S^n_{\rm inst})\;.\end{aligned} \tag{9}$$

Here $d\tilde\mu$ stands for the properly normalized integration measure for all the collective coordinates in the problem (bosonic and fermionic, adjoint and fundamental) [16-18], excepting the $N=2$ superspace position variables $(x_{0\alpha\dot\alpha}\,,\,\xi^1_\alpha\,,\,\xi^2_\alpha)$ which have been written out explicitly. As indicated, at leading order, $\bar\psi$ and $\bar\lambda$ are approximated by quantities $\bar\psi^{\rm LD}$ and $\bar\lambda^{\rm LD}$ defined as follows [1,17,18]: first, one solves the Euler-Lagrange equations for $\bar\psi$ and $\bar\lambda$ in the classical background of the ADHM multi-instanton with all its fermionic zero modes turned on (and parametrized by the collective coordinates described above); next, one projects the resulting $SU(2)$-valued configurations onto the unbroken $U(1)$ direction in the color space (this is the direction parallel to the adjoint VEV); and finally, one assumes that the insertion points x_i are far away from the instanton position x_0 and performs a long-distance (LD) expansion. For all the $N=2$ models, the result of this 3-step procedure may be expressed compactly as follows [18]:[4]

$$\bar\psi^{\rm LD}_{\dot\alpha}(x_i) \;=\; i\sqrt{2}\,\xi^{1\alpha}\,S_{\alpha\dot\alpha}(x_i,x_0)\,\frac{\partial}{\partial{\rm v}}\;+\;\cdots \tag{10a}$$

$$\bar\lambda^{\rm LD}_{\dot\alpha}(x_i) \;=\; -i\sqrt{2}\,\xi^{2\alpha}\,S_{\alpha\dot\alpha}(x_i,x_0)\,\frac{\partial}{\partial{\rm v}}\;+\;\cdots \tag{10b}$$

[3] See Eq. (7.32) of [17] for the explicit expression in the case of pure $N=2$ SYM theory, Eq. (5.20) of [18] for the incorporation of N_F (massive) fundamental hypermultiplets, and Eq. (15) of [9] for the incorporation of a single (massive) adjoint hypermultiplet.

[4] The three effective vertices (3) in $\mathcal{L}_{2\text{-deriv}}$ are precisely those for which the tail of the instanton dominates the d^4x_0 integration; likewise for the nine effective vertices (18) in $\mathcal{L}_{4\text{-deriv}}$.

Here $S_{\alpha\dot\alpha}$ is the Weyl spinor propagator,

$$S_{\alpha\dot\alpha}(x_i,x_0) \;=\; \sigma^m_{\alpha\dot\alpha}\partial_m\, G_0(x_i,x_0)\ , \quad G_0(x_i,x_0) \;=\; \frac{1}{4\pi^2(x_i-x_0)^2}\ , \tag{11}$$

and the derivative $\partial/\partial \mathrm{v}$ acts on $\exp(-S^n_{\mathrm{inst}})$, with the understanding that v and $\bar{\mathrm{v}}$ are always to be treated as independent variables. The omitted terms in (10) represent terms that fall off faster than $(x_i-x_0)^{-3}$, as well as terms that are independent of ξ^1 or ξ^2 and hence cannot saturate these integrations. Note that in the models with hypermultiplets, $\bar\psi^{\mathrm{LD}}$ and $\bar\lambda^{\mathrm{LD}}$ as given in (10) contain both linear and trilinear terms in Grassmann variables (hence would be tricky to derive using Feynman graphs rather than the methods of [18]). Substituting Eq. (10) into Eq. (9) and performing the ξ^i integrals yields

$$\int d^4x_0\, S^\alpha_{\ \dot\alpha}(x_1,x_0)S_{\alpha\dot\beta}(x_2,x_0)S^\gamma_{\ \dot\gamma}(x_3,x_0)S_{\gamma\dot\delta}(x_4,x_0)\,\frac{\partial^4}{\partial \mathrm{v}^4}\int d\tilde\mu\, e^{-S^n_{\mathrm{inst}}}\ . \tag{12}$$

This one recognizes as the position-space Feynman graph for the effective 4-fermion vertex $-\frac{i}{32\pi}\,\psi^2\lambda^2\mathcal{F}''''_n(\mathrm{v})$ with [18]:

$$\mathcal{F}_n(\mathrm{v}) \;\equiv\; \mathcal{F}(\mathrm{v})\Big|_{n\text{-inst}} \;=\; 8\pi i\int d\tilde\mu\, e^{-S^n_{\mathrm{inst}}}\ . \tag{13}$$

