Elementary results for the fundamental representation of SU(3)

Thomas L. Curtright[§] and Cosmas K. Zachos[‡] Department of Physics, University of Miami Coral Gables, FL 33124-8046, USA

[§]curtright@miami.edu [‡]zachos@anl.gov

Abstract

A general group element for the fundamental representation of SU(3) can always be expressed as a second order polynomial in a hermitian generating matrix H, with coefficients consisting of elementary trigonometric functions dependent on the sole invariant det (H), in addition to the group parameter.

In memoriam Yoichiro Nambu (1921-2015)

Consider an arbitrary 3×3 traceless hermitian matrix H. The Cayley-Hamilton theorem [1] gives

$$H^{3} = I \det(H) + \frac{1}{2} H \operatorname{tr}(H^{2}) , \qquad (1)$$

and therefore det $(H) = \operatorname{tr}(H^3)/3$. Note that an H^2 term is absent in the polynomial expansion of H^3 because of the trace condition, $\operatorname{tr}(H) = 0$. Also note, since $\operatorname{tr}(H^2) > 0$ for any nonzero hermitian H, this bilinear trace factor may be absorbed into the normalization of H, thereby setting the scale of the group parameter space.

We may now write the exponential of H as a matrix polynomial [2, 3]. As a consequence of (1) any such exponential can be expressed as a matrix polynomial second-order in H, with polynomial coefficients that depend on the displacement from the group origin as a "rotation angle" θ .

Moreover, the polynomial coefficients will also depend on invariants of the matrix H. These invariants can be expressed in terms of the eigenvalues of H, of course [2, 3, 4, 5, 6, 7]. Nevertheless, while the eigenvalues of H will be manifest in the final result given below, a deliberate diagonalization of H is not necessary. This is true for SU(3), for a normalized H, since there is effectively only one invariant: det(H). This invariant may be encoded cyclometrically as another angle. Define

$$\phi = \frac{1}{3} \left(\arccos\left(\frac{3}{2}\sqrt{3}\det\left(H\right)\right) - \frac{\pi}{2} \right) , \qquad (2)$$

whose geometrical interpretation will soon be clear. Inversely,

$$\det(H) = -\frac{2}{3\sqrt{3}} \sin(3\phi) .$$
 (3)

The result for any SU(3) group element generated by a traceless 3×3 hermitian matrix H is then

$$\exp\left(i\theta H\right) = \sum_{k=0,1,2} \left[H^2 + \frac{2}{\sqrt{3}} H \sin\left(\phi + \frac{2\pi k}{3}\right) - \frac{1}{3} I \left(1 + 2\cos\left(2\left(\phi + \frac{2\pi k}{3}\right)\right)\right) \right] \frac{\exp\left(\frac{2}{\sqrt{3}} i\theta \sin\left(\phi + \frac{2\pi k}{3}\right)\right)}{1 - 2\cos\left(2\left(\phi + \frac{2\pi k}{3}\right)\right)} \quad (4)$$

where we have set the scale for the θ parameter space by choosing the normalization

$$\operatorname{tr}\left(H^{2}\right) = 2 \ . \tag{5}$$

With this choice, the Cayley-Hamilton result (1) is just [9]

$$H^3 = H + I \det\left(H\right) \ . \tag{6}$$

The normalization (5) and the identity (6) are consistent with the Gell-Mann λ -matrices [8].

So expressed as a matrix polynomial, the group element depends on the sole invariant det (H) in addition to the group rotation angle θ . Both dependencies are in terms of elementary trigonometric functions when det (H)

is expressed as the angle ϕ , whose geometrical interpretation follows immediately from the three eigenvalues of H exhibited in the exponentials of (4). Those eigenvalues are the projections onto three mutually perpendicular axes of a single point on a circle formed by the intersection of the 0 = tr(H) eigenvalue plane with the $2 = \text{tr}(H^2)$ eigenvalue 2-sphere. The angle ϕ parameterizes that circle. Equivalently, the eigenvalues are the projections onto a single axis of three points equally spaced on a circle [10].

