Mathematical inequalities for some divergences

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Abstract. Divergences often play important roles for study in information science so that it is indispensable to investigate their fundamental properties. There is also a mathematical significance of such results. In this paper, we introduce some parametric extended divergences combining Jeffreys divergence and Tsallis entropy defined by generalized logarithmic functions, which lead to new inequalities. In addition, we give lower bounds for one-parameter extended Fermi-Dirac and Bose-Einstein divergences. Finally, we establish some inequalities for the Tsallis entropy, the Tsallis relative entropy and some divergences by the use of the Young's inequality.

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1 Introduction

For the study of multifractals, in 1988, Tsallis [27] introduced one-parameter extended entropy of Shannon entropy by

$$H_q(\mathbf{p}) \equiv -\sum_{j=1}^n p_j^q \ln_q p_j = \sum_{j=1}^n p_j \ln_q \frac{1}{p_j}, \ (q \ge 0, q \ne 1)$$
 (1)

where $\mathbf{p} = \{p_1, p_2, \dots, p_n\}$ is a probability distribution with $p_j > 0$ for all $j = 1, 2, \dots, n$ and the q-logarithmic function for x > 0 is defined by $\ln_q(x) \equiv \frac{x^{1-q}-1}{1-q}$ which uniformly converges to the usual logarithmic function $\log(x)$ in the limit $q \to 1$. Therefore Tsallis entropy converges to Shannon entropy in the limit $q \to 1$:

$$\lim_{q \to 1} H_q(\mathbf{p}) = H_1(\mathbf{p}) \equiv -\sum_{j=1}^n p_j \log p_j. \tag{2}$$

It is also known that Rényi entropy [18]

$$R_q(\mathbf{p}) \equiv \frac{1}{1-q} \log \left(\sum_{j=1}^n p_j^q \right) \tag{3}$$

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is one -parameter extension of Shannon entropy.

For two probability distributions $\mathbf{p} = \{p_1, p_2, \dots, p_n\}$ and $\mathbf{r} = \{r_1, r_2, \dots, r_n\}$ we have divergences based on these quantities (1) and (3). We denote by

$$D_q(\mathbf{p}||\mathbf{r}) \equiv \sum_{j=1}^n p_j^q (\ln_q p_j - \ln_q r_j) = -\sum_{j=1}^n p_j \ln_q \frac{r_j}{p_j}$$
(4)

Tsallis relative entropy. Tsallis relative entropy converges to the usual relative entropy (divergence, Kullback-Leibler information) in the limit $q \to 1$:

$$\lim_{q \to 1} D_q(\mathbf{p}||\mathbf{r}) = D_1(\mathbf{p}||\mathbf{r}) \equiv \sum_{j=1}^n p_j(\log p_j - \log r_j).$$
 (5)

We also denote by $R_q(\mathbf{p}||\mathbf{r})$ the Rényi relative entropy [18] defined by

$$R_q(\mathbf{p}||\mathbf{r}) \equiv \frac{1}{q-1} \log \left(\sum_{j=1}^n p_j^q r_j^{1-q} \right). \tag{6}$$

Obviously $\lim_{q\to 1} R_q(\mathbf{p}||\mathbf{r}) = D_1(\mathbf{p}||\mathbf{r}).$

The divergences can be considered to be a generalization of entropies in the sense that Shannon entropy can be reproduced by the divergence $\log n - D_1(\mathbf{p}||\mathbf{u})$ for the uniform distribution $\mathbf{u} = \{1/n, 1/n, \dots, 1/n\}$. Therefore the study of divergences it is important for the developments of information science. In this paper, we study several mathematical inequalities related to some generalized divergences.

2 Two parameter entropies and divergences

In this section and throughout the rest of the paper we consider $\mathbf{p} = \{p_1, p_2, \dots, p_n\}$ and $\mathbf{r} = \{r_1, r_2, \dots, r_n\}$ with $p_j > 0, r_j > 0$ for all $j = 1, 2, \dots, n$ to be probability distributions.

We start from the Tsallis quasilinear entropies and Tsallis quasilinear divergences as they were defined in [10].

Definition 2.1 ([10]) For a continuous and strictly monotonic function ψ on $(0, \infty)$ and $r \ge 0$ with $r \ne 1$ (the nonextensivity parameter), Tsallis quasilinear entropy (r-quasilinear entropy) is defined by

$$I_r^{\psi}(\mathbf{p}) \equiv \ln_r \psi^{-1} \left(\sum_{j=1}^n p_j \psi \left(\frac{1}{p_j} \right) \right). \tag{7}$$

In this context, as a particular case of Tsallis quasilinear entropy we have Sharma-Mittal information measure ([16],[20],[21]), that is for $\psi(x) = x^{1-q}$ we have:

$$I_r^{x^{1-q}}(\mathbf{p}) = \ln_r \left(\sum_{j=1}^n p_j^q \right)^{\frac{1}{1-q}} = \frac{\left(\sum_{j=1}^n p_j^q \right)^{\frac{1-r}{1-q}} - 1}{1-r} = H_{r,q}^{S-M}(\mathbf{p}).$$

We find that $H_{q,q}^{S-M}(\mathbf{p}) = H_q(\mathbf{p})$. Sharma-Mittal entropy is also seen in the literature as a two-parameter extension of Rényi entropy [19, Section 5]. This also gives rise to another case of interest

$$I_{\frac{2q-1}{q}}^{x^{1-q}}(\mathbf{p}) = \ln_{\frac{2q-1}{q}} \left(\sum_{j=1}^{n} p_j^q \right)^{\frac{1}{1-q}} = \frac{q}{1-q} \left[\left(\sum_{j=1}^{n} p_j^q \right)^{\frac{1}{q}} - 1 \right], \tag{8}$$

which coincides with Arimoto's entropy for $q = 1/\beta$, cf. [3], and with R-norm information measure, for R = q, cf. [4].

