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THE HENRICI HARMONIC ANALYZER AND DEVICES FOR EXTENDING AND FACILITATING ITS USE.*

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CONTENTS.

HARMONIC ANALYSIS.

THE harmonic method of analysis, based upon Fourier's Theorem, first published in "La Théorie Analytique de la Chaleur" (Paris, 1822), is of very great value in the investigation of many phenomena which can be represented by curves, and especially of wave motions which are represented by periodic curves. This theorem may be stated as follows: If any curve be given, having a wave-length l, the same curve can always be reproduced,

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and in one particular way only, by compounding simple harmonic curves of suitable amplitudes and phases, in general infinite in number, having the same axis and having wave-lengths of l, $\frac{1}{2}\,l$, $\frac{1}{3}\,l$, and successive aliquot parts of l. The given curve may have any arbitrary form whatever, including any number of straight portions, provided that the ordinate of the curve is always finite and that the projection on the axis of a point describing the curve moves always in the same direction.[1, 2] The practical significance of this theorem and the complete processes of harmonic analysis and synthesis have been explained in detail and graphically illustrated elsewhere by the author.[3, 4]

The mathematical expression for a curve, derived in accordance with Fourier's Theorem, consists of an infinite trigonometric series of sines and cosines, which may be stated as follows:

$$y=\frac{1}{l}\int_0^l y\,dx+\begin{cases}\left[\frac{2}{l}\int_0^l y\sin\frac{2\pi x}{l}dx\right]\sin\frac{2\pi x}{l}+\left[\frac{2}{l}\int_0^l y\sin\frac{4\pi x}{l}dx\right]\sin\frac{4\pi x}{l}+\cdots\\ \left[\frac{2}{l}\int_0^l y\cos\frac{2\pi x}{l}dx\right]\cos\frac{2\pi x}{l}+\left[\frac{2}{l}\int_0^l y\cos\frac{4\pi x}{l}dx\right]\cos\frac{4\pi x}{l}+\cdots\end{cases}\qquad \text{I}$$

in which y is the ordinate of the original complex curve at any point x on the base line, and l is the fundamental wave-length.

For the purposes of mathematical treatment it is convenient to express the abscissæ x in terms of angular measure, θ radians, with such a unit that the wave-length l equals 2π radians. The equation then has the form,

$$y=\frac{1}{2\pi}\int_0^{2\pi} y\,d\theta+\begin{cases}\left[\frac{1}{\pi}\int_0^{2\pi} y\sin\theta\,d\theta\right]\sin\theta+\left[\frac{1}{\pi}\int_0^{2\pi} y\sin 2\theta\,d\theta\right]\sin 2\theta+\cdots\\ \left[\frac{1}{\pi}\int_0^{2\pi} y\cos\theta\,d\theta\right]\cos\theta+\left[\frac{1}{\pi}\int_0^{2\pi} y\cos 2\theta\,d\theta\right]\cos 2\theta+\cdots\end{cases}\qquad \text{II}$$

If the first term is represented by a_0, and the coefficients of the sine and cosine terms, the quantities in square brackets are represented by a_1, b_1, a_2, etc., the equation will have the following symbolic form of simpler appearance:

$$y=a_o+\begin{cases}a_1\sin\theta+a_2\sin 2\theta+\ldots\\ b_1\cos\theta+b_2\cos 2\theta+\ldots\end{cases}\qquad \text{III}$$

The term a_0 is a constant and is equal to the distance from the line assumed as a base to the true axis of the curve; if the base coincides with the axis, $a_0 = 0$, and this term does not appear in the equation of the curve. This term has no relation to the form or significance of the curve, and usually its value is not required; it may be determined, however, by means of an ordinary planimeter, as described in the reference.[3]

The other terms of the equation occur in pairs, as $a_1 \sin \theta$, $b_1 \cos \theta$, etc., and each term, whether a sine or a cosine, represents a simple harmonic curve. The successive terms of the series of sines and of cosines repeat themselves with frequencies of 1, 2, 3, etc., which means that the curves have wave-lengths in the proportion of $1 : \frac{1}{2} : \frac{1}{3}$: etc.; such a succession of terms is said to form a *harmonic* series.

The coefficients of the various terms, the quantities in square brackets in equation I, which are represented by a_1, b_1, a_2, etc., in equation III, have the following general form, n being the *order* of the term and l the wave-length:[1, 2]

$$\frac{2}{l}\int_0^l y \sin \frac{2n\pi x}{l}\, dx.$$

Each coefficient is a number or factor indicating how much of the corresponding simple sine or cosine curve enters into the composite; that is, it shows the height or *amplitude* of the simple component.

The process of *analyzing a curve* consists of finding the particular numerical values of the coefficients of the Fourier equation so that it shall represent the curve. In general the number of terms required is indefinitely great, or even infinite, but in many instances a finite number of terms is sufficient. Whenever certain of the simple curves are not required, the corresponding coefficients are said to have the value zero, and their terms do not appear in the equation of the curve. In the study of sound waves the number of terms involved seldom exceeds thirty and often does not exceed ten. Fourier showed how the numerical values of the coefficients may be calculated, but the process is very long and tedious, requiring perhaps several days' work for a single curve. The great importance of harmonic analysis has caused the development of many methods, numerical, graphical, and mechanical, for facilitating the calculations.[3] The main part of each coefficient is a definite integral which is of the nature of an area, and various area-integrating machines, known in their simpler forms as *planimeters,* may be adapted to the evaluation of these coefficients. A complete apparatus arranged for mechanically deriving the Fourier equation of a curve is called a *harmonic analyzer.*

Perhaps the most convenient and precise harmonic analyzer yet devised is that of Henrici. An instrument of this type has been in use by the author since 1908 in the study of sound waves, and with it several thousand curves have been analyzed. The experience thus gained has led to the development of various instruments and methods for facilitating the analytical work, both with this particular form of analyzer and in general. It is the purpose of this paper to describe these devices.

HENRICI'S HARMONIC ANALYZER FOR TEN COMPONENTS.

The harmonic analyzer devised by Prof. O. Henrici, of London, is based upon the rolling sphere integrator, and its theory was first published in 1894.[5] These analyzers have been developed and mechanically perfected by Mr. G. Coradi, of Zurich; and, as constructed by him, they are not only of beautiful workmanship, but they are also calculating machines of high precision. An analyzer of this type, with five integrating apparatus, is shown in Fig. 1. The theory of the Henrici analyzer will be stated very briefly.

The general expressions for the coefficients of the sine and cosine terms of the Fourier equation, form II (represented by a_1, b_1, etc., in form III), integrated by parts, give the following equations, n being the order of the term:

$$a_n = \frac{1}{\pi}\int_0^{2\pi} y \sin n\theta d\theta$$

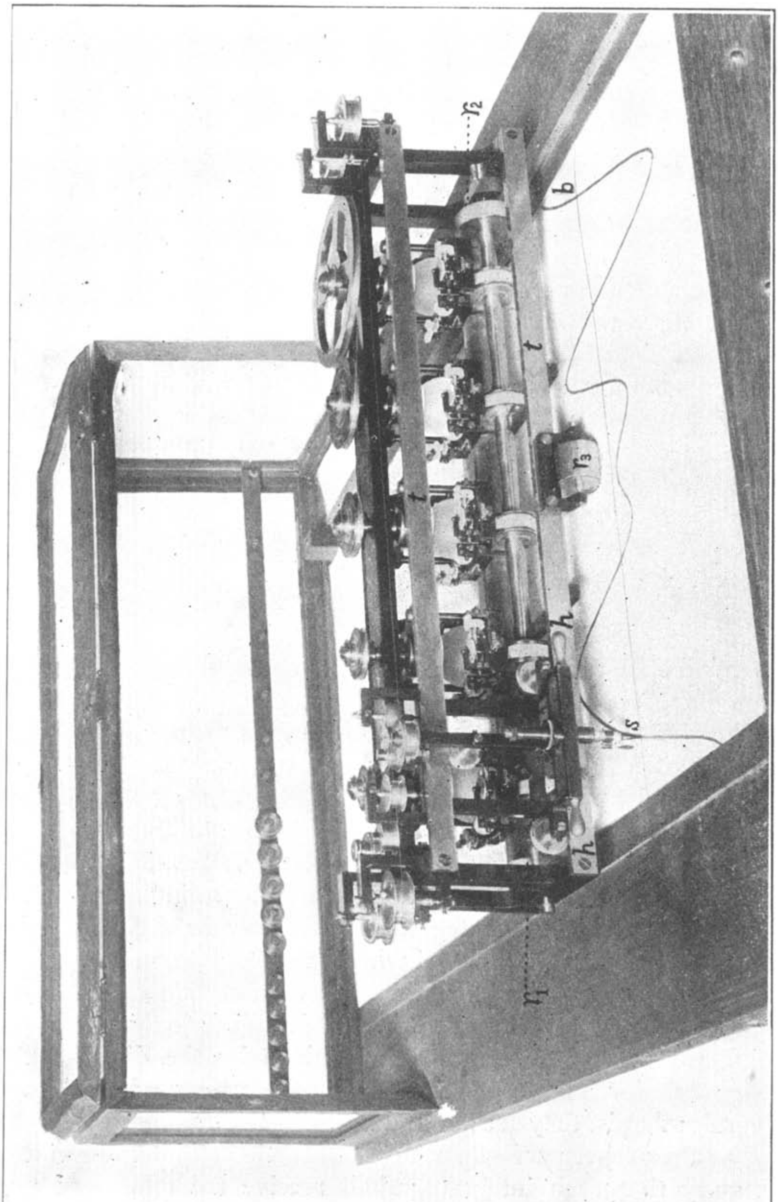
$$= \left[-\frac{1}{n\pi} y \cos n\theta \right]_0^{2\pi} + \frac{1}{n\pi}\int_0^{2\pi} \cos n\theta dy,$$

$$b_n = \frac{1}{\pi}\int_0^{2\pi} y \cos n\theta d\theta$$

$$= \left[\frac{1}{n\pi} y \sin n\theta \right]_0^{2\pi} - \frac{1}{n\pi}\int_0^{2\pi} \sin n\theta dy$$

If the curve is continuous, and if the initial and final values of y are equal, the terms in square brackets disappear. Nearly all the curves representing physical phenomena satisfy these conditions. If the curve has a discontinuity, it is made continuous

FIG. 1.



