# 'BUREAUCRATIC' SET SYSTEMS, AND THEIR ROLE IN PHYLOGENETICS 

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#### Abstract

We say that a collection $\mathcal{C}$ of subsets of $X$ is bureaucratic if every maximal hierarchy on $X$ contained in $\mathcal{C}$ is also maximum. We characterise bureaucratic set systems and show how they arise in phylogenetics. This framework has several useful algorithmic consequences: we generalize some earlier results and derive a polynomial-time algorithm for a parsimony problem arising in phylogenetic networks.


## 1. Bureaucratic sets and their characterization

We first recall some standard phylogenetic terminology (for more details, the reader can consult [6]). Recall that a hierarchy $\mathcal{H}$ on a finite set $X$ is a collection of sets with the property that the intersection of any two sets is either empty or equal to one of the two sets; we also assume that $X \in \mathcal{H}$.


Figure 1. (a): A rooted tree $T$ with leaf set $X=\{1,2,3,4,5\}$, and with cluster set $c(T)$ being equal to the hierarchy $\mathcal{H}$ consisting of the sets $\{1,2,3\},\{4,5\}$ and the trivial clusters. (b): A binary tree $T$ with a cluster set consisting of $\mathcal{H} \cup\{\{1,2\}\}$. (c): A binary and planar phylogenetic network $\mathcal{N}$ over $X=\{1,2,3,4\}$ with a soft-wired cluster set $\operatorname{sw}(\mathcal{N})$ consisting of $\{1,2\},\{2,3\},\{3,4\},\{1,2,3\},\{2,3,4\}$ and the trivial clusters.

A hierarchy is maximum if $|\mathcal{H}|=2|X|-1$, which is the largest possible cardinality. In this case $\mathcal{H}$ corresponds to the set of clusters $c(T)$ of some rooted binary tree $T$ with leaf set $X$ (a cluster of $T$ is the set of leaves that are separated from the root of the tree by any vertex). A maximum hierarchy necessarily contains $\{x\}$ for each $x \in X$, as well as $X$ itself; we will refer to these $|X|+1$ sets as the trivial clusters of $X$. More generally, any hierarchy containing all the trivial clusters corresponds to the clusters $c(T)$ of a rooted tree $T$ with leaf set $X$ (examples of these concepts are illustrated in Fig. $\mathbb{1}(\mathrm{a}),(\mathrm{b})$ ). Note that a hierarchy $\mathcal{H}$ is maximum if and only if (i) $\mathcal{H}$ contains all the trivial clusters, and (ii) each set $C \in \mathcal{H}$ of size greater than 1 can be written as a disjoint union $C=A \sqcup B$, for two (disjoint) sets $A, B \in \mathcal{H}$.

We now introduce a new notion.

[^0]Definition: We say that a collection $\mathcal{C}$ of subsets of a finite set $X$ is a bureaucracy if (i) $\mathcal{C} \neq \emptyset$ and $\emptyset \notin \mathcal{C}$, and (ii) every hierarchy $\mathcal{H} \subseteq \mathcal{C}$ can be extended to a maximum hierarchy $\mathcal{H}^{\prime}$ such that $\mathcal{H} \subseteq \mathcal{H}^{\prime} \subseteq \mathcal{C}$. In this case, we say that the collection is bureaucratic.

Simple examples of bureaucracies include two extreme cases: the set of clusters of a binary tree, and the set $\mathcal{P}(X)$ of all non-empty subsets of $X$. Notice that $\{\{a\},\{b\},\{c\},\{a, b\},\{a, b, c\}\}$ and $\{\{a\},\{b\},\{c\},\{b, c\},\{a, b, c\}\}$ are both bureaucratic subsets of $X=\{a, b, c\}$ but their intersection, $\{\{a\},\{b\},\{c\},\{a, b, c\}\}$, is not. In particular, for an arbitrary subset $Y$ of $X$ (e.g. $Y=\{\{a\},\{b\},\{c\},\{a, b, c\}\})$, there may not be a unique minimal bureaucratic subset of $X$ containing $Y$.