Definition 2.2 ([10]) For a continuous and strictly monotonic function ψ on $(0, \infty)$, the Tsallis quasilinear relative entropy is defined by

$$D_r^{\psi}(\mathbf{p}||\mathbf{r}) \equiv -\ln_r \psi^{-1} \left(\sum_{j=1}^n p_j \psi\left(\frac{r_j}{p_j}\right) \right). \tag{9}$$

Sharma-Mittal divergence ([2],[17]) becomes now a particular case of Tsallis quasilinear divergence:

$$D_r^{x^{1-q}}(\mathbf{p}||\mathbf{r}) = -\ln_r \left(\sum_{j=1}^n p_j \left(\frac{r_j}{p_j} \right)^{1-q} \right)^{\frac{1}{1-q}} = -\ln_r \left(\sum_{j=1}^n p_j^q r_j^{1-q} \right)^{\frac{1}{1-q}}$$

$$= \frac{-\left\{ \left[\left(\sum_{j=1}^n p_j^q r_j^{1-q} \right)^{\frac{1}{1-q}} \right]^{1-r} - 1 \right\}}{1-r} = \frac{1 - \left(\sum_{j=1}^n p_j^q r_j^{1-q} \right)^{\frac{1-r}{1-q}}}{1-r} = D_{r,q}^{S-M}(\mathbf{p}||\mathbf{r}).$$

By analogy to the entropy computation, we find the following Arimoto type divergence:

$$D_{\frac{2q-1}{q}}^{x^{1-q}}(\mathbf{p}||\mathbf{r}) = \frac{q}{1-q} \left[1 - \left(\sum_{j=1}^{n} p_j^q r_j^{1-q} \right)^{\frac{1}{q}} \right].$$
 (10)

Remark 2.3 In limit $r \to 1$ we have $I_r^{x^{1-q}}(\mathbf{p}) \to I_1^{x^{1-q}}(\mathbf{p}) = R_q(\mathbf{p})$ and $D_r^{x^{1-q}}(\mathbf{p}||\mathbf{r}) \to D_1^{x^{1-q}}(\mathbf{p}||\mathbf{r}) = R_q(\mathbf{p}||\mathbf{r})$. It is known that for $q \neq r$ the Sharma-Mittal divergence fails to conform to Shore-Johnson theorem [23, 24, 25], that is Sharma-Mittal divergence cannot be written as a f-divergence

$$D_{r,q}^{S-M}(\mathbf{p}||\mathbf{r}) = \sum_{j=1}^{n} p_j f\left(\frac{r_j}{p_j}\right),$$

for some function f. The previous limits give us a very intuitive way to conclude that Rényi divergence has a similar failure [2]. Also this enables us to say that the two-parameter extended relative entropy discussed in [9, Section 6] cannot be seen as a particular case of Sharma-Mittal divergence.

Remark 2.4 For x > 0 and $r \ge 0$ with $r \ne 1$, we define the r-exponential function as the inverse function of the r-logarithmic function by $\exp_r(x) \equiv \{1 + (1-r)x\}^{1/(1-r)}$, if 1 + (1-r)x > 0, otherwise it is undefined. Here is another connection among Sharma-Mittal entropy, Tsallis entropy and Rényi entropy [10];[22, (B.8)]:

$$\exp_r H_{r,q}^{S-M}(\mathbf{p}) = \exp_q H_q(\mathbf{p}) = \exp R_q(\mathbf{p}).$$

As for a connection among their divergences, we get

$$\exp_{2-r} \left(D_{r,q}^{S-M}(\mathbf{p}||\mathbf{r}) \right) = \left\{ 1 + (r-1)D_{r,q}^{S-M}(\mathbf{p}||\mathbf{r}) \right\}^{1/(r-1)} = \left\{ 1 + (q-1)D_q(\mathbf{p}||\mathbf{r}) \right\}^{1/(q-1)} \\ = \exp_{2-q} \left(D_q(\mathbf{p}||\mathbf{r}) \right) = \exp R_q(\mathbf{p}||\mathbf{r}).$$

Remark 2.5 The weighted quasilinear mean for some continuous and strictly monotonic function $\psi: I \to \mathbb{R}$ is defined by

$$M_{[\psi]}(x_1, x_2, \cdots, x_n) \equiv \psi^{-1} \left(\sum_{j=1}^n p_j \psi(x_j) \right),$$
 (11)

where $\sum_{j=1}^n p_j = 1$, $p_j > 0$, $x_j \in I$ for $j = 1, 2, \dots, n$. It is known that $M_{[\psi]}(x_1, x_2, \dots, x_n) = M_{[\varphi]}(x_1, x_2, \dots, x_n)$ if and only if ψ and φ are affine maps of each other, i.e. there exist constants a, b such that $\psi = a\varphi + b$ (cf. [1, page 141], cf. also [7]). We conclude that $M_{[x^{1-q}]}(x_1, x_2, \dots, x_n) = M_{[\ln q]}(x_1, x_2, \dots, x_n)$, a fact that yields $I_r^{x^{1-q}}(\mathbf{p}) = I_r^{\ln q}(\mathbf{p}) = H_{r,q}^{S-M}(\mathbf{p})$ and $D_r^{x^{1-q}}(\mathbf{p}||\mathbf{r}) = D_r^{\ln q}(\mathbf{p}||\mathbf{r}) = D_{r,q}^{S-M}(\mathbf{p}||\mathbf{r})$.

3 Jeffreys and Jensen-Shannon type divergences

3.1 Tsallis type divergences

We firstly review the definitions of two famous divergences.

Definition 3.1 ([8],[13]) The Jeffreys divergence is defined by

$$J_1(\mathbf{p}||\mathbf{r}) \equiv D_1(\mathbf{p}||\mathbf{r}) + D_1(\mathbf{r}||\mathbf{p}) \tag{12}$$

and the Jensen-Shannon divergence is defined as

$$JS_1(\mathbf{p}||\mathbf{r}) \equiv \frac{1}{2}D_1\left(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}\right) + \frac{1}{2}D_1\left(\mathbf{r}||\frac{\mathbf{p}+\mathbf{r}}{2}\right). \tag{13}$$

Analogously we may define the following divergences.

Definition 3.2 The Jeffreys-Tsallis divergence is

$$J_r(\mathbf{p}||\mathbf{r}) \equiv D_r(\mathbf{p}||\mathbf{r}) + D_r(\mathbf{r}||\mathbf{p}) \tag{14}$$

and the Jensen-Shannon-Tsallis divergence is

$$JS_r(\mathbf{p}||\mathbf{r}) \equiv \frac{1}{2}D_r\left(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}\right) + \frac{1}{2}D_r\left(\mathbf{r}||\frac{\mathbf{p}+\mathbf{r}}{2}\right). \tag{15}$$

We find that $J_r(\mathbf{p}||\mathbf{r}) = J_r(\mathbf{r}||\mathbf{p})$ and $JS_r(\mathbf{p}||\mathbf{r}) = JS_r(\mathbf{r}||\mathbf{p})$. That is, these divergences are symmetric in the above sense.

