

Algorithmic proofs of two theorems of Stafford

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Abstract

Two classical results of Stafford say that every (left) ideal of the n -th Weyl algebra A_n can be generated by two elements, and every holonomic A_n -module is cyclic, i.e. generated by one element. We modify Stafford's original proofs to make the algorithmic computation of these generators possible.

1 Introduction

Let k is a field of characteristic 0, and $A_n = A_n(k) = k\langle x_1, \dots, x_n, \partial_1, \dots, \partial_n \rangle$ be the n -th Weyl algebra, which is an associative k -algebra generated by x 's and ∂ 's with the relations $\partial_i x_i = x_i \partial_i + 1$ for all i . This algebra may be thought of as the algebra of linear differential operators with polynomial coefficients.

There are several things that are nice about the Weyl algebra. First of all the dimension theory can be developed for it; this is done, for example, in Chapter 1 of Björk [1]. It is shown that the Gelfand-Kirillov dimension of A_n equals $2n$, moreover, if M is a nontrivial A_n -module, then $n \leq \dim M \leq 2n$. The modules of dimension n (minimal possible dimension) constitute the *Bernstein class*.

One of the distinctive properties of the modules in Bernstein class, which are also called *holonomic* modules, is their finite length. Below we shall show that this property implies that **every holonomic module can be generated by one element**.

Another striking fact, which is very simple to state, but quite hard to prove, is that **for every left ideal of A_n there exist 2 elements that generate it**.

Both statements were proved by Stafford in [6], also these results appear in [1]. Unfortunately, the arguments given by Stafford can't be converted to algorithms straightforwardly. There are several obstacles to this, many of which one can overcome with the theory of Gröbner bases for Weyl algebras. However, the main difficulty is that both proofs contain an operation of taking an irreducible submodule of an A_n -module. To our best knowledge, there doesn't exist an algorithm for this; moreover, even if such algorithm is invented one should expect it to be quite involved.

We were able to modify the original proofs in such a way that computations are possible and implemented the corresponding algorithms in the computer algebra system *Macaulay 2* [2].

We have to mention that in their recent paper [3] Hillebrand and Schmale construct another effective modification of Stafford's proof which leads to an algorithm. We shall discuss the differences of their and our approaches in the last section.

2 Notation Table

For the convenience of the reader we provide the notation lookup table. All of the symbols listed below show up sooner or later in the paper along with more detailed definitions.

k	is	a (commutative) field of characteristic 0,
A_r	=	$A_r(k) = k\langle x_1, \dots, x_r, \partial_1, \dots, \partial_r \rangle$,
A	is	a simple ring of infinite length as a left module over itself,
D	is	a skew field of characteristic 0,
K	is	a commutative subfield of D ,
S	=	$D(x)\langle \partial \rangle$,
$S^{(m)}$	=	$S\varepsilon_1 + \dots + S\varepsilon_m$, a free S -module of rank m ,
$\delta_1, \dots, \delta_m$	is	a finite set of K -linearly independent elements in $K\langle x, \partial \rangle$,
$\sigma(\alpha, f)$	=	$\sum_{i=1}^m \alpha \delta_i f \varepsilon_i \in S^{(m)}$, ($\alpha \in S, f \in K\langle x, \partial \rangle$),
$P(\alpha, f)$	=	$S\sigma(\alpha, f)$, ideal of $S^{(m)}$,
\mathcal{D}_r	is	the quotient ring of A_r ,
\mathcal{R}_r	=	$\mathcal{D}_r(x_{r+1}, \dots, x_n)\langle \partial_{r+1}, \dots, \partial_n \rangle$,
\mathcal{S}_r	=	$\mathcal{D}_r(x_{r+1}, \dots, x_n)\langle \partial_{r+1} \rangle$.

With exception of some minor changes we tried to stick to the notation in [1].

3 Preliminaries

Several useful properties of Weyl algebras are discussed in this section. Also, we introduce a few rings that will come handy later on.

3.1 A_n is simple

To see that A_n is simple, i.e. has no nontrivial two-sided ideals, we notice that, for $f = \sum_i x^{\alpha_i} \partial^{\beta_i} \in A_n \setminus \{0\}$ in the standard form, $df/dx_r = \partial_r f - f \partial_r$ for $r = 1, \dots, n$, where $\partial f / \partial x_r$ is the formal derivative of the above expression of f with respect to x_r . Similarly, $df/d\partial_r = f x_r - x_r f$ for the formal derivative

with respect to ∂_r . Note that these formal derivatives as well as all the multiple derivatives of f belong to the two-sided ideal $A_n f A_n$.