In the next section we describe a more extensive list of examples, but first we describe some properties and provide a characterization of bureaucracies. In the following lemma, given two sets $A$ and $B$ from $\mathcal{C}$ we say that $B$ covers $A$ if $A \subsetneq B$ and there is no set $C \in \mathcal{C}$ with $A \subsetneq C \subsetneq B$.
Lemma 1. If $\mathcal{C}$ is bureaucratic then:
(i) For any pair $A, B \in \mathcal{C}$, if $B$ covers $A$ then $B-A \in \mathcal{C}$.
(ii) For any $C \in \mathcal{C}$ with $|C|>1$, we can write $C=A \sqcup B$ for (disjoint) sets $A, B \in \mathcal{C}$.

Proof. For Part (i), suppose that $A, B \in \mathcal{C}$ and that $B$ covers $A$. Let $\mathcal{H}=\{A, B\}$. Then $\mathcal{H}$ is a hierarchy that is contained within $\mathcal{C}$ and so there exists a maximum hierarchy $\mathcal{H}^{\prime} \subseteq \mathcal{C}$ that contains $\mathcal{H}$. Note that $A$ must be a maximal sub-cluster of $B$ in $\mathcal{H}^{\prime}$ (as otherwise $B$ does not cover $A$ ) which requires that $B-A$ is a cluster of $\mathcal{H}^{\prime}$ and thereby an element of $\mathcal{C}$.

For Part (ii), observe that the set $\mathcal{H}=\{C\}$ is a hierarchy, and the assumption that $\mathcal{C}$ is bureaucratic ensures the existence of a maximum hierarchy $\mathcal{H}^{\prime} \subseteq \mathcal{C}$ containing $\mathcal{H}$, and so $\mathcal{H}^{\prime}$ contains the required sets $A, B$.

Note that the conditions described in Parts (i) and (ii) of Lemma 1 while they are necessary for $\mathcal{C}$ to be a bureaucracy, are not sufficient. For example, let $X=\{1,2,3,4,5,6\}$ and let $\mathcal{C}$ be the union of

$$
\{\{1,2\},\{3,4\},\{5,6\},\{1,2,3\},\{4,5,6\},\{3,4,5\},\{1,2,6\},\{1,5,6\},\{2,3,4\}\}
$$

with the set of the seven trivial clusters. Then $\mathcal{C}$ satisfies Parts (i) and (ii) of Lemma 11 yet $\mathcal{C}$ is not bureaucratic since $\mathcal{H}=\{\{1,2\},\{3,4\},\{5,6\}\}$ does not extend to a maximum hierarchy on $X$ using just elements from $\mathcal{C}$.

Theorem 2. A collection $\mathcal{C}$ of subsets of $X$ is bureaucratic if and only if it satisfies the following two properties:

- $(\mathrm{P} 1) \mathcal{C}$ contains all trivial clusters of $X$.
- (P2) If $\left\{C_{1}, C_{2}, \ldots, C_{k}\right\} \subseteq \mathcal{C}$ are disjoint and have union $\cup_{i} C_{i}$ in $\mathcal{C}$ then there are distinct $i, j$ such that $C_{i} \cup C_{j} \in \mathcal{C}$.
Proof. First suppose that $\mathcal{C}$ is bureaucratic. Then $\mathcal{C}$ contains a maximal hierarchy; in particular, it contains all the trivial clusters, and so (P1) holds. For (P2), suppose that $\mathcal{C}^{\prime}$ is a collection of $k \geq 3$ disjoint subsets of $X$, each an element of $\mathcal{C}$, and $\bigcup \mathcal{C}^{\prime} \in \mathcal{C}$. Then $\mathcal{H}=\mathcal{C}^{\prime} \cup\left\{\bigcup \mathcal{C}^{\prime}\right\}$ is a hierarchy. Let $\mathcal{H}^{\prime} \subseteq \mathcal{C}$ be a maximal hierarchy on $X$ that contains $\mathcal{H}$ (this exists, since $\mathcal{C}$ is bureaucratic) and let $C$ be a minimal subset of $X$ in $\mathcal{H}^{\prime}$ that contains the union of at least two elements of $\mathcal{C}^{\prime}$. Since $\mathcal{H}^{\prime}$ is a binary hierarchy, and $\bigcup \mathcal{C}^{\prime} \in \mathcal{H}^{\prime}, C$ is precisely the union of exactly two elements of $\mathcal{C}^{\prime}$; since $C \in \mathcal{H}^{\prime} \subseteq \mathcal{C}$, this establishes (P2).