Two cases deserve special mention. On the one hand, the Rodrigues formula for SO(3) rotations about an axis \hat{n} , as generated by j = 1 spin matrices, is obtained for $\phi = 0 = \det(H)$. Thus

$$\exp(i\theta H)|_{\phi=0} = I + iH\sin\theta + H^2(\cos\theta - 1) \quad . \tag{7}$$

This is the Euler-Rodrigues result, upon identifying $H = \hat{n} \cdot \vec{J}$ (see [11, 12]). It provides an explicit embedding $SO(3) \subset SU(3)$. In fact, (7) is true if H is any *one* of the first seven Gell-Mann λ -matrices [8], or if H is a normalized linear combination of λ_{1-3} , or of λ_{4-7} . However, for generic linear combinations of λ_{1-7} , det (H) will *not* necessarily vanish, and the general result (4) must be used.

On the other hand,

$$\lambda_8 = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & 0 & 0\\ 0 & 1 & 0\\ 0 & 0 & -2 \end{pmatrix}$$
(8)

is the only one among Gell-Mann's choices for the 3×3 representation matrices for which $\phi \neq 0$, and for which two eigenvalues are degenerate. Obviously, det $(\lambda_8) = \frac{-2}{3\sqrt{3}}$, so $\phi = \pi/6$. In addition,

$$\lambda_8^2 = \frac{2}{3} I - \frac{1}{\sqrt{3}} \lambda_8 . (9)$$

Thus, directly from (4),

$$\exp(i\theta\lambda_8) = \frac{1}{3}\left(2I + \sqrt{3}\lambda_8\right)e^{\frac{1}{3}i\sqrt{3}\theta} + \frac{1}{3}\left(I - \sqrt{3}\lambda_8\right)e^{-i\frac{2}{\sqrt{3}}\theta} = \begin{pmatrix} \exp(i\theta/\sqrt{3}) & 0 & 0\\ 0 & \exp(i\theta/\sqrt{3}) & 0\\ 0 & 0 & \exp(-2i\theta/\sqrt{3}) \end{pmatrix}$$
(10)

as it should. Note that this particular example followed from (4) by *carefully* taking the limit as $\phi \to \pi/6$ of the k = 0 and k = 1 terms in that general expression (as necessitated by the degeneracy of the corresponding eigenvalues of λ_8) combined with the straightforward limit of the k = 2 term. That is to say,

$$\lim_{\phi \to \pi/6} \left(\left[\lambda_8^2 + \frac{2}{\sqrt{3}} \ \lambda_8 \sin\left(\phi + \frac{2\pi}{3}\right) - \frac{1}{3} \ I \left(1 + 2\cos\left(2\left(\phi + \frac{2\pi}{3}\right)\right) \right) \right] \frac{\exp\left(\frac{2}{\sqrt{3}} \ i\theta \sin\left(\phi + \frac{2\pi}{3}\right)\right)}{1 - 2\cos\left(2\left(\phi + \frac{2\pi}{3}\right)\right)} \right)$$
$$= \lim_{\phi \to \pi/6} \left(\left[\lambda_8^2 + \frac{2}{\sqrt{3}} \ \lambda_8 \sin\left(\phi\right) - \frac{1}{3} \ I \left(1 + 2\cos\left(2\phi\right) \right) \right] \frac{\exp\left(\frac{2}{\sqrt{3}} \ i\theta \sin\phi\right)}{1 - 2\cos(2\phi)} \right) = \left(\frac{1}{3} \ I + \frac{1}{2\sqrt{3}} \ \lambda_8 \right) e^{i\theta/\sqrt{3}} . \tag{11}$$

Finally, one readily verifies that the Laplace transform of (4) gives the resolvent in the standard form as a matrix polynomial [13, 14, 15],

$$\int_{0}^{\infty} e^{-t} \exp\left(itsH\right) dt = \frac{1}{I - isH} = \frac{1}{\det\left(I - isH\right)} \sum_{n=0}^{\operatorname{rank}(H)-1} \left(isH\right)^{n} \operatorname{Trunc}_{\operatorname{rank}(H)-1-n}\left[\det\left(I - isH\right)\right] , \quad (12)$$

where the truncation (as defined in [11]) is in powers of s. For the case at hand rank (H) = 3, again with tr (H) = 0 and tr $(H^2) = 2$, so

$$\frac{1}{I - isH} = \frac{1}{1 + s^2 + is^3 \det(H)} \left(\left(1 + s^2 \right) I + isH - s^2 H^2 \right) .$$
(13)