Henrici harmonic analyzer.

for purposes of analysis by joining the two points of discontinuity by a straight line; and if the final value of y is not equal to the initial value, that is, if the curve does not end on the base line, it is brought back to the base by a straight line parallel to the ordinate y. Under these conditions it can be proved that the integrals already given properly measure the coefficients of the Fourier equation of the curve. Further, if the base line is added to the path of integration, nothing is added to the value of the integral, since for this part of the path $dy = 0$, and the integral is taken around a closed curve.

Therefore the *Henrici integrals,* which define the values of the coefficients of the Fourier equation of the curve, have for the sine terms the form, n being the order of the term,

$$a_n = \frac{1}{n\pi}\int_{\theta=0}^{\theta=2\pi} \cos n\theta dy,$$

and for the cosine terms the form,

$$b_n = -\frac{1}{n\pi}\int_{\theta=0}^{\theta=2\pi} \sin n\theta dy.$$

It will be noticed that the coefficients of the sine terms contain cosine integrals, and *vice versa.*

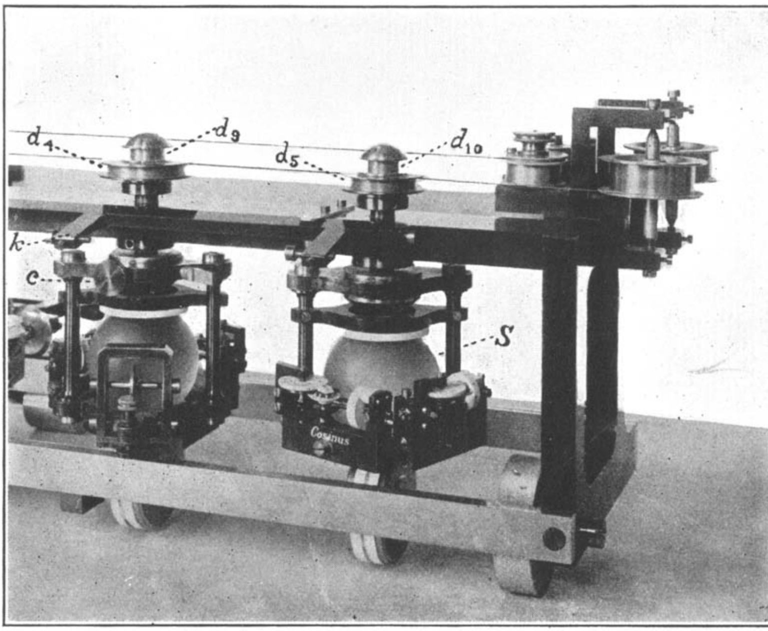
For the evaluation of any a coefficient, as the nth, it is necessary that each element of amplitude dy of the original curve shall be multiplied by the instantaneous value of $\cos n\theta$ and that the whole be integrated between the limits $\theta = 0$ and $\theta = 2\pi$; for the b coefficients the operation is the same, except that the resolving factor is $\sin n\theta$. The analyzer, then, is an integrating machine for evaluating these special integrals, and it may be arranged with one or more integrators, each of which at one operation determines one pair of integrals defining the term of a certain order. The integrations for the terms of various orders differ only as regards the factor n. The actual readings given by the dials of the integrators for any term are n times the values of the above integrals; that is, they are na_n and nb_n.

For the proper performance of its function of integration it is necessary that each integrator shall receive the effects of two rectangular motions which are related to the area of the curve in certain particular ways as required by the development of

Fourier's theorem. One of these movements is secured by the rolling of the instrument as a whole, back and forth parallel to itself, while the second is obtained from the movement of a carriage along a transverse track.

The analyzer is supported by three rollers, r_1, r_2, and r_3, Fig. 1. The curve to be analyzed, which must be drawn to a specified scale, is placed underneath, and is adjusted so that its axis (or base) is parallel to the track t; this condition may be determined by

FIG. 2.



The rolling sphere integrator.

inspection or by moving the carriage with the stylus s along the track and adjusting the curve until the stylus traces the given line. If no base line is given, any line touching the crests or troughs of two consecutive waves, or any parallel to such a line, may be used. The stylus is placed over the initial point of the curve, as shown; the stylus is adjustable in a transverse slot to facilitate this setting. The initial point may be chosen at convenience, since the analyses of a curve made from different starting points will (when fully reduced) differ only in the phases of the components. The handles h are grasped with the fingers and the stylus is caused to trace one exact wave-length of the curve, when it will again be on the base line, at the point b; the stylus is returned to the starting point s, along the assumed base line, completing the integration around a closed curve. The latter operation theoretically adds nothing to the value of the integral, while it eliminates errors which would be produced if the axes on which the integrators revolve do not pass through the centre of the axle a. The tracing of the curve will have required a combination of the two specified rectangular movements, and the manner of communicating these in the proper proportion to the integrators will be now described in detail.

FIG. 3.

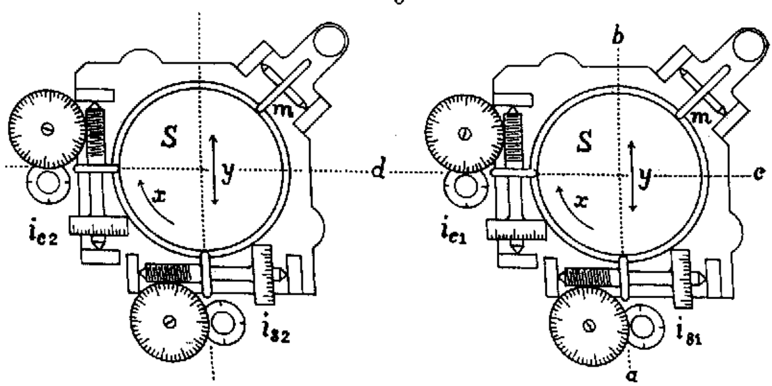


Diagram of two rolling sphere integrators in initial positions.

The rolling sphere integrator has for its essential parts a sphere of glass, S, Fig. 2, which when in use rests upon a celluloid roller underneath it at what may be called the south pole, and two integrating cylinders, i_{s1} and i_{c1}, Fig. 3, which rest against the equatorial circumference of the sphere and, while revolving around the sphere, always touch it at points 90 degrees apart. A third roller, m, holds the sphere against the integrating wheels with a spring pressure. When not in use, each sphere may be lifted from all the rollers and be held securely between celluloid clamping rings. In this condition the integrating cylinders are free to move, and each may be set so that its dials indicate zero, preparatory to the operation of tracing a curve.

The rollers r_1 and r_2, Fig. 1, and the five celluloid rollers underneath the spheres are all rigidly attached to the same axle a; the celluloid rollers, being smaller than the others, do not touch the paper.

As the curve is traced the machine rolls backward and forward, parallel to itself, the five spheres are *all rotated* by the *same amount,* and in the *same direction,* the direction of the amplitude of the curve, as indicated by the arrow y, Fig. 3. The amount of this rotation is always proportional to the amplitude y of the complex curve, and, therefore, at *each point* in the tracing, it is proportional to the instantaneous sum of the separate component amplitudes of the simple component curves.

The integrating cylinders i_s are for evaluating the sine coefficients and i_c for the cosines. Each pair of cylinders is attached

FIG. 4.

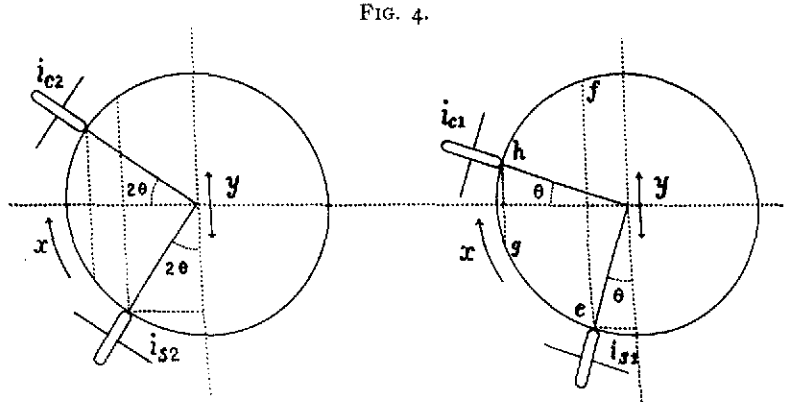


Diagram of rolling sphere integrators after the stylus has moved the distance $x = \theta$.

to a light frame which surrounds the equatorial region of the sphere, Figs. 2 and 3; the frame is supported from above and is capable of rotation independently of the sphere, on an axis in the prolongation of the north polar axis, as indicated by the arrows x. The initial or zero positions of the cylinders are shown in Fig. 1 and 3, the settings for which are facilitated by clamping screws c and spring stops k, Fig. 2. In this position $\theta = 0$, and when the instrument rolls in amplitude the sphere rolls in the plane ab and the integrating cylinder i_{s_1} records a rotation proportional to $dy \cos \theta$, and i_{c_1}, which touches the sphere at a point on the axis of rotation cd, has a rotation proportional to $dy \sin \theta$; that is, it is not rotated at all (for $\theta = 0$, $\cos \theta = 1$, and $\sin \theta = 0$).