Conversely, suppose that a collection $\mathcal{C}$ of subsets of $X$ satisfies (P1) and (P2), and that $\mathcal{H} \subseteq \mathcal{C}$ is a maximal hierarchy which is contained within $\mathcal{C}$. Suppose that $\mathcal{H}$ is not maximum (we will derive a contradiction). Then $\mathcal{H}$ contains a set $C$ that is the disjoint union of $k \geq 3$ maximal proper subsets $A_{1}, \ldots, A_{k}$, each belonging to $\mathcal{H}$ (and thereby $\mathcal{C}$ ). Applying (P2) to
$\mathcal{C}^{\prime}=\left\{A_{1}, \ldots, A_{k}\right\}$, there exists two sets, say $A_{i}, A_{j}$ for which $A_{i} \cup A_{j} \in \mathcal{C}$. So, if we let $\mathcal{H}^{\prime}=$ $\mathcal{H} \cup\left\{A_{i} \cup A_{j}\right\}$, then we obtain a larger hierarchy containing $\mathcal{H}$ that is still contained within $\mathcal{C}$, which is a contradiction. This completes the proof.

## 2. Examples of bureaucracies

We have mentioned two extreme cases of bureaucracies, namely the set of clusters of a binary $X$-tree and the full power set $\mathcal{P}(X)$. Here are some further examples.
(1) The set of intervals of $[n]=\{1,2, \ldots, n\}$ is a bureaucracy, where an interval is a set $[i, j]=\{k: i \leq k \leq j\}, 1 \leq i \leq j \leq n$.
Proof. Let $\mathcal{C}$ be the set of intervals of $[n]$. Then $\mathcal{C}$ contains the trivial clusters. Also, a disjoint collection $I_{1}, \ldots, I_{k}, k>2$ of intervals has union an interval if and only if every element of $[n]$ between $\min \bigcup I_{j}$ and $\max \bigcup I_{j}$ lies in (exactly) one interval, in which case the union of any pair of consecutive intervals is an interval, so (P2) holds. By Theorem [2, $\mathcal{C}$ is bureaucratic.

Similarly, if we order the elements of $X$ in any fashion, we can define the set of intervals on $X$ for that ordering by this construction (associating $x_{i}$ with $i$ ), and can thus obtain a bureaucracy.

A natural question at this point is the following: Does the extension of intervals in a 1-dimensional lattice (Example 1) to rectangles in a 2-dimensional lattice also necessarily lead to bureaucracies? The answer is 'no' because condition (P2) can be violated due to the existence of subdivisions of integral sized rectangles into $k>2$ disjoint squares of different integral sizes, the union of any two of which must therefore fail to be a rectangle (see e.g. [2]).
(2) Let $T$ be a rooted tree (generally not binary) with leaf set $X$ and let $\mathcal{C}$ be the set of all clusters compatible with all the clusters in $c(T)$. Then $\mathcal{C}$ is bureaucratic.
Proof. We have $\mathcal{C}=\left\{C \subseteq X: C \cap C^{\prime} \in\left\{C, C^{\prime}, \emptyset\right\}\right.$ for all $\left.C^{\prime} \in c(T)\right\} . \mathcal{C}$ is also the set of clusters that occur in at least one rooted phylogenetic $X$ - tree that refines $T$, that is:

$$
\mathcal{C}=\bigcup_{T^{\prime}: c(T) \subseteq c\left(T^{\prime}\right)} c\left(T^{\prime}\right)
$$