Similarly, in order to generate the n-instanton contribution to the effective vertices (4a-b) one analyzes the Green's functions (5a-b), respectively. These require the long-distance expression for the anti-self-dual part of the field strength [18]:[5]

$$v^{\mathrm{ASD,LD}}_{mn}(x_i) \;=\; \sqrt2\,\xi^1\sigma^{pq}\xi^2\, G_{mn,pq}(x_i,x_0)\,\frac{\partial}{\partial \mathrm{v}} \;+\;\cdots \tag{14}$$

where $G_{mn,pq}$ is the gauge-invariant propagator of $U(1)$ field strengths:

$$G_{mn,pq}(x_i,x_0) \;=\; \big(\eta_{mp}\partial_n\partial_q-\eta_{mq}\partial_n\partial_p-\eta_{np}\partial_m\partial_q+\eta_{nq}\partial_m\partial_p\big)G_0(x_i,x_0)\ . \tag{15}$$

The omitted terms in (14) include terms that fall off faster than $(x_i-x_0)^{-4}$, as well as terms containing fewer than two of the ξ^i modes and hence cannot saturate these integrations. An important property of $G_{mn,pq}$ is that it only connects v^{SD}_{mn} to v^{ASD}_{pq} and vice versa (just as $S_{\alpha\dot\alpha}$ only connects λ to $\bar\lambda$, and ψ to $\bar\psi$). This property follows from the identity

$$\bar\sigma^{pq\dot\alpha}{}_{\dot\beta}\, G_{mn,pq}(x) \;=\; \frac{2}{\pi^2x^6}\,\bar x^{\dot\alpha\alpha}\,\sigma_{mn\,\alpha}{}^{\beta}\,x_{\beta\dot\beta} \tag{16}$$

[5] The fact that the gauge field strength develops an anti-self-dual piece in perturbation theory around the instanton is detailed in Sec. 4.4 of [17].

which implies

$$0 \;=\; \bar\sigma^{mn}_{\dot\gamma\dot\delta}\,\bar\sigma^{pq}_{\dot\alpha\dot\beta}\,G_{mn,pq}(x) \;=\; \sigma^{mn}_{\gamma\delta}\,\sigma^{pq}_{\alpha\beta}\,G_{mn,pq}(x)\ . \tag{17}$$

Now the Green's functions (5a-b) may be calculated as before, by substituting the long-distance expressions (14) and (10) into the collective coordinate integration, and performing the ξ^i integrals explicitly. Thanks to Eq. (17), one indeed recovers the effective vertices (4a-b), with the n-instanton contribution to the prepotential still given by Eq. (13) as the reader can check [18].

3. Multi-instanton contribution to $\mathcal{H}(\Psi,\bar\Psi)$. A straightforward extension of these methods gives useful information about $\mathcal{L}_{\text{4-deriv}}$ too (as well as higher terms in the gradient expansion). As before, it is useful to expand $\mathcal{L}_{\text{4-deriv}}$ in component fields, and to focus on the following nine effective vertices[6]:

$$\mathcal{L}_{\text{4-deriv}} \;\supset\; 4\sum_{k,k'=1,2,3}\mathcal{V}^k\circ\bar{\mathcal{V}}^{k'}\circ\mathcal{H}(\mathrm{v},\bar{\mathrm{v}}). \tag{18}$$

Here $\mathcal{H}$ is the kernel in Eq. (2), the $\mathcal{V}^k$ are the holomorphic vertices (4), and the $\bar{\mathcal{V}}^k$ are their Hermitian conjugates (e.g., $\bar{\mathcal{V}}^2 = -\frac{i}{2\sqrt2}\,\bar\psi\bar\sigma^{mn}\bar\lambda\,v^{\rm ASD}_{mn}\,\partial^3/\partial\bar{\mathrm{v}}^3$). Again as before, these nine vertices are probed, respectively, by the nine antiholomorphic $\times$ holomorphic Green's functions

$$\mathbf{G}^{k,k'}(x_1,\ldots,x_{k+1},y_1,\ldots,y_{k'+1}) \;=\; \langle\bar{\mathcal{O}}^k(x_1,\ldots,x_{k+1})\,\mathcal{O}^{k'}(y_1,\ldots,y_{k'+1})\rangle\ , \tag{19}$$

