

Connections between Tsallis' formalisms employing the standard linear average energy and ones employing the normalized q -average energy

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Abstract

Tsallis' thermostatics with the standard linear average energy is revisited by employing S_{2-q} , which is the Tsallis entropy with q replaced by $2 - q$. We explore the connections among the S_{2-q} approach and the other different versions of Tsallis formalisms. It is shown that the normalized q -average energy and the standard linear average energy are related to each other. The relations among the Lagrange multipliers of the different versions are revealed. The relevant Legendre transform structures concerning the Lagrange multipliers associated with the normalization of probability are studied. It is shown that the generalized Massieu potential associated with S_{2-q} and the linear average energy is related to one associated with the normalized Tsallis entropy and the normalized q -average energy.

Key words: Tsallis entropy, escort probability, Legendre transform, Lagrange multiplier, Massieu potential

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1 Introduction

Nowadays Tsallis' thermostatics [1,2,3] is considered as one of the generalizations of the standard thermostatics [4] based on the Tsallis entropy

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$S_q \equiv \sum_i (p_i - p_i^q)/(q-1)$, where p_i stands for a probability of i -th state and q is a real parameter. For the sake of simplicity the Boltzmann constant is set to unity throughout this paper. In the $q \rightarrow 1$ limit, S_q reduces to the standard Boltzmann-Gibbs (BG) entropy $S = -\sum_i p_i \ln(p_i)$.

During the last decade, there have been vast numbers of basic studies and applications [5,6,7], and the formalism of Tsallis' thermostatics has been evolved. Tsallis' entropy was originally introduced [1] with the standard average energy $U^{(1)} = \sum_i p_i E_i$ as the internal energy constraint in the MaxEnt procedure. (Here and hereafter, we use the superscript (i) with $i = 1, 2, 3$ in order to distinguish the three different average energies in Tsallis' thermostatics.)

The second version [2] was proposed by replacing the energy constraint $U^{(1)}$ with the unnormalized q -average energy $U_q^{(2)} = \sum_i p_i^q E_i$ in order to restore the thermodynamic stability for all values of q at the expense of the invariance of the probability distribution function (pdf) under the uniform translation of energy spectrum.

The role of energy constraints ($U^{(1)}$, $U_q^{(2)}$, and $U_q^{(3)}$) within Tsallis' thermostatics was precisely studied [3], and the third (current) version was proposed by replacing the definition of the energy constraint with the normalized q -average energy $U_q^{(3)} = \sum_i p_i^q E_i / \sum_j p_j^q$, which is also expressed as the average energy w.r.t. the *so-called* escort probability $P_i \equiv p_i^q / \sum_j p_j^q$ [8]. Consequently Tsallis' thermostatics has the two types of probabilities (p_i and P_i), which coincide with each other in standard thermostatics ($q = 1$).

The correspondence between the two types of probabilities leads to the so called " $q \leftrightarrow 1/q$ "-duality [3,9]. Raggio [10] had already shown that the equivalence between the first and third versions of Tsallis' formalism by utilizing the " $q \leftrightarrow 1/q$ "-duality, i.e., maximizing S_q under the energy constraint of $U_q^{(3)}$ is equivalent to maximizing $S_{1/q}$ under that of $U^{(1)}$.

Through the efforts [11,12] to generalize the zeroth law of thermodynamics within Tsallis' thermostatics, it was revealed that the inverse temperature is not simply the Lagrange multiplier associated with the energy constraint. For this reason, the Tsallis variational problem and the Legendre transform structures have been extensively studied by e.g., *so-called optimal Lagrange multiplier* (OLM) formalism [13,14,15].

In the literature, some derivatives of Tsallis' entropy have been proposed. One of them is the normalized Tsallis entropy $S_q^N \equiv S_q / \sum_j p_j^q$ [16,17], and another is the escort Tsallis entropy S_q^E [18], which emerges from S_q by expressing p_i in terms of the escort probability P_i and then renaming P_i to p_i .

Since Tsallis' thermostatics has been still under development, there remain some fundamental questions to be answered. One of them is the choice of the energy average which is used in the MaxEnt procedure of Tsallis' thermostatics. Until now there are two main different opinions: one is to employ the standard average energy $U^{(1)}$; the other is to employ the normalized q -average energy $U_q^{(3)}$.