Now assume $x^\alpha \partial^\beta$ is the leading term of f with respect to some total degree monomial ordering. We are going to perform $|\alpha| + |\beta|$ differentiations: for all $i = 1, \dots, n$ differentiate f α_i times with respect to x_i and β_i times with respect to ∂_i . Under such operation the leading term becomes equal to $\prod_{i=1}^n \alpha_i! \beta_i!$ and all the other terms vanish. Since the derivatives of f don't leave $A_n f A_n$, we showed that there is a simple algorithm to find such $s_i, r_i \in A_n$ that

$$\sum_{i=1}^m s_i f r_i = 1.$$

Hence, $A_n f A_n = A_n$, so A_n is simple.

3.2 A_n is an Ore domain

Proposition 1 *A_n is an Ore domain, i.e. $A_n f \cap A_n g \neq 0$ and $f A_n \cap g A_n \neq 0$ for every $f, g \in A_n \setminus \{0\}$.*

Proof. See the proof of Proposition 8.4 in Björk [1]. ■

Let us point out that using Gröbner bases methods (see next subsection) we can find a left(right) common multiple of $f, g \in A_n \setminus \{0\}$, in other words we can find a nontrivial solution to the equations $af = bg$ and $fa = gb$ where a and b are unknowns.

3.3 Gröbner bases in A_n

As we mentioned before, the notion of Gröbner basis of a (left) ideal can be defined for Weyl algebras in the same way as it is defined in the case of polynomials. Moreover, Buchberger algorithm for computing Gröbner bases works, leading to algorithms for computing intersections of ideals, kernels of maps, syzygy modules, etc. A good reference on Gröbner bases for algebras of solvable type is [4].

3.4 More rings

There is a quotient ring D associated to every Ore domain A . Ring D is a skew field that can be constructed both as the ring of left fractions $a^{-1}b$ and as the ring of right fractions cd^{-1} , where $a, b, c, d \in A$. There is a detailed treatment of this issue in [1].

Let D be a skew field, we will be interested in the ring $S = D(x)\langle\partial\rangle$, which is a ring of differential operators with coefficients in $D(x)$. It is easy to see that S is simple.

Since the Weyl algebra A_r is an Ore domain, we can form its quotient ring, which we denote by \mathcal{D}_r . The S we are going to play with is $\mathcal{S}_r = \mathcal{D}_r(x_{r+2}, \dots, x_n)(x_{r+1})\langle\partial_{r+1}\rangle$. Let us state without proof a proposition which shall help us to compute Gröbner bases in \mathcal{S}_r .

Proposition 2 *Let $F = \{f_1, \dots, f_k\} \subset A_n$ is a generating set of left ideal I of S_r . Compute a Gröbner basis $G = \{g_1, \dots, g_m\}$ of $A_n \cdot F$ with respect to any monomial ordering eliminating ∂_{r+1} . Then G is contained in $S_r \cap A_n$ and is a Gröbner basis of I .*

4 Holonomic modules are cyclic

In this section we consider a simple ring A such that A has finite length as a left module over itself. Note that A_n is such a ring.

Theorem 3 *Every left A -module M of finite length is cyclic. In particular every holonomic A_n module is cyclic.*

Suppose we know how to compute a cyclic generator for every module M' of length less than l . For length 0 such generator would be 0.

Consider a module M of length l . Take $0 \neq \alpha \in M$. If $M = A\alpha$ then we are done. If not then since $l(M/A\alpha) < l$ by induction we can find β such that its image in $M/A\alpha$ is a cyclic generator. Now $M = A \cdot \{\alpha, \beta\}$ and what we need to prove is

Lemma 4 *Let M be a left A -module of finite length and $\alpha, \beta \in M$. Then there exists $\gamma \in M$ such that $A\gamma = A\alpha + A\beta$.*

Proof. Define two functions l_1 and l_2 for pair (α, β) .

$$\begin{aligned} l_1(\alpha, \beta) &= \text{length}(A\beta) \\ l_2(\alpha, \beta) &= \text{length}((A\alpha + A\beta)/A\alpha). \end{aligned}$$

Let also introduce an order $<$ on the set of pairs $(\alpha, \beta) \in M \times M$:

$$\begin{aligned} (\alpha', \beta') < (\alpha, \beta) &\Leftrightarrow (l_1(\alpha', \beta'), l_2(\alpha', \beta')) <_{lex} (l_1(\alpha, \beta), l_2(\alpha, \beta)) \\ &\Leftrightarrow l_1(\alpha', \beta') < l_1(\alpha, \beta) \\ &\quad \text{OR } (l_1(\alpha', \beta') = l_1(\alpha, \beta) \text{ AND } l_2(\alpha', \beta') < l_2(\alpha, \beta)) \end{aligned}$$

Suppose for any pair $(\alpha', \beta') < (\alpha, \beta)$, we can find $\gamma' \in M$ such that $A\gamma' = A \cdot \{\alpha', \beta'\}$.

