### EXPONENTIALLY MANY PERFECT MATCHINGS IN CUBIC GRAPHS

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ABSTRACT. We show that every cubic bridgeless graph G has at least  $2^{|V(G)|/3656}$  perfect matchings. This confirms an old conjecture of Lovász and Plummer.

This version of the paper uses a different definition of a burl from the journal version of the paper [7], and a different proof of Lemma 18. This simplifies the exposition of our arguments throughout the whole paper.

#### 1. Introduction

Given a graph G, let  $\mathcal{M}(G)$  denote the set of perfect matchings in G. A classical theorem of Petersen [15] states that every cubic bridgeless graph has at least one perfect matching, i.e.  $\mathcal{M}(G) \neq \emptyset$ . Indeed, it can be proven that any edge in a cubic bridgeless graph is contained in some perfect matching [14], which implies that  $|\mathcal{M}(G)| \geq 3$ .

In the 1970s, Lovász and Plummer conjectured that the number of perfect matchings of a cubic bridgeless graph G should grow exponentially with its order (see [12, Conjecture 8.1.8]). It is a simple exercise to prove that G contains at most  $2^{|V(G)|}$  perfect matchings, so we can state the conjecture as follows:

**Lovász-Plummer conjecture.** There exists a universal constant  $\epsilon > 0$  such that for any cubic bridgeless graph G,

$$2^{\epsilon|V(G)|} \le |\mathcal{M}(G)| \le 2^{|V(G)|}.$$

The problem of computing  $|\mathcal{M}(G)|$  is connected to problems in molecular chemistry and statistical physics (see e.g. [12, Section 8.7]). In general graphs, this problem is  $\sharp P$ -complete [17]. Thus we are interested in finding good bounds on the number of perfect matchings for various classes of graphs such as the bounds in the conjecture above.

For bipartite graphs,  $|\mathcal{M}(G)|$  is precisely the permanent of the graph biadjacency matrix. Voorhoeve proved the conjecture for cubic bipartite graphs in 1979 [18]; Schrijver later extended this result to all regular bipartite graphs [16]. We refer the reader to [11] for an exposition of this connection and of an elegant proof of Gurvits generalizing Schrijver's result. For fullerene graphs, a class of planar cubic graphs for which the conjecture relates to molecular stability and aromaticity of fullerene molecules, the problem was settled by Kardoš, Král', Miškuf and Sereni [9]. Chudnovsky and Seymour recently proved the conjecture for all cubic bridgeless planar graphs [1].

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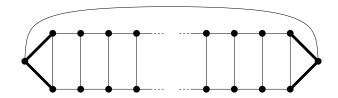


FIGURE 1. A graph cubic bridgeless graph G with  $m^*(G) = 1$ .

The general case has until now remained open. Edmonds, Lovász and Pulleyblank [4] proved that any cubic bridgeless G contains at least  $\frac{1}{4}|V(G)|+2$  perfect matchings (see also [13]); this bound was later improved to  $\frac{1}{2}|V(G)|$  [10] and then  $\frac{3}{4}|V(G)|-10$  [6]. The order of the lower bound was not improved until Esperet, Kardoš, and Král' proved a superlinear bound in 2009 [5]. The first bound, proved in 1982, is a direct consequence of a lower bound on the dimension of the perfect matching polytope, while the more recent bounds combine polyhedral arguments with analysis of brick and brace decompositions.

In this paper we solve the general case. To avoid technical difficulties when contracting sets of vertices, we henceforth allow graphs to have multiple edges, but not loops. Let m(G) denote  $|\mathcal{M}(G)|$ , and let  $m^{\star}(G)$  denote the minimum, over all edges  $e \in E(G)$ , of the number of perfect matchings containing e. Our result is the following:

**Theorem 1.** For every cubic bridgeless graph G we have  $m(G) \geq 2^{|V(G)|/3656}$ .

We actually prove that at least one of two sufficient conditions applies:

**Theorem 2.** For every cubic bridgeless graph G, at least one of the following holds:

- [S1]  $m^*(G) \ge 2^{|V(G)|/3656}$ , or
- [S2] there exist  $M, M' \in \mathcal{M}(G)$  such that  $M \triangle M'$  has at least |V(G)|/3656 components.

To see that Theorem 2 implies Theorem 1, we can clearly assume that [S2] holds since  $m^*(G) \leq m(G)$ . Choose  $M, M' \in \mathcal{M}(G)$  such that the set  $\mathcal{C}$  of components of  $M \triangle M'$  has cardinality at least |V(G)|/3656, and note that each of these components is an even cycle alternating between M and M'. Thus for any subset  $\mathcal{C}' \subseteq \mathcal{C}$ , we can construct a perfect matching  $M_{\mathcal{C}'}$  from M by flipping the edges on the cycles in  $\mathcal{C}'$ , i.e.  $M_{\mathcal{C}'} = M \triangle \bigcup_{C \in \mathcal{C}'} C$ . The  $2^{|\mathcal{C}|}$  perfect matchings  $M_{\mathcal{C}'}$  are distinct, implying Theorem 1.

We cannot discard either of the sufficient conditions [S1] or [S2] in the statement of Theorem 2. To see that [S2] cannot be omitted, consider the graph depicted in Figure 1 and observe that each of the four bold edges is contained in a unique perfect matching. To see that [S1] cannot be omitted, it is enough to note that there exist cubic graphs with girth logarithmic in their size (see [8] for a construction). Such graphs cannot have linearly many disjoint cycles, so condition [S2] does not hold.

## 1.1. Definitions and notation.

For a graph G and a set  $X \subseteq V(G)$ , G|X denotes the subgraph of G induced by X. For a set  $X \subseteq V(G)$ , let  $\delta(X)$  denote the set of edges with exactly one endpoint in X, and let  $E_X$  denote the set of edges with at least one endpoint in X, i.e.  $E_X = E(G|X) \cup \delta(X)$ . The set  $C = \delta(X)$  is called an *edge-cut*, or a *k-edge-cut*, where k = |C|, and X and  $V(G) \setminus X$  are the *sides* of C. A *k*-edge-cut is said to be *even* (resp. *odd*) if K is even (resp. odd). Observe that the parity of an edge-cut  $\delta(X)$  in a cubic graph is precisely that of |X|. An edge-cut  $\delta(X)$  is *cyclic* if both G|X and  $G|(V(G) \setminus X)$  contain a cycle. Observe that every 2-edge-cut in a cubic graph is cyclic. If G

contains no edge-cut (resp. cyclic edge-cut) of size less than k, we say that G is k-edge-connected (resp. cyclically k-edge-connected).

Observe that the number of perfect matchings of a graph is the product of the number of perfect matchings of its connected components. Hence, in order to prove Theorem 1, we restrict ourselves to connected graphs for the remainder of this paper (this means, for example, that we can consider the terms 2-edge-connected and bridgeless to be interchangeable, and the sides of a cut are well-defined).

1.2. Constants. Let  $x := \log(\frac{4}{3})/\log(2)$ . The following constants appear throughout the paper:

$$\alpha := \frac{x}{314}, \qquad \beta_1 := \frac{154x}{314}, \qquad \beta_2 := \frac{74x}{314}, \qquad \gamma := \frac{312x}{314}.$$

We avoid using the numerical values of these constants for the sake of clarity. Throughout the paper we make use of the following inequalities, which can be routinely verified:

(1) 
$$0 < \alpha \le \beta_2 \le \beta_1,$$
  
(2)  $1/3656 \le \frac{\alpha}{9\beta_1 + 3},$   
(3)  $\beta_2 + 6\alpha \le \beta_1,$   
(4)  $74\alpha \le \beta_2,$   
(5)  $146\alpha \le \beta_1,$   
(6)  $\beta_2 + 80\alpha \le \beta_1,$ 

$$6\alpha + \gamma \le \log(6)/\log(2),$$

(8) 
$$\gamma + 2\beta_1 + 7\alpha - \beta_2 \le 1$$
,

(9) 
$$6\alpha + 2\beta_1 \le \log(\frac{4}{3})/\log(2),$$

$$(10) 2\beta_1 + 4\alpha \le \gamma.$$

The integer 3656 is chosen minimum so that the system of inequalities above has a solution. Inequalities (4), (6), (9), and (10) are tight.

## 2. The proof of Theorem 2

In this section we sketch the proof of Theorem 2, postponing the proofs of two main lemmas until later sections. Our general approach to Theorem 2 is to reduce on cyclic 2-edge-cuts and cyclic 3-edge-cuts and prove inductively that either [S1] or [S2] holds. Dealing with [S1] is relatively straightforward – perfect matchings containing a given edge behave well with reductions on a cut, which is our main motivation for considering  $m^*(G)$ . To deal with [S2], we do not directly construct perfect matchings M and M' for which  $M \triangle M'$  has many components. Instead, we define a special type of vertex set in which a given random perfect matching is very likely to admit an alternating cycle. We call these sets burls and we call a set of disjoint burls a foliage – a large foliage will guarantee the existence of two perfect matchings with many components in their symmetric difference.