Abe and Bagci [19] have shown that the generalized relative entropy associated with $U_q^{(3)}$ has nice properties, which are superior to those associated with $U^{(1)}$. Di Sisto *et al.* [21] and Bashkirov [22] have independently shown that the modified treatment of the variational problem for the first version of Tsallis' thermostatics leads to the pdf which is analogous to the pdf of the third version.

There exists another duality which is called " $q \leftrightarrow 2 - q$ "-duality [20] in the q -deformed functions. Baldovin and Robledo [23] have observed that the maximization of S_{2-q} with the standard constraints $\sum_i p_i E_i = U^{(1)}$ and $\sum_i p_i = 1$ leads to the q -exponential pdf. They suggested, based on the " $q \leftrightarrow 2 - q$ "-duality, that the mutual S_q and S_{2-q} elegantly generalize the standard BG entropy, and pointed out that some features are equally expressed by both S_q and S_{2-q} , but some others appears only via the use of either S_q or S_{2-q} . Finally, in Ref. [24] Naudts has analyzed both dualities of Tsallis' thermostatics based on his generalized thermostatics [20,25], and proposed to replace S_q with S_{2-q} instead of introducing the normalized q -average energy $U_q^{(3)}$.

The purpose of this paper is twofold. Firstly, in order to study Naudts' proposition, and to understand a deeper relation between the formalisms employing $U^{(1)}$ and the formalisms employing $U_q^{(3)}$, we revisit Tsallis' thermostatics with $U^{(1)}$ by using S_{2-q} instead of S_q . The relationships among the Lagrange multipliers for the different versions are obtained.

Secondly, we study a generalization of Massieu's potential associated with either S_{2-q} and $U^{(1)}$ or S_q^N and $U_q^{(3)}$. The basic thermodynamic relations for these generalized potentials are discussed.

The plane of the paper is the following. In the next section we begin with the pdf obeying Tsallis' q -exponential, which maximizes S_{2-q} under the constraint of the linear energy $U^{(1)}$. It is shown that the original pdf is equivalent to that of the (modified) first version with q replaced by $2 - q$. In section 3, the escort pdf is introduced. By utilizing the averages w.r.t. the escort pdf, the original pdf is also shown to be equivalent to that of the third version and to that of the version which uses the normalized Tsallis entropy [16,17]. It is found that the standard linear average energy $U^{(1)}$ and the normalized q -average energy $U_q^{(3)}$ are related to each other as well as the corresponding Lagrange multipliers. In section 4, following the method used by Naudts to obtain the generalized free-energy [24,25], we obtain a generalization of the Massieu's potential [4] associated to the different formalisms. The final section is devoted to our conclusion.

2 S_{2-q} approach

Following the route developed by Naudts [25] and Abe [26] independently, we can obtain the generalized entropy optimized by a given pdf. We here take a

similar approach [27,28] in order to obtain the generalized entropy which is maximized by the Tsallis q -exponential pdf under the constraint of the linear average energy $U^{(1)}$. Let us begin with the following q -exponential pdf

$$p_i = \alpha \exp_q \left(-\beta^{(1)} E_i - \gamma^{(1)} \right), \quad (1)$$

where $\alpha, \beta^{(1)}$ and $\gamma^{(1)}$ are real parameters to be determined later. The Tsallis q -deformed exponential function $\exp_q(x)$ [1,5,6] is defined by

$$\exp_q(x) \equiv (1 + (1 - q)x)^{\frac{1}{1-q}}, \quad (2)$$

where q is a real parameter which characterizes the deformation. The inverse function of $\exp_q(x)$ is the q -logarithmic function defined by

$$\ln_q(x) \equiv \frac{x^{1-q} - 1}{1 - q}. \quad (3)$$

We choose the parameter α so that

$$\frac{d}{dx} \left\{ x \ln_q(x) \right\} = \ln_q \left(\frac{x}{\alpha} \right). \quad (4)$$

Then the parameter q is related with α by

$$\frac{1}{\alpha} = (2 - q)^{\frac{1}{1-q}}. \quad (5)$$

In addition, from Eq. (1), we readily see that

$$\ln_q \left(\frac{p_i}{\alpha} \right) = -\beta^{(1)} E_i - \gamma^{(1)}. \quad (6)$$

This relation and the property of Eq. (4) guarantee that the pdf given in Eq. (1) is the solution of the following MaxEnt procedure [25,26,27,28]

$$\frac{\delta}{\delta p_i} \left(S_{2-q} - \beta^{(1)} \sum_j p_j E_j - \gamma^{(1)} \sum_j p_j \right) = 0, \quad (7)$$

where

$$S_{2-q} = \frac{\sum_i (p_i^{2-q} - p_i)}{q - 1} = - \sum_i p_i \ln_q(p_i), \quad (8)$$

is the Tsallis entropy with q replaced by $2 - q$. We now see that the parameter $\beta^{(1)}$ is the Lagrange multiplier associated with the linear average energy

$$U^{(1)} \equiv \sum_i p_i E_i, \quad (9)$$

and $\gamma^{(1)}$ is that associated with the normalization of the pdf, $\sum_i p_i = 1$.