To show one of main results in this paper, we need the following lemma that has interest on its own.

Lemma 3.3 The function

$$f(x) = -\ln_r \frac{1 + \exp_q(-x)}{2}$$

is concave for $0 \le r \le q$.

Proof: The proof is a straightforward computation. The second derivative is given by

$$f''(x) = -2^{r-1} \left\{ 1 + (q-1)x \right\}^{\frac{2q-1}{1-q}} \left(1 + \left\{ 1 + (q-1)x \right\}^{\frac{1}{1-q}} \right)^{-r-1} \times \left(q + (q-r) \left\{ 1 + (q-1)x \right\}^{\frac{1}{1-q}} \right).$$

Therefore if $q \geq r$, then the function f(x) is concave.

We wish to note here that the above result yields the fact that under the same conditions the function $-\ln_r \frac{1+\exp_q(x)}{2}$ is also concave, as the composition of a concave function with an affine one.

Lemma 3.4 Tsallis divergence satisfies

$$D_r\left(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}\right) \leq \frac{1}{2}D_{\frac{1+r}{2}}(\mathbf{p}||\mathbf{r}).$$

Proof: From the famous inequality between the arithmetic and geometric means, we have

$$\frac{p_j + r_j}{2} \ge \sqrt{p_j r_j}$$

for all $j = 1, 2, \dots, n$. This implies that

$$D_{r}\left(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}\right) = -\sum_{j=1}^{n} p_{j} \ln_{r}\left(\frac{\frac{p_{j}+r_{j}}{2}}{p_{j}}\right) \leq -\sum_{j=1}^{n} p_{j} \ln_{r}\left(\frac{\sqrt{p_{j}r_{j}}}{p_{j}}\right) = -\sum_{j=1}^{n} p_{j} \ln_{r}\left(\sqrt{\frac{r_{j}}{p_{j}}}\right)$$

$$= -\sum_{j=1}^{n} p_{j} \frac{\left(\sqrt{\frac{r_{j}}{p_{j}}}\right)^{1-r} - 1}{1-r} = -\sum_{j=1}^{n} p_{j} \frac{\left(\frac{r_{j}}{p_{j}}\right)^{1-\frac{1+r}{2}} - 1}{1-r} = -\frac{1}{2} \sum_{j=1}^{n} p_{j} \frac{\left(\frac{r_{j}}{p_{j}}\right)^{1-\frac{1+r}{2}} - 1}{1-\frac{1+r}{2}}$$

$$= \frac{1}{2} D_{\frac{1+r}{2}}(\mathbf{p}||\mathbf{r}).$$

Hence we derive the following result.

Theorem 3.5 It holds that

$$JS_r(\mathbf{p}||\mathbf{r}) \le \min \left\{ -\ln_r \frac{1 + \exp_q\left(-\frac{1}{2}J_q(\mathbf{p}||\mathbf{r})\right)}{2}, \frac{1}{4}J_{\frac{1+r}{2}}(\mathbf{p}||\mathbf{r}) \right\}$$
(16)

for $0 \le r \le q$.

Proof: According to Lemma 3.3,

$$JS_{r}(\mathbf{p}||\mathbf{r}) \equiv \frac{1}{2} \left(-\sum_{j=1}^{n} p_{j} \ln_{r} \frac{1 + \exp_{q} \ln_{q} \left(\frac{r_{j}}{p_{j}} \right)}{2} - \sum_{j=1}^{n} r_{j} \ln_{r} \frac{1 + \exp_{q} \ln_{q} \left(\frac{p_{j}}{r_{j}} \right)}{2} \right)$$

$$\leq \frac{1}{2} \left(-\ln_{r} \frac{1 + \exp_{q} \sum_{j=1}^{n} p_{j} \ln_{q} \left(\frac{r_{j}}{p_{j}} \right)}{2} - \ln_{r} \frac{1 + \exp_{q} \sum_{j=1}^{n} r_{j} \ln_{q} \left(\frac{p_{j}}{r_{j}} \right)}{2} \right)$$

$$= \frac{1}{2} \left(-\ln_{r} \frac{1 + \exp_{q} \left(-D_{q}(\mathbf{p}||\mathbf{r}) \right)}{2} - \ln_{r} \frac{1 + \exp_{q} \left(-D_{q}(\mathbf{r}||\mathbf{p}) \right)}{2} \right). \tag{17}$$

Then

$$JS_r(\mathbf{p}||\mathbf{r}) \le -\ln_r \frac{1 + \exp_q\left(\frac{-D_q(\mathbf{p}||\mathbf{r}) - D_q(\mathbf{r}||\mathbf{p})}{2}\right)}{2} = -\ln_r \frac{1 + \exp_q\left(-\frac{1}{2}J_q(\mathbf{p}||\mathbf{r})\right)}{2}.$$

We apply Lemma 3.4 whence it follows

$$JS_r(\mathbf{p}||\mathbf{r}) \leq \frac{1}{4} \left(D_{\frac{1+r}{2}}(\mathbf{p}||\mathbf{r}) + D_{\frac{1+r}{2}}(\mathbf{r}||\mathbf{p}) \right).$$

Thus the proof is completed.

Remark 3.6 For q = r, $r \to 1$ we have $JS_r(\mathbf{p}||\mathbf{r}) \to JS_1$, $J_r(\mathbf{p}||\mathbf{r}) \to J_1$ and the inequality (16) gives us

$$JS_1(\mathbf{p}||\mathbf{r}) \le \min \left\{ -\log \frac{1 + \exp\left(-\frac{1}{2}J_1(\mathbf{p}||\mathbf{r})\right)}{2}, \frac{1}{4}J_1(\mathbf{p}||\mathbf{r}) \right\}.$$

Since

$$-\log\frac{1+\exp\left(-x\right)}{2} \le \frac{x}{2}$$

we get the main results in [6]:

$$JS_1(\mathbf{p}||\mathbf{r}) \le -\log \frac{1 + \exp\left(-\frac{1}{2}J_1(\mathbf{p}||\mathbf{r})\right)}{2} \le \frac{1}{4}J_1(\mathbf{p}||\mathbf{r}). \tag{18}$$

3.2 Dual symmetric divergences

In this subsection, we introduce another type divergences and then we highlight some inequalities for them.