### 2.1. Burls, twigs, and foliage weight.

Consider a subset  $X \subseteq V(G)$ . Let  $\mathcal{M}(G,X)$  denote the family of subsets M of  $E_X$  (the edges with at least one endpoint in X) such that every vertex of X is incident with exactly one edge of M. Note that some elements of  $\mathcal{M}(G,X)$  might not be matchings in G (if two edges of  $\delta(X)$  share a vertex from  $V(G) \setminus X$ ). However, for any  $M \in \mathcal{M}(G,X)$ ,  $M \cap E(G|X)$  is a matching.

A probability distribution  $\mathbf{M}$  on  $\mathcal{M}(G,X)$  is balanced if for any edge  $e \in E_X$ ,  $\Pr[e \in \mathbf{M}] = \frac{1}{3}$ . It follows from Edmonds' characterization of the perfect matching polytope [3] that if G is cubic and bridgeless, there exists a balanced probability distribution on  $\mathcal{M}(G,V(G)) = \mathcal{M}(G)$ . For any  $X \subseteq V(G)$ , the restriction of this distribution to  $E_X$  yields a balanced probability distribution on  $\mathcal{M}(G,X)$ . The following easy fact will be used several times throughout the proof:

Claim 3. Let G be a cubic bridgeless graph and consider  $Y \subseteq X \subseteq V(G)$  such that  $C = \delta(Y)$  is a 3-edge-cut in G. For any balanced probability distribution  $\mathbf{M}$  on  $\mathcal{M}(G,X)$ , and any  $M \in \mathcal{M}(G,X)$  such that  $\Pr[\mathbf{M} = M] > 0$ , we have  $|M \cap C| = 1$ .

Given some  $M \in \mathcal{M}(G,X)$ , a cycle of G|X is M-alternating if it has even length and half of its edges are in M (it alternates edges in M and edges not in M). Let a(G,X,M) denote the maximum number of disjoint M-alternating cycles in G|X (equivalently, the maximum number of components of  $M \triangle M'$ , for  $M' \in \mathcal{M}(G,X)$ ).

We define a burl as a vertex set  $X \subseteq V(G)$  such that for any balanced probability distribution  $\mathbf{M}$  on  $\mathcal{M}(G,X)$ ,  $\mathbb{E}[a(G,X,\mathbf{M})] \geq \frac{1}{3}$ . Note that if X is a burl, any set  $Y \supset X$  is also a burl, since any balanced probability distribution on  $\mathcal{M}(G,Y)$  induces a balanced probability distribution on  $\mathcal{M}(G,X)$ . We would like to insist on the fact that we consider the whole set  $\mathcal{M}(G,X)$ , and not only  $\{M \cap E_X, M \in \mathcal{M}(G)\}$ . This way, being a burl is really a local property of X and is completely independent of the structure of  $G|(V(G)\setminus X)$ . This aspect of burls will be fundamental in the proof of Theorem 2.

A collection of disjoint vertex sets  $\{X_1, \ldots, X_k\}$  is a *foliage* if each  $X_i$  is a burl. Assume that G contains such a collection of disjoint sets, and consider a balanced probability distribution  $\mathbf{M}$  on  $\mathcal{M}(G, V(G)) = \mathcal{M}(G)$ . This distribution induces balanced probability distributions  $\mathbf{M}_{X_i}$  on  $\mathcal{M}(G, X_i)$ , for each  $1 \leq i \leq k$ . By definition of a burl, we have  $\mathbb{E}[a(G, X_i, \mathbf{M}_{X_i})] \geq \frac{1}{3}$  for each each  $1 \leq i \leq k$ . By linearity of expectation, the maximum number of disjoint alternating cycles of  $\mathbf{M}$  is then expected to be at least k/3. We get the following key fact as a consequence:

**Corollary 4.** If a cubic bridgeless graph G contains a foliage  $\mathcal{X}$ , then there exist perfect matchings  $M, M' \in \mathcal{M}(G)$  such that  $M \triangle M'$  has at least  $|\mathcal{X}|/3$  components.

We now introduce a special class of burls. Let G be a cubic bridgeless graph and let  $X \subseteq V(G)$ . We say that X is a 2-twig if  $|\delta(X)| = 2$ , and X is a 3-twig if  $|\delta(X)| = 3$  and  $|X| \ge 5$  (that is, X is neither a triangle, nor a single vertex). A twig in G is a 2- or 3-twig. Before we prove that every twig is a burl, we need a simple lemma.

**Lemma 5.** Let G be a cubic bridgeless graph. Then

- (1)  $m(G-e) \geq 2$  for every  $e \in E(G)$ , and
- (2)  $m(G) \ge 4$  if  $|V(G)| \ge 6$ . In particular, for any  $v \in V(G)$  there is an  $e \in \delta(\{v\})$  contained in at least two perfect matchings.

*Proof.* The first item follows from the classical result mentioned in the introduction: every edge of a cubic bridgeless graph is contained in a perfect matching. The second is implied by the bound  $m(G) \ge \frac{1}{4}|V(G)| + 2$  from [4].

**Lemma 6.** Every twiq X in a cubic bridgeless graph G is a burl.

*Proof.* Let **M** be a balanced probability distribution on  $\mathcal{M}(G,X)$ .

If X is a 2-twig, let H be obtained from G|X by adding an edge e joining the two vertices incident with  $\delta(X)$ . Then H is cubic and bridgeless. By applying Lemma 5(1) to H, we see that G|X contains at least one M-alternating cycle for every  $M \in \mathcal{M}(G,X)$  such that  $M \cap \delta(X) = \emptyset$ .

Note that since H is cubic, |X| is even, and thus M either contains the two edges of  $\delta(X)$ , or none of them. Since  $\mathbf{M}$  is balanced,  $\Pr[\mathbf{M} \cap \delta(X) = \emptyset] \ge 1 - 1/3 = 2/3$ . Hence  $\mathbb{E}[a(G, X, \mathbf{M})] \ge \frac{2}{3}$  and we conclude that X is a burl.

Suppose now that X is a 3-twig. Let  $\delta(X) = \{e_1, e_2, e_3\}$ . Let H be obtained from G by identifying all the vertices in V(G) - X (removing loops but preserving multiple edges). We apply Lemma 5(2) to H, which is again cubic and bridgeless. It follows that for some  $1 \le i \le 3$ , the edge  $e_i$  is in at least two perfect matchings of H. Therefore G|X contains at least one M-alternating cycle for every  $M \in \mathcal{M}(G,X)$  such that  $M \cap \delta(X) = \{e_i\}$ . By Claim 3,  $\Pr[\mathbf{M} \cap \delta(X) = \{e_i\}] = \Pr[e_i \in \mathbf{M}] = 1/3$ . It implies that  $\mathbb{E}[a(G,X,\mathbf{M})] \ge \frac{1}{3}$  and thus X is a burl.

The weight of a foliage  $\mathcal{X}$  containing k twigs is defined as  $fw(\mathcal{X}) := \beta_1 k + \beta_2(|\mathcal{X}| - k)$ , that is each twig has weight  $\beta_1$  and each non-twig burl has weight  $\beta_2$ . Let fw(G) denote the maximum weight of a foliage in a graph G.

# 2.2. Reducing on small edge-cuts.

We now describe how we reduce on 2-edge-cuts and 3-edge-cuts, and consider how these operations affect  $m^*(G)$  and foliages. Let C be a 3-edge-cut in a cubic bridgeless graph G. The two graphs  $G_1$  and  $G_2$  obtained from G by identifying all vertices on one of the sides of the edge-cut (removing loops but preserving multiple edges) are referred to as C-contractions of G and the vertices in  $G_1$  and  $G_2$  created by this identification are called new.

We need a similar definition for 2-edge-cuts. Let  $C = \{e, e'\}$  be a 2-edge-cut in a cubic bridgeless graph G. The two C-contractions  $G_1$  and  $G_2$  are now obtained from G by deleting all vertices on one of the sides of C and adding an edge joining the remaining ends of e and e'. The resulting edge is now called new.

In both cases we say that  $G_1$  and  $G_2$  are obtained from G by a cut-contraction. The next lemma provides some useful properties of cut-contractions.