Let the ideals $L(\alpha)$ and $L(\beta)$ in A be the annihilators of α and β respectively. Since $\text{length}(A) = \infty$, we know that $L(\alpha) \neq 0$; pick any element $0 \neq f \in L(\alpha)$. Since A is simple we can find $s_i, r_i \in A$, $i = 1, \dots, M$ such that

$$\sum_{i=1}^m s_i f r_i = 1. \tag{1}$$

Consider two cases:

1. There is some $r = r_i$ such that $L(\beta) + L(\alpha)r = A$.

2. The opposite is true.

Case 1. We can write $1 = E_\alpha r + E_\beta$ for some $E_\alpha, E_\beta \in A$ such that $E_\alpha \alpha = 0$ and $E_\beta \beta = 0$. Let $\gamma = \alpha + r\beta$.

Now we can get β from γ :

$$\beta = (E_\alpha r + E_\beta) \beta = E_\alpha r \beta = E_\alpha \alpha + E_\alpha r \beta = E_\alpha \gamma.$$

Hence $\beta \in A\gamma$ and since $\alpha = \gamma - r\beta$ the module $M = A\alpha + A\beta$ is indeed generated by γ .

Case 2. From (1) it follows that $\sum L(\beta) + A fr_i = A$, hence, $\sum A(fr_i \beta) = A\beta$, so there is $r = r_i$ such that

$$A(fr\beta) \not\subseteq A\alpha. \quad (2)$$

Since we are not in case 1, $L(\beta) + A fr \subset L(\beta) + L(\alpha)r \neq A$. Take this modulo $L(\beta)$ to get

$$A(fr\beta) \cong (L(\beta) + A fr) / L(\beta) \subsetneq A / L(\beta) \cong A\beta, \quad (3)$$

so $A(fr\beta)$ is proper in $A\beta$.

The last statement implies $l_1(\alpha, fr\beta) < l_1(\alpha, \beta)$, hence, $(\alpha, fr\beta) < (\alpha, \beta)$, so by induction hypothesis we can find $\gamma' \in M$ such that $A\gamma' = A(fr\beta) + A\alpha$.

Now (2) guarantees that $l_2(\gamma', \beta) < l_2(\alpha, \beta)$, and by induction we can find γ for which

$$A\gamma = A\gamma' + A\beta = A(fr\beta) + A\alpha + A\beta = A\alpha + A\beta.$$

■

Remark 5 *There is an algorithm that finds a cyclic generator for a holonomic left module over a Weyl algebra, since every step in the proof of the Lemma 4 is computable. The most non-trivial and time consuming operation is producing the annihilators $L(\alpha + r\beta)$ and $L(fr\beta)$ in the proof of Lemma 4 provided $L(\alpha)$ and $L(\beta)$. This is done using Gröbner bases technique.*

We have programmed the algorithm corresponding to the proof of Theorem 3 using *Macaulay 2*.

Example. Let us view the ring of polynomials $k[x]$ as an A_1 -module under the natural action of differential operators. It has an irreducible module, because starting with a nonzero polynomial f we can obtain a nonzero constant by differentiating it $\deg(f)$ times. The module $M = k[x]^3$ is the direct sum of 3 copies of $k[x]$, is holonomic ($\text{length}(M) = 3$) and is generated by vectors $(1, 0, 0), (0, 1, 0), (0, 0, 1)$. Our algorithm produces a cyclic generator $\gamma = (x^2, x, 1)$ and its A_1 -annihilator $L(\gamma) = A_1 \partial^3$.

5 Ideals are 2-generated

In this section we give an effective proof of

Theorem 6 *Every left ideal of the Weyl algebra A_n can be generated by two elements.*

Proof for A_1 . In this case the theorem follows from the fact that module A_1/J is holonomic for any nonzero ideal J of A_1 .

Indeed, let I be a left ideal of A_1 . Pick $f \in I$ and set $J = A_1 f$. Then I/J is a submodule of the holonomic module A_1/J , hence, is holonomic. By Theorem 3 there is $\bar{g} \in I/J$ such that $A_1 \bar{g} = I/J$. Find a lifting $g \in A_1$ such that $\bar{g} = g \bmod J$. Elements f and g generate I . ■

However, the theorem for $n > 1$ makes a much tougher challenge.

5.1 Lemmas for S

Let us explore some properties of $S = D(x)\langle\partial\rangle$, the ring of linear differential operators with coefficients in rational expressions in x over a skew field D .

