Vacuum Mass Spectra for SU(N) Self-Dual Chern-Simons-Higgs Systems*

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Abstract

We study the $SU(N)$ self-dual Chern-Simons-Higgs systems with adjoint matter coupling, and show that the sixth order self-dual potential has $p(N)$ gauge inequivalent degenerate minima, where $p(N)$ is the number of partitions of N. We compute the masses of the gauge and scalar excitations in these different vacua, revealing an intricate mass structure which reflects the self-dual nature of the model.

1 Introduction

Relativistic self-dual Chern-Simons-Higgs systems in $2+1$ dimensions have been shown to possess many remarkable properties. In the abelian theories [1], with the scalar potential of a particular sixth order form, the energy functional is bounded below by a Bogomol'nyi style bound [2]. This lower bound is saturated by topological solitons and nontopological vortices [3]. Furthermore, the self-dual structure of the Chern-Simons-Higgs system is related at a fundamental level to an $N=2$ supersymmetry in $2+1$ dimensions [4, 5, 6]. The self-dual structure of these abelian Chern-Simons-Higgs systems has been shown to extend to nonabelian Chern-Simons-Higgs systems with a global $U(1)$ symmetry [7], once again with a special sixth order scalar potential. However, while the self-dual structure generalizes in a relatively straightforward manner, the analysis of the nonabelian self-duality equations themselves is significantly more complicated. This complication is further compounded by

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the many different choices: of gauge group, of representation, of matter coupling, etc... . Matter fields in the defining representation were studied in [8], while the adjoint matter coupling, in which one can treat the gauge and matter fields in the same representation, has been studied in [7, 9]. Recently, the $SU(3)$ self-dual Chern-Simons-Higgs system with adjoint coupling has been investigated in detail, with a systematic analysis of the three distinct degenerate vacua [10]. In this paper, I present the mass spectra of the various gauge inequivalent degenerate vacua of the $SU(N)$ self-dual Chern-Simons-Higgs system with adjoint coupling. While I concentrate on $SU(N)$, the approach is readily generalizable to other compact gauge groups. In the $SU(N)$ case, the number of gauge inequivalent minima is equal to the number, $p(N)$, of partitions of N. The mass spectra in these vacua reveal a remarkably intricate structure, reflecting the self-duality symmetry in conjunction with the Chern-Simons Higgs mechanism [11].

In Section 2 I introduce the model and briefly review the derivation of the relativistic self-dual Chern-Simons equations. The potential minima are found by solving an algebraic embedding problem, and this fact is exploited in Section 3 to provide a complete and constructive classification of the gauge inequivalent degenerate vacua. In Section 4 I analyze the masses of the gauge and scalar excitations in these various vacua, and discuss some of the interesting features which arise. The most involved nontrivial vacuum is the maximal symmetry breaking case, for which the complete $SU(N)$ mass spectrum is presented. The mass matrices of the real fields are discussed in Section 5. Section 6 is devoted to some conclusions and suggestions for further investigation.

2 Relativistic Self-Duality Equations

Consider the following Lagrange density in $2+1$ dimsensional spacetime

$$\mathcal{L} = -\mathrm{tr}\left(\left(D_\mu\phi\right)^\dagger D^\mu\phi\right) - \kappa\epsilon^{\mu\nu\rho}\mathrm{tr}\left(\partial_\mu A_\nu A_\rho + \frac{2}{3}A_\mu A_\nu A_\rho\right) - V\left(\phi, \phi^\dagger\right) \tag{1}$$

where the gauge invariant scalar field potential $V(\phi,\phi^\dagger)$ is

$$V\left(\phi,\phi^\dagger\right) = \frac{1}{4\kappa^2}\mathrm{tr}\left(\left([\,[\,\phi,\phi^\dagger\,],\phi\,] - v^2\phi\right)^\dagger\,\left([\,[\,\phi,\phi^\dagger\,],\phi\,] - v^2\phi\right)\right). \tag{2}$$

The covariant derivative is $D_\mu \equiv \partial_\mu + [A_\mu, \quad]$, the space-time metric is taken to be $g_{\mu\nu} = \mathrm{diag}\,(-1,1,1)$, and tr refers to the trace in a finite dimensional representation of the compact simple Lie algebra $\mathcal{G}$ to which the gauge fields A_μ and the charged matter fields ϕ and $\phi^\dagger$ belong. Most of the discussion will focus on the Lie algebra of $SU(N)$, but the generalization to an arbitrary compact simple Lie algebra is straightforward and is indicated at the appropriate points. The v^2 parameter appearing in the potential (2) plays the role of a mass parameter.