where the $\bar{\mathcal{O}}^k$ are given in (5) and the $\mathcal{O}^{k'}$ are their Hermitian conjugates, $k,k'=1,2,3$. We now need, in addition to Eqs. (10) and (14), the long-distance expansions of the fields ψ_α, λ_α and $v^{\rm SD}_{mn}$. These are easily derived from the full $SU(2)$ expressions [14,15,17]:

$$(v^{\rm SD}_{mn})^{\dot\alpha}{}_{\dot\beta} \;=\; 4\bar U^{\dot\alpha\alpha}\,b\,\sigma_{mn\,\alpha}{}^{\beta}\,f\,\bar b\,U_{\beta\dot\beta} \tag{20a}$$

$$(\lambda_\alpha)^{\dot\alpha}{}_{\dot\beta} \;=\; \bar U^{\dot\alpha\gamma}\mathcal{M}_\gamma f\,\bar b\,U_{\alpha\dot\beta} \;-\; \bar U^{\dot\alpha}{}_{\alpha}\,bf\mathcal{M}^{\gamma T}U_{\gamma\dot\beta} \tag{20b}$$

Here $\dot\alpha$ and $\dot\beta$ are color $SU(2)$ indices, the ADHM quantities U, f and b are as defined in Sec. 6 of [17], and $\mathcal{M}_\gamma$ is the Grassmann collective coordinate matrix (6); for $(\psi_\alpha)^{\dot\alpha}{}_{\dot\beta}$, substitute $\mathcal{N}_\gamma$ for $\mathcal{M}_\gamma$. Projecting onto the unbroken $U(1)$ direction (which we assume for

[6] In $N=1$ language these nine vertices are all contained in the last term in Eq. (4.7) in [6], which in our $d^2\theta$ conventions reads $\frac14\int d^2\theta d^2\bar\theta\,W^2\bar W^2\partial^4\mathcal{H}(\Phi,\bar\Phi)/\partial^2\Phi\partial^2\bar\Phi$.

definiteness to lie in the τ^3 direction in color space) and utilizing the asymptotic formulae listed at the end of Sec. 6 of [17], one then obtains the long-distance expressions

$$\begin{aligned} v^{\text{SD,LD}}_{mn}(y_i) &= \frac{4i}{(y_i-x_0)^6}\sum_{k=1}^n \text{tr}_2\,\bar w_k\tau^3 w_k\,(\bar y_i-\bar x_0)\,\sigma_{mn}\,(y_i-x_0)\ +\cdots \\ &= 2i\pi^2\,G_{mn,pq}(y_i,x_0)\sum_{k=1}^n \text{tr}_2\,\bar w_k\tau^3 w_k\,\bar\sigma^{pq}\ +\cdots\ , \qquad (21a) \\ \lambda^{\text{LD}}_\alpha(y_i) &= 4i\pi^2\,S_{\alpha\dot\alpha}(y_i,x_0)\sum_{k=1}^n \bar w_k^{\dot\alpha\beta}\,(\tau^3)_\beta{}^\gamma\,\mu_{k\gamma}\ +\cdots\ , \qquad (21b) \\ \psi^{\text{LD}}_\alpha(y_i) &= 4i\pi^2\,S_{\alpha\dot\alpha}(y_i,x_0)\sum_{k=1}^n \bar w_k^{\dot\alpha\beta}\,(\tau^3)_\beta{}^\gamma\,\nu_{k\gamma}\ +\cdots\ , \qquad (21c) \end{aligned}$$

omitting terms with a faster falloff. Here w_k, μ_k and ν_k are the top-row elements of the collective coordinate matrices a, $\mathcal{M}$ and $\mathcal{N}$, respectively (see Eq. (6)); the second equality in Eq. (21a) follows from Eq. (16).