Suppose that $\mathcal{H} \subseteq \mathcal{C}$ is a hierarchy on $X$. Then $\mathcal{H} \cup c(T)$ is also a hierarchy on $X$ since every element of $\mathcal{H}$ is compatible with every element of $c(T)$. Let $\mathcal{H}^{\prime}$ be any maximal hierarchy on $X$ containing $\mathcal{H}$. Then since $c(T) \subseteq \mathcal{H}^{\prime}$, we have $\mathcal{H}^{\prime} \subseteq \mathcal{C}$, and so, by definition, $\mathcal{C}$ is a bureaucracy.
(3) Let $\mathcal{C}$ be a collection of subsets of $X$ that includes the trivial clusters and which satisfies the condition:

$$
\begin{equation*}
A, B \in \mathcal{C} \text { and } A \cap B \neq \emptyset \Rightarrow A \cup B \in \mathcal{C} \tag{1}
\end{equation*}
$$

Then $\mathcal{C}$ is bureaucratic if and only if $\mathcal{C}$ satisfies the covering condition in Lemman(i).
Condition (1) is a weakening of the condition required for a 'patchwork' set system on $X$ due to Andreas Dress and Sebastian Böcker (see e.g. [6], where the covering condition of Lemman(i) leads to an 'ample patchwork').
Proof. The 'only if' part follows from Lemma (i). Conversely, suppose that (1) holds for a set system $\mathcal{C}$ that includes all the trivial clusters of $X$ and that satisfies the covering condition of Lemma 1 (i). Suppose that $\mathcal{H} \subseteq \mathcal{C}$ is a maximal hierarchy contained within $\mathcal{C}$. We show that $\mathcal{H}$ is maximum. Suppose that this is not the case - we will derive a contradiction (by constructing a larger hierarchy $\mathcal{H}^{\prime}$ containing $\mathcal{H}$ but still lying within
$\mathcal{C})$. The assumption that $\mathcal{H}$ is not maximum implies that there exists a set $B \in \mathcal{H}$ which is the union of three or more disjoint sets $A_{1}, A_{2}, A_{3}, \ldots, A_{k}$, where $A_{i} \in \mathcal{H}$ (since the rooted tree associated with $\mathcal{H}$ has a vertex of degree $k \geq 3$ ). We consider two cases:
(i) $B$ covers none of the sets from $A_{1}, A_{2}, A_{3}, \ldots, A_{k}$.
(ii) $B$ covers one of the sets from $A_{1}, A_{2}, A_{3}, \ldots, A_{k}$.

We first show that Case (i) cannot arise under Condition (11). Suppose to the contrary Case (i) arises. Then for $i=1, \ldots k$ there exists a set $C_{i} \in \mathcal{C}$ that contains $A_{i}$ and which is covered by $B$. For any pair $i, j$ with $i \neq j$, if $\left(B-C_{i}\right) \cap C_{j}=\emptyset$ then $C_{j} \subseteq C_{i}$. On the other hand, if $\left(B-C_{i}\right) \cap C_{j} \neq \emptyset$ then, by Condition (1), $\left(B-C_{i}\right) \cup C_{j} \in \mathcal{C}$, which means that $B=\left(B-C_{i}\right) \cup C_{j}$ (otherwise $\left(B-C_{i}\right) \cup C_{j}$ an element of $\mathcal{C}$ strictly containing $C_{j}$ and strictly contained by $B$ ) and so $C_{i} \subseteq C_{j}$. Thus Case (i) requires that either $C_{i} \subseteq C_{j}$ or $C_{j} \subseteq C_{i}$, which implies (again by the assumption that $B$ covers $C_{i}$ and $B$ covers $C_{j}$ ) that $C_{i}=C_{j}$. Since this identity holds for all distinct pairs $i, j$ it follows that $C_{1}, C_{2}, \ldots, C_{k}$ are the same set $C$ and this set contains $\bigcup_{i=1}^{k} A_{i}$ (since $A_{i} \subset C_{i}$ ). But then $B=\bigcup_{i=1}^{k} A_{i} \subseteq C$ which contradicts the assumption that $B$ covers $C_{1}(=C)$.

