# Elementary results for the fundamental representation of $\operatorname{SU}(3)$ 

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#### Abstract

A general group element for the fundamental representation of $S U(3)$ can always be expressed as a second order polynomial in a hermitian generating matrix $H$, with coefficients consisting of elementary trigonometric functions dependent on the sole invariant $\operatorname{det}(H)$, in addition to the group parameter.


In memoriam Yoichiro Nambu (1921-2015)
Consider an arbitrary $3 \times 3$ traceless hermitian matrix $H$. The Cayley-Hamilton theorem [1] gives

$$
\begin{equation*}
H^{3}=I \operatorname{det}(H)+\frac{1}{2} H \operatorname{tr}\left(H^{2}\right) \tag{1}
\end{equation*}
$$

and therefore $\operatorname{det}(H)=\operatorname{tr}\left(H^{3}\right) / 3$. Note that an $H^{2}$ term is absent in the polynomial expansion of $H^{3}$ because of the trace condition, $\operatorname{tr}(H)=0$. Also note, since $\operatorname{tr}\left(H^{2}\right)>0$ for any nonzero hermitian $H$, this bilinear trace factor may be absorbed into the normalization of $H$, thereby setting the scale of the group parameter space.

We may now write the exponential of $H$ as a matrix polynomial [2, 3]. As a consequence of (11) any such exponential can be expressed as a matrix polynomial second-order in $H$, with polynomial coefficients that depend on the displacement from the group origin as a "rotation angle" $\theta$.

Moreover, the polynomial coefficients will also depend on invariants of the matrix $H$. These invariants can be expressed in terms of the eigenvalues of $H$, of course [2, 3, 4, 5, 6, 7]. Nevertheless, while the eigenvalues of $H$ will be manifest in the final result given below, a deliberate diagonalization of $H$ is not necessary. This is true for $S U(3)$, for a normalized $H$, since there is effectively only one invariant: $\operatorname{det}(H)$. This invariant may be encoded cyclometrically as another angle. Define

$$
\begin{equation*}
\phi=\frac{1}{3}\left(\arccos \left(\frac{3}{2} \sqrt{3} \operatorname{det}(H)\right)-\frac{\pi}{2}\right) \tag{2}
\end{equation*}
$$

whose geometrical interpretation will soon be clear. Inversely,

$$
\begin{equation*}
\operatorname{det}(H)=-\frac{2}{3 \sqrt{3}} \sin (3 \phi) \tag{3}
\end{equation*}
$$

The result for any $S U(3)$ group element generated by a traceless $3 \times 3$ hermitian matrix $H$ is then

$$
\begin{equation*}
\exp (i \theta H)=\sum_{k=0,1,2}\left[H^{2}+\frac{2}{\sqrt{3}} H \sin \left(\phi+\frac{2 \pi k}{3}\right)-\frac{1}{3} I\left(1+2 \cos \left(2\left(\phi+\frac{2 \pi k}{3}\right)\right)\right)\right] \frac{\exp \left(\frac{2}{\sqrt{3}} i \theta \sin \left(\phi+\frac{2 \pi k}{3}\right)\right)}{1-2 \cos \left(2\left(\phi+\frac{2 \pi k}{3}\right)\right)} \tag{4}
\end{equation*}
$$

where we have set the scale for the $\theta$ parameter space by choosing the normalization

$$
\begin{equation*}
\operatorname{tr}\left(H^{2}\right)=2 \tag{5}
\end{equation*}
$$

With this choice, the Cayley-Hamilton result (1) is just 9

$$
\begin{equation*}
H^{3}=H+I \operatorname{det}(H) \tag{6}
\end{equation*}
$$

The normalization (5) and the identity (6) are consistent with the Gell-Mann $\lambda$-matrices [8].
So expressed as a matrix polynomial, the group element depends on the sole invariant $\operatorname{det}(H)$ in addition to the group rotation angle $\theta$. Both dependencies are in terms of elementary trigonometric functions when det $(H)$
is expressed as the angle $\phi$, whose geometrical interpretation follows immediately from the three eigenvalues of $H$ exhibited in the exponentials of (4). Those eigenvalues are the projections onto three mutually perpendicular axes of a single point on a circle formed by the intersection of the $0=\operatorname{tr}(H)$ eigenvalue plane with the $2=\operatorname{tr}\left(H^{2}\right)$ eigenvalue 2 -sphere. The angle $\phi$ parameterizes that circle. Equivalently, the eigenvalues are the projections onto a single axis of three points equally spaced on a circle 10 .