Definition 3.7 The dual symmetric Jeffreys-Tsallis divergence and the dual symmetric Jensen-Shannon-Tsallis divergence are defined by

$$J_r^{(ds)}(\mathbf{p}||\mathbf{r}) \equiv D_r(\mathbf{p}||\mathbf{r}) + D_{2-r}(\mathbf{r}||\mathbf{p})$$
(19)

respectively

$$JS_r^{(ds)}(\mathbf{p}||\mathbf{r}) \equiv \frac{1}{2} \left[D_r \left(\mathbf{p}||\frac{\mathbf{p} + \mathbf{r}}{2} \right) + D_{2-r} \left(\mathbf{r}||\frac{\mathbf{p} + \mathbf{r}}{2} \right) \right]. \tag{20}$$

As one can see directly from the definition, we find that $J_r^{(ds)}(\mathbf{p}||\mathbf{r}) = J_{2-r}^{(ds)}(\mathbf{r}||\mathbf{p})$ and $JS_r^{(ds)}(\mathbf{p}||\mathbf{r}) = JS_{2-r}^{(ds)}(\mathbf{r}||\mathbf{p})$. See [22] and references therein for additive duality $r \leftrightarrow 2-r$ in Tsallis statistics.

Then we get the following upper bound for $JS_r^{(ds)}(\mathbf{p}||\mathbf{r})$.

Proposition 3.8 For $0 \le r \le 2$, we have

$$JS_r^{(ds)}(\mathbf{p}||\mathbf{r}) \le \frac{1}{4}J_{\frac{1+r}{2}}^{(ds)}(\mathbf{p}||\mathbf{r}). \tag{21}$$

Proof: We infer from Lemma 3.4 that

$$D_{2-r}(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}) \leq \frac{1}{2}D_{\frac{3-r}{2}}(\mathbf{p}||\mathbf{r}).$$

Consequently

$$JS_r^{(ds)}(\mathbf{p}||\mathbf{r}) \le \frac{1}{2} \left(D_{\frac{1+r}{2}}(\mathbf{p}||\mathbf{r}) + D_{\frac{3-r}{2}}(\mathbf{p}||\mathbf{r}) \right) = \frac{1}{4} J_{\frac{1+r}{2}}^{(ds)}(\mathbf{p}||\mathbf{r}).$$

This completes the proof.

In order to derive further results regarding the dual symmetric divercences, we need the following lemmas.

Lemma 3.9 The function $\exp_q x$ is monotone increasing in q, for $x \ge 0$.

Proof: We have

$$\frac{\mathrm{d}\exp_q x}{\mathrm{d}q} = \frac{\{1 + (1-q)x\}^{\frac{q}{1-q}} h_q(x)}{(1-q)^2},$$

where

$$h_q(x) \equiv (q-1)x + \{1 + (1-q)x\} \log \{1 + (1-q)x\}.$$

Then

$$\frac{\mathrm{d}h_q(x)}{\mathrm{d}x} = (1-q)\log\{1 + (1-q)x\} \ge 0$$

for $x \ge 0$ and $q \ge 0$. Therefore $h_q(x) \ge h_q(0) = 0$. Thus we have $\frac{\mathrm{d} \exp_q x}{\mathrm{d} q} \ge 0$, as asserted.

Lemma 3.10 For $1 < r \le 2$ and x > 0, we have

$$-\ln_{2-r} x \le -\ln_r x$$

and

$$\exp_{2-r} x \le \exp_r x.$$

Proof: Since we have $x^{1-r} + x^{r-1} \ge 2$, which implies $x^{r-1} - 1 \ge 1 - x^{1-r}$, we have for $1 < r \le 2$,

$$\ln_{2-r} x = \frac{x^{r-1} - 1}{r - 1} \ge \frac{1 - x^{1-r}}{r - 1} = \ln_r x.$$

The second inequality is a consequence of Lemma 3.9.

Our next result reads as follows.

Theorem 3.11 The following inequality holds

$$\max\left\{JS_r^{(ds)}(\mathbf{r}||\mathbf{p}), JS_r^{(ds)}(\mathbf{p}||\mathbf{r})\right\} \le -\ln_r \frac{1 + \exp_q\left(-\frac{1}{2}J_q(\mathbf{p}||\mathbf{r})\right)}{2},\tag{22}$$

for all $1 < r \le 2$ and $r \le q$.

Proof: By Jensen's inequality, applying Lemma 3.3, we have

$$JS_r^{(ds)}(\mathbf{p}||\mathbf{r}) \equiv \frac{1}{2} \left(-\sum_{j=1}^n p_j \ln_r \frac{1 + \exp_q \ln_q \left(\frac{r_j}{p_j}\right)}{2} - \sum_{j=1}^n r_j \ln_{2-r} \frac{1 + \exp_q \ln_q \left(\frac{p_j}{r_j}\right)}{2} \right)$$

$$\leq \frac{1}{2} \left(-\ln_r \frac{1 + \exp_q \left(-D_q(\mathbf{p}||\mathbf{r})\right)}{2} - \ln_{2-r} \frac{1 + \exp_q \left(-D_q(\mathbf{r}||\mathbf{p})\right)}{2} \right).$$

Thus, via Lemma 3.10, it turns out that

$$JS_r^{(ds)}(\mathbf{p}||\mathbf{r}) \leq \frac{1}{2} \left(-\ln_r \frac{1 + \exp_q(-D_q(\mathbf{p}||\mathbf{r}))}{2} - \ln_r \frac{1 + \exp_q(-D_q(\mathbf{r}||\mathbf{p}))}{2} \right)$$

$$\leq -\ln_r \frac{1 + \exp_q\left(-\frac{1}{2}J_q(\mathbf{p}||\mathbf{r})\right)}{2}.$$

Further we also have $0 \le 2 - r < 1$ and the computation is similar for $JS_{2-r}^{(ds)}(\mathbf{p}||\mathbf{r})$, hence we get (using the additive duality)

$$JS_r^{(ds)}(\mathbf{r}||\mathbf{p}) = JS_{2-r}^{(ds)}(\mathbf{p}||\mathbf{r}) \le -\ln_r \frac{1 + \exp_q\left(-\frac{1}{2}J_q(\mathbf{p}||\mathbf{r})\right)}{2}.$$

Remark 3.12 For $q = r, r \to 1$ we have $JS_r^{(ds)}(\mathbf{p}||\mathbf{r}) \to JS_1^{(ds)}(\mathbf{p}||\mathbf{r}) = JS_1(\mathbf{p}||\mathbf{r}), J_r^{(ds)}(\mathbf{p}||\mathbf{r}) \to J_1^{(ds)}(\mathbf{p}||\mathbf{r}) = J_1(\mathbf{p}||\mathbf{r})$ and the inequality (22) yields again the left side inequality of (18).