We can now calculate, for example, the n-instanton contribution to the effective 8-fermi vertex

$$\frac{1}{16}\,\psi^2\,\lambda^2\,\bar\psi^2\,\bar\lambda^2\,\frac{\partial^4}{\partial \text{v}^4}\,\frac{\partial^4}{\partial \bar{\text{v}}^4}\,\mathcal{H}(\text{v},\bar{\text{v}})\ . \qquad (22)$$

Inserting

$$\bar\psi^{\text{LD}}_{\dot\alpha}(x_1)\,\bar\psi^{\text{LD}}_{\dot\beta}(x_2)\,\bar\lambda^{\text{LD}}_{\dot\gamma}(x_3)\,\bar\lambda^{\text{LD}}_{\dot\delta}(x_4)\,\psi^{\text{LD}}_\alpha(y_1)\,\psi^{\text{LD}}_\beta(y_2)\,\lambda^{\text{LD}}_\gamma(y_3)\,\lambda^{\text{LD}}_\delta(y_4) \qquad (23)$$

into the collective coordinate integration and performing the ξ^i integrals leaves

$$\begin{aligned} &\int d^4x_0\,\epsilon^{\kappa\lambda}S_{\kappa\dot\alpha}(x_1,x_0)S_{\lambda\dot\beta}(x_2,x_0)\,\epsilon^{\rho\sigma}S_{\rho\dot\gamma}(x_3,x_0)S_{\sigma\dot\delta}(x_4,x_0) \\ &\qquad\times\ \epsilon^{\dot\kappa\dot\lambda}S_{\alpha\dot\kappa}(y_1,x_0)S_{\beta\dot\lambda}(y_2,x_0)\,\epsilon^{\dot\rho\dot\sigma}S_{\gamma\dot\rho}(y_3,x_0)S_{\delta\dot\sigma}(y_4,x_0) \qquad (24) \\ &\times\ \frac{\partial^4}{\partial\text{v}^4}\int d\tilde\mu\,e^{-S^n_{\text{inst}}}\sum_{k,k',l,l'=1}^n \tfrac14(4i\pi^2)^4\,(\nu_k\tau^3 w_k\bar w_{k'}\tau^3\nu_{k'})(\mu_l\tau^3 w_l\bar w_{l'}\tau^3\mu_{l'}) \end{aligned}$$

Again, we recognize this expression as the position-space Feynman graph for a local $\psi^2\lambda^2\bar\psi^2\bar\lambda^2$ vertex with an effective coupling given by the last line of Eq. (24). A comparison with the canonical form (22) gives a formal expression for the n-instanton contribution to $\mathcal{L}_{4\text{-deriv}}$, valid to leading semiclassical order:

$$\frac{\partial^4}{\partial\bar{\text{v}}^4}\,\mathcal{H}(\text{v},\bar{\text{v}})\Big|_{n\text{-inst}} = 64\pi^8\int d\tilde\mu\,e^{-S^n_{\text{inst}}}\sum_{k,k',l,l'=1}^n (\nu_k\tau^3 w_k\bar w_{k'}\tau^3\nu_{k'})(\mu_l\tau^3 w_l\bar w_{l'}\tau^3\mu_{l'}) \qquad (25)$$

This is the analog of Eq. (13) for the prepotential. Somewhat different (although necessarily consistent) formal expressions for $\partial^2\mathcal{H}/\partial\bar{\rm v}^2$ and $\partial^3\mathcal{H}/\partial\bar{\rm v}^3$ may be derived in the same way, by examining the Green's functions $\mathbf{G}^{k,1}$ and $\mathbf{G}^{k,2}$, respectively. Exchanging v and $\bar{\rm v}$ in Eq. (25) gives the n-antiinstanton contribution. There may also in general be mixed n-instanton, m-antiinstanton contributions to $\mathcal{H}$ (unlike $\mathcal{F}$ due to holomorphicity), but these lie beyond the scope of this paper.

As a simple illustration, let us calculate the 1-instanton contribution to $\mathcal{H}$ in the case of pure $N=2$ SYM theory. In that case the instanton action reads [17]:

$$S^{n=1}_{\rm inst} \;=\; 4\pi^2|{\rm v}|^2|w|^2 - 2\sqrt{2}\,i\pi^2\bar{\rm v}\mu^\alpha\,(\tau^3)_\alpha{}^\beta\,\nu_\beta\ , \tag{26}$$

where $|{\rm v}| = \sqrt{{\rm v}\bar{\rm v}}$. Also [17]:

$$\int d\tilde{\mu} \;=\; \frac{\Lambda^4}{2\pi^4}\int d^4w\, d^2\mu\, d^2\nu\ , \tag{27}$$

with Λ the dynamically generated Pauli-Villars scale. The resulting integration in (25) is elementary, and gives[7]

$$\mathcal{H}({\rm v},\bar{\rm v})\Big|_{1\text{-inst}} \;=\; -\frac{1}{8\pi^2}\frac{\Lambda^4}{{\rm v}^4}\,\log\bar{\rm v} \tag{28}$$

in accord with an earlier prediction of Yung's, arrived at using completely different reasoning [12]. In contrast, for $N_F>0$, the first nonvanishing contribution to $\mathcal{H}$ is at the 2-instanton level, due to a discrete $\mathbb{Z}_2$ symmetry that forbids all odd-instanton contributions [2,18]; it may be calculated straightforwardly using the methods of [17,18].