When the stylus, in tracing the curve, has moved in the direction of the x axis by an amount equal to θ, the integrating cylinders for the first component must be revolved (as described later) around the first sphere through an angle θ, Fig. 4, the integrators for the second component must revolve through an angle 2θ, and for any higher component the cylinders must be revolved through an angle $n\theta$. In the position shown in Fig. 4 the integrator i_{s_1}, which has moved through the angle θ, now touches the sphere on the circumference of a circle of diameter *ef*, and as the sphere rolls in amplitude by the amount dy, the cylinder receives a rotation proportional to $dy \cos \theta$; the integrator i_{c_1} touches the sphere on the circumference of the circle *gh* and it is rotated by an amount proportional to $dy \sin \theta$. The integrators i_{s_2} and i_{c_2} have at the same instant been revolved through the angle 2θ and they receive rotations proportional to $dy \cos 2\theta$ and $dy \sin 2\theta$, respectively. The integrators arranged for terms of other orders operate in a corresponding manner.

For the evaluation of the first component of a curve, the fundamental, the sine and cosine integrating cylinders must be carried around the sphere *exactly* one revolution while the stylus traces one fundamental wave-length of the curve, and for the second component the cylinders must revolve twice in the fundamental wave-length, and for the third component the cylinders must revolve three times in the same interval, etc. This revolution of the cylinders is accomplished by attaching a wire to the carriage of the stylus, which passes around fixed guide pulleys and is wrapped once around suitable disks, d_1, d_2, etc., Figs. 2 and 5, attached to the upper parts of the spindles which support the frames of the cylinders. One integrator may be used for evaluating any number of terms (one at a time) by supplying a series of disks suitable for turning the apparatus once, twice, three times, and so on, while the stylus traces the fundamental wave-length. Several integrators may be used simultaneously, the disks being of such diameters that the cylinders of the successive apparatus are revolved one, two, three, four, and five times, etc. The instrument being described, having five integrators, gives five pairs of coefficients with one tracing. Two disks are attached to each integrator, so that after one tracing the wire may be slackened and be removed from the lower series of disks d_4 and d_5. Fig. 2, and be wrapped around the upper series, d_9 and d_{10}, causing the integrators to revolve six, seven, eight, nine, and ten times, respectively, for one wave-length, so that with two tracings ten pairs of coefficients are determined.

The relations described remain true as the curve is traced, and the motion of the stylus is continuously resolved into the two rectangular components: one in the direction of the amplitude, producing rolling of the spheres, and the other in the direction of the length of the curve, revolving the integrating cylinders around the several spheres. The integrating cylinders are rolled in such a manner as to add, algebraically, the properly resolved components of the amplitude increments; or, in other words, they perform the process of true, continuous integration as required for the evaluation of the Henrici integrals. The operations of selective analysis and summation are secured by the beautifully related mechanical movements, which proceed so smoothly that they seem almost mysterious. Because of the design and workmanship, which produce continuous movements with the complete elimination of lost motion, remarkable precision is attained, as is illustrated by numerical examples in the latter part of this paper.

The maker of the analyzer has so proportioned and adjusted the sizes of the various parts of the integrators that the readings on the dials are not only *proportional* to the coefficients of the various terms of the Fourier equation, but there are also automatically incorporated in the readings a proportionality factor, the factor π, and the algebraic signs, so that the readings are *exactly equal* to the *amplitudes expressed in millimetres,* without any reduction except for the factor n as mentioned below. The dials indicate 0.1 millimetre directly and 0.01 millimetre by estimation, and they have a capacity for 2000 millimetres before repeating.

The readings of the first set of dials are the amplitudes of the fundamental sine and cosine curves. In the fundamental wave-length there are, of course, two wave-lengths of the second component, and in tracing the curve the second integrator operates over the two waves of this order and at the end its dials show the sum of the amplitudes for the two waves, or twice the amplitude for this component; the third integrator has integrated three wave-lengths, etc. Therefore, at the conclusion of the tracing of the complex curve, any integrator for the nth term indicates n times the corresponding amplitude, which accounts mechanically for the presence of the factor n in the denominator of the Henrici integrals.

The maker has chosen 400 millimetres (about 15¾ inches) for the wave-length; this provides a frame for the instrument which accommodates five integrators, and when the stylus is moved across the wave it gives a movement to the wire which is suitable for putting the integrators into proper action. If the wave-length were less, the disks for the wire would have to be correspondingly smaller, resulting in increased difficulty of revolution of the integrating cylinders, as mentioned in the next section. However, the instrument may be adapted to any specified shorter wave-length by providing new disks for the wire adjusted to have effective circumferences which produce complete revolutions of the integrators in the order of the natural numbers.

When only certain particular components are required, a special set of disks may be provided to evaluate these. For instance, in electric alternating-current waves there are present only odd-numbered terms, and the disks may be arranged for one, three, five, seven, nine, etc., turns of the integrators.

In the development of the theory of the analyzer the general integration contains other parts than the ones so far considered; it is shown that when the integration is taken around a closed curve under conditions such as are involved in the practical operation of the analyzer, these other integrals are eliminated. It is interesting to note that these terms are also present in the mechanical integrations, and that they are automatically eliminated from the readings only at the conclusion of the tracing.

While one wave-length of a complex curve is being traced, the integrating cylinders of any sphere are rotated by larger or smaller amounts, depending upon the instantaneous amplitude. Finally, when the end of the exact wave-length has been reached and each cylinder has been revolved around its sphere an exact integral number of times, the rotation of the cylinder then registered is proportional to the true amplitude of the sine or cosine component whose wave-length is equal to the distance the stylus moved transversely during *one* complete revolution of the cylinder around the sphere; the effects of all other harmonic component curves will have been automatically eliminated from this particular integrator. Further, the sphere of an integrator arranged for evaluating a component of a certain order which in a particular curve has the value zero (as in one of the test curves described later) actually rolls just as much in the amplitude direction as does any other sphere of the series; the integrating cylinders, revolving around their sphere n times, at several different times in the process of analysis show large readings which are zeroized only at the *end* of the tracing.

When there are *inharmonic* components in the curve, the method of analysis and the instrument here described are not directly applicable. If, however, such a curve is traced, the integrators will indicate the amplitudes of the respective terms of the infinite series of *harmonic* components which may be considered equivalent to the inharmonic components *in so far as the terms of the specified orders are involved.*

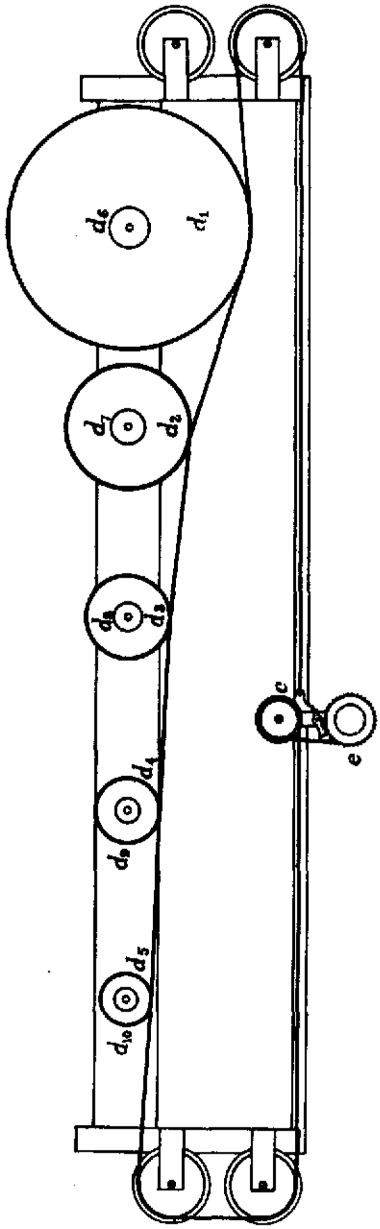
EXTENSION OF THE ANALYZER FOR THIRTY COMPONENTS.

When arrangements were being made for the construction of an analyzer suitable for the investigation of sound waves, Mr. Coradi, the maker, upon being asked whether an instrument could be provided which would determine more than ten components, replied as follows: "The Henrici analyzer has never been arranged for more than ten elements. The diameters of the disks of the eleventh and following elements would become too small to be able to put the integrating apparatus in movement in a correct manner."

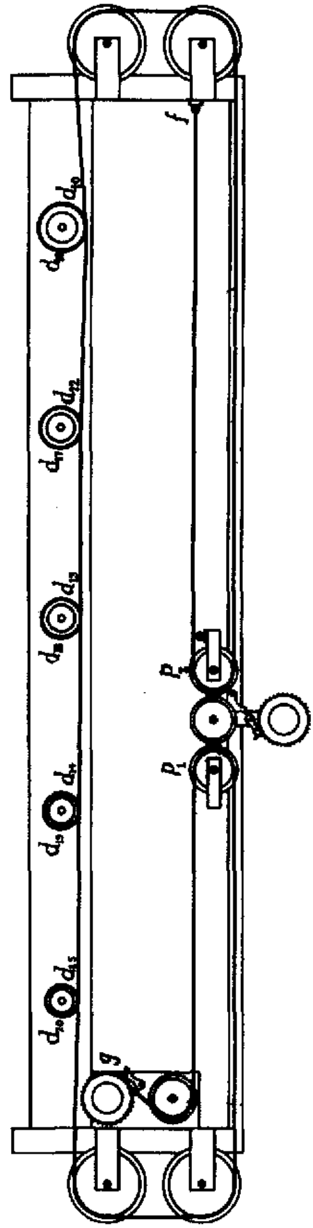
In accordance with this opinion, and because at that time it was thought that sound waves in general would not contain more than ten components, an instrument was purchased arranged for ten components. This analyzer was found sufficient and very satisfactory for the study of curves representing the sounds of the flute. When other sounds, as those of the clarinet and oboe, were investigated, it appeared that at least twenty components must be considered. This necessity led, in 1910, to an addition to the analyzer, extending its range to thirty components. Six years of use, during which thousands of analyses have been made, has shown that the arrangement is very practicable. The new device provides simply for doubling the motion of the wire caused by the travel of the tracing stylus, which provision makes possible the use of larger disks for revolving the integrators, thus obviating the difficulty mentioned by Mr. Coradi.