Remark 3.13 The inequality (22) does not hold for $0 \le r \le q < 1$, in general. We have the following counter-example. We consider the probability distributions $\mathbf{p} = \{2/5, 2/5, 1/5\}$ and $\mathbf{r} = \{1/10, 1/10, 4/5\}$. Then for r = q = 0.1, we have

$$-\ln_r \frac{1 + \exp_q\left(-\frac{1}{2}J_q(\mathbf{p}||\mathbf{r})\right)}{2} - JS_r^{(ds)}(\mathbf{p}||\mathbf{r}) \simeq -0.141646.$$

Open problem 3.14 *Prove, disprove or find conditions such that the following inequality holds:*

$$JS_r^{(ds)}(\mathbf{p}||\mathbf{r}) \le -\ln_r \frac{1 + \exp_r \left(-\frac{1}{2}J_r^{(ds)}(\mathbf{p}||\mathbf{r})\right)}{2}, \ r \in [0, 2] \setminus \{1\}.$$

We have not yet found any counter-example of (23). One may try to follow the same argument as in the proof of Theorem 3.11. This means that one should prove

$$\frac{1}{2} \left(-\ln_r \frac{1 + \exp_r(-D_r(\mathbf{p}||\mathbf{r}))}{2} - \ln_{2-r} \frac{1 + \exp_{2-r}(-D_{2-r}(\mathbf{r}||\mathbf{p}))}{2} \right)$$

$$\leq -\ln_r \frac{1 + \exp_r\left(-\frac{1}{2}J_r^{(ds)}(\mathbf{p}||\mathbf{r})\right)}{2}.$$
(24)

For $0 \le r < 1$, we have considered already over 100 particular cases without finding any counter-example for (24). For $1 < r \le 2$ we have the following counter-example. Assume r = 1.3, $\mathbf{p} = \{0.14, 0.01, 0.85\}$ and $\mathbf{r} = \{0.07, 0.48, 0.45\}$. Then the right hand side in (24) minus the left hand side in (24) approximately equals -0.0125861. Therefore in the case of $1 < r \le 2$ the proof of (23) (if it holds) couldn't begin with Jensen's inequality as a first step.

3.3 More quasilinear divergences

We generalize the above definitions.

Definition 3.15 Let the quasilinear Jeffreys-Tsallis divergence be

$$J_r^{\psi}(\mathbf{p}||\mathbf{r}) \equiv D_r^{\psi}(\mathbf{p}||\mathbf{r}) + D_r^{\psi}(\mathbf{r}||\mathbf{p}),$$

respectively the quasilinear Jensen-Shannon-Tsallis divergence be

$$JS_r^{\psi}(\mathbf{p}||\mathbf{r}) \equiv \frac{1}{2} \left[D_r^{\psi}(\mathbf{p}||\frac{\mathbf{p}+\mathbf{r}}{2}) + D_r^{\psi}(\mathbf{r}||\frac{\mathbf{p}+\mathbf{r}}{2}) \right].$$

The above quasilinear divergences are symmetric in the sense that we have $J_r^{\psi}(\mathbf{p}||\mathbf{r}) = J_r^{\psi}(\mathbf{r}||\mathbf{p})$ and $JS_r^{\psi}(\mathbf{p}||\mathbf{r}) = JS_r^{\psi}(\mathbf{r}||\mathbf{p})$. For $\psi(x) = x^{1-q}$ we obtain $J_r^{x^{1-q}}(\mathbf{p}||\mathbf{r}) = J_r(\mathbf{p}||\mathbf{r})$ and $JS_r^{x^{1-q}}(\mathbf{p}||\mathbf{r}) = JS_r(\mathbf{p}||\mathbf{r})$.

Proposition 3.16 Let ψ be a continuous and strictly monotonic function on $(0, \infty)$. Suppose that $\psi\left(\frac{1+\psi^{-1}(x)}{2}\right)$ is concave. Then

$$JS_r^{\psi}(\mathbf{p}||\mathbf{r}) \le -\ln_r \frac{1 + \exp_q\left(-\frac{1}{2}J_q^{\psi}(\mathbf{p}||\mathbf{r})\right)}{2},$$

for all $0 \le r \le q$.

Proof: Since

$$JS_r^{\psi}(\mathbf{p}||\mathbf{r}) \equiv -\frac{1}{2} \ln_r \psi^{-1} \left(\sum_{j=1}^n p_j \psi \left(\frac{1 + \psi^{-1} \left(\psi \left(\frac{r_j}{p_j} \right) \right)}{2} \right) \right)$$
$$-\frac{1}{2} \ln_r \psi^{-1} \left(\sum_{j=1}^n r_j \psi \left(\frac{1 + \psi^{-1} \left(\psi \left(\frac{p_j}{r_j} \right) \right)}{2} \right) \right),$$

by Jensen's inequality, due to the monotonicity and from Lemma 3.3, we just compute

$$JS_{r}^{\psi}(\mathbf{p}||\mathbf{r}) \leq \frac{1}{2} \left(-\ln_{r} \frac{1 + \psi^{-1} \left(\sum_{j=1}^{n} p_{j} \psi\left(\frac{r_{j}}{p_{j}}\right) \right)}{2} - \ln_{r} \frac{1 + \psi^{-1} \left(\sum_{j=1}^{n} r_{j} \psi\left(\frac{p_{j}}{r_{j}}\right) \right)}{2} \right)$$

$$= \frac{1}{2} \left(-\ln_{r} \frac{1 + \exp_{q} \left(-D_{q}^{\psi}(\mathbf{p}||\mathbf{r}) \right)}{2} - \ln_{r} \frac{1 + \exp_{q} \left(-D_{q}^{\psi}(\mathbf{r}||\mathbf{p}) \right)}{2} \right)$$

$$\leq -\ln_{r} \frac{1 + \exp_{q} \left(-\frac{1}{2} J_{q}^{\psi}(\mathbf{p}||\mathbf{r}) \right)}{2}.$$

4 Fermi-Dirac and Bose-Einstein type divergences

As one-parameter extension of Fermi-Dirac entropy and Bose-Einstein entropy (see also [11, 26]), that is of

$$I_1^{FD}(\mathbf{p}) \equiv -\sum_{j=1}^n p_j \log p_j - \sum_{j=1}^n (1 - p_j) \log(1 - p_j)$$

and

$$I_1^{BE}(\mathbf{p}) \equiv -\sum_{j=1}^n p_j \log p_j + \sum_{j=1}^n (1+p_j) \log(1+p_j),$$

the Fermi-Dirac-Tsallis entropy was introduced in [26]. Similarly, we may define the Bose-Einstein-Tsallis entropy.