From Eqs. (4), (6) and (8), we find

$$\begin{aligned}\frac{dS_{2-q}}{d\beta^{(1)}} &= - \sum_i \frac{d}{dp_i} (p_i \ln_q(p_i)) \frac{dp_i}{d\beta^{(1)}} = - \sum_i \ln_q\left(\frac{p_i}{\alpha}\right) \frac{dp_i}{d\beta^{(1)}} \\ &= \sum_i \left(\beta^{(1)} E_i + \gamma^{(1)}\right) \frac{dp_i}{d\beta^{(1)}} = \beta^{(1)} \frac{dU^{(1)}}{d\beta^{(1)}},\end{aligned}\quad (10)$$

under the 'no work' condition, i.e., $dE_i = 0, \forall E_i$. In the last step we used $\sum_i (dp_i/d\beta^{(1)}) = 0$, which follows from the normalization of p_i . We then obtain the thermodynamic Legendre relation [29,30]

$$\frac{dS_{2-q}}{dU^{(1)}} = \beta^{(1)}. \quad (11)$$

At this point, let us confirm Eq. (1) is equivalent to the pdf of the first version [1,3,21,22] in Tsallis' thermostatics. By taking the average of Eq. (6) w.r.t. p_i , we have

$$\left\langle \ln_q\left(\frac{p_i}{\alpha}\right) \right\rangle = -\beta^{(1)} U^{(1)} - \gamma^{(1)}, \quad (12)$$

and the l.h.s. is further expressed as

$$\left\langle \ln_q\left(\frac{p_i}{\alpha}\right) \right\rangle = 1 - (2-q)S_{2-q}. \quad (13)$$

By combining the above two equations, we obtain

$$\gamma^{(1)} = (2-q)S_{2-q} - 1 - \beta^{(1)} U^{(1)}. \quad (14)$$

Substituting this $\gamma^{(1)}$ into Eq. (1) and utilizing the identity

$$\exp_q(x+y) = \exp_q(x) \cdot \exp_q\left(\frac{y}{1+(1-q)x}\right), \quad (15)$$

it follows

$$\begin{aligned}p_i &= \alpha \exp_q\left(1 - (2-q)S_{2-q}\right) \cdot \exp_q\left(\frac{-\beta^{(1)}(E_i - U^{(1)})}{1 + (1-q)[1 - (2-q)S_{2-q}]}\right) \\ &= \frac{1}{\bar{Z}_{2-q}^{(1)}} \cdot \exp_q\left(\frac{-\beta^{(1)}(E_i - U^{(1)})}{(2-q) \sum_i p_i^{2-q}}\right),\end{aligned}\quad (16)$$

where we introduced the generalized partition function as

$$\bar{Z}_{2-q}^{(1)} \equiv \left[\alpha \exp_q(1 - (2-q)S_{2-q})\right]^{-1} = \left(\sum_i p_i^{2-q}\right)^{\frac{1}{q-1}}. \quad (17)$$

From the normalization of p_i , it follows that

$$\bar{Z}_{2-q}^{(1)} = \sum_i \exp_q\left(\frac{-\beta^{(1)}(E_i - U^{(1)})}{(2-q) \sum_j p_j^{2-q}}\right). \quad (18)$$