The scheme of the original wire arrangement is shown in Fig. 5. The wire is attached to the tracing carriage at c, and is threaded around the guide pulleys and around the integrator

FIGS. 5 AND 6.



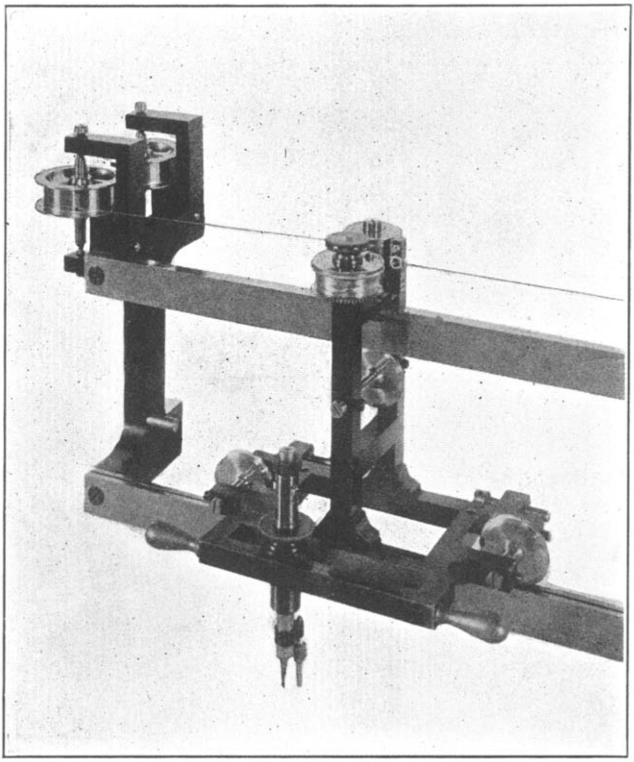
Original scheme for wire for ten components.



New scheme for wire for thirty components.

disks, as shown, and is brought back to the carriage, where it is held taut by winding on the supply drum e, which is held in place by a pawl and ratchet. The effective circumference of the disk d_1 for the first component is 400 millimetres, the wave-length, while the effective circumference of the disk for the tenth component is 40 millimetres, giving a diameter of less than 13 millimetres. The analyzer is always used in its original form for evaluating the first ten components.

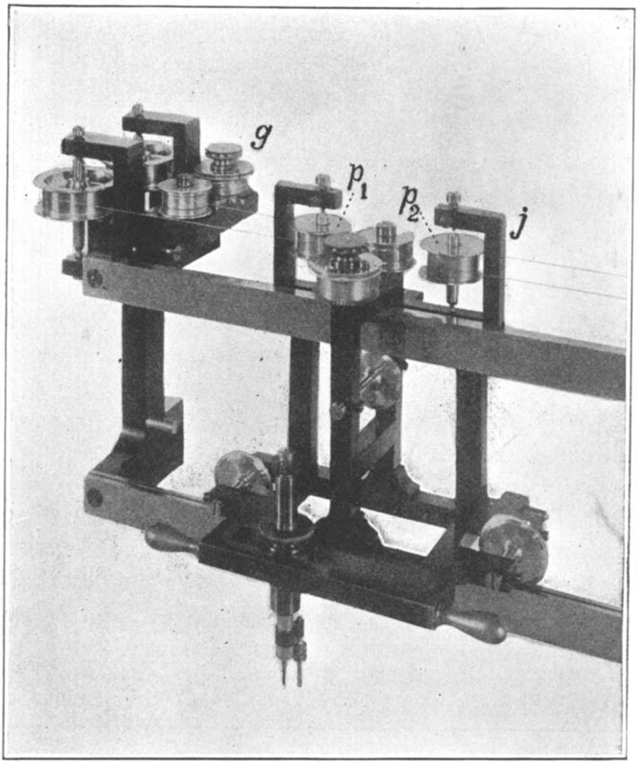
FIG. 7.



Tracing carriage in original form.

When components of higher order are to be evaluated, a new set of five double-disks is attached to the several integrators, and the wire is arranged according to the new scheme shown in Fig. 6. The wire is attached to the frame at f, and is threaded around a free pulley p_2 on the carriage, around the end pulleys and disks as shown, and around a second free pulley p_1 on the carriage to the new winding drum at g. Any travel of the carriage now causes a movement of the wire of double the amount. For the eleventh component the effective circumference of the disk d_{11} is $2 \times 400 \div 11 = 72.7$ millimetres, and for the twentieth component it is $2 \times 400 \div 20 = 40$ millimetres, the same as for the tenth component in the original wire scheme.

FIG. 8.



Tracing carriage with new attachments.

A third set of five double-disks has been provided for the components from 21 to 30, the effective circumference of the latter being $2 \times 400 \div 30 = 26.6$ millimetres. When the wire is prepared as described in the next section, the determination of thirty components is entirely practicable. However, since the wire stretches under the increased pull required to revolve the integrators of higher orders, the last five components are usually determined by two tracings in groups of three and two, respectively, instead of in one group of five.

The nature of the additions to the analyzer to adapt it to the determination of the components of higher orders is shown by a comparison of Figs. 7 and 8: Fig. 7 is a view of the tracing carriage and one end of the frame in the original form, and Fig. 8 is a view of the same parts with the additions. The new parts consist of the rectangular frame *j*, Fig. 8, rigidly attached to the tracing carriage, and the two free pulleys, p_1 and p_2, and of the duplicate guide pulley and winding drum with their support attached to the end of the frame at *g*. There are two new hooks attached to the opposite end of the frame in the position indicated by *f*, Fig. 6. There are also the two new sets of five each of double-disks for the integrators, which are shown on the rack at the back of the view given in Fig. 1.

PREPARATION OF WIRE FOR THE ANALYZER.

The transverse movement of the stylus of the analyzer during the process of tracing a curve is transmitted to the integrators by means of a wire. The wire provided by the maker of the instrument was of copper, silver plated, having a diameter of about 0.27 millimetre (about No. 29 of the Brown & Sharpe wire gauge) and a breaking strength of about 4.5 pounds. Such wire serves fairly well when determining the first ten components, though it is rather soft and subject to stretching, which causes the phases of the revolving integrators to lag slightly. The pull on the wire increases in proportion to the number of revolutions of the disks, and this, added to the rolling action on the wire as it turns the disks, causes so much stretching that the copper wire is not suitable for components above ten.

Experiments have been made with wire of the following materials: Copper, iron, steel, nickel, hard brass, soft brass, German silver, platinoid, and manganin. The steel and hard-drawn brass wires, while free from stretching, were too stiff and springy: where the wire was wrapped around the disks the tendency was to open out and allow slipping. Most of the other wires proved to be too soft and likely to stretch. Soft German silver seemed to be the only suitable material.

The preparation of the wire is as follows: The stock is soft

German silver wire of No. 28 Brown & Sharpe gauge, as obtained from the usual supply dealers, which has a diameter of about 0.31 millimetre. One end of a piece of the wire about 70 inches long is fastened in a vise and the other end is wrapped around the hook of a spring balance. The wire is subjected to a slowly-increasing pull up to about 8 pounds, which is continued for two minutes. This causes a stretch of about 4 inches, after which the wire seems to acquire a permanent set and it may be subjected to an additional pull of one or two pounds without further permanent elongation. This stretching is a kind of tempering process. The wire is next drawn through a steel draw-plate, reducing the diameter to about 0.29 millimetre, and increasing the length to about 84 inches. Finally the wire is drawn through a second and very accurate draw-plate which reduces the diameter to 0.28 millimetre, the length now being about 90 inches; it has a mild spring temper and a breaking strength of about 13 pounds. The wire thus prepared may be used on the analyzer for more than a hundred analyses before it breaks; for the higher-element integrations there is a slight stretch, perhaps amounting to one millimetre, which must be taken up by the winding drum after each such tracing.

A chain, such as the fusee chain of watches, would, no doubt, be very suitable in place of the wire; such chain is used with entire satisfaction in the 32-element harmonic synthesizer described by the author in this JOURNAL for January, 1916. For use with the analyzer a small-sized chain about eight feet long would be required; it is necessary to unthread the chain frequently, and there would be great danger of its becoming kinked and broken. Wire is much more convenient and has been found satisfactory.

THE ANALYZING BOARD.

The analyzer in operation rolls back and forth in the direction of the amplitude of the curve, and it may be placed on any flat surface, as a table, and roll over the paper on which the curve is drawn. To secure a uniform and rectilinear rolling, and to facilitate the operation in general, a special board has been arranged, having for its foundation a drawing-board 30 by 42 inches in size. Two strips of hard wood about one inch wide and one-fourth inch thick are set into the board about an eighth inch, so that the upper surfaces of the strips are about an eighth inch above the surface of the board; these strips are about 21 inches apart, forming tracks on which the rollers r_1 and r_2, Fig. 1, may run.

A third strip of hard wood is placed against the right edge of the right track, forming a guide for the roller r_2. Two stops are provided near the front edge of the board, against which the analyzer is rolled, setting the instrument as a whole with its axis perpendicular to the tracks. A ledge is placed on the front of the board to prevent the analyzer from accidentally rolling off, and a glass house for the analyzer and attachments is permanently arranged as shown in Fig. 1.

ENLARGING THE CURVES FOR ANALYSIS.

Curves which are to be analyzed with the Henrici analyzer must be drawn to a standard wave-length of exactly 400 millimetres, as has been explained. If a wave is drawn exactly to the wave-length of 200 millimetres, then the integrators for the components 2, 4, 6, etc., of the 400-millimetre series will, of course, give the components 1, 2, 3, etc., for the wave 200 millimetres long, and similarly for other aliquot parts of 400 millimetres.