4. Nonrenormalization theorem for the $N=2$, $N_F=4$ model. To make further progress, we note a second general property of $S^n_{\rm inst}$ [17,18,9]: when the hypermultiplet masses are zero, all dependence on v and $\bar{\rm v}$ can be eliminated from $S^n_{\rm inst}$ by performing the collective coordinate rescaling

$$\begin{gathered} a\to a/|{\rm v}|\ ,\quad \mathcal{M}\to\mathcal{M}/\sqrt{\bar{\rm v}}\ ,\quad \mathcal{N}\to\mathcal{N}/\sqrt{\bar{\rm v}}\ ,\\ \mathcal{K}\to\mathcal{K}/\sqrt{\rm v}\ ,\quad \tilde{\mathcal{K}}\to\tilde{\mathcal{K}}/\sqrt{\rm v}\ ,\quad \mathcal{R}\to\mathcal{R}/\sqrt{\rm v}\ ,\quad \tilde{\mathcal{R}}\to\tilde{\mathcal{R}}/\sqrt{\rm v}\ . \end{gathered} \tag{29}$$

Let us concentrate, first, on the $N=2$ models with $0\le N_F\le 4$ flavors of massless fundamental hypermultiplets and no adjoint hypermultiplets. In these cases the rescaling (29) implies

$$d\tilde{\mu}\ \to\ |{\rm v}|^{4-8n}(\sqrt{\bar{\rm v}}\,)^{8n-4}(\sqrt{\rm v}\,)^{2nN_F}\cdot d\tilde{\mu} \;=\; {\rm v}^{2-(4-N_F)n}\cdot d\tilde{\mu} \tag{30}$$

[7] Note that only mixed derivatives of $\mathcal{H}$ with respect to both v and $\bar{\rm v}$ enter $\mathcal{L}_{4\text{-deriv}}$ so that $\mathcal{H}$ itself can be written in a variety of equivalent ways.

so that

$$\mathcal{H}(\mathrm{v},\bar{\mathrm{v}})\Big|_{n\text{-inst}} \sim \frac{\log\bar{\mathrm{v}}}{\mathrm{v}^{(4-N_F)n}}\ , \qquad \mathcal{F}(\mathrm{v})\Big|_{n\text{-inst}} \sim \frac{1}{\mathrm{v}^{(4-N_F)n-2}} \tag{31}$$

as follows from Eqs. (25) and (13), respectively. In particular, for the special case $N_F = 4$, one has simply $\mathcal{H}\big|_{n\text{-inst}} \sim \log\bar{\mathrm{v}}$ so that the effective component vertices contained in $\mathcal{L}_{4\text{-deriv}}$ (all of which involve differentiating $\mathcal{H}$ with respect to both v and $\bar{\mathrm{v}}$) automatically vanish; likewise for the antiinstanton case with $\mathrm{v} \leftrightarrow \bar{\mathrm{v}}$. Thus we confirm the nonperturbative nonrenormalization theorem of Dine and Seiberg in this model.[8] Giving any of the hypermultiplets a mass spoils the argument, since m_i rescales to m_i/v, and this rescaled mass can be pulled down from the exponent.

5. Nonrenormalization theorem for the $N = 4$ model. Next we consider the $N = 4$ theory, i.e., $N = 2$ SYM coupled to a single massless adjoint hypermultiplet. In this model, after spontaneous symmetry breakdown, the low-energy dynamics involves a larger set of massless fields, corresponding to a single $N = 4$ $U(1)$ multiplet. Concomitantly, S^n_{inst} is now independent of four additional Grassmann collective coordinates: the 'trace' components of the $n \times n$ matrices $\mathcal{R}'_\gamma$ and $\tilde{\mathcal{R}}'_\gamma$ introduced in Eq. (7) [1,9]. Respectively, these components constitute the third and fourth supersymmetry modes, ξ^3_γ and ξ^4_γ. Now the collective coordinate integration takes the form

$$\int d^4x_0\, d^2\xi^1 d^2\xi^2 d^2\xi^3 d^2\xi^4\, \mathcal{B}_n(\mathrm{v},\bar{\mathrm{v}})\ , \tag{32}$$

where $\mathcal{B}_n$ is the n-instanton contribution to what one might call the "anteprepotential" in analogy to Eq. (13):

$$\mathcal{B}_n(\mathrm{v},\bar{\mathrm{v}}) \;=\; \int D\hat{\mu}\, e^{-S^n_{\text{inst}}}\ . \tag{33}$$