**Lemma 7.** Let G be a graph, let C be a 2- or a 3-edge-cut in G, and let  $G_1$  and  $G_2$  be the two C-contractions. Then

- (1)  $G_1$  and  $G_2$  are cubic bridgeless graphs,
- (2)  $m^*(G) \ge m^*(G_1) m^*(G_2)$ , and
- (3) For i = 1, 2 let  $\mathcal{X}_i$  be a foliage in  $G_i$  such that for every  $X \in \mathcal{X}_i$ , if |C| = 3 then X does not contain the new vertex, and if |C| = 2 then  $E(G_i|X)$  does not contain the new edge. Then  $\mathcal{X}_1 \cup \mathcal{X}_2$  is a foliage in G. In particular, we have  $fw(G) \geq fw(G_1) + fw(G_2) 2\beta_1$ .

Proof.

- (1) This can be confirmed routinely.
- (2) Consider first the case of the contraction of a 2-edge-cut  $C = \delta(X)$  in G. Let e be an edge with both ends in  $X = V(G_1)$ . Every perfect matching of  $G_1$  containing e combines either with  $m^*(G_2)$  perfect matchings of  $G_2$  containing the new edge of  $G_2$ , or with  $2m^*(G_2)$  perfect matchings of  $G_2$  avoiding the new edge of  $G_2$ . If e lies in C, note that perfect matchings of  $G_1$  and  $G_2$  containing the new edges can be combined into perfect matchings of G containing G. Hence, e is in at least  $m^*(G_1) m^*(G_2)$  perfect matchings of G.

Now consider a 3-edge-cut  $C = \delta(X)$ . If e has both ends in  $X \subset V(G_1)$ , perfect matchings of  $G_1$  containing e combine with perfect matchings of  $G_2$  containing either of the 3 edges of C. If e is in C, perfect matchings containing e in  $G_1$  and  $G_2$  can also be combined into perfect matchings of G. In any case, e is in at least  $m^*(G_1) m^*(G_2)$  perfect matchings of G.

(3) In this case the elements of  $\mathcal{X}_1 \cup \mathcal{X}_2$  are disjoint subsets of V(G). Consider some  $X \in \mathcal{X}_1 \cup \mathcal{X}_2$ , and assume without loss of generality that  $X \in \mathcal{X}_1$ . Since X does not contain the new vertex of  $G_1$  (if |C| = 3), or the new edge (if |C| = 2), each balanced probability distribution on  $\mathcal{M}(G, X)$  is also a balanced probability distribution on  $\mathcal{M}(G_1, X)$ , so  $\mathcal{X}_1 \cup \mathcal{X}_2$  is a foliage in G. Since  $\beta_1 \geq \beta_2$ , this implies  $fw(G) \geq fw(G_1) + fw(G_2) - 2\beta_1$ .

It is not generally advantageous to reduce on a 3-edge-cut arising from a triangle, unless this reduction leads to a chain of similar reductions. Thus we wish to get rid of certain triangles from the outset. We say that a triangle sharing precisely one edge with a cycle of length three or four in a graph G is relevant, and otherwise it is irrelevant. A graph G is pruned if it contains no irrelevant triangles. The following easy lemma shows that we can prune a bridgeless cubic graph by repeated cut-contraction without losing too many vertices.

**Lemma 8.** Let G be a cubic bridgeless graph, and let k be the size of maximum collection of vertex-disjoint irrelevant triangles in G. Then one can obtain a pruned cubic bridgeless graph G' from G with  $|V(G')| \ge |V(G)| - 2k$  by repeatedly contracting irrelevant triangles.

Proof. We proceed by induction on k. Let a graph G'' be obtained from G by contracting an irrelevant triangle T. The graph G'' is cubic and bridgeless by Lemma 7(1). Since T is irrelevant in G, the unique vertex of G'' obtained by contracting T is not in a triangle in G''. Therefore if T is a collection of vertex disjoint irrelevant triangles in G'' then  $T \cup \{T\}$  is such a collection in G. (After the contraction of an irrelevant triangle, triangles that were previously irrelevant might become relevant, but the converse is not possible.) It follows that  $|T| \leq k - 1$ . By applying the induction hypothesis to G'', we see that the lemma holds for G.

**Corollary 9.** Let G be a cubic bridgeless graph. Then we can obtain a cubic bridgeless pruned graph G' from G with  $|V(G')| \ge |V(G)|/3$  by repeatedly contracting irrelevant triangles.

We wish to restrict our attention to pruned graphs, so we must make sure that the function  $m^{\star}(G)$  and the maximum size of a foliage does not increase when we contract a triangle.

**Lemma 10.** Let G' be obtained from a graph G by contracting a triangle. Then  $m^*(G') \leq m^*(G)$  and the maximum size of a foliage in G' is at most the maximum size of a foliage in G.

Proof. Let xyz be the contracted triangle, and let  $e_x$ ,  $e_y$ , and  $e_z$  be the edges incident with x, y, z and not contained in the triangle in G. Let t be the vertex of G' corresponding to the contraction of xyz. Every perfect matching M' of G' has a canonical extension M in G: assume without loss of generality that  $e_x$  is the unique edge of M' incident to t. Then M consists of the union of M' and yz. Observe that perfect matchings in G containing yz necessarily contain  $e_x$ , so every edge of G is contained in at least  $m^*(G')$  perfect matchings.

Now consider a burl X' in G' containing t. We show that  $X = X' \cup \{x, y, z\} \setminus t$  is a burl in G. Let  $\mathbf{M}$  be a balanced probability distribution on  $\mathcal{M}(G,X)$ . By Claim 3 and the remark above, we can associate a balanced probability distribution  $\mathbf{M}'$  on  $\mathcal{M}(G',X')$  to  $\mathbf{M}$  such that  $\mathbb{E}[a(G,X,\mathbf{M})] = \mathbb{E}[a(G',X',\mathbf{M}')]$ . Since X' is a burl in G', this expectation is at least  $\frac{1}{3}$  and X is a burl in G.

Since a burl avoiding t in G' is also a burl in G, it follows from the analysis above that the maximum size of a foliage cannot increase when we contract a triangle.

### 2.3. Proving Theorem 2.

We say that G has a core if we can obtain a cyclically 4-edge-connected graph G' with  $|V(G')| \ge 6$  by applying a (possibly empty) sequence of cut-contractions to G (recall that this notion was defined in the previous subsection).

We will deduce Theorem 2 from the next two lemmas. This essentially splits the proof into two cases based on whether or not G has a core.

**Lemma 11.** Let G be a pruned cubic bridgeless graph. Let  $Z \subseteq V(G)$  be such that  $|Z| \ge 2$  and  $|\delta(Z)| = 2$ , or  $|Z| \ge 4$  and  $|\delta(Z)| = 3$ . Suppose that the  $\delta(Z)$ -contraction G' of G with  $Z \subseteq V(G')$  has no core. Then there exists a foliage  $\mathcal{X}$  in G with  $\bigcup_{X \in \mathcal{X}} X \subseteq Z$  and

$$fw(\mathcal{X}) \ge \alpha |Z| + \beta_2.$$

By applying Lemma 11 to a cubic graph G without a core and  $Z = V(G) \setminus \{v\}$  for some  $v \in V(G)$ , we obtain the following.

Corollary 12. Let G be a pruned cubic bridgeless graph without a core. Then

$$fw(G) \ge \alpha(|V(G)| - 1) + \beta_2.$$

On the other hand, if G has a core, we will prove that either fw(G) is linear in the size of G or every edge of G is contained in an exponential number of perfect matchings.

**Lemma 13.** Let G be a pruned cubic bridgeless graph. If G has a core then

$$m^{\star}(G) > 2^{\alpha|V(G)|-fw(G)+\gamma}$$
.

We finish this section by deriving Theorem 2 from Lemmas 11 and 13.

Proof of Theorem 2. Let  $\epsilon := 1/3656$ . By Corollary 9 there exists a pruned cubic bridgeless graph G' with  $|V(G')| \ge |V(G)|/3$  obtained from G by repeatedly contracting irrelevant triangles. Suppose first that G' has a core. By Corollary 9 and Lemmas 10 and 13, condition [S1] holds as long as  $\epsilon |V(G)| \le \alpha |V(G)|/3 - fw(G')$ . Therefore we assume  $fw(G') \ge (\frac{\alpha}{3} - \epsilon)|V(G)|$ . It follows from the definition of fw(G') that G' has a foliage containing at least  $(\frac{\alpha}{3} - \epsilon)|V(G)|/\beta_1$  burls. If G' has no core then by Corollary 12 and the fact that  $\alpha \le \beta_2$ ,  $fw(G') \ge \alpha (|V(G')| - 1) + \beta_2 \ge \alpha |V(G')|$ , so G' contains a foliage of size at least  $\alpha |V(G')|/\beta_1 \ge \alpha |V(G)|/3\beta_1$ . In both cases condition [S2] holds by Corollary 4 and Lemma 10, since Equation (2) tells us that  $3\epsilon \le (\frac{\alpha}{3} - \epsilon)/\beta_1$ .