Definition 4.1 The Fermi-Dirac-Tsallis entropy is given by

$$I_r^{FD}(\mathbf{p}) \equiv \sum_{j=1}^n p_j \ln_r \frac{1}{p_j} + \sum_{j=1}^n (1 - p_j) \ln_r \frac{1}{1 - p_j}$$
 (25)

and the Bose-Einstein-Tsallis entropy is defined as

$$I_r^{BE}(\mathbf{p}) \equiv \sum_{j=1}^n p_j \ln_r \frac{1}{p_j} - \sum_{j=1}^n (1 + p_j) \ln_r \frac{1}{1 + p_j}.$$
 (26)

Based on the above extensions, we may introduce Fermi-Dirac-Tsallis divergence and Bose-Einstein-Tsallis divergence in the following way.

Definition 4.2 Let

$$D_r^{FD}(\mathbf{p}||\mathbf{r}) \equiv -\sum_{j=1}^n p_j \ln_r \frac{r_j}{p_j} - \sum_{j=1}^n (1 - p_j) \ln_r \frac{1 - r_j}{1 - p_j}$$
 (27)

and

$$D_r^{BE}(\mathbf{p}||\mathbf{r}) \equiv -\sum_{j=1}^n p_j \ln_r \frac{r_j}{p_j} + \sum_{j=1}^n (1+p_j) \ln_r \frac{1+r_j}{1+p_j}.$$
 (28)

Then D_r^{FD} is called the Fermi-Dirac-Tsallis divergence and D_r^{BE} is called the Bose-Einstein-Tsallis divergence.

Lemma 4.3 For 0 < x, y < 1 we have

$$-x\ln_{r}\frac{y}{x} - (1-x)\ln_{r}\frac{1-y}{1-x} \ge \frac{4^{r}}{r+1}\left[y^{r+1}\left(1-x\right)^{r} + x^{r}\left(1-y\right)^{r+1} - x^{r}\left(1-x\right)^{r}\right] \ge 0. \quad (29)$$

Proof: Following the idea of [5, Lemma 11.6.1] we denote

$$f(x,y) \equiv -x \ln_r \frac{y}{x} - (1-x) \ln_r \frac{1-y}{1-x} - \frac{4^r}{r+1} \left[y^{r+1} \left(1-x \right)^r + x^r \left(1-y \right)^{r+1} - x^r \left(1-x \right)^r \right].$$

We get

$$\frac{\mathrm{d}f(x,y)}{\mathrm{d}y} = \left[\frac{1}{y^r(1-y)^r} - 4^r\right] \left[y^r(1-x)^r - x^r(1-y)^r\right].$$

We can easily check that $\frac{1}{y(1-y)} \ge 4$ under the assumption 0 < y < 1. For $y \le x$ (which implies $y(1-x) \le x(1-y)$) we establish that the function f is decreasing in its second variable, hence $f(x,y) \ge f(x,x) = 0$. Clearly for the case $y \ge x$ we have similarly $\frac{\mathrm{d}f(x,y)}{\mathrm{d}y} \ge 0$, which leads again $f(x,y) \ge f(x,x) = 0$.

Our next step is to take

$$g(x,y) \equiv y^{r+1} (1-x)^r + x^r (1-y)^{r+1} - x^r (1-x)^r$$
.

For $y \geq x$ we may write that

$$\frac{\mathrm{d}g(x,y)}{\mathrm{d}y} = (r+1)\left[y^r(1-x)^r - x^r(1-y)^r\right] \ge 0.$$

Therefore $g(x,y) \ge g(x,x) = 0$. For the case of $y \le x$, one can show that $g(x,y) \ge 0$ by the similar way.

Proposition 4.4 The Fermi-Dirac-Tsallis divergence satisfies

$$D_r^{FD}(\mathbf{p}||\mathbf{r}) \ge \frac{4^r}{r+1} \sum_{j=1}^n \left[r_j^{r+1} (1-p_j)^r + p_j^r (1-r_j)^{r+1} - p_j^r (1-p_j)^r \right] \ge 0.$$
 (30)

Proof: Via Lemma 4.3, putting $x = p_j$ and $y = r_j$, then taking the sum on both sides, it follows the claimed result.

Lemma 4.5 For 0 < x, y < 1 we have

$$-x\ln_r \frac{y}{x} + (1+x)\ln_r \frac{1+y}{1+x} \ge \frac{1}{2^r(r+1)} \left[(1+x)^r y^{r+1} - (1+y)^{r+1} x^r + (1+x)^r x^r \right] \ge 0.$$
 (31)

Proof: Consider the function

$$f(x,y) \equiv -x \ln_r \frac{y}{x} + (1+x) \ln_r \frac{1+y}{1+x} - \frac{1}{2^r (r+1)} \left[(1+x)^r y^{r+1} - (1+y)^{r+1} x^r + (1+x)^r x^r \right].$$

Differentiating f yields

$$\frac{\mathrm{d}f(x,y)}{\mathrm{d}y} = \left[(1+x)^r y^r - (1+y)^r x^r \right] \left[\frac{1}{y^r (1+y)^r} - \frac{1}{2^r} \right].$$

Obviously one has $y(1+y) \leq 2$ provided 0 < y < 1. For the case $x \geq y$ (i.e. $x(1+y) \geq y(1+x)$), we have $\frac{\mathrm{d}f(x,y)}{\mathrm{d}y} \leq 0$ so that $f(x,y) \geq f(x,x) = 0$. One checks that for $x \leq y$ we get $\frac{\mathrm{d}f(x,y)}{\mathrm{d}y} \geq 0$, so that $f(x,y) \geq f(x,x) = 0$.

Next, we put

$$g(x,y) \equiv (1+x)^r y^{r+1} - (1+y)^{r+1} x^r + (1+x)^r x^r.$$

For $y \geq x$

$$\frac{\mathrm{d}g(x,y)}{\mathrm{d}y} = (r+1)\left[y^r(1+x)^r - x^r(1+y)^r\right] \ge 0,$$

whence $g(x,y) \ge g(x,x) = 0$. Further, for the case $y \le x$, one can prove similarly that $g(x,y) \ge 0$ holds, which ends the proof.