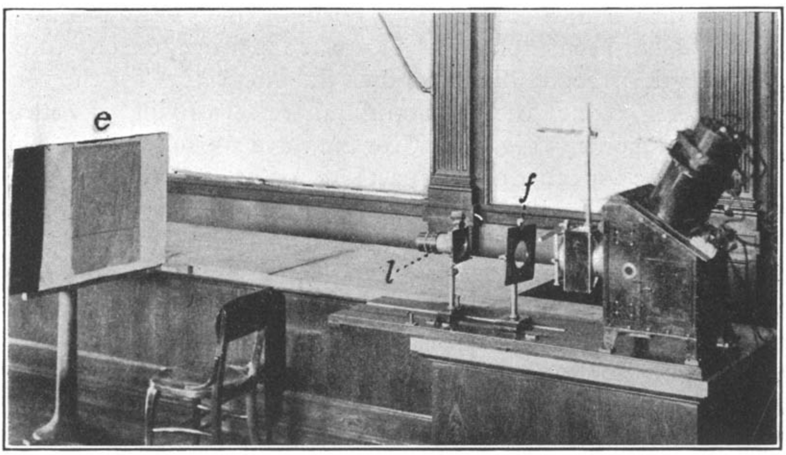
The curves provided by observation are rarely, if ever, of the exact wave-length required, and they must be redrawn. This may be done by plotting numerical data to the proper scale, or the curve may be enlarged with the help of proportional dividers, a pantograph, or other draughting instrument. A photographic camera or enlarging apparatus may be useful. When the original curve has been photographically recorded on a plate or film, a projecting lantern may be the most convenient enlarging arrangement.

The photographs of sound waves obtained with the phonodeik [3] are commonly made on films five inches wide and the wave-lengths vary from 25 millimetres to 100 millimetres. These curves are enlarged with the apparatus shown in Fig. 9, consisting of an optical bench projection apparatus having a lens l of large size, with the addition of a special film holder f and an easel e.

The film holder f consists of a frame with an opening covered by a glass plate; this frame is adjustable in two rectangular directions by means of micrometer screws, and is capable of rotation about an axis perpendicular to its own plane. A smaller frame with a glass plate is hinged to the larger one. The film is placed between the two glass plates, and the image of the curve is projected on the easel. The frame is adjusted till the image is in a suitable position for the tracing.

The easel consists of a drawing-board so mounted on a stand that the optical axis of the projection apparatus is perpendicular to the board at its centre. The easel rolls on ball-bearing castors and has a guide resting against a track (the edge of the long table, as shown in the illustration), which permits the easel to be moved forward and backward, the board remaining at all times in the proper relation to the projecting lantern. On the easel are ruled vertical lines 400 millimetres apart and also horizontal lines. The distance of the easel from the film and the position of the lens are adjusted till the image of one wave-length of the curve is exactly 400 millimetres long and is in focus. The wave-length is best

FIG. 9.



Lantern and easel for enlarging curves.

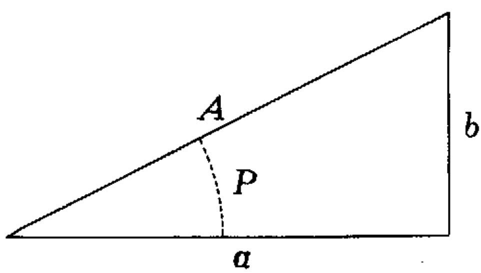
determined by the points where two consecutive waves cross the axis (or any base line) in the same phase; if no base line is given, the sharp crests or troughs, or other definite points of two consecutive waves which are in the same phase, may be used. A piece of paper, 19 by 24 inches in size, is placed on the easel and the wave is traced with a pencil, one wave-length only being required; the axis or base line is drawn, and the curve thus obtained is ready for analysis.

The drawing of all curves to this large scale, on a standard size of paper, facilitates comparison and filing. The harmonic synthesizer [4] used in this work draws curves of the same size, and a direct comparison of the original and the synthesized curves can always be made. When the operation of enlarging is performed in routine manner, the time required is less than five minutes for each curve.

MACHINE FOR CALCULATING AMPLITUDES AND PHASES.

The equation of a curve as given by harmonic analysis is in the form of a double series of sines and cosines, as explained in the first part of this paper. Another form of the equation, consisting of a series of sines only (or of cosines only), with epochs or phases for the several terms, is more suitable for expressing the results of a physical investigation. The latter form presumably corresponds to the physical phenomena represented, while the former is a mathematical expression resulting from the Fourier method of derivation. It is therefore necessary to reduce the data given by the analyzer to determine the amplitudes and phases of the components of a curve, and this is carried out in accordance with the following principle.[2]

FIG. 10.



Amplitude and phase relations of component and resultant harmonic motions.

The Fourier equation, form III, having on one side a constant term and a series of pairs of sine and cosine terms, may be reduced to the following equivalent form:

$$y = A_0 + A_1 \sin(\theta + P_1) + A_2 \sin(2\theta + P_2) + A_3 \sin(3\theta + P_3) + \ldots \qquad \text{IV}$$

if $a_0 = A_0$ and if each pair of terms, n being the order of the term, is reduced as follows:

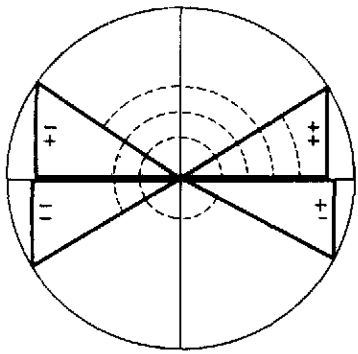
$$a_n \sin n\theta + b_n \cos n\theta = A_n \sin(n\theta + P_n).$$

The relations expressed in this equation are shown by trigonometry to be true when

$$A_n = \sqrt{a^2{}_n + b^2{}_n}, \text{ and } P_n = \tan^{-1}\frac{b_n}{a_n},$$

and these conditions are involved in the relations of the parts of a right triangle. If the base of a triangle is made equal to a and its altitude equal to b, Fig. 10, then the length of the hypothenuse is

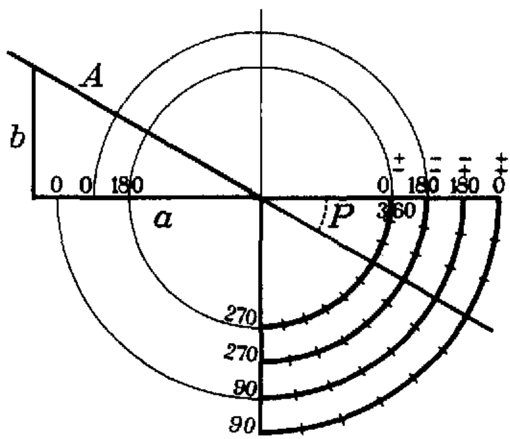
FIG. 11.



Phase angles in four quadrants.

equal to A, as expressed above, and the angle which the hypothenuse makes with the base is equal to P. It is necessary, therefore, to reduce each pair of coefficients given by the analyzer to deter-

FIG. 12.

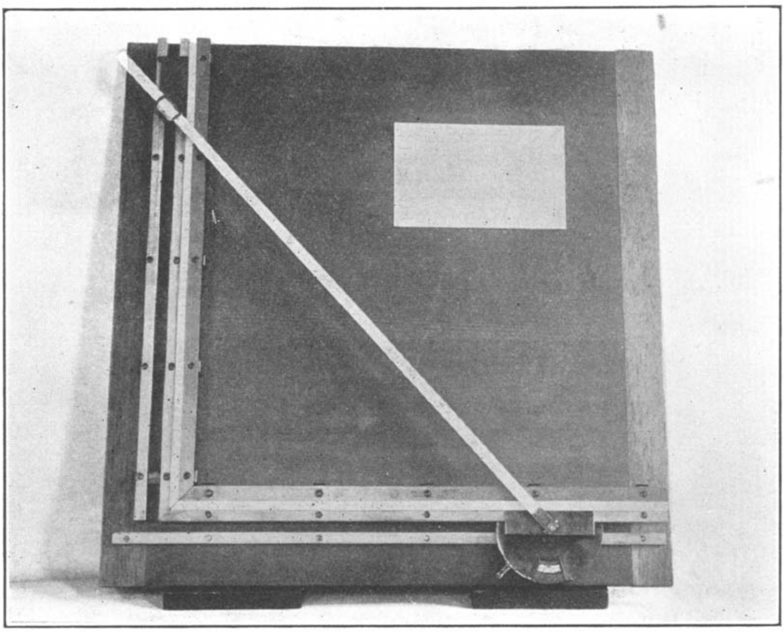


Scheme for measuring phases in one quadrant.

mine A_n and P_n as given by the last two equations. According to the values of n, these equations give the quantity A_1 the amplitude of the first component or fundamental, P_1 its phase, A_2 the amplitude of the second component (octave), P_2 its phase, and so on. An equivalent expression for the analysis involving cosines only, with phases, can be obtained in a similar manner.

The values of the amplitudes and phases may be obtained by numerical calculation as described in the next section, but it has been found possible to facilitate greatly the reduction by using a machine which has been devised and constructed in our own laboratory. This amplitude-and-phase calculator is essentially a machine for solving right triangles, with the addition of a special angle measurer.

FIG. 13.

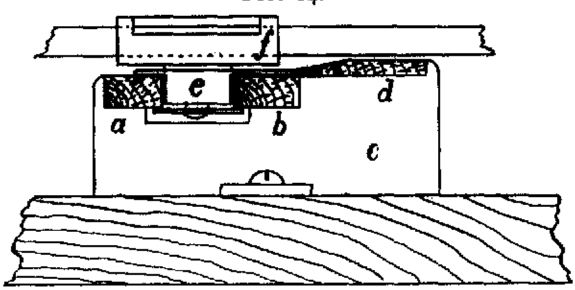


Machine for calculating amplitudes and phases in harmonic analysis.