Note that if we replace $2 - q$ with q in Eq. (16), it becomes the pdf which Di Sisto *et al.* [21] and Bashkirov [22] have independently obtained by modifying the treatment of the first version. We remark that the pdf of the original first version [1] is written in

$$p_i = \frac{\exp_q(-\beta^* E_i)}{\sum_i \exp_q(-\beta^* E_i)}, \quad (19)$$

where β^* is not the Lagrange multiplier associated with the energy constraint. From Eq. (1) and utilizing the identity (15), we readily obtain the relation among β^* , $\beta^{(1)}$ and $\gamma^{(1)}$ as

$$\beta^* = \frac{\beta^{(1)}}{1 - (1 - q)\gamma^{(1)}}. \quad (20)$$

3 Connections with the other versions of Tsallis' thermostatics

Let us now introduce the escort probability P_i w.r.t. p_i in the sense of Naudts' generalized thermostatics [24,25]. For the q -exponential pdf, the P_i can be written by

$$P_i \equiv \frac{1}{Z_q} \cdot \frac{d \exp_q(x)}{dx} \Big|_{x=\ln_q(p_i)} = \frac{p_i^q}{Z_q} \quad (21)$$

where Z_q is the normalization factor and we used $d \exp_q(x)/dx = [\exp_q(x)]^q$. From the normalization of P_i , we have $Z_q = \sum_j p_j^q$. We see then that the escort probability of Eq. (21) is nothing but the *so-called* q -escort probability [3,8]

$$P_i = \frac{p_i^q}{\sum_j p_j^q}, \quad (22)$$

and that the average energy w.r.t. P_i is the normalized q -average energy [3]

$$\langle E_i \rangle_P \equiv \sum_i E_i P_i = U_q^{(3)}, \quad (23)$$

where $\langle \cdots \rangle_P$ stands for the average value w.r.t. the escort probability P_i .

Next let us confirm the original pdf of Eq. (1) is also equivalent to that of the third version [3] in Tsallis' thermostatics. The key point is that $\gamma^{(1)}$ is expressed not only in terms of $U^{(1)}$ as Eq. (14) but also in terms of $U_q^{(3)}$. By taking the average of the both sides of Eq. (6) w.r.t. P_i , we obtain

$$\left\langle \ln_q \left(\frac{p_i}{\alpha} \right) \right\rangle_P = -\beta^{(1)} U_q^{(3)} - \gamma^{(1)}, \quad (24)$$

and the l.h.s. is further expressed as

$$\left\langle \ln_q \left(\frac{p_i}{\alpha} \right) \right\rangle_P = 1 - (2 - q) S_q^N, \quad (25)$$

where

$$S_q^N \equiv \frac{S_q}{\sum_j p_j^q} = \frac{1 - \frac{\sum_i p_i}{\sum_j p_j^q}}{1 - q}, \quad (26)$$

is the normalized Tsallis entropy [16,17]. We then have

$$\gamma^{(1)} = (2 - q)S_q^N - 1 - \beta^{(1)}U_q^{(3)}. \quad (27)$$

Substituting this $\gamma^{(1)}$ into Eq. (1) and utilizing Eq. (15), we obtain

$$\begin{aligned} p_i &= \alpha \exp_q \left(1 - (2 - q)S_q^N \right) \cdot \exp_q \left(\frac{-\beta^{(1)}(E_i - U_q^{(3)})}{1 + (1 - q)\{1 - (2 - q)S_q^N\}} \right) \\ &= \frac{1}{\bar{Z}_q^{(3)}} \exp_q \left(\frac{-\beta^{(1)} \sum_k p_k^q}{2 - q} (E_i - U_q^{(3)}) \right), \end{aligned} \quad (28)$$

where $\bar{Z}_q^{(3)}$ is the q -generalized partition function, and from the normalization of p_i , it can be written as

$$\bar{Z}_q^{(3)} = \sum_i \exp_q \left(\frac{-\beta^{(1)} \sum_j p_j^q}{2 - q} (E_i - U_q^{(3)}) \right). \quad (29)$$

From Eq. (28) we readily confirm the following known relation [3]

$$\bar{Z}_q^{(3)} = \left[\alpha \exp_q \left(1 - (2 - q)S_q^N \right) \right]^{-1} = \left(\sum_i p_i^q \right)^{\frac{1}{1-q}}. \quad (30)$$

By comparing Eq. (17) with Eq. (30), it follows that

$$\bar{Z}_q^{(1)} = \bar{Z}_q^{(3)}. \quad (31)$$

This result is a consequence of “ $q \leftrightarrow 2 - q$ ”-duality.