Thus only Case (ii) can arise. In this case, suppose that $B$ covers $A_{i}$. By assumption that $\mathcal{C}$ satisfies the covering condition described in Lemma $\mathbb{1}(\mathrm{i}), B-A_{i} \in \mathcal{C}$ holds, and so we can take $\mathcal{H}^{\prime}=\mathcal{H} \cup\left\{B-A_{i}\right\}$ which provides the required contradiction.
(4) Let $G=(X, E)$ be a connected graph. Let $\mathcal{C}$ be the set of subsets $Y \subseteq X$ such that $G[Y]$ is connected (where $G[Y]$ is the subgraph formed by deleting vertices not in $Y$, together with their incident edges). Then $\mathcal{C}$ is bureaucratic.

Observe that taking $G$ to be a linear graph recovers Example (1).
Proof. First note that $\mathcal{C}$ satisfies (P1), since $G$ itself is connected, as is each vertex by itself. Now suppose that $A_{1}, \ldots, A_{k}, k>2$, are disjoint clusters in $\mathcal{C}$ whose union, $A$, is also in $\mathcal{C}$. As $G[A]$ is connected, at least two clusters $A_{i}, A_{j}$ must contain adjacent vertices, in which case $G\left[A_{i} \cup A_{j}\right]$ is connected and $A_{i} \cup A_{j} \in \mathcal{C}$. The result now follows by Theorem 2 ,

An alternative proof is to apply Example (3) and note that $\mathcal{C}$ satisfies Condition (1) and the covering condition of Lemma 1 (i).

## 3. Algorithmic applications

3.1. Maximum weight hierarchies. In general, the problem of finding the largest hierarchy contained within a set of clusters is NP-hard [3]. The problem becomes trivial in a bureaucratic collection since all maximal hierarchies are maximum. Less obvious, however, is the fact that finding a hierarchy with maximum weight can also be solved in polynomial time.

Theorem 3. Let $\mathcal{C}$ be a bureaucratic collection of clusters on $X$ and let $w: \mathcal{C} \longrightarrow \mathbb{R}$ be a weight function on $\mathcal{C}$. The problem of finding the hierarchy $\mathcal{H} \subseteq \mathcal{C}$ such that $w(\mathcal{H})=\sum_{A \in \mathcal{H}} w(A)$ is maximized can be solved in polynomial time.
Proof. If there are any clusters $A \in \mathcal{C}$ with negative weight $w(A)$ then set their weights to zero. It follows then that the weight of any maximum hierarchy $\mathcal{H} \subseteq \mathcal{C}$ equals the weight of the maximum weight hierarchy contained within $\mathcal{H}$. The 'Hunting for Trees' algorithm of [1 can now be used to recover the maximum hierarchy of maximum weight.
3.2. Parsimony problems on networks. Consider a set $\mathcal{C}$ of clusters on $X$ and let $f: X \rightarrow \mathcal{A}$ be a function that assigns each element $x \in X$ a state $f(x)$ in a finite set $\mathcal{A}(f$ is referred to in phylogenetics as a (discrete) character). Suppose we have a non-negative function $\delta$ on $\mathcal{A} \times \mathcal{A}$
where $\delta(a, b)$ assigns a penalty score for changing state $a$ to $b$ for each pair $a, b \in \mathcal{A}$ (the default option is to to take $\delta(a, b)=1$ for all $a \neq b$ and $\delta(a, a)=0$ for all $a$ ).