From this resolvent one immediately obtains a matrix polynomial for the simple Cayley transform representation [1] of the corresponding SU(3) group elements [4],

$$\frac{I+isH}{I-isH} = \frac{1}{1+s^2+is^3\det(H)} \left(\left(1+s^2-is^3\det(H)\right)I + 2isH - 2s^2H^2 \right) .$$
(14)

The Laplace transform (12) can be inverted, in standard fashion [16], to obtain (4) in terms of the impulse response of the transfer function given by the prefactor in (13). Explicitly,

$$\exp(i\theta H) = \left(H^2 - iH\frac{d}{d\theta} - I\left(1 + \frac{d^2}{d\theta^2}\right)\right) \sum_{k=0,1,2} \frac{\exp\left(\frac{2}{\sqrt{3}}i\theta\sin(\phi + 2\pi k/3)\right)}{1 - 2\cos^2(\phi + 2\pi k/3)} .$$
 (15)

Acknowledgements This work was supported in part by NSF Award PHY-1214521, and in part by a University of Miami Cooper Fellowship.

References

- [1] The Collected Mathematical Papers of Arthur Cayley, Cambridge University Press (1889). See Vol. I. pp 28-35 for the Cayley transform, and see Vol. II. pp 475-496 for the Cayley-Hamilton theorem. Also see https://en.wikipedia.org/wiki/Cayley-Hamilton_theorem and https://en.wikipedia.org/wiki/Cayley_transform.
- [2] The general method to express any analytic matrix function of a finite, diagonalizable matrix as a polynomial in the matrix, through the use of projection matrices, is due to J J Sylvester, Phil.Mag. 16 (1883) 267-269. Also see https://en.wikipedia.org/wiki/Sylvester's_formula.
- [3] Y Lehrer, "On functions of matrices" Rendiconti del Circolo Matematico di Palermo 6 (1957) 103-108; Y Lehrer–Ilamed, "On the direct calculations of the representations of the three-dimensional pure rotation group" Proc.Camb.Phil.Soc. 60 (1964) 61–66 (especially see Remark (1), Eqn(10)).
- [4] A J MacFarlane, A Sudbery, and P H Weisz, "On Gell-Mann's λ-Matrices, d- and f-Tensors, Octets, and Parametrizations of SU(3)" Commun.Math.Phys. 11 (1968) 77-90.
- [5] S P Rosen, "Finite Transformations in Various Representations of SU(3)" J.Math.Phys. 12 (1971) 673-681.
- [6] D Kusnezov, "Exact matrix expansions for group elements of SU(N)" J.Math.Phys. 36 (1995) 898-906.
- [7] A Laufer, "The exponential map of GL(N)" J.Phys.A:Math.Gen. 30 (1997) 5455.
- [8] M Gell-Mann and Y Ne'eman, *The Eightfold Way, W A Benjamin (1964)*. Also see https://en.wikipedia.org/wiki/Gell-Mann_matrices.
- [9] The reader may use (3) and (6) to check that the coefficients of the three exponentials in (4) are indeed projection matrices.
- [10] R W D Nickalls, "Viète, Descartes and the cubic equation" Mathematical Gazette 90 (2006) 203–208. Also see https://en.wikipedia.org/wiki/Cubic_function#Three_real_roots.
- [11] T L Curtright, D B Fairlie, and C K Zachos, "A Compact Formula for Rotations as Spin Matrix Polynomials" SIGMA 10 (2014) 084. e-Print: arXiv:1402.3541 [math-ph]
- [12] T L Curtright and T S Van Kortryk, "On rotations as spin matrix polynomials" J.Phys.A: Math.Theor. 48 (2015) 025202. e-Print: arXiv:1408.0767 [math-ph]
- [13] M X He and P E Ricci, "On Taylor's formula for the resolvent of a complex matrix" Computers and Mathematics with Applications 56 (2008) 2285–2288.
- [14] T S Van Kortryk, "Cayley transforms of su (2) representations" e-Print: arXiv:1506.00500 [math-ph]
- [15] T L Curtright, "More on Rotations as Spin Matrix Polynomials" e-Print: arXiv:1506.04648 [math-ph]
- [16] https://en.wikipedia.org/wiki/Laplace_transform#Inverse_Laplace_transform