Let K be a commutative subfield of D , let $\delta_1, \dots, \delta_m$ be a finite set of K -linearly independent elements in $K\langle x, \partial \rangle \subset S$, and let $S^{(m)} = S\varepsilon_1 + \dots + S\varepsilon_m$ be a free S -module of rank m .

Also define $\sigma(\alpha, f) \in S^{(m)}$ to be the following sum $\sigma(\alpha, f) = \sum_{i=1}^m \alpha \delta_i f \varepsilon_i$, and $P(\alpha, f) = S\sigma(\alpha, f)$ the submodule of $S^{(m)}$ generated by $\sigma(\alpha, f)$. Note that $\sigma(\alpha, f)$ is S -linear in α and respects addition in f .

Lemma 7 *Let $0 \neq \alpha \in S$ and let M be an S -submodule of $S^{(m)}$ generated by $\{\sigma(\alpha, f) | f \in K\langle x, \partial \rangle\}$. Then $M = S^{(m)}$.*

Proof. Without loss of generality let us assume that $\alpha \in D\langle x, \partial \rangle$: if not we can always find such $p \in D[x]$ that $p\alpha \in D\langle x, \partial \rangle$.

Fix a monomial ordering that respects the total degree in x and ∂ . For vector $v = \sum v_i \varepsilon_i \in (D\langle x, \partial \rangle)^{(m)}$ denote by $\text{lm}(v)$ the largest of the the leading monomials of the components v_i of v in this ordering.

Now start with vector $v = v^{(0)} = \sigma(\alpha, 1)$; its components $v_i = \alpha \delta_i$ are D -linearly independent. Note that computing expressions $\pi(v) = \partial v - v\partial$ and $\chi(v) = vx - xv$ has an effect of differentiating each component of v formally with respect to x and ∂ respectively. These operations lower the total degree of v by 1 if the differentiation is done with respect to a variable that is present in $\text{lm}(v)$. Also, it is not hard to see that they keep us in module M ; for example, for v_0 we have $\pi(v_0) = \partial v_0 - v_0 \partial = \partial \sigma(\alpha, 1) - \sigma(\alpha, \partial)$.

Run the following algorithm: initialize $v := v_0$, while $\text{lm}(v)$ contains an x set $v := \pi(v)$, then while $\text{lm}(v)$ contains a ∂ we set $v := \chi(v)$. Since each step lowers the total degree of v by 1, this procedure terminates producing vector $w \in M$ of total degree 0.

Hence, $w = w_{i_1}\varepsilon_{i_1} + \dots + w_{i_t}\varepsilon_{i_t}$ where $0 \neq w_{i_j} \in D$ for $j = 1, \dots, t$. Via multiplying on the left by the inverse of w_{i_1} we can get the relation

$$\varepsilon_{i_1} = a_2\varepsilon_{i_2} + \dots + a_t\varepsilon_{i_t} \quad (4)$$

with $a_j \in D$ for $j = 2, \dots, t$.

Now take $v^{(0)}$ and reduce it using (4). We get vector $v^{(1)}$ whose i_1 -th component is 0 and the remaining components are D -linearly independent, since the components of $v^{(0)}$ are.

Repeat the above algorithm for $v = v^{(1)}$ and so on. At the end we get a vector which is a scalar multiple of ε_i for some i , hence $e_i \in M$. Using relations (4) we see that all basis vectors ε_j , for $j = 1, \dots, m$, are in M . ■

Remark 8 From the proof it follows that given a submodule M of $S^{(m)}$ and $\alpha \in S$ one can find $f \in K\langle x, \partial \rangle$ such that $\sigma(\alpha, f) \notin M$ algorithmically.

The next lemma is central in the proof of the result. Note that every step of the proof of the lemma can be carried out algorithmically.

Lemma 9 Let M be an S -submodule of $S^{(m)} = S\varepsilon_1 + \dots + S\varepsilon_m$ such that $\text{length}(S^{(m)}/M) < \infty$. We can find $f \in K\langle x, \partial \rangle$ such that $S^{(m)} = M + P(\alpha, f)$.

Proof. Let $l = \text{length}(S^{(m)}/M)$. Assume the assertion is proved for all M' such that $\text{length}(S^{(m)}/M') < l$. Remark 8 says that we can find an $f \in K\langle x, \partial \rangle$ such that $\sigma(\alpha, f)$ doesn't belong to M .