By the way, the pdf of the third version [3] is written by

$$p_i = \frac{1}{\bar{Z}_q^{(3)}} \exp_q \left(-\frac{\beta^{(3)}}{\sum_j p_j^q} (E_i - U_q^{(3)}) \right), \quad (32)$$

which can be obtained as the solution of the following MaxEnt problem

$$\frac{\delta}{\delta p_i} \left(S_q - \beta^{(3)} \frac{\sum_j p_j^q E_j}{\sum_k p_k^q} - \gamma^{(3)} \sum_j p_j \right) = 0, \quad (33)$$

where $\beta^{(3)}$ and $\gamma^{(3)}$ are the Lagrange multipliers associated with the normalized q -average energy $U_q^{(3)}$ and the normalization of the pdf, respectively. From Eq. (33) it follows [29,30]

$$\beta^{(3)} = \frac{dS_q}{dU_q^{(3)}}. \quad (34)$$

By comparing Eq. (28) with Eq. (32) we find that the both pdfs are equivalent each other, and that $\beta^{(1)}$ and $\beta^{(3)}$ are related by

$$\beta^{(3)} = \frac{(\sum_j p_j^q)^2}{2-q} \beta^{(1)}. \quad (35)$$

From Eq. (33) we have

$$\frac{qp_i^{q-1} - 1}{1-q} - \beta^{(3)} \frac{qp_i^{q-1}}{\sum_j p_j^q} (E_i - U_q^{(3)}) - \gamma^{(3)} = 0. \quad (36)$$

Multiplying the both sides of this equation by p_i and taking summation, we obtain

$$\gamma^{(3)} = qS_q - 1. \quad (37)$$

We next consider the relations with the pdf for the normalized Tsallis entropy given in Eq. (26) [16,17]

$$p_i = \frac{1}{\bar{Z}_q^{N(3)}} \exp_q \left(-\beta^{N(3)} \cdot \sum_j p_j^q \cdot (E_i - U_q^{(3)}) \right), \quad (38)$$

where

$$\bar{Z}_q^{N(3)} = \left(\sum_j p_j^q \right)^{\frac{1}{1-q}}, \quad (39)$$

and from the normalization of p_i , it follows that

$$\bar{Z}_q^{N(3)} = \sum_i \exp_q \left(-\beta^{N(3)} \cdot \sum_j p_j^q \cdot (E_i - U_q^{(3)}) \right). \quad (40)$$

p_i can be obtained as the solution of the following MaxEnt problem

$$\frac{\delta}{\delta p_i} \left(S_q^N - \beta^{N(3)} \cdot \frac{\sum_j p_j^q E_j}{\sum_k p_k^q} - \gamma^{N(3)} \cdot \sum_j p_j \right) = 0, \quad (41)$$

where $\beta^{N(3)}$ and $\gamma^{N(3)}$ are the Lagrange multipliers associated with the normalized q -average energy $U_q^{(3)}$ and the normalization of the pdf, respectively. From Eq. (41) we have [29,30]

$$\beta^{N(3)} = \frac{dS_q^N}{dU_q^{(3)}}, \quad (42)$$

and by comparing Eq. (28) with Eq. (38) we find that the both pdfs are equivalent each other, if

$$\beta^{N(3)} = \frac{\beta^{(1)}}{2-q}. \quad (43)$$

From Eq. (41) it follows

$$\frac{1}{(\sum_j p_j^q)^2} \cdot \left(\frac{qp_i^{q-1} - \sum_k p_k^q}{1-q} \right) - \beta^{N(3)} \cdot \frac{qp_i^{q-1}}{\sum_j p_j^q} (E_i - U_q^{(3)}) - \gamma^{N(3)} = 0. \quad (44)$$

Multiplying the both sides of this equation by p_i and taking summation, we obtain

$$\gamma^{N(3)} = (1-q)S_q^N - 1 = -\frac{1}{\sum_i p_i^q}. \quad (45)$$

Until here, we have considered the pdfs of the three different versions in which the combinations of the entropies and average energies are: i) S_{2-q} and $U^{(1)}$; ii) S_q and $U_q^{(3)}$; and iii) S_q^N and $U_q^{(3)}$. It is thus natural to consider the pdf for the combination of the normalized Tsallis entropy with q replaced by $2-q$,

$$S_{2-q}^N \equiv \frac{S_{2-q}}{\sum_j p_j^{2-q}}, \quad (46)$$

and $U^{(1)}$. The associated pdf can be written as

$$p_i = \frac{1}{\bar{Z}_{2-q}^{N(1)}} \exp_q \left(-\frac{\beta^{N(1)}}{2-q} \sum_i p_i^{2-q} (E_i - U^{(1)}) \right), \quad (47)$$

which can be obtained as the solution of the following MaxEnt problem

$$\frac{\delta}{\delta p_i} \left(S_{2-q}^N - \beta^{N(1)} \cdot \sum_j E_j p_j - \gamma^{N(1)} \cdot \sum_j p_j \right) = 0, \quad (48)$$