The analyzer gives directly the numerical values of the coefficients, na_n and nb_n, of the several terms of the general Fourier equation, and each coefficient may have either the positive or the negative sign. For given numerical values of the coefficients there is but one value of the resultant amplitude, $A = \sqrt{a^2 + b^2}$, but the phase $P = \tan^{-1}\frac{b}{a}$ may have any one of four different values, according to the combination of signs, as shown in Fig. 11: For $+a$, $+b$, the angle will have a value between 0° and 90°; for $-a$, $+b$, its value is between 90° and 180°; for $-a, -b$, it lies between 180° and 270°; and for $+a, -b$, it is between 270° and 360°. A machine was constructed on the plan of this diagram which measured around a circle; later this was reduced to a quadrant machine with a special angle measurer having four graduations of a quadrant each numbered to correspond to the four possible combinations of algebraic signs, as shown in Fig. 12. Since a single graduated arc may be provided with two sets of numbers, one on each side, two graduations are sufficient, and the machine now in use is so constructed.

The amplitude-and-phase calculator, Fig. 13, the base of which is a drawing-board 24 inches square, consists of two grooves at right angles to each other, provided with linear scales 50 centimetres long, graduated to single millimetres; in the grooves are

FIG. 14.



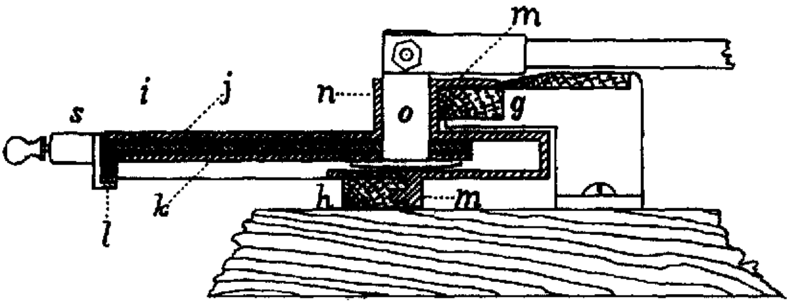
Cross-section of vertical groove.

slides carrying the graduated hypothenuse bar, which is 73 centimetres long. One end of the bar is attached to the angle measurer, while the other end slides through a support to which is attached an index for showing the length of the hypothenuse.

The vertical groove, which is about half an inch wide, is made by fastening two strips of hard wood, *a* and *b*, Fig. 14, to brass supports *c*; the bevel-edged graduated scale is shown at *d*, and the slider at *e*. A spring on the under side of the slider gives a smooth and firm movement. The hypothenuse bar slides through the cloth-lined casing *f*, which is pivoted over the centre of the groove.

The zero end of the hypothenuse bar is pivoted over the horizontal groove and is attached to the axis of the graduated quadrant, so that the quadrant is turned through the angle which gives the phase as indicated by the stationary index lines described below. The two guides, g and h, Fig. 15, which form the horizontal

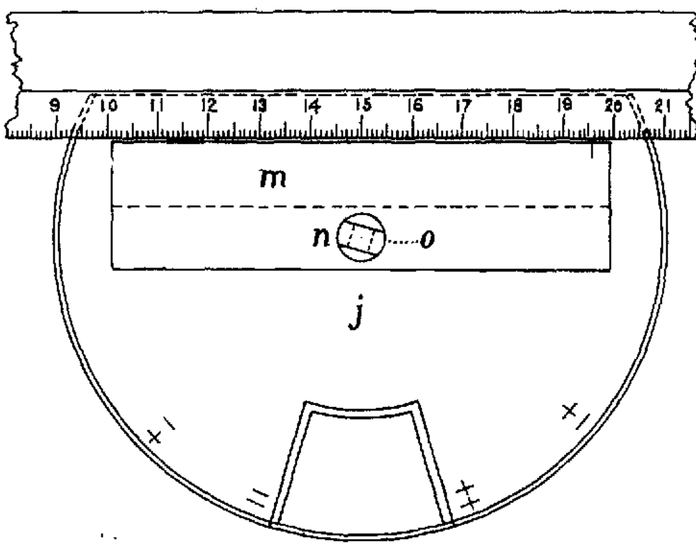
FIG. 15.



Cross-section of angle measurer.

groove, are at different levels to allow the angle measurer i to pass over the lower guide.

FIG. 16.

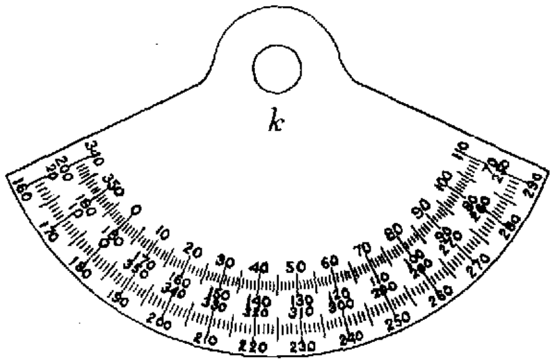


Top plan of angle measurer.

The novel feature of the instrument is the angle measurer, which consists of three parts, the outer casing j, Fig. 15, the graduated quadrant k, and the index cover l. The outer casing j, Figs. 15 and 16, carries on its upper and lower faces the guides m

which cause it to slide accurately in the groove. This casing also provides a bearing *n* through which passes the stud *o* of the graduated quadrant, so that the centre of the graduation is always

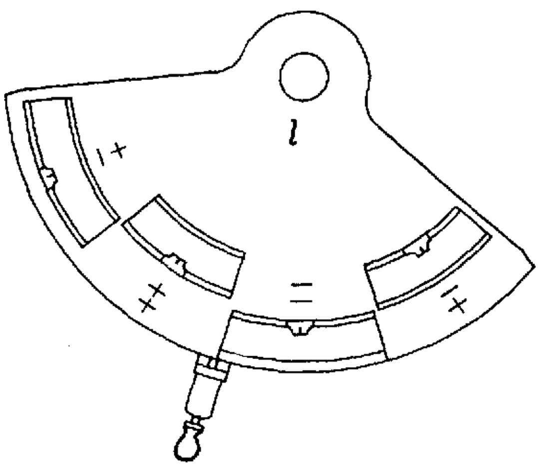
FIG. 17.



The graduated quadrant.

on the centre line of the groove. To the upper end of the stud is attached the zero end of the hypothenuse bar, the nominal zero being over the centre of the horizontal groove.

FIG. 18.



Index-cover for the quadrant.

The graduation of the quadrant, shown in Fig. 17, is in accordance with the scheme of Fig. 12; the graduation is extended a few degrees beyond the quadrant on each side to facilitate reading.

The adjustable index cover, Fig. 18, has four narrow apertures, which, when combined with the aperture in the outer casing as shown in Fig. 16, allow one graduated arc and any one, and only one, of the four sets of numbers to be seen at one time, according to the position of the cover. A spring stop s, Figs. 15 and 18, is attached to the cover; there are four positions for this catch, marked for the four possible combinations of the algebraic signs of the coefficients *a and b:* $\substack{+\\+}$, $\pm$, $\mp$, and $=$; when the cover is set, one reads the true phase angle without any reduction.

The graduations for the horizontal and vertical scales begin about 1½ inches from the junction of the grooves, thus bringing the indexes to convenient locations on the sliders. There are several geometrical methods of locating and verifying the several indexes for the linear and circular graduations; when the angle measurer is placed under the vertical groove and is so adjusted that the vertical slider can be moved from one end to the other of its groove with *no change* in the angle of the hypothenuse bar, then the horizontal slider must be in its zero position, and the index must indicate zero on the horizontal scale; the indexes for the hypothenuse bar and for the vertical slider must both show the *same* readings throughout the motion of the slider; the indexes on the cover of the quadrant must show 90° for the $\substack{+\\+}$ and for the $\pm$ positions, and 270° for the $=$ and for the $\mp$ positions. Other similar tests must be made, which will be evident when required.

The linear scales are 50 centimetres long; often the values of a and b given by the analysis are less than 50 millimetres, in which cases the reduction is carried out by using a centimetre on the machine for a millimetre of the analysis; by reading to single millimetres the result is obtained to tenths of a millimetre. If the values of a and b are between 50 millimetres and 100 millimetres, it is a simple procedure to use a half of the values for the settings, and to double the hypothenuse reading for the result. If the values of a and b are still larger, their actual values in millimetres are set off on the machine and the readings are made to tenths of a millimetre. The phases may be read to a tenth of a degree, though usually it is sufficient to read to the nearest half degree.

It is often desirable to state the relative phases of the various components when the phase of the first component is 0°. These relative phases are obtained by subtracting from the phase of each component n times the value of the phase of the first component (n being the order of the component). A special machine has been devised for transforming the calculated phases into the relative phases when $P_1 = 0°$, but the machine has never been constructed, since the harmonic synthesizer previously referred to gives the relative phases without extra work.[4] In the operation of verifying the analysis of a curve by synthesis, the amplitudes and phases of each component are set up on the synthesizer; it is then only necessary to turn the handle of the synthesizer until the phase dial of the first component indicates $0°$, when the direct readings of the phase dials of the other components are the relative phases desired.

CARDS AND CHARTS FOR RECORDS OF ANALYSES.

The comparison and filing of the results of the analyses of many curves is facilitated by using a card form for the record; five such forms have been tried, two of which will be described. Each card, 5 by 8 inches in size, contains the data for ten components; white cards are used for the components from 1 to 10, buff cards for components 11 to 20, and salmon-colored cards for components 21 to 30; blue cards are provided for averages, special data, etc.

When the reduction is to be made by numerical calculation, the form of card shown in Fig. 19 is convenient. The readings of the dials of the analyzer are recorded in the horizontal lines na_n and nb_n; the squares of these numbers are taken from printed tables, such as Crelle's "Rechentafeln," or Barlow's Tables;[6] the sums of the two square numbers for the several components are placed in the line $[nA_n]^2$, and the square roots of these sums, taken from the printed tables, are placed on the next lower line; the latter numbers divided by the respective values of n are the true amplitudes of the harmonic components and are recorded in the line labelled *Amplitude*. The logarithms of the numbers nb_n and na_n are placed in the appropriate lines, and the differences of the logarithms corresponding to each component are the logarithmic tangents of the phase angles, log tan p, and the corresponding angles (taken in the proper quadrants according to the algebraic signs as shown in Fig. 11) are the true phases of the components.