For $t \in S$, $g \in K\langle x, \partial \rangle$ let us define two S -modules

$$\begin{aligned} N_1 &= M + P_1, \text{ where } P_1 = P(\alpha, g), \\ N_2 &= M + P_2, \text{ where } P_2 = P(t\alpha, g). \end{aligned}$$

Claim. There is a module M' such that $M \subset M' \subset M + P(\alpha, f)$, $t \in S$, and $g \in K\langle x, \partial \rangle$ for which

$$\begin{aligned} t\sigma(\alpha, f) &\in M, \\ M' + P(t\alpha, g) &= S^{(m)}, \\ N_1 &= N_2. \end{aligned}$$

To prove this we employ (second) induction on $\text{length}(M'/M)$. We start with $M' = M + P(\alpha, f)$. We can find $0 \neq t \in S$ such that $t\alpha \sum \delta_i f \varepsilon_i \in M$; it follows from S being Ore. By the first induction hypothesis, for M' and $t\alpha$ there exists $g \in K\langle x, \partial \rangle$ such that $M' + P(t\alpha, g) = S^{(m)}$. Notice that $N_1 \supset N_2$ and $M' + P_i = S^{(m)}$ for $i = 1, 2$. Also for $i = 1, 2$ we have

$$S^{(m)}/N_i = (M' + P_i)/(M + P_i) = M'/(M + M' \cap P_i).$$

If $\text{length}(S^{(m)}/N_1) = \text{length}(S^{(m)}/N_2)$ then $N_1 = N_2$ and we are done. We are done as well if $N_1 = S^{(m)}$. If both conditions above fail, by looking at the

right hand side of 5.1 we determine that $M'' = M + M' \cap P_1$ both contains M and is contained in M' properly, plus $\text{length}(M''/M) < \text{length}(M'/M)$. Set $M' := M''$ and repeat the above procedure.

To finish the proof of the lemma we take M', t, g as in the claim and assert that $N' = M + P(\alpha, f + g)$ equals $S^{(m)}$. Indeed, $\sigma(t\alpha, f + g) = t\sigma(\alpha, f) + \sigma(t\alpha, g) = \sigma(t\alpha, g)$ modulo M , so $N_2 \subset N'$. But $N_1 = N_2$, thus $\sigma(\alpha, g) \in N'$, hence, $\sigma(\alpha, f) = \sigma(\alpha, f + g) - \sigma(\alpha, g) \in N'$. Now we see that $M' \subset N'$ and $P_2 \subset N'$. Since $M' + P_2 = S^{(m)}$, we proved $N' = S^{(m)}$. ■

5.2 Lemmas for \mathcal{R}_r

At this stage we shall specify the components in the definition of $S = D(x)\langle\partial\rangle$. We set $D = \mathcal{D}_r(x_{r+2}, \dots, x_n)$, $x = x_{r+1}$ and $\partial = \partial_{r+1}$, so that new S is equal to $\mathcal{S}_r = \mathcal{D}_r(x_{r+1}, x_{r+2}, \dots, x_n)\langle\partial_{r+1}\rangle$ which is a subring of \mathcal{R}_r . Also the commutative subfield K of D that showed up before is replaced by the k , the coefficient field from the definition of $A_n = A_n(k)$.

Proposition 10 *Let $\delta_1, \dots, \delta_m$ be a finite set of K -linearly independent elements in $K\langle x_{r+1}, \partial_{r+1} \rangle$ and let $0 \neq \rho \in A_{r+1}[x_{r+2}, \dots, x_n]$. Let $S^{(m+1)} = S\varepsilon_0 + S\varepsilon_1 + \dots + S\varepsilon_m$ be a free S -module of rank $m+1$. And let $S^{(m+1)}\rho \subset S^{(m+1)}$ be its S -submodule generated by $\{\rho\varepsilon_0, \rho\varepsilon_1, \dots, \rho\varepsilon_m\}$. Then there exists some $f \in K$ such that*

$$S^{(m+1)} = S^{(m+1)}\rho + S(\varepsilon_0 + \delta_1 f \varepsilon_1 + \dots + \delta_m f \varepsilon_m).$$

Proof. Follows from Lemma 9 ■

Lemma 11 *Let $q \in A_r[x_{r+1}, \dots, x_n]$ and let a_1, \dots, a_t be a finite set in A_n .*

Then there exists some $0 \neq \rho \in A_r[x_{r+1}, \dots, x_n]$ such that $\rho a_j \in A_n q$ for all j .

Proof. See the proof of Lemma 8.5 in Björk [1]. ■

Let us point out that once we know the statement of the lemma is true, we can compute the required ρ by finding a Gröbner basis of the module of syzygies of the columns of the matrix

$$\begin{pmatrix} a_1 & q & 0 & \dots & 0 \\ a_2 & 0 & q & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots \\ a_t & 0 & 0 & \dots & q \end{pmatrix}$$

with respect to a monomial order that eliminates $\partial_{r+1}, \dots, \partial_n$ and such that $\varepsilon_1 > \varepsilon_2 > \dots > \varepsilon_{t+1}$ where $\varepsilon_1, \varepsilon_2, \dots, \varepsilon_{t+1}$ is the basis (ε_i corresponds to the i -th column) of the free module A_n^{t+1} containing our submodule of syzygies. Such Gröbner basis is guaranteed (by Lemma 11) to contain some syzygy producing the relation $\rho\varepsilon_1 + b_2\varepsilon_2 + \dots + b_{t+1}\varepsilon_{t+1} = 0$ where $\rho \in A_r[x_{r+1}, \dots, x_n]$, $b_i \in A_n$ for $i = 2, \dots, n$. It is not hard to see that this is the ρ we need.