The Euler-Lagrange equations of motion obtained from the Lagrange density (1) are

$$D_\mu D^\mu \phi \;=\; \frac{\partial V}{\partial \phi^\dagger} \tag{3}$$

$$-\kappa \epsilon^{\mu\nu\rho} F_{\nu\rho} \;=\; i J^\mu \tag{4}$$

where $F_{\nu\rho} \equiv \partial_\nu A_\rho - \partial_\rho A_\nu + [\, A_\nu, A_\rho \,]$ is the gauge curvature, and the nonabelian current J^μ is given by

$$J^\mu \equiv -i \left([\, \phi^\dagger, D^\mu \phi \,] - [\, (D^\mu \phi)^\dagger, \phi \,] \right) \tag{5}$$

Note that the currrent J_μ is covariantly conserved, $D_\mu J^\mu = 0$, while the gauge invariant current

$$V^\mu = -i \mathrm{tr} \left(\phi^\dagger D^\mu \phi - (D^\mu \phi)^\dagger \phi \right) \tag{6}$$

is ordinarily conserved: $\partial_\mu V^\mu = 0$.

The energy density for this system can be expressed as [7, 9, 10]

$$\begin{aligned} \mathcal{E} \;=\; & \mathrm{tr} \left(\left(D_0 \phi - \frac{i}{2\kappa} \left([\, [\, \phi, \phi^\dagger \,], \phi \,] - v^2 \phi \right) \right)^\dagger \left(D_0 \phi - \frac{i}{2\kappa} \left([\, [\, \phi, \phi^\dagger \,], \phi \,] - v^2 \phi \right) \right) \right) \\ & + \mathrm{tr} \left((D_- \phi)^\dagger D_- \phi \right) + \frac{i v^2}{2\kappa} \mathrm{tr} \left(\phi^\dagger (D_0 \phi) - (D_0 \phi)^\dagger \phi \right) \end{aligned} \tag{7}$$

where $D_- = D_1 - i D_2$, and κ has been chosen positive. The first two terms in (7) are manifestly positive and the third gives a lower bound for the energy density, which may be written in terms of the time component, V^0, of the gauge invariant current defined in (6):

$$\mathcal{E} \geq \frac{v^2}{2\kappa} V^0 \tag{8}$$

This lower bound is saturated when the following two conditions (each first order in spacetime derivatives) hold:

$$D_- \phi \;=\; 0 \tag{9}$$

$$D_0 \phi \;=\; \frac{i}{2\kappa} \left([\, [\, \phi, \phi^\dagger \,], \phi] - v^2 \phi \right) \tag{10}$$

The consistency condition of these two equations states that

$$\begin{aligned} (D_0 D_- - D_- D_0)\, \phi \;&\equiv\; [F_{0-}, \phi] \\ &=\; -\frac{i}{2\kappa} [[\phi, (D_+ \phi)^\dagger], \phi] \\ &=\; \frac{1}{2\kappa} [J_-, \phi] \end{aligned} \tag{11}$$

which expresses the gauge field Euler-Lagrange equation of motion, $F_{0-} = \frac{1}{2\kappa} J_-$, for the spatial component of the current. The other gauge field equation, $F_{+-} = \frac{1}{\kappa} J_0$, may be re-expressed, using equation (10), in a form not involving explicit time derivatives. We thus arrive at the "*relativistic self-dual Chern-Simons equations*":

$$D_- \phi \;=\; 0 \tag{12}$$

$$F_{+-} \;=\; \frac{1}{\kappa^2}[v^2\phi - [[\phi, \phi^\dagger], \phi], \phi^\dagger] \tag{13}$$

Note that if we ignore the quartic ϕ term in (13), which corresponds to taking the nonrelativistic limit [9], then these relativistic self-duality equations (12, 13) reduce to the nonrelativistic self-dual Chern-Simons equations studied in [12, 13, 14].