where $\beta^{N(1)}$ and $\gamma^{N(1)}$ are the Lagrange multipliers associated with the linear average energy $U^{(1)}$ and the normalization of the pdf, respectively. By comparing this equation with Eq. (16), we find that the both pdf are equivalent each other, if

$$\beta^{N(1)} = \frac{\beta^{(1)}}{(\sum_i p_i^{2-q})^2}, \quad (49)$$

and

$$\bar{Z}_{2-q}^{(1)} = \bar{Z}_{2-q}^{N(1)}. \quad (50)$$

From Eq. (48) it follows

$$-\frac{1}{(q-1)\sum_j p_j^{2-q}} \left(1 - (2-q) \frac{\sum_k p_k}{\sum_\ell p_\ell^{2-q}} p_i^{1-q} \right) - \beta^{N(1)} E_i - \gamma^{N(1)} = 0. \quad (51)$$

Multiplying both sides of this equation by p_i and taking the summation, we obtain

$$\gamma^{N(1)} = (q-1) S_{2-q}^N - 1 = -\frac{1}{\sum_i p_i^{2-q}}. \quad (52)$$

Summing up, the pdfs of the different versions of Tsallis' thermostatics are related one another. Note that $U_q^{(3)}$ is automatically introduced as the average energy w.r.t. the escort probability. This is a consequence of the " $q \leftrightarrow 1/q$ "-duality. In other words, $U_q^{(3)}$ is accompanied with $U^{(1)}$. From Eqs. (14) and (27), we see that they are related by

$$S_{2-q} - \left(\frac{\beta^{(1)}}{2-q} \right) U^{(1)} = S_q^N - \beta^{N(3)} U_q^{(3)}. \quad (53)$$

By taking the derivative of both sides of this equation w.r.t. $\beta^{(1)}$, we obtain

$$(1-q) \beta^{(1)} \frac{dU^{(1)}}{d\beta^{(1)}} = U^{(1)} - U_q^{(3)}. \quad (54)$$

In our opinion, it is thus meaningless asking which of the two average energies is correct. They cannot exclude each other.

4 Generalized Massieu potential and associated Legendre structures

Let us first remind you of some basic relations concerning with Massieu's potential [4] in the standard BG thermostatics. Massieu's potential Φ is defined as the Legendre transform of the standard BG entropy $S(U)$ which is a function of internal energy U ,

$$\Phi(\beta) \equiv S - \beta U. \quad (55)$$

Massieu's potential is thus a function of the Lagrange multiplier β associated with energy constraint, whereas Helmholtz free energy F is defined as the Legendre transform of the internal energy $U(S)$ which is a function of S ,

$$F(T) \equiv U - TS, \quad (56)$$

and consequently F is a function of temperature T . Since the temperature T and the Lagrange multiplier β are related by $T = 1/\beta$ in the standard BG thermostatics, Massieu's potential is related with free energy as

$$\Phi = -\frac{F}{T}. \quad (57)$$

By differentiating the both side of Eq. (55) and utilizing the relation

$$\frac{dS(\beta)}{d\beta} = \beta \frac{dU(\beta)}{d\beta}, \quad (58)$$

we readily obtain

$$\frac{d\Phi}{d\beta} = -U. \quad (59)$$

Eqs. (55), (58), and (59) are basic relations concerning with Massieu's potential in the standard BG thermostatics.

We next review the relation between Massieu's potential and the Lagrange multiplier γ associated with the normalization of probability in the standard MaxEnt procedure,

$$\frac{\delta}{\delta p_i} \left(S - \beta \sum_j p_j E_j - \gamma \sum_j p_j \right) = 0. \quad (60)$$

Its solution is the well-known BG pdf

$$p_i^{\text{BG}} = \exp(-\beta E_i - \gamma - 1) = \frac{1}{Z} \exp(-\beta E_i), \quad (61)$$

where the partition function $Z = \exp(1 + \gamma)$ is introduced. By substituting Eq. (61) into the BG entropy S , we obtain

$$S = - \sum_i p_i^{\text{BG}} \ln p_i^{\text{BG}} = \sum_i p_i^{\text{BG}} (\beta E_i + \gamma + 1) = \beta U + \gamma + 1. \quad (62)$$

Comparing this with Eq. (55), we have the known relations

$$\Phi = 1 + \gamma = \ln Z. \quad (63)$$

We thus see that the Lagrange multiplier γ is related to Massieu's potential Φ , and that Eq. (59) is equivalent to the well-known relation $d \ln Z / d\beta = -U$.