Two cases deserve special mention. On the one hand, the Rodrigues formula for $S O(3)$ rotations about an axis $\widehat{n}$, as generated by $j=1$ spin matrices, is obtained for $\phi=0=\operatorname{det}(H)$. Thus

$$
\begin{equation*}
\left.\exp (i \theta H)\right|_{\phi=0}=I+i H \sin \theta+H^{2}(\cos \theta-1) \tag{7}
\end{equation*}
$$

This is the Euler-Rodrigues result, upon identifying $H=\widehat{n} \cdot \vec{J}$ (see [11, 12]). It provides an explicit embedding $S O(3) \subset S U(3)$. In fact, (7) is true if $H$ is any one of the first seven Gell-Mann $\lambda$-matrices [8, or if $H$ is a normalized linear combination of $\lambda_{1-3}$, or of $\lambda_{4-7}$. However, for generic linear combinations of $\lambda_{1-7}$, $\operatorname{det}(H)$ will not necessarily vanish, and the general result (4) must be used.

On the other hand,

$$
\lambda_{8}=\frac{1}{\sqrt{3}}\left(\begin{array}{ccc}
1 & 0 & 0  \tag{8}\\
0 & 1 & 0 \\
0 & 0 & -2
\end{array}\right)
$$

is the only one among Gell-Mann's choices for the $3 \times 3$ representation matrices for which $\phi \neq 0$, and for which two eigenvalues are degenerate. Obviously, $\operatorname{det}\left(\lambda_{8}\right)=\frac{-2}{3 \sqrt{3}}$, so $\phi=\pi / 6$. In addition,

$$
\begin{equation*}
\lambda_{8}^{2}=\frac{2}{3} I-\frac{1}{\sqrt{3}} \lambda_{8} \tag{9}
\end{equation*}
$$

Thus, directly from (4),

$$
\exp \left(i \theta \lambda_{8}\right)=\frac{1}{3}\left(2 I+\sqrt{3} \lambda_{8}\right) e^{\frac{1}{3} i \sqrt{3} \theta}+\frac{1}{3}\left(I-\sqrt{3} \lambda_{8}\right) e^{-i \frac{2}{\sqrt{3}} \theta}=\left(\begin{array}{ccc}
\exp (i \theta / \sqrt{3}) & 0 & 0  \tag{10}\\
0 & \exp (i \theta / \sqrt{3}) & 0 \\
0 & 0 & \exp (-2 i \theta / \sqrt{3})
\end{array}\right)
$$

as it should. Note that this particular example followed from (4) by carefully taking the limit as $\phi \rightarrow \pi / 6$ of the $k=0$ and $k=1$ terms in that general expression (as necessitated by the degeneracy of the corresponding eigenvalues of $\lambda_{8}$ ) combined with the straightforward limit of the $k=2$ term. That is to say,

$$
\begin{align*}
& \lim _{\phi \rightarrow \pi / 6}\left(\left[\lambda_{8}^{2}+\frac{2}{\sqrt{3}} \lambda_{8} \sin \left(\phi+\frac{2 \pi}{3}\right)-\frac{1}{3} I\left(1+2 \cos \left(2\left(\phi+\frac{2 \pi}{3}\right)\right)\right)\right] \frac{\exp \left(\frac{2}{\sqrt{3}} i \theta \sin \left(\phi+\frac{2 \pi}{3}\right)\right)}{1-2 \cos \left(2\left(\phi+\frac{2 \pi}{3}\right)\right)}\right) \\
= & \lim _{\phi \rightarrow \pi / 6}\left(\left[\lambda_{8}^{2}+\frac{2}{\sqrt{3}} \lambda_{8} \sin (\phi)-\frac{1}{3} I(1+2 \cos (2 \phi))\right] \frac{\exp \left(\frac{2}{\sqrt{3}} i \theta \sin \phi\right)}{1-2 \cos (2 \phi)}\right)=\left(\frac{1}{3} I+\frac{1}{2 \sqrt{3}} \lambda_{8}\right) e^{i \theta / \sqrt{3}} . \tag{11}
\end{align*}
$$

Finally, one readily verifies that the Laplace transform of (44) gives the resolvent in the standard form as a matrix polynomial [13, 14, 15,

$$
\begin{equation*}
\int_{0}^{\infty} e^{-t} \exp (i t s H) d t=\frac{1}{I-i s H}=\frac{1}{\operatorname{det}(I-i s H)} \sum_{n=0}^{\operatorname{rank}(H)-1}(i s H)^{n} \operatorname{Trunc}_{\operatorname{rank}(H)-1-n}[\operatorname{det}(I-i s H)] \tag{12}
\end{equation*}
$$

where the truncation (as defined in [11) is in powers of $s$. For the case at hand $\operatorname{rank}(H)=3$, again with $\operatorname{tr}(H)=0$ and $\operatorname{tr}\left(H^{2}\right)=2$, so

$$
\begin{equation*}
\frac{1}{I-i s H}=\frac{1}{1+s^{2}+i s^{3} \operatorname{det}(H)}\left(\left(1+s^{2}\right) I+i s H-s^{2} H^{2}\right) \tag{13}
\end{equation*}
$$