#### 3. Cut decompositions

In this section we study cut decompositions of cubic bridgeless graphs. We mostly follow notation from [1], however we consider 2- and 3-edge-cuts simultaneously. Cut decompositions play a crucial role in the proof of Lemma 11 in the next section.

Let G be a graph. A non-trivial cut-decomposition of G is a pair  $(T, \phi)$  such that:

- T is a tree with  $E(T) \neq \emptyset$ ,
- $\phi: V(G) \to V(T)$  is a map, and
- $|\phi^{-1}(t)| + \deg_T(t) \ge 3$  for each  $t \in V(T)$ .

For an edge f of T, let  $T_1$ ,  $T_2$  be the two components of  $T \setminus f$ , and for i = 1, 2 let  $X_i = \phi^{-1}(T_i)$ . Thus  $(X_1, X_2)$  is a partition of V(G) that induces an edge-cut denoted by  $\phi^{-1}(f)$ . If  $|\phi^{-1}(f)| \in \{2, 3\}$  for each  $f \in E(T)$  we call  $(T, \phi)$  a small-cut-decomposition of G.

Let  $(T,\phi)$  be a small-cut-decomposition of a 2-edge-connected cubic graph G, and let  $T_0$  be a subtree of T such that  $\phi^{-1}(V(T_0)) \neq \emptyset$ . Let  $T_1, \ldots, T_s$  be the components of  $T \setminus V(T_0)$ , and for  $1 \leq i \leq s$  let  $f_i$  be the unique edge of T with an end in  $V(T_0)$  and an end in  $V(T_i)$ . For  $0 \leq i \leq s$ , let  $X_i = \phi^{-1}(V(T_i))$ . Thus  $X_0, X_1, \ldots, X_s$  form a partition of V(G). Let G' be the graph obtained from G as follows. Set  $G_0 = G$ . For  $i = 1, \ldots, s$ , take  $G_{i-1}$  and let  $G_i$  be the  $(\phi^{-1}(f_i))$ -contraction containing  $X_0$ . Now let G' denote  $G_s$ . Note that G' is cubic. We call G' the hub of G at  $T_0$  (with

respect to  $(T, \phi)$ ). If  $t_0 \in V(T)$  and  $\phi^{-1}(t_0) \neq \emptyset$ , by the hub of G at  $t_0$  we mean the hub of G at  $T_0$ , where  $T_0$  is the subtree of T with vertex set  $\{t_0\}$ .

Let  $\mathcal{Y}$  be a collection of disjoint subsets of V(G). We say that a small-cut-decomposition  $(T,\phi)$  of G refines  $\mathcal{Y}$  if for every  $Y \in \mathcal{Y}$  there exists a leaf  $v \in V(T)$  such that  $Y = \phi^{-1}(v)$ . Collections of subsets of V(G) that can be refined by a small-cut decomposition are characterized in the following easy lemma.

**Lemma 14.** Let G be a cubic bridgeless graph. Let  $\mathcal{Y}$  be a collection of disjoint subsets of V(G). Then there exists a small-cut-decomposition refining  $\mathcal{Y}$  if  $|Y| \geq 2$  and  $|\delta(Y)| \in \{2,3\}$  for every  $Y \in \mathcal{Y}$ , and either

- (1)  $\mathcal{Y} = \emptyset$  and G is not cyclically 4-edge-connected, or
- (2)  $\mathcal{Y} = \{Y\}$ , and  $|V(G) \setminus Y| > 1$ , or
- (3)  $|\mathcal{Y}| \ge 2$ .

*Proof.* We only consider the case  $|\mathcal{Y}| \geq 3$ , as the other cases are routine. Take T to be a tree on  $|\mathcal{Y}| + 1$  vertices with  $|\mathcal{Y}|$  leaves  $\{v_Y \mid Y \in \mathcal{Y}\}$  and a non-leaf vertex  $v_0$ . The map  $\phi$  is defined by  $\phi(u) = v_Y$ , if  $u \in Y$  for some  $Y \in \mathcal{Y}$ , and  $\phi(u) = v_0$ , otherwise. Clearly,  $(T, \phi)$  refines  $\mathcal{Y}$  and is a small-cut-decomposition of G.

We say that  $(T, \phi)$  is  $\mathcal{Y}$ -maximum if it refines  $\mathcal{Y}$  and |V(T)| is maximum among all small-cut decompositions of G refining  $\mathcal{Y}$ . The following lemma describes the structure of  $\mathcal{Y}$ -maximum decompositions. It is a variation of Lemma 4.1 and Claim 1 of Lemma 5.3 in [1].

**Lemma 15.** Let G be a cubic bridgeless graph. Let Y be a collection of disjoint subsets of V(G) and let  $(T, \phi)$  be a Y-maximum small-cut-decomposition of G. Then for every  $t \in V(T)$  either  $\phi^{-1}(t) = \emptyset$ , or  $\phi^{-1}(t) \in \mathcal{Y}$ , or the hub of G at t is cyclically 4-edge-connected.

Proof. Fix  $t \in V(T)$  with  $\phi^{-1}(t) \neq \emptyset$  and  $\phi^{-1}(t) \notin \mathcal{Y}$ . Let  $f_1, \ldots, f_k$  be the edges of T incident with t, and let  $T_1, \ldots, T_k$  be the components of  $T \setminus \{t\}$ , where  $f_i$  is incident with a vertex  $t_i$  of  $T_i$  for  $1 \leq i \leq k$ . Let  $X_0 = \phi^{-1}(t)$ , and for  $1 \leq i \leq k$  let  $X_i = \phi^{-1}(V(T_i))$ . Let G' be the hub of G at t, and let G'' be the graph obtained from G' by subdividing precisely once every new edge e corresponding to the cut-contraction of a cut C with |C| = 2. The vertex on the subdivided edge e is called the new vertex corresponding to the cut-contraction of C, by analogy with the new vertex corresponding to the cut-contraction of a cyclic 3-edge-cut.

Note that G' is cyclically 4-edge-connected if and only if G'' is cyclically 4-edge-connected. Suppose for the sake of contradiction that  $C = \delta(Z)$  is a cyclic edge-cut in G'' with  $|C| \leq 3$ . Then  $|C| \in \{2,3\}$  by Lemma 7(1), as G'' is a subdivision of G' and G' can be obtained from G by repeated cut-contractions. Let T' be obtained from T by by splitting t into two vertices t' and t'', so that  $t_i$  is incident to t' if and only if the new vertex of G'' corresponding to the cut-contraction of  $\phi^{-1}(f_i)$  is in Z. Let  $\phi'(t') = X_0 \cap Z$ ,  $\phi'(t'') = X_0 \setminus Z$ , and  $\phi'(s) = \phi(s)$  for every  $s \in V(T') \setminus \{t', t''\}$ .

We claim that  $(T', \phi')$  is a small-cut-decomposition of G contradicting the choice of T. It is only necessary to verify that  $|\phi^{-1}(s)| + \deg_{T'}(s) \ge 3$  for  $s \in \{t', t''\}$ . We have  $|\phi^{-1}(t')| + \deg_{T'}(t') - 1 = |Z \cap V(G'')| \ge 2$  as C is a cyclic edge-cut in G''. It follows that  $|\phi^{-1}(t')| + \deg_{T'}(t') \ge 3$  and the same holds for t'' by symmetry.  $\Box$ 

We finish this section by describing a collection  $\mathcal{Y}$  to which we will be applying Lemma 15 in the sequel. In a cubic bridgeless graph G a union of the vertex set of a relevant triangle with the vertex set of a cycle of length at most four sharing an edge with it is called a *simple twig*. Note that simple twigs corresponding to distinct relevant triangles can intersect, but one can routinely verify that each simple twig intersects a simple twig corresponding to at most one other relevant triangle. An

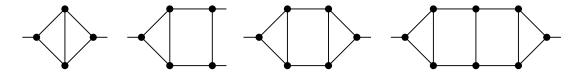


FIGURE 2. Isomorphism classes of subgraphs induced by elementary twigs.

elementary twig is either a simple twig, that intersects no simple twig corresponding to a relevant triangle not contained in it, or the union of two intersecting simple twigs, corresponding to distinct relevant triangles. An elementary twig is, indeed, a twig, unless it constitutes the vertex set of the entire graph. Figure 2 shows all possible elementary twigs. The next corollary follows immediately from the observations above and Lemmas 14 and 15.