Here $D\hat{\mu}$ is the properly normalized integration measure for all collective coordinates in the problem excepting the $N = 4$ superspace position variables $(x_0, \xi^1_\gamma, \xi^2_\gamma, \xi^3_\gamma, \xi^4_\gamma)$. As before, these eight unbroken ξ^i_γ modes must be saturated by the insertion of an appropriate set of fields, for instance the eight antifermions

$$\mathbf{G}^8(x_1,\ldots,x_8) \;=\; \langle \bar\psi_{\dot\alpha}(x_1)\,\bar\psi_{\dot\beta}(x_2)\,\bar\lambda_{\dot\gamma}(x_3)\,\bar\lambda_{\dot\delta}(x_4)\,\bar\chi_{\dot\kappa}(x_5)\,\bar\chi_{\dot\lambda}(x_6)\,\bar{\tilde\chi}_{\dot\rho}(x_7)\,\bar{\tilde\chi}_{\dot\sigma}(x_8)\rangle\ , \tag{34}$$

where χ and $\tilde\chi$ are the adjoint hypermultiplet Higgsinos associated with the collective coordinate matrices $\mathcal{R}$ and $\tilde{\mathcal{R}}$, respectively. Now the action S^n_{inst} in the $N = 4$ model

[8] Notice that, in contrast to $\mathcal{H}$, in the $N_F = 4$ model one also has $\mathcal{F}\big|_{n\text{-inst}} \sim \mathrm{v}^2$ so that the effective $U(1)$ complexified coupling $\tau_{\text{eff}} = \mathcal{F}''(\mathrm{v})$ actually receives contributions from all (even) instanton numbers; see Refs. [18,8] for a discussion of this point.

has the discrete symmetry $\{\mathcal{M},\mathcal{N},\mathrm{v}\}\leftrightarrow\{\mathcal{R},\tilde{\mathcal{R}},\bar{\mathrm{v}}\}$ [9]. This symmetry, together with the long-distance expressions (10), implies

$$\bar{\tilde{\chi}}^{\text{LD}}_{\dot\alpha}(x_i) \;=\; i\sqrt{2}\,\xi^{3\alpha}\,S_{\alpha\dot\alpha}(x_i,x_0)\,\frac{\partial}{\partial\bar{\mathrm{v}}}\;+\;\cdots \tag{35a}$$

$$\bar{\chi}^{\text{LD}}_{\dot\alpha}(x_i) \;=\; -i\sqrt{2}\,\xi^{4\alpha}\,S_{\alpha\dot\alpha}(x_i,x_0)\,\frac{\partial}{\partial\bar{\mathrm{v}}}\;+\;\cdots \tag{35b}$$

From Eqs. (10) and (32)-(35) it follows that $\mathbf{G}^8\big|_{n\text{-inst}}\;\propto\;\partial^8\mathcal{B}_n/\partial\mathrm{v}^4\partial\bar{\mathrm{v}}^4$. However, the rescaling argument (29) implies that

$$D\hat\mu\;\rightarrow\;|\mathrm{v}|^{4-8n}(\sqrt{\bar{\mathrm{v}}}\,)^{8n-4}(\sqrt{\mathrm{v}}\,)^{8n-4}\cdot D\hat\mu\;=\;D\hat\mu\;, \tag{36}$$

so that actually $\mathcal{B}_n(\mathrm{v},\bar{\mathrm{v}})$ is a constant, independent of v and $\bar{\mathrm{v}}$. Thus all (multi-)instanton contributions to $\mathbf{G}^8$ vanish, as predicted by Dine and Seiberg in this model as well.

Interestingly, in the $N=8$ theory in three space-time dimensions, obtained by dimensional reduction of the $N=4$ theory in 4D, this nonrenormalization theorem for the higher derivative terms no longer holds; indeed the non-vanishing one-instanton contribution to the corresponding 4-derivative/8-fermion term has been calculated in [19], and all higher multi-instanton contributions have been obtained in closed form in [20]. The significance of such instanton corrections for Matrix theory was discussed last week in [21].