At the self-dual point, we can use equation (10) to express the energy density as

$$\mathcal{E}_{\rm SD} = \frac{v^2}{2\kappa^2} {\rm tr}\left(\phi^\dagger\left(v^2\phi - [[\phi, \phi^\dagger], \phi]\right)\right) \tag{14}$$

Recall that all solutions to the **nonrelativistic** self-duality equations correspond to the static zero-energy solutions to the Euler-Lagrange equations of motion [13]. Here, in the relativistic theory, the situation is rather different. First, the lower bound (8) on the energy density is not necessarily zero, and the solutions of (10) are time dependent. Furthermore, unlike in the nonrelativistic case, it is possible to have nontrivial solutions for ϕ while having $F_{+-} = 0$. These solutions do have zero energy, and are gauge equivalent to solutions of the *algebraic* equation

$$[[\phi, \phi^\dagger], \phi] = v^2\phi. \tag{15}$$

Solutions of this equation also correspond to the minima of the potential (2), and these potential minima are clearly degenerate.

A class of solutions to the self-duality equations (13) is given by the following zero energy solutions of the Euler-Lagrange equations:

$$\begin{aligned} \phi &= g^{-1}\phi_{(0)} g \\ A_\pm &= g^{-1}\partial_\pm g \\ A_0 &= g^{-1}\partial_0 g \end{aligned} \tag{16}$$

where $\phi_{(0)}$ is any solution of (15), and $g = g(\vec{x}, t)$ takes values in the gauge group. It is clear that these solutions satisfy $D_0\phi = 0$, $D_-\phi = 0$, $F_{+-} = 0$, as well as the algebraic equation (15), which implies that they are self-dual, and that they have zero magnetic field and zero charge density.

While this class of solutions looks somewhat trivial, it is still important because the solutions, $\phi_{(0)}$, of the algebraic equation (15) classify the minima of the potential V, and the finite *nonzero* energy solutions of the self-duality equations must be gauge equivalent to such a solution at infinity:

$$\phi \to g^{-1}\phi_{(0)} g \qquad \text{as} \quad r \to \infty \tag{17}$$

3 Classification of Minima

As has been pointed out in the context of the $SU(2)$ and $SU(3)$ models [7, 10], equation (15) is just the $SU(2)$ commutation relation, once a factor of v has been absorbed into the field ϕ. For a general gauge algebra, finding the solutions to (15) is the classic Dynkin problem [15] of embedding $SU(2)$ into a general Lie algebra.[1]

It is clear that in order to satisfy (15) for a general gauge algebra, $\phi = \phi_{(0)}$ must be a linear combination of the step operators for the *positive* roots of the algebra. Further, since we have the freedom of global gauge invariance, we can choose representative gauge inequivalent solutions $\phi_{(0)}$ to be linear combinations of the step operators of the positive *simple* roots. It is therefore convenient to work in the Chevalley basis [16] for the gauge algebra (for ease of presentation we shall present formulas for the simply-laced algebras). In the Chevalley basis, the Cartan subalgebra elements, H_a, and the simple root step operators, E_a, have the following simple commutation relations

$$\begin{aligned} [H_a, H_b] &= 0 \\ [E_a, E_{-b}] &= \delta_{a\,b} H_a \\ [H_a, E_{\pm b}] &= \pm K_{b\,a} E_{\pm b} \end{aligned} \tag{18}$$

where a and b take values $1 \dots r$ (r is the rank of the algebra), and $K_{a\,b}$ is the Cartan matrix which encapsulates the inner products of the simple roots $\vec{\alpha}^{(a)}$:

$$K_{a\,b} \equiv 2\, \frac{\vec{\alpha}^{(a)} \cdot \vec{\alpha}^{(b)}}{\vec{\alpha}^{(b)} \cdot \vec{\alpha}^{(b)}} \tag{19}$$

The step operators satisfy $E_{-a} = E_a^\dagger$, and the generators are normalized in the Chevalley basis as:

$$\begin{aligned} \mathrm{tr}\,(H_a H_b) &= K_{a\,b} \\ \mathrm{tr}\,(E_a E_{-b}) &= \delta_{a\,b} \\ \mathrm{tr}\,(H_a E_{\pm b}) &= 0 \end{aligned} \tag{20}$$

In this paper we concentrate on the gauge algebra $SU(N)$, for which the Cartan matrix K is the $(N-1) \times (N-1)$ symmetric tridiagonal matrix:

$$K = \begin{pmatrix} 2 & -1 & 0 & \dots & & 0 \\ -1 & 2 & -1 & 0 & & \\ 0 & -1 & 2 & -1 & 0 & \\ \vdots & & & & & \vdots \\ 0 & \dots & & 0 & -1 & 2 \end{pmatrix} \tag{21}$$

[1]It is interesing to note that this type of embedding problem also plays a significant role in the theory of spherically symmetric magnetic monopoles and the Toda molecule equations [17].