When the calculation of amplitudes and phases is to be made by machine, as described in the preceding article, the form of card shown in Fig. 20 is more convenient. The readings from the

FIG. 19.

CASE SCHOOL OF APPLIED SCIENCE — DEPARTMENT OF PHYSICS

ANALYSIS OF SOUND-WAVES

No. 1690 Source Organ Pipe—"Reedless Oboe." Date April 23, 1915

1-10 Tone C_3 Abs. N 260 Purpose analysis.

component, n	1	2	3	4	5	6	7	8	9	10
na_n	+ 22.6	+ 99.6	+ 101.1	+ 76.1	+ 44.5	+ 49.1	+ 44.8	+ 24.9	− 11.5	− 7.1
nb_n	+ 94.1	− 87.2	− 42.5	− 7.5	− 26.1	− 10.6	− 3.8	− 66.7	− 36.9	− 21.7
$\|na_n\|^2$	510.76	9920.16	10221.21	5791.21	1980.25	2410.81	2007.04	620.01	132.25	50.41
$[nb_n]^2$	8854.81	7603.84	1806.25	56.25	681.21	112.36	14.44	4448.89	1361.61	470.89
$\|nA_n\|^2$	9365.57	17524.00	12027.46	5847.46	2661.46	2523.17	2021.48	5068.90	1493.86	521.30
nA_n	96.8	132.4	109.7	76.5	51.6	50.2	45.0	71.2	38.7	22.8
log nb_n	1.9736	1.9405	1.6284	0.8751	1.4166	1.0253	0.5798	1.8241	1.5670	1.3365
log na_n	1.3541	1.9983	2.0048	1.8814	1.6484	1.6911	1.6513	1.3962	1.0607	0.8513
log tan p	0.6195	9.9422	9.6236	8.9937	9.7682	9.3342	8.9285	0.4279	0.5063	0.4842
p	76.°	319.°	337.°	354.°	330.°	348.°	355.°	290.°	252.°	252.°
Amplitude	96.8	66.2	36.6	19.1	10.3	8.4	6.4	8.9	4.3	2.3
Phase										
Intensity										

Remarks — Sum — Analyzed R. F. H. — Synthesized

Card for record of analysis, for numerical reduction.

FIG. 20.

CASE SCHOOL OF APPLIED SCIENCE — DEPARTMENT OF PHYSICS

ANALYSIS OF SOUND-WAVES

No. 1690 Source Organ Pipe—"Reedless Oboe." Date April 23, 1915

1-10 Tone C_3 Abs. N 260 Purpose analysis.

component, n	1	2	3	4	5	6	7	8	9	10
na_n	+ 22.6	+ 99.6	+ 101.1	+ 76.1	+ 44.5	+ 49.1	+ 44.8	+ 24.9	− 11.5	− 7.1
nb_n	+ 94.1	− 87.2	− 42.5	− 7.5	− 26.1	− 10.6	− 3.8	− 66.7	− 36.9	− 21.7
nA_n	96.8	132.3	109.4	76.3	51.6	50.2	45.0	71.1	38.7	22.8
k_n, ( )										
nA_nk_n										
$[nA_nk_n]^2$										
A_n	96.8	66.2	36.5	19.1	10.3	8.4	6.4	8.9	4.3	2.3
P_n	76.°	319.°	337.°	354.°	330.°	347.°	355.°	290.°	252.°	252.°
A_nk_n										
Amplitude, %										
Phase										
Intensity, %										

Remarks — Sum — Analyzed R. F. H. — Synthesized

Card for record of analysis, for reduction by machine.

analyzer are recorded as before in the lines labelled na_n and nb_n. The numbers for each component are set off on the amplitude-and-phase calculator, and the readings of the hypothenuse bar are recorded on the line nA_n, and the phases (the index cover being set to correspond to the signs of a and b in each case) are read directly and recorded on the line P_n. The values of nA_n being divided by the respective values of n give the true amplitudes which are recorded on the line A_n. This completes the analysis. The other data for which provision is made on the card illustrated do not belong to the process of harmonic analysis, but are connected with the correction and reduction of the analyses of sound waves by a particular method which is fully described elsewhere.[3]

The scheme to be adopted for the graphic presentation of the results of a harmonic analysis will depend upon the nature of the phenomena represented. In the investigation of sound waves, the results may be given in terms of the intensity or loudness of the separate components, instead of the amplitude, and the relative values may be shown by plotting on a logarithmic scale of frequencies corresponding to the tones of the musical scale. A description of special charts and plotting scales suitable for such representations has been given in the explanation of the analysis of sound waves just referred to.

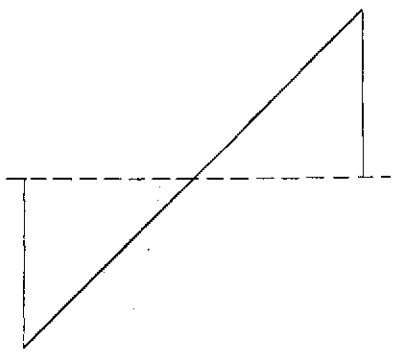
PRECISION ATTAINED IN HARMONIC ANALYSIS.

Harmonic analysis is usually applied to curves obtained by plotting observed numbers as coördinates or by graphically recording vibrations with the aid of some form of oscillograph. The forms of such curves can hardly be those of the resultants of a few simple harmonic components. Though the phenomenon represented by a curve is made up of a finite number of periodic components, yet the curve itself may be slightly distorted and will require an infinite series of components, many being of very small magnitude, for its exact reproduction. The methods of analysis by numerical reduction of the measures of the coördinates of a curve have the precision of ordinary graphical methods of investigation; however, many of these methods become much less precise, the higher the order of the component being evaluated.[7, 8]

The Henrici harmonic analyzer probably gives the values of the components with greater precision than it is practicable to obtain by any other method. Numerous tests have been made to determine its precision, and the results of several trials will be explained. Each curve was drawn to the standard size of one wave-length equal to 400 millimetres, and each was analyzed to thirty components. The numerical results given in the tables are the actual amplitudes in millimetres of the several indicated components, as given by the mechanical analyzer. For Table I the readings were taken to hundredths of a millimetre, while for Table II they were taken only to tenths of a millimetre, a degree of refinement quite sufficient for the tracing of curves drawn on paper. The phases of the components, which are obtained from the readings, are not given at this time, since they are determined with the same order of precision as are the amplitudes.

Table I gives several analyses of each of two test curves. The

FIG. 21.



A test curve, $y = x$.

analyses of the first set are for a straight line inclined at an angle of 45°, $y = x$, Fig. 21, which has an infinite series of components, represented by the Fourier equation (the wave-length being 400),

$$y = \frac{400}{\pi}(\sin x - \tfrac{1}{2}\sin 2x + \tfrac{1}{3}\sin 3x - \dots).$$

The computed values of the components are given in the column headed *calc;* the columns M – 1 and M – 2 give the values of the components obtained from two analyses made by D. C. M., the tracing being done with great care; the column S – 1 gives the values obtained by D. H. S., when the tracing was done in the usual manner of routine analysis. The average departure of the means of the two analyses by D. C. M. from the true values is 0.022 millimetre, which is one twenty-thousandth of the wave-length, or one ten-thousandth of the amplitude of the original curve, or one six-thousandth of the amplitude of the fundamental component. The maximum departure in the single analysis by D. H. S. is 0.25 millimetre in the value of the nineteenth component.

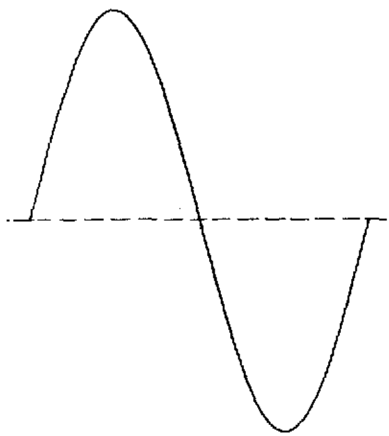
TABLE I

Analyses of Special Test Curves.

Comp.	Amplitudes of components in millimetres						
	Straight line, $y = x$				Sine curve, $y = \sin x$		
n	Calc.	M – 1	M – 2	S – 1	H – 1	H – 2	H – 3
1	127.32	127.30	127.30	126.90	250.03	250.27	250.03
2	63.66	63.55	63.60	63.35	0.97	0.94	0.71
3	42.44	42.47	42.50	42.37	0.69	0.56	0.72
4	31.83	31.85	31.82	31.75	0.27	0.28	0.28
5	25.46	25.50	25.46	25.46	0.19	0.22	0.22
6	21.22	21.17	21.17	21.24	0.31	0.29	0.20
7	18.19	18.16	18.14	18.14	0.20	0.12	0.10
8	15.91	15.94	15.97	15.90	0.08	0.11	0.05
9	14.15	14.12	14.15	14.14	0.20	0.03	0.11
10	12.73	12.75	12.70	12.73	0.05	0.11	0.04
11	11.57	11.60	11.56	11.65	0.18	0.10	0.13
12	10.61	10.64	10.67	10.50	0.18	0.10	0.09
13	9.79	9.81	9.80	9.73	0.04	0.08	0.04
14	9.09	9.09	9.17	9.00	0.12	0.14	0.09
15	8.49	8.50	8.49	8.45	0.14	0.07	0.11
16	7.96	7.91	7.97	8.02	0.12	0.05	0.05
17	7.49	7.51	7.40	7.44	0.13	0.04	0.08
18	7.07	7.10	7.02	6.93	0.04	0.04	0.03
19	6.70	6.71	6.70	6.95	0.19	0.02	0.11
20	6.37	6.43	6.41	6.21	0.06	0.09	0.10
21	6.06	6.04	6.06	6.09	0.15	0.06	0.10
22	5.79	5.81	5.78	5.81	0.06	0.14	0.12
23	5.54	5.53	5.46	5.48	0.10	0.06	0.09
24	5.30	5.23	5.23	5.36	0.09	0.08	0.18
25	5.09	5.10	5.07	5.17	0.11	0.08	0.04
26	4.90	4.91	4.90	4.99	0.03	0.03	0.10
27	4.72	4.78	4.73	4.70	0.07	0.07	0.13
28	4.55	4.61	4.52	4.40	0.03	0.01	0.11
29	4.39	4.41	4.39	4.36	0.11	0.10	0.10
30	4.24	4.25	4.26	4.12	0.04	0.10	0.18

The analyses of the second set given in Table I are for a simple sine curve, Fig. 22, $y = \sin x$, three successive analyses, all made by R. F. H., being shown in the columns marked H – 1, H – 2, and H – 3. This curve which was drawn by the harmonic synthesizer was presumed to have but one component, a fundamental of an amplitude of 250 millimetres. While all the higher components have very small values, components two and three have values which are not due to inaccuracies in reading. It is probable that these small readings correspond to real components existing in the curve; a very slight distortion might introduce an infinite series of very small components. The reality of the components is indicated by the fact that the successive analyses give values differing but little from each other. The average of all the readings of the three analyses for all components above the sixth

FIG. 22.