From this resolvent one immediately obtains a matrix polynomial for the simple Cayley transform representation [1] of the corresponding $S U(3)$ group elements [4],

$$
\begin{equation*}
\frac{I+i s H}{I-i s H}=\frac{1}{1+s^{2}+i s^{3} \operatorname{det}(H)}\left(\left(1+s^{2}-i s^{3} \operatorname{det}(H)\right) I+2 i s H-2 s^{2} H^{2}\right) \tag{14}
\end{equation*}
$$

The Laplace transform (12) can be inverted, in standard fashion [16], to obtain (4) in terms of the impulse response of the transfer function given by the prefactor in (13). Explicitly,

$$
\begin{equation*}
\exp (i \theta H)=\left(H^{2}-i H \frac{d}{d \theta}-I\left(1+\frac{d^{2}}{d \theta^{2}}\right)\right) \sum_{k=0,1,2} \frac{\exp \left(\frac{2}{\sqrt{3}} i \theta \sin (\phi+2 \pi k / 3)\right)}{1-2 \cos 2(\phi+2 \pi k / 3)} \tag{15}
\end{equation*}
$$

Acknowledgements This work was supported in part by NSF Award PHY-1214521, and in part by a University of Miami Cooper Fellowship.

## References

[1] The Collected Mathematical Papers of Arthur Cayley, Cambridge University Press (1889). See Vol. I. pp 28-35 for the Cayley transform, and see Vol. II. pp 475-496 for the CayleyHamilton theorem. Also see https://en.wikipedia.org/wiki/Cayley-Hamilton_theorem and https://en.wikipedia.org/wiki/Cayley_transform.
[2] The general method to express any analytic matrix function of a finite, diagonalizable matrix as a polynomial in the matrix, through the use of projection matrices, is due to J J Sylvester, Phil.Mag. 16 (1883) 267-269. Also see https://en.wikipedia.org/wiki/Sylvester's_formula.
[3] Y Lehrer, "On functions of matrices" Rendiconti del Circolo Matematico di Palermo 6 (1957) 103-108;
Y Lehrer-Ilamed, "On the direct calculations of the representations of the three-dimensional pure rotation group" Proc.Camb.Phil.Soc. 60 (1964) 61-66 (especially see Remark (1), Eqn(10)).
[4] A J MacFarlane, A Sudbery, and P H Weisz, "On Gell-Mann's $\lambda$-Matrices, $d$ - and $f$-Tensors, Octets, and Parametrizations of SU(3)" Commun.Math.Phys. 11 (1968) 77-90.
[5] S P Rosen, "Finite Transformations in Various Representations of SU(3)" J.Math.Phys. 12 (1971) 673-681.
[6] D Kusnezov, "Exact matrix expansions for group elements of SU(N)" J.Math.Phys. 36 (1995) 898-906.
[7] A Laufer, "The exponential map of GL(N)" J.Phys.A:Math.Gen. 30 (1997) 5455.
[8] M Gell-Mann and Y Ne'eman, The Eightfold Way, W A Benjamin (1964). Also see https://en.wikipedia.org/wiki/Gell-Mann_matrices.
[9] The reader may use (3) and (6) to check that the coefficients of the three exponentials in (4) are indeed projection matrices.
[10] R W D Nickalls, "Viète, Descartes and the cubic equation" Mathematical Gazette 90 (2006) 203-208. Also see https://en.wikipedia.org/wiki/Cubic_function\#Three_real_roots.
[11] T L Curtright, D B Fairlie, and C K Zachos, "A Compact Formula for Rotations as Spin Matrix Polynomials" SIGMA 10 (2014) 084. e-Print: arXiv:1402.3541 [math-ph]
[12] T L Curtright and T S Van Kortryk, "On rotations as spin matrix polynomials" J.Phys.A: Math.Theor. 48 (2015) 025202, e-Print: arXiv:1408.0767 [math-ph]
[13] M X He and P E Ricci, "On Taylor's formula for the resolvent of a complex matrix" Computers and Mathematics with Applications 56 (2008) 2285-2288.
[14] T S Van Kortryk, "Cayley transforms of $s u(2)$ representations" e-Print: arXiv:1506.00500 [math-ph]
[15] T L Curtright, "More on Rotations as Spin Matrix Polynomials" e-Print: arXiv:1506.04648 [math-ph]
[16] https://en.wikipedia.org/wiki/Laplace_transform\#Inverse_Laplace_transform