**Corollary 16.** Let G be a cubic bridgeless graph that is not cyclically 4-edge-connected with  $|V(G)| \geq 8$ . Then there exists a collection  $\mathcal{Y}$  of pairwise disjoint elementary twigs in G such that every relevant triangle in G is contained in an element of  $\mathcal{Y}$ . Further, there exists a  $\mathcal{Y}$ -maximum small-cut-decomposition  $(T,\phi)$  of G and for every  $t \in V(T)$  either  $\phi^{-1}(t) = \emptyset$ , or  $\phi^{-1}(t)$  is an elementary twig, or the hub of G at t is cyclically 4-edge-connected.

### 4. Proof of Lemma 11.

The proof of Lemma 11 is based on our ability to find burls locally in the graph. The following lemma is a typical example.

**Lemma 17.** Let G be a cubic bridgeless graph and let  $X \subseteq V(G)$  be such that  $|\delta(X)| = 4$  and  $m(G|X) \ge 2$ . Then X is a burl.

*Proof.* Note that if  $M \in \mathcal{M}(G,X)$  contains no edges of  $\delta(X)$  then G|X contains an M-alternating cycle. Let  $\mathbf{M}$  be a balanced probability distribution on  $\mathcal{M}(G,X)$ . As  $M \cap \delta(X)$  is even for every  $M \in \mathcal{M}(G,X)$  we have

$$\tfrac{4}{3} = \mathbb{E}\left[|\mathbf{M} \cap \delta(X)|\right] \ge 2\Pr[\mathbf{M} \cap \delta(X) \ne \emptyset].$$

Therefore  $\Pr[\mathbf{M} \cap \delta(X) = \emptyset] \ge 1/3$ . Hence,  $\mathbb{E}[a(G, X, \mathbf{M})] \ge \frac{1}{3}$  and so X is a burl.

The proof of Lemma 11 relies on a precise study of the structure of small-cut trees for graphs with no core. The following two lemmas indicate that long paths in such trees necessarily contain some burls.

**Lemma 18.** Let  $(T, \phi)$  be a small-cut-decomposition of a cubic bridgeless graph G, and let P be a path in T with |V(P)| = 10. If we have

- $\deg_T(t) = 2$  for every  $t \in V(P)$ ,
- the hub of G at t is isomorphic to  $K_4$  for every  $t \in V(P)$ , and
- $|\phi^{-1}(f)| = 3$  for every edge  $f \in E(T)$  incident to a vertex in V(P),

then  $\phi^{-1}(P)$  is a burl.

Proof. Let  $P' = v_{-1}v_0 \dots v_9 v_{10}$  be a path in T such that  $P = v_0 \dots v_9$ . Let  $f_i = v_{i-1}v_i$  and let  $C_i = \{e_1^i, e_2^i, e_3^i\} = \phi^{-1}(f_i), 0 \le i \le 10$ . Let  $X := \phi^{-1}(V(P))$ . It is easy to see that  $\phi^{-1}(v_i)$  contains precisely two vertices joined by an edge,  $0 \le i \le 9$ .

We assume without loss of generality that G|X contains no cycles of length 4, as otherwise the lemma holds by Lemma 17. Let A be the set of ends of edges in  $C_0$  outside of X, and let B be the set of ends of edges in  $C_{10}$  outside of X. Observe that  $E_X$  consists of 3 internally vertex-disjoint



FIGURE 3. If there are three consecutive pairs of pairwise distinct indices contained in G|X, there is always an M-alternating cycle for any  $M \in \mathcal{M}(G,X)$  using just the vertical edges (left); the same is true for any four consecutive pairs (right).

paths from A to B, as well as one edge in  $G|\phi^{-1}(\{v_i\})$  for each  $0 \le i \le 9$ . Let  $R_1$ ,  $R_2$  and  $R_3$  be these three paths from A to B, and let  $u_j$  and  $v_j$  be the ends of  $R_j$  in A and B, respectively, for j = 1, 2, 3. For  $0 \le i \le 9$ , we have  $\phi^{-1}(v_i) = \{x_i, y_i\}$  so that  $x_i \in V(R_j), y_i \in V(R_{j'})$  for some  $\{j, j'\} \subseteq \{1, 2, 3\}$  with  $j \ne j'$ ; let the index  $\sigma_i$  of  $v_i$  be defined as  $\{j, j'\}$ . Since there is no 4-cycle in X,  $\sigma_i \ne \sigma_{i-1}$  for  $1 \le i \le 9$ . Let the type  $\psi_i$  of  $v_i$  (for  $1 \le i \le 8$ ) be defined as 0 if  $\sigma_{i-1} = \sigma_{i+1}$ , otherwise let  $\psi_i = 1$ .

Let i, j, k be integers such that  $0 \le i < j < k \le 10$  which will be determined later. Let  $X_1 := \phi^{-1}(\{v_i, \dots, v_{j-1}\}), X_2 := \phi^{-1}(\{v_j, \dots, v_{k-1}\}), X_0 = X_1 \cup X_2$ .

Let **M** be a balanced probability distribution on  $\mathcal{M}(G,X)$ , let  $Z_0$  ( $Z_1$ ,  $Z_2$ ) be the maximum number of disjoint **M**-alternating cycles in  $G|X_0$  ( $G|X_1$ ,  $G|X_2$ , respectively). Let  $Y_\ell = |\mathbf{M} \cap C_\ell|$ , for every  $\ell$ , and let  $Y = Y_i + Y_j + Y_k$ . Since **M** is balanced, we have  $\mathbb{E}(Y) = 3$ ; moreover,  $Y_i \equiv Y_j \equiv Y_k$  (mod 2). Therefore,  $\Pr(Y = 1) = 0$ ; Y = 3 if and only if  $Y_i = Y_j = Y_k = 1$ ; and Y = 2 if and only if  $\{Y_i, Y_j, Y_k\} = \{2, 0, 0\}$ .

Assume that i, j, k fulfill the following conditions:

- (1)  $\Pr(Z_1 = 0 \mid Y_i = Y_j = 0) = 0$ ,  $\Pr(Z_2 = 0 \mid Y_j = Y_k = 0) = 0$ , and  $\Pr(Z_0 = 0 \mid Y_i = Y_k = 0) = 0$ ;
- (2) for at least one of the cuts  $C_i$ ,  $C_j$  or  $C_k$ , say  $C_t$ , there exists an edge  $e \in C_t$  such that for at least one of the two corresponding graphs among  $G|X_0$ ,  $G|X_1$ ,  $G|X_2$ , say  $G|X_s$ , there is an alternating cycle in  $G|X_s$  for any element of  $\mathcal{M}(G,X)$  containing e, provided  $Y_i = Y_j = Y_k = 1$ .

First, we derive  $\mathbb{E}(Z_0) \geq \frac{1}{3}$  from these assumptions, then we prove the existence of such a triple i, j, k.

Observe that the first condition yields  $\mathbb{E}(Z_0 \mid Y = 0) \geq 2$  and  $\mathbb{E}(Z_0 \mid Y = 2) \geq 1$ . Since  $\mathbb{E}(Y) = 3$ , we have  $3 \cdot \Pr(Y = 0) + \Pr(Y = 2) \geq \Pr(Y \geq 4)$ . This gives  $\Pr(Y \neq 3) \leq 4 \cdot \Pr(Y = 0) + 2 \cdot \Pr(Y = 2)$ , and hence  $\mathbb{E}(Z_0 \mid Y \neq 3) \geq \frac{1}{2}$ . Let  $C_t = \{e_1, e_2, e_3\}$ , where  $e = e_1$ . Let  $p_i = \Pr[\mathbf{M} \cap C_t = \{e_i\} \land Y = 3]$ , i = 1, 2, 3. Clearly  $p_1 + p_2 + p_3 = \Pr(Y = 3)$ . On the other hand, since  $\mathbf{M}$  is balanced,  $\frac{1}{3} - p_1 \leq \frac{1}{3} - p_2 + \frac{1}{3} - p_3$  (all elements of  $\mathcal{M}(G, X)$  containing  $e_1$  together with some other edge from  $C_t$  contain  $e_2$  or  $e_3$ ). Hence,  $p_1 \geq \frac{1}{2} \cdot (\Pr(Y = 3) - \frac{1}{3})$ .

Altogether, in this case

$$\mathbb{E}(Z_0) = \mathbb{E}(Z_0 \mid Y \neq 3) \cdot \Pr(Y \neq 3) + \mathbb{E}(Z_0 \mid Y = 3) \cdot \Pr(Y = 3) \ge$$

$$\ge \frac{1}{2} \cdot (1 - \Pr(Y = 3)) + \frac{1}{2} \cdot \left(\Pr(Y = 3) - \frac{1}{3}\right) = \frac{1}{3}.$$

Now we prove that there is always i, j, k such that both conditions (1) and (2) are satisfied. Note that (1) is satisfied if j - i = 3 and  $\psi_{i+1} = 1$ , or if  $j - i \ge 4$ ; the same holds for j and k (to observe this, see Figure 3).