Proposition 4.6 The Bose-Einstein-Tsallis divergence satisfies

$$D_r^{BE}(\mathbf{p}||\mathbf{r}) \ge \frac{1}{2^r (r+1)} \sum_{j=1}^n \left[(1+p_j)^r r_j^{r+1} - (1+r_j)^{r+1} p_j^r + (1+p_j)^r p_j^r \right] \ge 0.$$
 (32)

Proof: According to Lemma 4.5, putting $x = p_j$ and $y = r_j$, then taking the sum on both sides, it follows the claimed result.

Remark 4.7 Proposition 4.4 and Proposition 4.6 give refined lower bounds for the Fermi-Dirac-Tsallis divergence and the Bose-Einstein-Tsallis divergence, respectively. At the same time, they assure the nonnegativity of $D_r^{FD}(\mathbf{p}||\mathbf{r})$ and $D_r^{BE}(\mathbf{p}||\mathbf{r})$. Tsus we easily find that the following inequality for the Tsallis relative entropy holds

$$D_r(\mathbf{p}||\mathbf{r}) \ge \max \left\{ \sum_{j=1}^n (1-p_j) \ln_r \frac{1-r_j}{1-p_j}, -\sum_{j=1}^n (1+p_j) \ln_r \frac{1+r_j}{1+p_j} \right\}.$$

Corollary 4.8 The following inequalities hold

$$D_1^{FD}(\mathbf{p}||\mathbf{r}) \ge 2\sum_{j=1}^n (p_j - r_j)^2$$

and

$$D_1^{EB}(\mathbf{p}||\mathbf{r}) \ge \frac{1}{4} \sum_{j=1}^{n} (p_j - r_j)^2$$
.

Here $D_1^{FD}(\mathbf{p}||\mathbf{r})$ is called the Fermi-Dirac divergence, respectively $D_1^{BE}(\mathbf{p}||\mathbf{r})$ is called the Bose-Einstein divergence and their definition corresponds to the limit $r \to 1$ in Definition 4.1.

Proof: Put $r \to 1$ in Proposition 4.4 and Proposition 4.6.

5 Young's inequality and Tsallis entropies with finite sum

We establish more inequalities involving Tsallis entropy and Tsallis relative entropy applying Young's inequality.

Lemma 5.1 (Young's inequality) Let $m, n \ge 0$ and $p, q \in \mathbb{R}$ such that $\frac{1}{p} + \frac{1}{q} = 1$. If p < 0 (then 0 < q < 1) or 0 (then <math>q < 0), then one has $\frac{m^p}{p} + \frac{n^q}{q} \le mn$.

Lemma 5.2 (i) Let $p, q \in \mathbb{R}$ satisfying $\frac{1}{1-p} + \frac{1}{1-q} = 1$. If p > 1 and 0 < q < 1, or if 0 and <math>q > 1, then

$$\ln_p x + \ln_q y \le xy - 1.$$

(ii) Let $p, q \in \mathbb{R}$ satisfying $\frac{1}{p-1} + \frac{1}{q-1} = 1$. If p < 1 and 1 < q < 2, or if 1 and <math>q < 1, then

$$\ln_p \frac{1}{x} + \ln_q \frac{1}{y} \ge -xy + 1.$$

Proof:

(i) Using Lemma 5.1, we obtain

$$\ln_p x + \ln_q y = \frac{x^{1-p} - 1}{1 - p} + \frac{y^{1-q} - 1}{1 - q} \le xy - 1.$$

(ii) Lemma 5.1 leads to

$$\ln_p \frac{1}{x} + \ln_q \frac{1}{y} = \frac{x^{p-1} - 1}{1 - p} + \frac{y^{q-1} - 1}{1 - q} = -\left(\frac{x^{p-1}}{p - 1} + \frac{y^{q-1}}{q - 1}\right) + \left(\frac{1}{p - 1} + \frac{1}{q - 1}\right) \ge -xy + 1.$$

Then we have the following proposition.

Proposition 5.3 (i) Let $p, q \in \mathbb{R}$ satisfying $\frac{1}{1-p} + \frac{1}{1-q} = 1$. If 1 and <math>0 < q < 1, or if 0 and <math>1 < q < 2, then

$$D_p(\mathbf{p}||\mathbf{r}) + H_{2-q}(\mathbf{p}) \ge 1 - \sum_{j=1}^n p_j r_j$$
 (33)

and

$$D_{2-p}(\mathbf{p}||\mathbf{r}) + H_q(\mathbf{p}) \le \sum_{j=1}^n \frac{p_j}{r_j} - 1.$$
 (34)

(ii) Let $p, q \in \mathbb{R}$ satisfying $\frac{1}{p-1} + \frac{1}{q-1} = 1$. If 0 and <math>1 < q < 2 or if 1 and <math>0 < q < 1, then

$$D_p(\mathbf{p}||\mathbf{r}) + H_{2-q}(\mathbf{p}) \le \sum_{j=1}^n \frac{p_j}{r_j} - 1$$
(35)

and

$$D_{2-p}(\mathbf{p}||\mathbf{r}) + H_q(\mathbf{p}) \ge 1 - \sum_{j=1}^{n} p_j r_j.$$
 (36)

Proof:

(i) In (i) of Lemma 5.2, since we have $\ln_q y = -\ln_{2-q} \frac{1}{y}$ for all y > 0, we get

$$\ln_p x + \ln_q y = \ln_p x - \ln_{2-q} \frac{1}{y} \le xy - 1.$$

Putting $x = \frac{r_j}{p_j}$ and $y = p_j$ and multiplying $-p_j$ and then taking the sum on both sides, it follows

$$-\sum_{j=1}^{n} p_j \ln_p \frac{r_j}{p_j} + \sum_{j=1}^{n} p_j \ln_{2-q} \frac{1}{p_j} \ge \sum_{j=1}^{n} (p_j - p_j r_j),$$

which implies the inequality (33). We also have the inequality (34) from

$$\ln_p x + \ln_q y = -\ln_{2-p} \frac{1}{x} + \ln_q y \le xy - 1.$$

(ii) Using (ii) of Lemma 5.2 we have two inequalities (35) and (36) by the similar way to the proof of (i).