Lemma 12 Let $0 \neq q \in A_{r+1}[x_{r+2}, \dots, x_n]$ and let $u, v \in A_n$ with $v \neq 0$. Then there is some $f \in A_n$ such that $\mathcal{R}_r = \mathcal{R}_r q + \mathcal{R}_r(u + vf)$.

Proof. Consider the following subring of A_n obtained by "removing" x_{r+1} and ∂_{r+1} :

$$A_{r+1}^\wedge = k\langle x_1, \dots, x_r, x_{r+2}, \dots, x_n, \partial_1, \dots, \partial_r, \partial_r + 2, \dots, \partial_n \rangle.$$

Now $A_n = A_{r+1}^\wedge \otimes_k k\langle x_{r+1}, \partial_{r+1} \rangle$, so we can write $v = \delta_1 g_1 + \dots + \delta_m g_m$ where $\delta_1, \dots, \delta_m$ are elements of $k\langle x_{r+1}, \partial_{r+1} \rangle$ linearly independent over k and $g_1, \dots, g_m \in A_{r+1}^\wedge$. The ring A_{r+1}^\wedge is simple, since it is a Weyl algebra, thus we can find such $h_1, \dots, h_l \in A_{r+1}^\wedge$ that

$$A_{r+1}^\wedge = \sum_{i=1}^m \sum_{j=0}^l A_{r+1}^\wedge g_i h_j.$$

Since A_{r+1}^\wedge is a subring of \mathcal{R}_r it means that $\mathcal{R}_r = \sum \sum \mathcal{R}_r g_i h_j$.

Sublemma. For any $b_1, \dots, b_m \in A_{r+1}^\wedge$ there exists some $f \in k\langle x_{r+1}, \partial_{r+1} \rangle$ such that

$$\mathcal{R}_r q + \mathcal{R}_r u + \mathcal{R}_r b_1 + \dots + \mathcal{R}_r b_m = \mathcal{R}_r q + \mathcal{R}_r(u + \delta_1 f b_1 + \dots + \delta_m f b_m).$$

Proof. It follows from Lemma 11 that there is $0 \neq \rho \in A_r[x_{r+1}, \dots, x_n]$ such that $\rho b_1, \dots, \rho b_m \in A_n q$ as well as $\rho u \in A_n q$. With the help from Proposition 10 we get $f \in k\langle x_{r+1}, \partial_{r+1} \rangle$ such that $S^{(m+1)} = S^{(m+1)}\rho + S(\varepsilon_0 + \delta_1 f \varepsilon_1 + \dots + \delta_m f \varepsilon_m)$ and since S is a subring of \mathcal{R}_r we have

$$\mathcal{R}_r^{(m+1)} = \mathcal{R}_r^{(m+1)}\rho + \mathcal{R}_r(\varepsilon_0 + \delta_1 f \varepsilon_1 + \dots + \delta_m f \varepsilon_m). \quad (5)$$

Now map $\varepsilon_0 \mapsto u$ and $\varepsilon_i \mapsto b_i$ for all i ; this map from $\mathcal{R}_r^{(m+1)}$ to \mathcal{R}_r has its image equal to $\mathcal{R}_r q + \mathcal{R}_r u + \mathcal{R}_r b_1 + \dots + \mathcal{R}_r b_m$ and maps the right hand side of (5) to a subset of $\mathcal{R}_r q + \mathcal{R}_r(u + \delta_1 f b_1 + \dots + \delta_m f b_m)$, because $\rho u, \rho b_1, \dots, \rho b_m \in A_n q$. Moreover these two expressions are equal, since it is easy to see that the latter is contained in the former as well. ■

Proof of lemma continued. We apply our Sublemma to $b_i = g_i h_1$ ($i = 1, \dots, m$) to get $f_1 \in k\langle x_{r+1}, \partial_{r+1} \rangle$ such that

$$\mathcal{R}_r q + \mathcal{R}_r u + \sum_{j=1}^m \mathcal{R}_r g_j h_1 = \mathcal{R}_r q + \mathcal{R}_r(u + \sum_{j=1}^m \delta_j f_1 g_j h_1).$$