Consider the sequence  $\Psi$  of 8 types occurring in X.

Suppose  $\Psi$  contains 1111 as a subsequence, say e.g.  $\psi_2 = \psi_3 = \psi_4 = \psi_5 = 1$ . Then for i = 1, j = 4, k = 7 the condition (1) clearly holds. The condition (2) holds for  $G|X_0$  in this case, see Figure 4, left. One can prove that the following subsequences are feasible as well by drawing triples

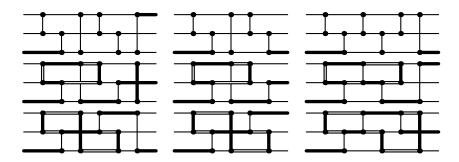


FIGURE 4. If  $\Psi$  contains 1111 (left), 111 (center) or 0001 (right) as a subsequence, then for each perfect matching containing the bottom-leftmost edge such that Y=2 there is an alternating cycle. Observe that there is always one case out of three which is not possible.

of figures (we omit the details): 00011, 01011, 100000, 100010, 100100, 101000, 101010, 100110, 10110, 1000010, 1001010, 1010010, 00000000.

Similarly, if  $\psi_1 = \psi_2 = \psi_3 = 1$ , then we may pick i = 0, j = 5, and k = 9 (or even k = 8 if  $\psi_6 = 1$ ). In this case the condition (2) holds for  $G|X_1$ , see Figure 4, center. Analogously it works with 0001. It means 111\*\*1 and 111\*\*\*\*\* are feasible as well, and so are 0001\*\*1 and 0001\*\*\*\*\*.

It remains to prove that  $\Psi$  always contains at least one feasible subsequence, which is a routine case analysis.

**Lemma 19.** Let  $(T, \phi)$  be a small-cut-decomposition of a cubic bridgeless graph G. Let  $t_1, t_2 \in V(T)$  be a pair of adjacent vertices of degree 2. Suppose that  $|\phi^{-1}(f)| = 2$  for every edge  $f \in E(T)$  incident to  $t_1$  or  $t_2$ . Then  $\phi^{-1}(\{t_1, t_2\})$  is a burl.

Proof. Let  $t_0t_1t_2t_3$  be a subpath of T and let  $C_i = \phi^{-1}(t_{i-1}t_i)$  for i = 1, 2, 3 be an edge-cut of size 2. Assume that both  $G|\phi^{-1}(t_1)$  and  $G|\phi^{-1}(t_2)$  have at most one perfect matching. By Lemma 17 it suffices to show that  $G|\phi^{-1}(\{t_1,t_2\})$  has at least two perfect matchings. As the hub  $G_1$  over  $t_1$  is cubic and bridgeless it contains at least 2 perfect matching avoiding any edge. Let  $e_1, e_2 \in E(G_1)$  be the edges in  $E(G_1) - E(G)$  corresponding to  $C_1$ - and  $C_2$ -contraction, respectively. By assumption, at most one perfect matching of  $G_1$  avoids both  $e_1$  and  $e_2$ . It follows that either two perfect matchings of  $G_1$  avoid  $e_1$  and contain  $e_2$ , or one avoids  $e_1$  and  $e_2$  and one avoids  $e_1$  and contains  $e_2$ . Let  $G_2$  be the hub over  $e_1$ . The symmetric statement holds for  $e_2$ . In any case, the perfect matchings in  $e_1$  and  $e_2$  can be combined to obtain at least two perfect matchings of  $e_1$  and  $e_2$  and one avoids  $e_1$  and  $e_2$  and  $e_3$  and  $e_4$  and  $e_5$  and  $e_6$  are the perfect matchings in  $e_1$  and  $e_2$  can be combined to obtain at least two perfect matchings of  $e_1$  and  $e_2$  and  $e_3$  and  $e_4$  and  $e_5$  are the perfect matchings of  $e_1$  and  $e_2$  and  $e_3$  are the perfect matchings of  $e_4$  and  $e_5$  are the perfect matchings of  $e_4$  and  $e_5$  and  $e_6$  are the perfect matchings of  $e_6$  and  $e_7$  are the perfect matchings of  $e_7$  and  $e_8$  are the perfect matchings of  $e_7$  and  $e_8$  are the perfect matchings of  $e_1$  and  $e_2$  and  $e_3$  are the perfect matchings of  $e_1$  and  $e_2$  are t

From the definition of a small-cut-decomposition, we immediately get the following corollary:

**Corollary 20.** Let  $(T, \phi)$  be a small-cut-decomposition of a cubic bridgeless graph G, and let P be a path in T in which every vertex has degree 2. Suppose there exist three edges  $f_1$ ,  $f_2$ ,  $f_3$  of T incident to vertices of P such that  $|\phi^{-1}(f_1)| = |\phi^{-1}(f_2)| = |\phi^{-1}(f_3)| = 2$ . Then  $\phi^{-1}(P)$  is a burl.

Let  $B_3$  denote the cubic graph consisting of two vertices joined by three parallel edges. Lemmas 18 and 19 imply the following.

Corollary 21. Let  $(T, \phi)$  be a small-cut-decomposition of a cubic bridgeless graph G and let P be a path in T with |V(P)| = 32. If for every  $t \in V(P)$ ,  $\deg_T(t) = 2$  and the hub of G at t is isomorphic to  $K_4$  or  $B_3$ , then  $\phi^{-1}(P)$  is a burl.

*Proof.* If at least three edges incident to vertices in V(P) correspond to edge-cuts of size 2 in G then the corollary holds by Corollary 20. Otherwise, since there are 33 edges of T incident to vertices of P, there must be 11 consecutive edges incident to vertices in P corresponding to edge-cuts of size 3. In this case, the result follows from Lemma 18.

Proof of Lemma 11. We proceed by induction on |Z|. If  $|Z| \leq 6$  then Z is a twig. In this case the lemma holds since  $\beta_1 \geq \beta_2 + 6\alpha$  by (3). We assume for the remainder of the proof that  $|Z| \geq 7$ . It follows that G' is not cyclically 4-edge-connected, as G' has no core. Therefore Corollary 16 is applicable to G'. Let  $\mathcal{Y}$  be a collection of disjoint elementary twigs in G' such that every relevant triangle in G' is contained in an element of  $\mathcal{Y}$ , and let  $(T, \phi)$  be a  $\mathcal{Y}$ -maximum small-cut decomposition of G'. By Corollary 16, the hub at every  $t \in V(T)$  with  $|\phi^{-1}(t)| \neq \emptyset$  is either an elementary twig, in which case t is a leaf of T, or is cyclically 4-edge-connected, in which case it is isomorphic to either  $K_4$  or  $B_3$ .

In calculations below we will make use of the following claim: If  $\deg_T(t) = 2$  for some  $t \in V(T)$ , then  $|\phi^{-1}(t)| \leq 2$ . If this is not the case, the hub at t is isomorphic to  $K_4$ , and at least three of its vertices must be vertices of G. It follows that there is an edge  $f \in E(T)$  incident to t for which  $|\phi^{-1}(f)| = 2$ . Let  $v \in \phi^{-1}(t)$  be a vertex incident to an edge in  $\phi^{-1}(f)$ . Then  $C = \phi^{-1}(f) \triangle \delta(v)$  is a 3-edge-cut in G. As in the proof of Lemma 15 we can split t into two vertices t', t'' with  $\phi^{-1}(t') = \{v\}$  and  $\phi^{-1}(t'') = \phi^{-1}(t) \setminus v$ . We now have  $\phi^{-1}(t't'') = C$  and the new small-cut-decomposition contradicts the maximality of  $(T, \phi)$ . This completes the proof of the claim.

Let  $t_0 \in V(T)$  be such that  $\phi^{-1}(t_0)$  contains the new vertex or one of the ends of the new edge in G'. Since G is pruned, G' contains at most one irrelevant triangle, and if such a triangle exists, at least one of its vertices lies in  $\phi^{-1}(t_0)$ . As a consequence, for any leaf  $t \neq t_0$  of T,  $\phi^{-1}(t)$  is a twig. Let  $t^* \in V(T) \setminus \{t_0\}$  be such that  $\deg_T(t^*) \geq 3$  and, subject to this condition, the component of  $T \setminus \{t^*\}$  containing  $t_0$  is maximal. If  $\deg_T(t) \leq 2$  for every  $t \in V(T) \setminus \{t_0\}$ , we take  $t^* = t_0$  instead. Let  $T_1, \ldots, T_k$  be all the components of  $T \setminus \{t^*\}$  not containing  $t_0$ . By the choice of  $t^*$ , each  $T_i$  is a path. If  $|V(T_i)| \geq 33$  for some  $1 \leq i \leq k$  then let T' be the subtree of  $T_i$  containing a leaf of T and exactly 32 other vertices. Let f be the unique edge in  $\delta(T')$ . Let H (resp. H') be the  $\phi^{-1}(f)$ -contraction of G (resp. G') containing  $V(G') \setminus \phi^{-1}(T')$ , and let Z' consist of  $V(H') \cap Z$  together with the new vertex created by  $\phi^{-1}(f)$ -contraction (if it exists). If H is not pruned then it contains a unique irrelevant triangle and we contract it, obtaining a pruned graph. By the induction hypothesis, either  $|Z'| \leq 6$  or we can find a foliage  $\mathcal{X}'$  in Z' with  $fw(\mathcal{X}') \geq \alpha(|Z'| - 2) + \beta_2$ . If  $|Z'| \leq 6$  let  $\mathcal{X}' := \emptyset$ .