Let us now focus on the generalizations of these basic relations within Tsallis' thermostatics. Naudts [24] has already shown that there exists a generalized (Helmholtz) free energy associated with the average energy w.r.t. the escort probability in his generalized thermostatics, which contains Tsallis' thermostatics as a special case. For a generalized entropy based on a deformed logarithmic function, the Lagrange multiplier associated with an energy constraint is generally not equivalent to the inverse temperature $1/T$. In our formalism it is thus appropriate to introduce a generalized Massieu potential, which is a function of the Lagrange multiplier $\beta^{(1)}$ or $\beta^{\text{N}(3)}$, instead of the generalized free energy, which is a function of temperature. Following the same

method used by Naudts to derive the generalized free energy [24], let us derive the generalized Massieu potential.

By utilizing Eqs. (5) and (15), we rewrite the original pdf of Eq. (1) as

$$\begin{aligned} p_i &= \exp_q \left(\ln_q(\alpha) \right) \cdot \exp_q \left(-\beta^{(1)} E_i - \gamma^{(1)} \right) \\ &= \exp_q \left(\frac{-1}{2-q} \left(\beta^{(1)} E_i + 1 + \gamma^{(1)} \right) \right) = \exp_q \left(-\beta^{N(3)} E_i - \Phi_q^{N(3)} \right), \end{aligned} \quad (64)$$

where in the last step we introduced the quantity

$$\Phi_q^{N(3)} \equiv \frac{1 + \gamma^{(1)}}{2 - q}, \quad (65)$$

which reduces to Eq. (63) in the limit of $q \rightarrow 1$.

By differentiating the both sides of $\sum_i p_i = 1$ w.r.t. $\beta^{N(3)}$, and utilizing Eqs. (21), (23) and (64), we have

$$\begin{aligned} 0 &= \sum_i \frac{dp_i}{d\beta^{N(3)}} = - \sum_i \left(E_i + \frac{d\Phi_q^{N(3)}}{d\beta^{N(3)}} \right) \frac{d \exp_q(x)}{dx} \Big|_{x = -\beta^{N(3)} E_i - \Phi_q^{N(3)}} \\ &= - \sum_i \left(E_i + \frac{d\Phi_q^{N(3)}}{d\beta^{N(3)}} \right) Z_q P_i = -Z_q \left(U_q^{(3)} + \frac{d\Phi_q^{N(3)}}{d\beta^{N(3)}} \right). \end{aligned} \quad (66)$$

We then obtain

$$\frac{d\Phi_q^{N(3)}}{d\beta^{N(3)}} = -U_q^{(3)}. \quad (67)$$

By comparing this relation with Eq. (59), we find that $\Phi_q^{N(3)}$ is the generalized Massieu potential associated with the escort average energy $U_q^{(3)}$. In the limit of $q \rightarrow 1$, Eq. (67) of course reduces to Eq. (59).

Now a natural question arises at this point: what are the generalizations of Eqs. (55) and (58)? In other words, what is a generalized entropy whose Legendre transform is $\Phi_q^{N(3)}$? From Eq. (27) and the definition of $\Phi_q^{N(3)}$ given by Eq. (65), we obtain

$$\Phi_q^{N(3)} = S_q^N - \beta^{N(3)} \cdot U_q^{(3)}, \quad (68)$$

which shows that the generalized Massieu potential $\Phi_q^{N(3)}(\beta^{N(3)})$ is the Legendre transform of $S_q^N(U_q^{(3)})$. By differentiating the both sides of Eq. (68) w.r.t. $\beta^{N(3)}$ and utilizing Eq. (67), we obtain

$$\frac{dS_q^N}{d\beta^{N(3)}} = \beta^{N(3)} \frac{dU_q^{(3)}}{d\beta^{N(3)}}. \quad (69)$$

Eqs. (68) and (69) lead us to consider S_q^N as the generalized entropy associated with the Massieu potential $\Phi_q^{N(3)}$.