A test curve, $y = \sin x$.

is 0.09 millimetre, while the largest single value is 0.2 millimetre. The latter quantity is less than the width of the line which represents the curve. As already mentioned, the rolling spheres of the integrators for these higher components actually roll just as much in the amplitude direction as does the sphere for the first component, which has an amplitude of 250 millimetres; and the integrating cylinders, revolving around their spheres, at several different times in the process of analyzing the curve show large readings, which readings for a zero component are zeroized only at the end of the tracing; therefore it is not surprising that the final readings differ from zero by one- or two-tenths of a millimetre.

Table II gives three values of the analyses of each of two curves of unknown composition, the photographed curves of the sound waves from a clarinet, Fig. 23, and from an oboe, Fig. 24.

TABLE II

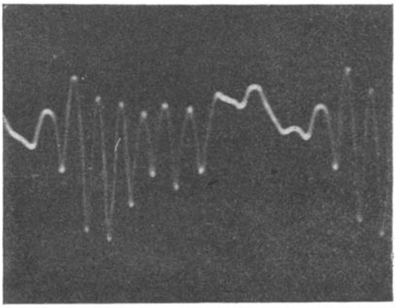
Analyses of Sound Waves

Comp.	Amplitudes of components in millimetres					
	Clarinet curve No. 85			Oboe curve No. 81		
n	M − 1	M − 2	S − 1	M − 1	M − 2	S − 1
1	30.3	30.3	30.7	90.5	90.5	90.9
2	7.2	7.2	6.9	77.5	77.1	76.8
3	20.8	20.6	20.8	37.8	37.8	37.4
4	2.2	2.2	2.0	31.2	31.2	31.9
5	1.1	1.3	1.5	42.6	43.1	42.7
6	5.8	5.8	6.0	12.9	12.4	12.6
7	7.1	7.2	6.6	46.3	46.9	46.6
8	6.9	7.2	7.3	34.6	34.5	34.5
9	17.0	16.8	17.2	14.9	15.5	15.2
10	11.8	11.6	11.9	6.2	5.9	6.1
11	23.8	24.1	23.2	0.7	0.7	0.7
12	34.9	35.3	35.2	3.3	3.7	3.5
13	20.0	20.3	20.8	5.0	5.1	5.0
14	8.9	8.8	9.6	0.6	1.1	0.8
15	2.2	2.2	2.7	4.3	4.5	4.4
16	1.2	1.0	0.4	3.9	4.0	3.9
17	1.4	1.4	0.9	2.9	2.9	2.9
18	0.8	1.0	0.9	1.5	1.4	1.5
19	2.4	2.4	2.3	0.8	0.3	0.6
20	2.3	2.1	2.3	2.5	2.6	2.5
21	3.7	3.5	3.7	2.4	2.4	2.5
22	3.5	3.2	4.1	0.4	0.4	0.4
23	3.1	2.8	2.0	1.7	1.5	1.6
24	7.3	7.0	6.3	1.6	1.1	1.3
25	5.3	5.4	4.5	0.8	0.5	0.7
26	2.8	2.5	2.7	0.3	0.3	0.3
27	0.7	0.8	0.9	0.5	0.5	0.5
28	0.2	0.2	0.2	0.9	0.9	0.9
29	1.0	1.1	1.0	0.8	0.7	0.8
30	0.3	0.3	0.3	0.5	0.2	0.3

These analyses were all made with only the ordinary care of routine work. The variation is of the order of 0.2 millimetre, which is less than the width of the line of the curve.

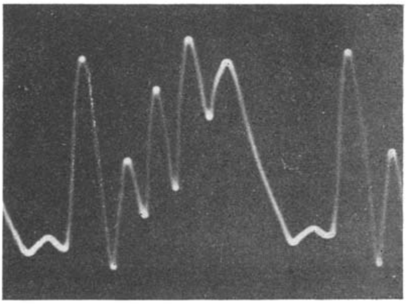
The conclusion is that the Henrici harmonic analyzer possesses an inherent accuracy much greater than can be taken advantage of in graphic work. The results are always accurate to a fraction of the width of the line representing the curve, and this precision

FIG. 23.



Photograph of the sound wave from a clarinet.

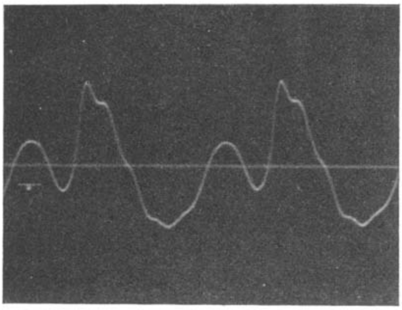
FIG. 24.



Photograph of the sound wave from an oboe.

is maintained uniformly for all the components, even to the thirtieth. If a curve could be traced exactly, the analyzer would undoubtedly give results of high precision, equalling one part in five thousand or more. Such accuracy is not approached by other

FIG. 25.



Photograph of the sound wave from an organ pipe.

instruments or methods of easy calculation with which the writer is acquainted.

Table III gives the results of three analyses, by different methods, of the same curve, the sound wave from an organ pipe,

Fig. 25: the first is purely mechanical (time required, thirteen minutes), the second is by the complete numerical reduction from thirty-six measured ordinates (time required, ten hours), and the third is the reduction from eighteen measured ordinates by means of a prepared schedule (time required for eight components, three hours).

TABLE III

Analyses of Organ-pipe Curve No. 1690 by Three Methods.

Amplitudes of components in millimetres			
Comp. n	Henrici analyzer	Steinmetz arithmetical	Grover schedule
1	96.8	97.0	96.6
2	66.2	66.0	66.5
3	36.5	36.7	37.2
4	19.1	19.0	19.9
5	10.3	10.4	13.6
6	8.4	8.9	9.7
7	6.4	7.2	8.4
8	8.9	9.0	12.0
9	4.3	4.1	
10	2.3	3.1	

TIME EFFICIENCY OF ANALYSIS BY MACHINE.

The analytical investigation of sound waves by the author has required the performance of hundreds of thousands of numerical operations, and an effort has been made to determine the most efficient methods of reduction. Some of the results of this study are as follows:

Addition.—The additions mostly required in the work referred to are of groups of ten (or twenty) numbers, each number having from one to four digits. A large number of such groups were added by two different computers, mentally and with each of three different machines. A typical result only will be given. R. F. H. added ten columns of ten numbers each mentally in 515 seconds, while the same additions with two machines required 834 seconds and 821 seconds respectively: L. W. S. required 510 seconds for the mental operations and 930 seconds for the machine additions. The fact that some of the machines recorded the numbers added is of no value in this work. The errors are few, and were no more and no less frequent in the mental operations than in the machine work.

Multiplication.—The multiplication of two numbers of from three to five digits each was performed with Crelle's "Rechentafeln" and with a Brunsviga calculating machine by R. F. H. With the tables 163 products were secured per hour, and with the machine 175 products per hour.

Division.—The only operation of division required in our work is the finding of the percentages of each of ten or twenty components, the sum of the components being given. This is carried out with a 20-inch slide rule, requiring three minutes to determine the percentages of twenty components.

Squares.—The squares of numbers of three digits are most easily taken from Crelle's "Rechentafeln," which, on two opposite pages, gives the squares of all numbers of from one to three figures.

Enlarging Curves.—The redrawing of a photographed curve to the standard size required for analysis, having a wave-length of 400 millimetres, requires, on the average, 4.1 minutes with the apparatus described.

Amplitudes and Phases.—The reduction of the coefficients obtained with the Henrici analyzer to obtain the amplitudes and phases of the components of a curve, by the numerical method represented by the card form shown in Fig. 19, using Crelle's "Rechentafeln" for the squares and square-roots, and four-place logarithm tables for the phases, requires twenty-eight minutes for ten components. The same reduction is made with the machine shown in Fig. 12, according to the card form of Fig. 20, in five minutes.

Synthesis.—The synthesis of a curve of ten components, forming a complete verification of the analysis, can be performed by the machine previously described, for ten components, in five minutes; for thirty components, in twelve minutes.

Analysis.—With the Henrici analyzer one tracing of a curve gives the data for five components; the time required for tracing a curve twice, including the changing of the wire, and for reading and recording the results, for ten components is ten minutes. Since an analysis is considered complete only when the data have been reduced to give the amplitudes and phases, the time for an analysis is then about fifteen minutes. As described in "The Science of Musical Sounds," pp. 122 and 136, the analysis and reduction of a certain curve by machine required thirteen minutes;

the analysis of the same curve by a method described by Steinmetz [2] required ten hours for ten components; the analysis with the help of prepared tabular forms as described by Grover [8] required three hours for the determination of eight components, the largest number for which a form was available.

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