Let t' be a vertex of T' which is not a leaf in T. Since  $\deg_T(t') = 2$ ,  $|\phi^{-1}(t')| \neq \emptyset$ . Therefore  $\phi^{-1}(t')$  is isomorphic to  $B_3$  or  $K_4$  and we can apply Corollary 21. This implies that  $\phi^{-1}(T')$  contains an elementary twig and a burl that are vertex-disjoint, where the elementary twig is the preimage of the leaf. Further, we have  $|\phi^{-1}(T')| \leq 8 + 2 \cdot 32 = 72$ , since an elementary twig has size at most 8 and the preimage of every non-leaf vertex of T' has size at most 2 by the claim above. By Lemma 7(3), we can obtain a foliage  $\mathcal{X}$  in Z by adding the twig and the burl to  $\mathcal{X}'$  and possibly removing a burl (which can be a twig) containing the new element of H' created by  $\phi^{-1}(f)$ -contraction. It follows that if  $|Z'| \geq 7$  then

$$fw(\mathcal{X}) \ge \alpha(|Z'| - 2) + 2\beta_2 \ge (\alpha|Z| + \beta_2) - 74\alpha + \beta_2 \ge \alpha|Z| + \beta_2,$$

by (4), as desired. If  $|Z'| \le 6$  then  $|Z| \le 78$  and

$$fw(\mathcal{X}) \ge \beta_1 + \beta_2 \ge 78\alpha + \beta_2 \ge \alpha |Z| + \beta_2,$$

by **(5)**.

It remains to consider the case when  $|V(T_i)| \leq 32$  for every  $1 \leq i \leq k$ . Suppose first that  $t^* \neq t_0$  and that  $|\phi^{-1}(T_0)| \geq 7$ , where  $T_0$  denotes the component of  $T \setminus t^*$  containing  $t_0$ . Let  $f_0$  be the edge incident to  $t^*$  and a vertex of  $T_0$ . We form the graphs H, H' and a set Z' by a  $\phi^{-1}(f_0)$ -contraction as in the previous case, and possibly contract a single irrelevant triangle. As before, we find a foliage  $\mathcal{X}'$  in Z' with  $fw(\mathcal{X}') \geq \alpha(|Z'|-2) + \beta_2$ . Note that  $\phi^{-1}(T_i)$  contains a twig for every  $1 \leq i \leq k$ . By Lemma 7(3), we now obtain a foliage  $\mathcal{X}$  in Z from  $\mathcal{X}'$ , adding  $k \geq 2$  twigs and possibly removing one burl (which can be a twig) from  $\mathcal{X}'$ . We have  $|\phi^{-1}(T_i)| \leq 8 + 31 \cdot 2 = 70$  for every  $1 \leq i \leq k$ , and  $|\phi^{-1}(t^*)| \leq 4$ . Therefore  $|Z| \leq |Z'| + 70k + 4$ . It follows from (5) that

$$fw(\mathcal{X}) \ge \alpha(|Z'|-2) + \beta_2 + (k-1)\beta_1 \ge \alpha|Z| + \beta_2 - 76\alpha + (k-1)(\beta_1 - 70\alpha) \ge \alpha|Z| + \beta_2.$$

Now we can assume  $t^* = t_0$  or  $|\phi^{-1}(T_0)| \le 6$ . First suppose  $t^* \ne t_0$  but  $|\phi^{-1}(T_0)| \le 6$ . Then again  $|\phi^{-1}(t^*)| \le 4$ , so we have  $|Z| \le 70k+10$ . Let  $\mathcal{X}$  be the foliage consisting of twigs in  $T_1, \ldots, T_k$ . Thus by (6), we have

$$fw(\mathcal{X}) = k\beta_1 \ge (\alpha |Z| + \beta_2) + k(\beta_1 - 70\alpha) - 10\alpha - \beta_2 \ge \alpha |Z| + \beta_2.$$

Finally we can assume  $t^* = t_0$ . Then  $|\phi^{-1}(t^*)| \le 4$ , unless k = 1 and  $\phi^{-1}(t^*)$  is an elementary twig. In either case,  $|Z| \le 70k + 8$  and the equation above applies.

### 5. Proof of Lemma 13

The following lemma is a direct consequence of a theorem of Kotzig, stating that any graph with a unique perfect matching contains a bridge (see [6]).

**Lemma 22.** Every edge of a cyclically 4-edge-connected cubic graph with at least six vertices is contained in at least two perfect matchings.

Let G be a cubic graph. For a path  $v_1v_2v_3v_4$ , the graph obtained from G by splitting along the path  $v_1v_2v_3v_4$  is the cubic graph G' obtained as follows: remove the vertices  $v_2$  and  $v_3$  and add the edges  $v_1v_4$  and  $v_1'v_4'$  where  $v_1'$  is the neighbor of  $v_2$  different from  $v_1$  and  $v_3$  and  $v_4'$  is the neighbor of  $v_3$  different from  $v_2$  and  $v_4$ . The idea of this construction (and its application to the problem of counting perfect matchings) originally appeared in [18]. We say that a perfect matching M of G is a canonical extension of a perfect matching M' of G' if  $M \triangle M' \subseteq E(G) \triangle E(G')$ , i.e. M and M' agree on the edges shared by G and G'.

**Lemma 23.** Let G be a cyclically 4-edge-connected cubic graph with  $|V(G)| \ge 6$ . If G' is the graph obtained from G by splitting along some path  $v_1v_2v_3v_4$ , then

- (1) G' is cubic and bridgeless:
- (2) G' contains at most 2 irrelevant triangles;
- (3)  $fw(G) > fw(G') 2\beta_1$ ;
- (4) Every perfect matching M' of G' avoiding the edge  $v_1v_4$  has a canonical extension in G.

Proof.

- (1) The statement is a consequence of an easy lemma in [5], stating that the cyclic edge-connectivity can drop by at most two after a splitting.
- (2) Since G is cyclically 4-edge-connected and has at least six vertices, it does not contain any triangle. The only way an irrelevant triangle can appear in G' is that  $v_1$  and  $v_4$  (or  $v_1'$  and  $v_4'$ ) have precisely one common neighbor (if they have two common neighbors, the two arising triangles share the new edge  $v_1v_4$  or  $v_1'v_4'$  and hence, are relevant).

- (3) At most two burls from a foliage of G' contain  $\{v_1, v_4\}$  or  $\{v'_1, v'_4\}$ . Therefore, a foliage of G can be obtained from any foliage of G' by removing at most two burls (observe that this is precisely here that we use the fact that being a burl is a local property, independent of the rest of the graph).
- (4) The canonical extension is obtained (uniquely) from  $M' \cap E(G)$  by adding either  $v_2v_3$  if  $v_1'v_4' \notin M'$  or  $\{v_1'v_2, v_3v_4'\}$  if  $v_1'v_4' \in M'$ .

*Proof of Lemma 13.* We proceed by induction on |V(G)|. The base case |V(G)| = 6 holds by Lemma 22 and (7).

For the induction step, consider first the case that G is cyclically 4-edge-connected. Fix an edge  $e = uv \in E(G)$ . Our goal is to show that e is contained in at least  $2^{\alpha|V(G)|-fw(G)+\gamma}$  perfect matchings.

Let  $w \neq u$  be a neighbor of v and let  $w_1$  and  $w_2$  be the two other neighbors of w. Let  $x_i, y_i$  be the neighbors of  $w_i$  distinct from w for i = 1, 2. Let  $G_1, \ldots, G_4$  be the graphs obtained from  $G_i$  by splitting along the paths  $vww_1x_1$ ,  $vww_1y_1$ ,  $vww_2x_2$  and  $vww_2y_2$ . Let  $G_i'$  be obtained from  $G_i$  by contracting irrelevant triangles for  $i = 1, \ldots, 4$ . By Lemma 23(2) we have  $|V(G_i')| \geq |V(G)| - 6$ .