Remark 5.4 We have a pair of additive duality $(p, 2-q) \leftrightarrow (2-p, q)$ between (i) and (ii) of Proposition 5.3.

A cross-entropy type formula [15] of two probability distributions is the following:

$$H(\mathbf{p}, \mathbf{r}) = D_1(\mathbf{p}||\mathbf{r}) + H_1(\mathbf{p}).$$

One may see the left side terms in Proposition 5.3 as some generalizations of $H(\mathbf{p}, \mathbf{r})$.

Corollary 5.5 The following inequalities holds:

$$0 \le 1 - \sum_{j=1}^{n} p_j r_j \le H(\mathbf{p}, \mathbf{r}) \le \sum_{j=1}^{n} \frac{p_j}{r_j} - 1.$$

Proof: In Proposition 5.3, we take $q \to 1$.

Corollary 5.6 The following inequalities hold:

$$0 \le 1 - \sum_{j=1}^{n} p_j^2 \le H_1(\mathbf{p}) \le n - 1.$$

Proof: In Corollary 5.5, we take $\mathbf{r} = \mathbf{p}$.

Proposition 5.7 Let $p, q \in \mathbb{R}$ satisfying $\frac{1}{p-1} + \frac{1}{q-1} = 1$. If p < 1 and 1 < q < 2, or if 1 and <math>q < 1, then

$$I_p^{FD}(\mathbf{p}) + I_q^{FD}(\mathbf{p}) \ge 3 \sum_{j=1}^n p_j (1 - p_j).$$

Proof: From Lemma 5.2, (ii), putting $x = y = p_j$ and multiplying p_j and then taking the sum on both sides, it follows

$$\sum_{j=1}^{n} p_j \ln_p \frac{1}{p_j} + \sum_{j=1}^{n} p_j \ln_q \frac{1}{p_j} \ge -\sum_{j=1}^{n} p_j^3 + 1.$$

Putting $x = y = 1 - p_j$ and multiplying $1 - p_j$ and then taking the sum on both sides, it follows

$$\sum_{j=1}^{n} (1 - p_j) \ln_p \frac{1}{1 - p_j} + \sum_{j=1}^{n} (1 - p_j) \ln_q \frac{1}{1 - p_j} \ge -\sum_{j=1}^{n} (1 - p_j)^3 + n - 1.$$

Summing up these two inequalities we get

$$I_p^{FD}(\mathbf{p}) + I_q^{FD}(\mathbf{p}) \ge n - \sum_{j=1}^n p_j^3 - \sum_{j=1}^n (1 - p_j)^3 = 3 \sum_{j=1}^n p_j (1 - p_j).$$

We also find that the following interesting inequalities on finite sum hold true.

Proposition 5.8 For two probability distributions $\mathbf{p} = \{p_1, p_2, \dots, p_n\}$ and $\mathbf{r} = \{r_1, r_2, \dots, r_n\}$, we have the following relations.

(i) If
$$0 \le q < 1$$
, then we have $\sum_{j=1}^{n} p_j^q r_j^{1-q} \le 1 \le \sum_{j=1}^{n} p_j^{2-q} r_j^{q-1}$.

(ii) If
$$1 < q \le 2$$
, then we have $\sum_{j=1}^{n} p_j^q r_j^{1-q} \ge 1 \ge \sum_{j=1}^{n} p_j^{2-q} r_j^{q-1}$.

Proof: From the nonnegativity of Tsallis relative entropy $D_q(\mathbf{p}||\mathbf{r}) \ge 0$ and $D_{2-q}(\mathbf{p}||\mathbf{r}) \ge 0$, we have the statements.

6 Concluding remarks

We close this paper giving further generalized entropy and divergence by the use of two-parameter extended logarithmic function.

Definition 6.1 For a continuous and strictly monotonic function ψ on $(0, \infty)$ and $r, q \ge 0$ with $r, q \ne 1$, the (r, q)-quasilinear entropy is defined by

$$I_{r,q}^{\psi}(\mathbf{p}) \equiv \ln_{r,q} \psi^{-1} \left(\sum_{j=1}^{n} p_j \psi \left(\frac{1}{p_j} \right) \right). \tag{37}$$

Here the two-parameter extended logarithmic function [19] is given by $\ln_{r,q}(x) = \ln_q \exp \ln_r(x)$. Correspondingly, the inverse function of $\ln_{r,q}$ is denoted by $\exp_{r,q}$. For $\psi(x) = \ln_{r,q}(x)$ we recover the entropy used in [19, Section 4].

For $\psi(x) = x^{1-r}$, we have an extension of Tsallis entropy

$$I_{r,q}^{x^{1-r}}(\mathbf{p}) = \ln_q \exp H_r(\mathbf{p}) \equiv H_{r,q}(\mathbf{p}).$$

For $\psi(x) = x^{1-p}$, we have

$$I_{r,q}^{x^{1-p}}(\mathbf{p}) = \ln_q \exp H_{r,p}^{S-M}(\mathbf{p}) \equiv H_{r,q,p}(\mathbf{p})$$

that extends Sharma-Mittal entropy to a three-parameter entropy.

Definition 6.2 For a continuous and strictly monotonic function ψ on $(0, \infty)$ and $r, q \ge 0$ with $r, q \ne 1$, the (r, q)-quasilinear divergence is defined by

$$D_{r,q}^{\psi}(\mathbf{p}||\mathbf{r}) \equiv -\ln_{r,q} \psi^{-1} \left(\sum_{j=1}^{n} p_j \psi\left(\frac{r_j}{p_j}\right) \right). \tag{38}$$

For $\psi(x) = x^{1-r}$, we get the following extension of Tsallis relative entropy

$$D_{r,q}^{x^{1-r}}(\mathbf{p}||\mathbf{r}) = \ln_q \exp D_r(\mathbf{p}||\mathbf{r}) \equiv D_{r,q}(\mathbf{p}||\mathbf{r}).$$

For $\psi(x) = x^{1-p}$, we have

$$D_{r,q}^{x^{1-p}}(\mathbf{p}||\mathbf{r}) = \ln_q \exp D_{r,p}^{S-M}(\mathbf{p}||\mathbf{r}) \equiv D_{r,q,p}(\mathbf{p}||\mathbf{r})$$

that extends Sharma-Mittal divergence to a three-parameter divergence.

For a three parametrization extension of the logarithmic function see for instance [12] and the references cited therein. With such extensions the quasilinear entropies can be analogously extended to three parametric classes too. This is not the purpose of the present paper.

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