Now we'd like to point out some observations. Firstly, it is worth noting that $\Phi_q^{N(3)}$ is not associated with $U^{(1)}$ but associated with $U_q^{(3)}$. In fact, from Eq. (14), $\Phi_q^{N(3)}$ can be also expressed as

$$\Phi_q^{N(3)} = S_{2-q} - \frac{\beta^{(1)}}{2-q} \cdot U^{(1)}. \quad (70)$$

By taking the derivative of the both sides of this equation w.r.t. $\beta^{(1)}$ and utilizing Eq. (10), we have

$$\frac{d\Phi_q^{N(3)}}{d\beta^{(1)}} = -\frac{U^{(1)}}{2-q} + \left(\frac{1-q}{2-q}\right) \beta^{(1)} \frac{dU^{(1)}}{d\beta^{(1)}}. \quad (71)$$

This is not a form invariant generalization of Eq. (59), whereas Eq. (67) is a natural generalization.

However, we can construct an appropriate generalization in order to overcome this difficulty by utilizing the fact that $U^{(1)}$ and $U_q^{(3)}$ are related to each other as shown in Eq. (53) or Eq. (54). Let us define the Massieu potential associated with $U^{(1)}$ as

$$\Phi_{2-q} \equiv \Phi_q^{N(3)} - \left(\frac{1-q}{2-q}\right) \beta^{(1)} \cdot U^{(1)} = \frac{1+\gamma^{(1)}}{2-q} - \left(\frac{1-q}{2-q}\right) \beta^{(1)} \cdot U^{(1)}. \quad (72)$$

Substituting Eq. (14) or Eq. (70) into this equation leads to

$$\Phi_{2-q} = S_{2-q} - \beta^{(1)} \cdot U^{(1)}, \quad (73)$$

which shows that the generalized Massieu potential $\Phi_{2-q}(\beta^{(1)})$ is the Legendre transform of $S_{2-q}(U^{(1)})$. Furthermore from Eqs. (71) and (72), we obtain

$$\frac{d\Phi_{2-q}}{d\beta^{(1)}} = -U^{(1)}, \quad (74)$$

which can be also derived by utilizing Eq. (54) in the same way as we have derived Eq. (67).

Secondly, in the literature, the generalized free energy $F_q^{(3)}$ [3] associated with the normalized q -average energy $U_q^{(3)}$ is known as

$$F_q^{(3)} \equiv U_q^{(3)} - \frac{1}{\beta^{(3)}} \cdot S_q^{(3)}. \quad (75)$$

The corresponding Massieu potential can be written as

$$\Phi_q^{(3)} \equiv -\beta^{(3)} F_q^{(3)} = S_q^{(3)} - \beta^{(3)} \cdot U_q^{(3)}. \quad (76)$$

For the version which utilizes the standard linear energy average $U^{(1)}$ as the constraint in the MaxEnt procedure, the Lagrange multiplier $\gamma^{(1)}$, which is associated with the normalization of the pdf, is related with the generalized Massieu potential in the same way of the standard BG thermostatics.

Note that $\Phi_q^{N(3)}$ has no relation to $\gamma^{N(3)}$. In fact both the generalized Massieu potentials $\Phi_q^{N(3)}$ and Φ_{2-q} are related only with $\gamma^{(1)}$. However, the situation is different for the versions utilizing the normalized q -average energy $U_q^{(3)}$, for which the reference energy is forced to shift from zero to $U_q^{(3)}$ [3]. Consequently the Lagrange multipliers $\gamma^{(3)}$ and $\gamma^{N(3)}$ are related only with S_q (as shown in Eq. (37)) and S_q^N (as Eq. (45)), respectively.

It is thus difficult to obtain the $\Phi_q^{(3)}$ or $F_q^{(3)}$ by applying our method, which is based on the relation between the Lagrange multiplier associated with the normalization of pdf and Massieu' potential.

5 Conclusions

Based on the Tsallis' thermostatics with the standard linear average energy $U^{(1)}$ and S_{2-q} , we have explored the connections among the pdfs of the two different kinds of Tsallis' formalism: one employs $U^{(1)}$, and the other employs the normalized q -average energy $U_q^{(3)}$ as the energy constraint.

We have shown the relations among the Lagrange multipliers associated with energy constraints of the different versions. It is revealed that the standard linear average energy $U^{(1)}$ and the normalized q -average energy $U_q^{(3)}$ are related to each other.

Furthermore we have studied the relevant thermodynamic Legendre relations concerning with the Lagrange multiplier $\gamma^{(1)}$ associated with the normalization of the pdf. By utilizing the relation between $U^{(1)}$ and $U_q^{(3)}$, we have constructed the generalized Massieu potential associated with either $U^{(1)}$ or $U_q^{(3)}$.

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