Suppose first that one of the resulting graphs, without loss of generality  $G_1^i$ , does not have a core. By Corollary 12, Lemma 10 and Lemma 23, we have

$$\alpha|V(G)| \le \alpha(|V(G_1')| + 6) \le fw(G_1') + 7\alpha - \beta_2 \le fw(G_1) + 7\alpha - \beta_2 \le fw(G_1) + 7\alpha - \beta_2 \le fw(G_1) + 6\alpha \le fw(G_1') + 6\alpha \le$$

Therefore

$$\alpha |V(G)| - fw(G) + \gamma \le \gamma + 2\beta_1 + 7\alpha - \beta_2 \le 1$$

by (8) and the lemma follows from Lemma 22.

We now assume that all four graphs  $G'_1, \ldots, G'_4$  have a core. By Lemma 23(4), every perfect matching containing e in  $G_i$  canonically extends to a perfect matching containing e in G. Let S be the sum of the number of perfect matchings of  $G_i$  containing e, for  $i \in \{1, 2, 3, 4\}$ . By induction hypothesis and Lemmas 10 and 23,  $S \ge 4 \cdot 2^{\alpha(|V(G)|-6)-fw(G)-2\beta_1+\gamma}$ . On the other hand, a perfect matching M of G containing e is the canonical extension of a perfect matching containing e in precisely three of the graphs  $G_i$ ,  $i \in \{1, 2, 3, 4\}$ . For instance if  $w_1y_1, ww_2 \in M$ , then  $G_2$  is the only graph (among the four) that does not have a perfect matching M' that canonically extends to M (see Figure 5). As a consequence, there are precisely S/3 perfect matchings containing e in G. Therefore,

$$m^{\star}(G) \ge \frac{4}{3} \cdot 2^{\alpha(|V(G)|-6)-fw(G)-2\beta_1+\gamma} \ge 2^{\alpha|V(G)|-fw(G)+\gamma},$$

by (9), as desired.

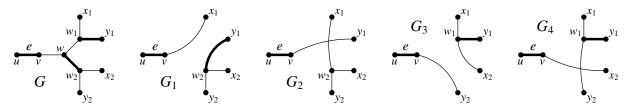


FIGURE 5. Perfect matchings in only three of the  $G_i$ 's canonically extend to a given perfect matching of G containing e.

It remains to consider the case when G contains a cyclic edge-cut C of size at most 3. Suppose first that for such edge-cut C, both C-contractions  $H_1$  and  $H_2$  have a core. Then, by Lemma 7(3),

 $fw(G) \ge fw(H_1) + fw(H_2) - 2\beta_1$  and, by induction hypothesis, applied to  $H_1$  and  $H_2$  (after possibly contracting one irrelevant triangle in each) and Lemma 7,

$$m^*(G) \ge m^*(H_1)m^*(H_2) \ge 2^{\alpha|V(G)|-4\alpha-fw(G)-2\beta_1+2\gamma} \ge 2^{\alpha|V(G)|-fw(G)+\gamma}$$

by (10), as desired. Finally, if for every cyclic edge-cut C of size at most 3 only one C-contraction has a core, we apply Corollary 16 to G. Let  $(T,\phi)$  be the resulting small-cut-decomposition of G. There exists a unique vertex  $t \in V(T)$  such that the hub H of G at t is cyclically 4-edge-connected with  $|V(H)| \geq 6$ . Let  $T_1, \ldots, T_k$  be the components of T - t and let  $Z_i = \phi^{-1}(V(T_i))$ . We apply Lemma 11 to  $Z_1, \ldots, Z_k$ . Note that Lemma 11 is indeed applicable, as G is pruned, and therefore every triangle in G belongs to an elementary twig. Consequently, no edge-cut corresponding to an edge of  $(T, \phi)$  separates exactly 3 vertices of G.

Let  $\mathcal{X}_1, \mathcal{X}_2, \ldots, \mathcal{X}_k$  be the foliages satisfying the lemma. Let  $\mathcal{X}_0$  be the maximal foliage in H avoiding new vertices and edges created by contraction of the edge-cuts  $\delta(Z_1), \ldots, \delta(Z_k)$ . Then  $fw(\mathcal{X}_0) \geq fw(H) - k\beta_2$ , as H contains no twigs (it is cyclically 4-edge-connected). Since  $\mathcal{X}_0 \cup \mathcal{X}_1 \cup \ldots \cup \mathcal{X}_k$  is a foliage in G we have

$$fw(G) \ge fw(H) - k\beta_2 + \sum_{i=1}^{k} fw(\mathcal{X}_i) \ge fw(H) + \alpha \sum_{i=1}^{k} |Z_i|,$$

by the choice of  $\mathcal{X}_1, \ldots, \mathcal{X}_k$ . It remains to observe that

$$m^{\star}(G) \geq m^{\star}(H) \geq 2^{\alpha|V(H)| - fw(H) + \gamma} \geq 2^{\alpha(|V(G)| - \sum_{i=1}^{k} |Z_i|) - fw(H) + \gamma} \geq 2^{\alpha|V(G)| - fw(G) + \gamma}$$

by the above.  $\Box$ 

### 6. Concluding remarks

6.1. **Improving the bound.** We expect that the bound in Theorem 1 can be improved at the expense of more careful case analysis. In particular, it is possible to improve the bound on the length of the path in Corollary 21. We have chosen not to do so in an attempt to keep the argument as straightforward as possible.

In [2] it is shown that for some constant c > 0 and every integer n there exists a cubic bridgeless graph on at least n vertices with at most  $c2^{n/17.285}$  perfect matchings.

6.2. Number of perfect matchings in k-regular graphs. In [12, Conjecture 8.1.8] the following generalization of the conjecture considered in this paper is stated. A graph is said to be *matching-covered* if every edge of it belongs to a perfect matching.

**Conjecture 24.** For  $k \geq 3$  there exist constants  $c_1(k), c_2(k) > 0$  such that every k-regular matching-covered graph contains at least  $c_2(k)c_1(k)^{|V(G)|}$  perfect matchings. Furthermore,  $c_1(k) \rightarrow \infty$  as  $k \rightarrow \infty$ .

While our proof does not seem to extend to the proof of this conjecture, the following weaker statement can be deduced from Theorem 1. We are grateful to Paul Seymour for suggesting the idea of the following proof.

**Theorem 25.** Let G be a k-regular (k-1)-edge-connected graph on n vertices for some  $k \geq 4$ . Then

$$\log_2 m(G) \ge (1 - \frac{1}{k})(1 - \frac{2}{k})\frac{n}{3656}.$$

*Proof.* It follows by Edmonds' characterization of the perfect matching polytope [3] that there exists a probability distribution  $\mathbf{M}$  on  $\mathcal{M}(G)$  such that for every edge  $e \in E(G)$ ,  $\Pr[e \in \mathbf{M}] = \frac{1}{k}$ .

We choose a triple of perfect matchings of G as follows. Let  $M_1 \in \mathcal{M}(G)$  be arbitrary. We have

$$\mathbb{E}[|\mathbf{M} \cap M_1|] = \frac{n}{2k}.$$

Therefore we can choose  $M_2 \in \mathcal{M}(G)$  so that  $|M_2 \cap M_1| \leq \frac{n}{2k}$ . Let  $Z \subseteq V(G)$  be the set of vertices not incident with an edge of  $M_1 \cap M_2$ . Then  $|Z| \geq (1 - \frac{1}{k}) n$ . For each  $v \in Z$  we have

$$\Pr[\mathbf{M} \cap \delta(\{v\}) \cap (M_1 \cup M_2) = \emptyset] = 1 - \frac{2}{k}.$$

Therefore the expected number of vertices whose three incident edges are in  $\mathbf{M}$ ,  $M_1$  and  $M_2$  respectively, is at least  $(1-\frac{1}{k})(1-\frac{2}{k})n$ . It follows that we can choose  $M_3 \in \mathcal{M}(G)$  so that the subgraph G' of G with  $E(G') = M_1 \cup M_2 \cup M_3$  has at least  $(1-\frac{1}{k})(1-\frac{2}{k})n$  vertices of degree three. Note that G' is by definition matching-covered. It follows that the only bridges in G' are edges joining pairs of vertices of degree one. Let G'' be obtained from G' by deleting vertices of degree one and replacing by an edge every maximal path in which all the internal vertices have degree two. The graph G'' is cubic and bridgeless and therefore by Theorem 1 we have

$$\log_2 m(G) > \log_2 m(G') \ge \log_2 m(G'') \ge \frac{1}{3656} |V(G'')| \ge (1 - \frac{1}{k})(1 - \frac{2}{k}) \frac{n}{3656}$$

as desired.  $\Box$ 

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