Given any rooted $X$-tree $T$, with vertex set $V$ and $\operatorname{arc}$ set $E$, let $l(f, T, \delta)$ denote the parsimony score of $f$ on $T$ relative to $\delta$; that is,

$$
l(f, T, \delta)=\min _{F: V \rightarrow \mathcal{A}, F \mid X=f}\left\{\sum_{(u, v) \in E} \delta(F(u), F(v))\right\}
$$

In words, $l(f, T, \delta)$ is the minimum sum of $\delta$-penalty scores that are required in order to extend $f$ to an assignment of states to all the vertices of $T$. This quantity can be calculated for a given $T$ by well-known dynamic programming techniques (see e.g. [6]). Let $l(f, \mathcal{C}, \delta)$ (respectively, $\left.l_{\text {bin }}(f, \mathcal{C})\right)$ denote the minimal value of $l(f, T, \delta)$ among all trees $T$ (respectively, all binary trees) that have their clusters in $\mathcal{C}$. Then we have the following general result.

Theorem 4. Suppose that $\mathcal{C}$ is contained within a bureaucratic collection $\mathcal{C}^{\prime}$ of subsets of $X$ and $f: X \rightarrow \mathcal{A}$. There is an algorithm for computing $l(f, \mathcal{C}, \delta)$ with running time polynomial in $n=|X|,|\mathcal{A}|$ and $\left|\mathcal{C}^{\prime}\right|$. Moreover, the algorithm can be extended to construct a rooted phylogenetic $X$-tree having all its clusters in $\mathcal{C}$ and with parsimony score equal to $l(f, \mathcal{C}, \delta)$ in polynomial time.

Proof. For any subset $Y$ of $X$, let

$$
\delta_{Y}(a, b)= \begin{cases}\delta(a, b), & \text { if } Y \in \mathcal{C} \\ 0, & \text { if } Y \notin \mathcal{C} \text { and } a=b \\ \infty, & \text { otherwise }\end{cases}
$$

and for any rooted phylogenetic $X$-tree $T$, let

$$
l^{\prime}(f, T, \delta):=\min _{F: V \rightarrow \mathcal{A}, F \mid X=f}\left\{\sum_{(u, v) \in E} \delta_{c(v)}(F(u), F(v))\right\}
$$

where $c(v)$ is the cluster of $T$ associated with $v$.
Let $l^{\prime}\left(f, \mathcal{C}^{\prime}, \delta\right)$ (respectively, $l_{\text {bin }}^{\prime}\left(f, \mathcal{C}^{\prime}, \delta\right)$ ) be the minimal value of $l^{\prime}(f, T, \delta)$ over all trees (respectively, all binary trees) with clusters in $\mathcal{C}^{\prime}$. By the definition of $\delta_{Y}$, we have:

$$
\begin{equation*}
l(f, \mathcal{C}, \delta)=l^{\prime}\left(f, \mathcal{C}^{\prime}, \delta\right) \tag{2}
\end{equation*}
$$

and by the assumption that $\mathcal{C}^{\prime}$ is bureaucratic we have:

$$
\begin{equation*}
l^{\prime}\left(f, \mathcal{C}^{\prime}, \delta\right)=l_{\mathrm{bin}}^{\prime}\left(f, \mathcal{C}^{\prime}, \delta\right) \tag{3}
\end{equation*}
$$

We now describe how $l_{\text {bin }}^{\prime}\left(f, \mathcal{C}^{\prime}, \delta\right)$ can be efficiently calculated by dynamic programming.
For an element $a \in \mathcal{A}$ and $Y \in \mathcal{C}^{\prime}$, let $L^{\prime}(Y, a)$ be the minimum value of $l^{\prime}(f \mid Y, T, \delta)$ across all binary trees $T$ having leaf set $Y$ and clusters in $\mathcal{C}^{\prime}$, in which the root is assigned state $a$.