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References

[1] N. Seiberg, Phys. Lett. B206 (1988) 75.

[2] N. Seiberg and E. Witten, *Electric-magnetic duality, monopole condensation, and confinement in $N = 2$ supersymmetric Yang-Mills theory*, Nucl. Phys. B426 (1994) 19, (E) B430 (1994) 485 hep-th/9407087; and *Monopoles, duality and chiral symmetry breaking in $N = 2$ supersymmetric QCD*, Nucl. Phys B431 (1994) 484, hep-th/9408099.

[3] M.A. Shifman and A.I. Vainshtein, Nucl. Phys. B277 (1986) 456;
M. Dine and Shirman, Phys. Rev. D50 (1994) 5389, hep-th/9405155.

[4] G. Sierra and P.K. Townsend, in *Supersymmetry and Supergravity 1983*, proc. XIXth Winter School, Karpacz, ed. B. Milewski, World Scientific 1983, p.396;
S.J. Gates, Nucl. Phys. B238 (1984) 349;
B. de Wit, P.G. Lauwers, R. Philippe, S.-Q. Su and A. Van Proeyen, Phys. Lett. B134 (1984) 37.

[5] R. Grimm, M. Sohnius and J. Wess, Nucl. Phys. B133 (1978) 275.

[6] M. Henningson, *Extended Superspace, Higher Derivatives and $SL(2, \mathbb{Z})$ Duality*, Nucl. Phys. B458 (1996) 445, hep-th/9507135.

[7] B. de Wit, M.T. Grisaru and M. Rocek, *Nonholomorphic Corrections to the One-loop $N = 2$ Super Yang-Mills Action*, Phys. Lett. B374 (1996) 297, hep-th/9601115.

[8] N. Dorey, V.V. Khoze and M.P. Mattis, *On $N = 2$ supersymmetric QCD with 4 flavors*, hep-th/9611016.

[9] N. Dorey, V.V. Khoze and M.P. Mattis, *On mass-deformed $N = 4$ supersymmetric Yang-Mills theory*, Phys. Lett. B396 (1997) 141, hep-th/9612231.

[10] A. Pickering and P. West, *The One Loop Effective Superpotential and Nonholomorphicity*, Phys. Lett. B383 (1996) 54, hep-th/9604147.

[11] U. Lindstrom, F. Gonzalez-Rey, M. Rocek and R. von Unge, *On $N = 2$ Low-Energy Effective Actions*, Phys. Lett. B388 (1996) 581, hep-th/9607089.

[12] A. Yung, *Instanton-induced Effective Lagrangian in the Seiberg-Witten Model*, Nucl. Phys. B485 (1997) 38, hep-th/9605096; and *Higher Derivative Terms in the Effective Action of $N = 2$ SUSY QCD from Instantons*, hep-th/9705181.

[13] M. Dine and N. Seiberg, *Comments on Higher Derivative Operators in Some SUSY Field Theories*, hep-th/9705057.

[14] M. Atiyah, V. Drinfeld, N. Hitchin and Yu. Manin, Phys. Lett. A65 (1978) 185.

[15] E. Corrigan, P. Goddard and S. Templeton, Nucl. Phys. B151 (1979) 93;
E. Corrigan, D. Fairlie, P. Goddard and S. Templeton, Nucl. Phys. B140 (1978) 31.

[16] H. Osborn, Ann. Phys. 135 (1981) 373.

[17] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in $N = 2$ supersymmetric gauge theory*, Phys. Rev. D54 (1996) 2921, hep-th/9603136.

[18] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-instanton calculus in $N = 2$ supersymmetric gauge theory. II. Coupling to matter*, Phys. Rev. D54 (1996) 7832, hep-th/9607202.

[19] J. Polchinski and P. Pouliot, *Membrane Scattering with M-momentum Transfer*, hep-th/9704029.

[20] N. Dorey, V.V. Khoze and M.P. Mattis, *Multi-Instantons, Three-Dimensional Gauge Theory and the Gauss-Bonnet-Chern Theorem*, hep-th/9704197.

[21] T. Banks, W. Fischler, N. Seiberg and L. Susskind, *Instantons, Scale Invariance and Lorentz Invariance in Matrix Theory*, hep-th/9705190.