Since $v = \delta_1 g_1 + \dots + \delta_m g_m$ and since f_1 commutes with all g_i , the last equation transforms into

$$\mathcal{R}_r q + \mathcal{R}_r u + \sum \mathcal{R}_r g_i h_1 = \mathcal{R}_r q + \mathcal{R}_r(u + v f_1 h_1).$$

Now reapply the Sublemma with u replaced by $u + v f_1 h_1$ and $b_i = g_i h_2$ ($i = 1, \dots, m$). As in the first step we get

$$\begin{aligned} & \mathcal{R}_r q + \mathcal{R}_r u + \sum \mathcal{R}_r g_i h_1 + \sum \mathcal{R}_r g_i h_2 \\ &= \mathcal{R}_r q + \mathcal{R}_r(u + v f_1 h_1) + \sum \mathcal{R}_r g_i h_2 \\ &= \mathcal{R}_r q + \mathcal{R}_r(u + v f_1 h_1 + f_2 h_2) \end{aligned}$$

for some $f_2 \in k\langle x_{r+1}, \partial_{r+1} \rangle$. After l many steps we arrive at

$$\mathcal{R}_r = \mathcal{R}_r q + \mathcal{R}_r u + \sum_{i=1}^m \sum_{j=1}^l \mathcal{R}_r g_i h_j = \mathcal{R}_r q + \mathcal{R}_r (u + v \sum_{j=1}^l f_j h_j),$$

which proves the lemma with $f = f_1 h_1 + \dots + f_l h_l$. ■

The following lemma follows from the previous one.

Lemma 13 *Let $0 \leq r \leq n-1$ and let $0 \neq q \in A_{r+1}[x_{r+2}, \dots, x_n]$ and let $u, v \in A_n$ with $v \neq 0$. Then there is some $f \in A_n, q' \in A_r[x_{r+1}, \dots, x_n]$ such that $q' \in A_n q + A_n(u + vf)$.*

Proof. It is easy to see that this lemma is equivalent to the previous one. ■

5.3 Final chords

Proposition 14 (r) *Let $0 \leq r \leq n$, there is some $q_r \in A_r[x_{r+1}, \dots, x_n]$ and $d_r, e_r \in A_n$ such that $q_r c \in A_n(a + d_r c) + A_n(b + e_r c)$.*

Proof. The statement is true for $r = n$, since A_n is Ore and $A_n c \cap (A_n a + A_n b)$.

Fix r . Assume that the statement is true for $r+1, \dots, n$, then there exist $q_{r+1}, d_{r+1}, e_{r+1}$ such that $q_{r+1} c \in A_n a' + A_n b'$, where $a' = a + d_{r+1} c$ and $b' = b + e_{r+1} c$. Hence we can write $q_{r+1} c = h_1 a' + h_2 b'$, where we can take $h_1 h_2 \neq 0$ since $A_n a' \cap A_n b' \neq 0$. Also since $h_1 A_n \cap h_2 A_n \neq 0$ we can also find g_1, g_2 satisfying $h_1 g_1 + h_2 g_2 = 0$, and since $A_n q_{r+1} c \cap A_n b' \neq 0$ there are s, t such that $s q_{r+1} c = t b'$. Using Lemma 13 to $q = q_{r+1}$ with $u = 0$ and $v = t g_2$, we get $q_r = q'$ and f such that $q_r = p_1 q_{r+1} + p_2 t g_2 f$ for some p_1, p_2 . Summarizing, there exist such $h_1, h_2, g_1, g_2, s, t, p_1, p_2 \in A_n \setminus \{0\}$ that

$$\begin{aligned} q_r &= p_1 q_{r+1} + p_2 t g_2 f \\ q_{r+1} c &= h_1 a' + h_2 b' \\ h_1 g_1 + h_2 g_2 &= 0 \\ s q_{r+1} c &= t b' \end{aligned}$$

Using these 4 equations, make the following calculation: (In each section the underlined terms sum up to 0.)

$$\begin{aligned} q_r c &= p_1 q_{r+1} c + p_2 t g_2 f c \\ &= p_1 q_{r+1} c - \underline{p_2 s q_{r+1} c} \\ &+ p_2 t g_2 f c + \underline{p_2 t b'} \\ &= (p_1 - p_2 s) q_{r+1} c + p_2 t (b' + g_2 f c) \\ &= (p_1 - p_2 s) (h_1 a' + h_2 b') + p_2 t (b' + g_2 f c) \\ &= (p_1 - p_2 s) h_1 a' + \underline{(p_1 - p_2 s) h_1 g_1 f c} \\ &+ (p_1 - p_2 s) h_2 b' + \underline{(p_1 - p_2 s) h_2 g_2 f c} + p_2 t (b' + g_2 f c) \\ &= (p_1 - p_2 s) h_1 (a' + g_1 f c) + ((p_1 - p_2 s) h_2 + p_2 t) (b' + g_2 f c). \end{aligned}$$