For $|Y|=1$, say $Y=\{y\}$, we have

$$
L^{\prime}(Y, a)= \begin{cases}0, & \text { if } f(y)=a \\ \infty, & \text { otherwise }\end{cases}
$$

and for $|Y|>1$, we have

$$
\begin{equation*}
L^{\prime}(Y, a)=\min _{Y_{1}, Y_{2} \in \mathcal{C}^{\prime}, a_{1}, a_{2} \in \mathcal{A}}\left\{L^{\prime}\left(Y_{1}, a_{1}\right)+\delta_{Y_{1}}\left(a, a_{1}\right)+L^{\prime}\left(Y_{2}, a_{2}\right)+\delta_{Y_{2}}\left(a, a_{2}\right): Y_{1} \sqcup Y_{2}=Y\right\} \tag{4}
\end{equation*}
$$

Now,

$$
l_{\text {bin }}^{\prime}\left(f, \mathcal{C}^{\prime} \delta\right)=\min _{a \in \mathcal{A}} L^{\prime}(X, a)
$$

Notice that when one evaluates $L^{\prime}(X, a)$ using the above recursion (Eqn. (4)), it is sufficient to compute $L^{\prime}(Y, a)$ for just the sets $Y \in \mathcal{C}^{\prime}$ rather than all subsets of $X$.

Thus, in view of Eqns. (2) and (3), one can compute $l(f, \mathcal{C}, \delta)$ in time polynomial in $n=$ $|X|,|\mathcal{A}|$ and $\left|\mathcal{C}^{\prime}\right|$. Moreover, by suitable book-keeping along the way, one can construct a rooted binary phylogenetic $X$-tree with clusters in $\mathcal{C}^{\prime}$ and with a parsimony score equal to $l_{\text {bin }}\left(f, \mathcal{C}^{\prime}, \delta\right)$; by collapsing all edges of this tree that have a $\delta$-score equal to 0 we obtain a rooted phylogenetic $X$-tree with clusters in $\mathcal{C}$ and with parsimony score equal to $l(f, \mathcal{C}, \delta)$.

We note that this result has been described in the particular case when $\mathcal{C}$ is the bureaucracy described in example (2) above, and where $f$ maps to a set $A$ with only two elements [5]. We provide a second application, to phylogenetic networks, based on Example (1) above, of intervals as bureaucratic set systems.

Let $\mathcal{N}$ be a rooted binary phylogenetic network on $X$. We say that $\mathcal{N}$ is planar if it can be drawn in the plane so that all the leaves and the root all lie on the outer face. Let $s w(\mathcal{N})$ denote the set of 'soft-wired' clusters in $\mathcal{N}$ (the union of the cluster sets of all trees embedded in $\mathcal{N}$; see e.g. [4). A simple example is shown in Fig. [(c).

Corollary 5. Suppose that $\mathcal{N}$ is a binary and planar phylogenetic network on $X$, and $f: X \rightarrow \mathcal{A}$. There is an algorithm for computing $l(f, s w(\mathcal{N}))$ with running time polynomial time in $n$.
Proof. Let $x_{1}, \ldots, x_{n}$ be the ordering of $X$ given by their positions around the outer face in a planar embedding of $\mathcal{N}$, where $x_{1}$ and $x_{n}$ come immediately after and before the root. Then any tree $T$ embedded in $\mathcal{N}$ can be ordered so that the leaves are in order $x_{1}, \ldots, x_{n}$, implying that the clusters of $T$ are all of the form $\left\{x_{i}, x_{i+1}, \ldots, x_{j}\right\}$ for some $1 \leq i \leq j \leq n$. It follows that the set $\operatorname{sw}(\mathcal{N})$ is contained in the set of intervals of $X=\left\{x_{1}, \ldots, x_{n}\right\}$ (Example 1, above). The corollary now follows from Theorem 4.

We end this paper by posing a computational problem.
Question. Is there an algorithm for deciding whether or not $\mathcal{C}$ is bureaucratic that runs in time polynomial in $|\mathcal{C}|$ and $|X|$ ?

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