Thus, with $d_r = d_{r+1} + g_1 f c$ and $e_r = e_{r+1} + g_2 f c$ the conclusion of the proposition holds. ■

The proposition above (for $r = 0$) shows that by “elimination” of variables ∂_i one at a time we can get such $d, e \in A_n$ that $q_0 c \in A_n(a + dc) + A_n(b + ec)$ where $q_0 \in k[x_1, \dots, x_n]$. This proves a 50% version of Theorem 6:

Theorem 15 *Every ideal of $k(x_1, \dots, x_n)\langle \partial_1, \dots, \partial_n \rangle$ can be generated by two elements.*

To go other 50% of the way one has to do a similar kind of “elimination” of x_i -s. This amounts to making copies of all lemmas that we stated for a slightly different set of rings. The trickiest part is considering ring $\mathcal{S}_r' = k(x_1, \dots, x_r)\langle x_{r+1}, \partial_{r+1} \rangle$ instead of \mathcal{S}_r . In other words instead of a ring of type $D(x)\langle \partial \rangle$ where D is a skew field, we have to consider the first Weyl algebra $A_1(\mathcal{K})$ where \mathcal{K} is a (commutative) field. Fortunately, analogues of Lemmas 7 and 9 for the latter ring can be effectively proved along the same lines.

Examples. (1) Consider A_3 . For $a = \partial_1, b = \partial_2, c = \partial_3$ one can show that $A_3 \cdot \{a, b, c\} = A_3 \cdot \{a, b + x_1 c\}$. Indeed, the following calculation displays it:

$$c = (-x_1 \partial_3 - \partial_2)a + \partial_1(b + x_1 c).$$

(2) Another example is produced by our algorithm implemented in *Macaulay 2*. Let $a = \partial_1 + x_3, b = \partial_2^2 + x_2 + x_3^2, c = \partial_3 + x_1$. Then the ideal $A_3 \cdot \{a, b, c\}$ is generated by $\partial_1 + x_3$ and $\partial_2^2 + (x_1^2 x_3 + x_1) \partial_3 + x_1^3 x_3 + x_1^2 + x_3^2 + x_2$.

(3) In case of A_1 we can construct a more efficient algorithm based on the proof of Theorem 6 given for this special case. Here is a *Macaulay 2* script computing 2 generators for the annihilating ideal $I \subset A_1(\mathbb{Q})$ of the set of polynomials $\{ax^4 + bx^6 + cx^8 + dx^{10} \mid a, b, c, d \in \mathbb{Q}\} \subset \mathbb{Q}[x]$.

```
i1 : load "D-modules.m2"; load "stafford.m2";
i3 : R = QQ[x,D, WeylAlgebra=>{x=>D}];
i4 : L = {4,6,8,10};
i5 : I = ideal gens gb intersect apply(L, i->PolyAnn x^i);
o5 = ideal (x D4 - 22x3 D3 + 207x2 D2 - 975x*D + 1920, D11, ...)
o5 : Ideal of R
i6 : time J = ideal stafford I
-- used 73.08 seconds
o6 = ideal (x D4 - 22x3 D3 + 207x2 D2 - 975x*D + 1920,
          x3 D15 + x2 D15 + 15x2 D14 + 9x*D14 + 58x*D13
          + 15D13 + 50D12 + D11)
o6 : Ideal of R
```

```
i7 : I == J
o7 = true
```

6 Conclusion

The implementations of the algorithms constructed along the lines of the proofs of Theorems 3 and 6 in *Macaulay 2* work only on rather small examples for quite obvious reason: the expression swell in Gröbner bases computations.

Let us comment on the differences of algorithm of Hillebrand and Schmale [3] and ours. Their algorithm takes care of (weaker) Theorem 15. As a step it includes enumerating a certain infinite subset of polynomials in one variable and testing them to satisfy a certain property, where the testing procedure involves Gröbner bases computations. Although we believe that their argument could be extended to build an algorithm for 100% of Stafford’s theorem, it looks as the “test set” for the remaining 50% will be significantly more complicated. Hence, our constructive approach at every step of the algorithm seems to be more practical. Having programmed Hillebrand and Schmale’s algorithm as well, we have to point out, that it faces the same type of expression swell as our program, hence the comparison of performance is just a theoretical question at this point.

Finally, let us mention that the algorithm for finding a cyclic generator of a holonomic module is already included in the *D-modules package for Macaulay 2* [5]; eventually, the algorithms for finding two generators of a A_n -ideal will be added to the package as well.

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