Expand $\phi_{(0)}$ in terms of the positive simple root step operators as:

$$\phi_{(0)} = \sum_{a=1}^{N-1} \phi^a_{(0)} E_a \tag{22}$$

Then $[\phi_{(0)}, \phi^\dagger_{(0)}]$ is diagonal,

$$[\phi_{(0)}, \phi^\dagger_{(0)}] = \sum_{a=1}^{N-1} |\phi^a_{(0)}|^2 H_a. \tag{23}$$

The commutation relations in (18) then imply that

$$[[\phi_{(0)}, \phi^\dagger_{(0)}], \phi_{(0)}] = \sum_{a=1}^{N-1} \sum_{b=1}^{N-1} |\phi^a_{(0)}|^2 \phi^b_{(0)} K_{b\,a} E_b \tag{24}$$

which, like $\phi_{(0)}$, is once again a linear combination of just the simple root step operators. Thus, for suitable choices of the coefficients $\phi^a_{(0)}$ it is possible for the $SU(N)$ algebra element $\phi_{(0)}$ to satisfy the $SU(2)$ commutation relation $[[\phi, \phi^\dagger], \phi] = \phi$.

For example, one can always choose $\phi_{(0)}$ proportional to a *single* step operator, which by global gauge invariance can always be taken to be E_1 :

$$\phi_{(0)} = \frac{1}{\sqrt{2}} E_1 \tag{25}$$

In the other extreme, the $SU(N)$ "maximal embedding" case, with *all* $N-1$ step operators involved in the expansion (22), the solution for $\phi_{(0)}$ is :[2]

$$\phi_{(0)} = \frac{1}{\sqrt{2}} \sum_{a=1}^{N-1} \sqrt{a(N-a)}\; E_a \tag{26}$$

All other solutions for $\phi_{(0)}$, intermediate between the two extremes (25) and (26), can be generated by the following systematic procedure. If one of the simple root step operators, say E_b, is omitted from the summation in (22) then this effectively decouples the $E_{\pm a}$'s with $a < b$ from those with $a > b$. Then the coefficients for the $(b-1)$ step operators E_a with $a < b$ are just those for the maximal embedding (see equation (26)) in $SU(b)$, and the coefficients for the $(N-b-1)$ E_a's with $a > b$ are those for the maximal embedding in $SU(N-b)$:

$$\phi_{(0)} = \frac{1}{\sqrt{2}} \sum_{a=1}^{b-1} \sqrt{a(b-a)} E_a + \frac{1}{\sqrt{2}} \sum_{a=b+1}^{N-1} \sqrt{a(N-b-a)} E_a \tag{27}$$

[2]In general, the squares of the coefficients are the coefficients, in the simple root basis, of (one half times) the sum of *all* positive roots of the algebra.

Diagrammatically, we can represent the maximal embedding case (26) with the Dynkin diagram of $SU(N)$:

$$\underbrace{o - o - o - \ldots - o - o}_{N-1} \tag{28}$$

which shows the $N-1$ simple roots of the algebra, each connected to its nearest neighbours by a single line. Omitting the b^{th} simple root step operator from the sum in (22) can be conveniently represented as breaking the Dynkin diagram in two by deleting the b^{th} dot:

$$\underbrace{o - o - \ldots - o}_{b-1} \;\times\; \underbrace{o - \ldots - o}_{N-b-1} \tag{29}$$

With this deletion of the b^{th} dot, the $SU(N)$ Dynkin diagram breaks into the Dynkin diagram for $SU(b)$ and that for $SU(N-b)$. Since the remaining simple root step operators decouple into a Chevalley basis for $SU(b)$ and another for $SU(N-b)$, the coefficients required for the summation over the first $b-1$ step operators are just those given in (26) for the maximal embedding in $SU(b)$, while the coefficients for the summation over the last $N-b-1$ step operators are given by the maximal embedding for $SU(N-b)$, as indicated in (27).

It is clear that this process may be repeated with further roots being deleted from the Dynkin diagram, thereby subdividing the original $SU(N)$ Dynkin diagram, with its $N-1$ consecutively linked dots, into subdiagrams of $\leq N-1$ consecutively linked dots. The final diagram, with M deletions made, can be characterized, up to gauge equivalence, by the $M+1$ lengths of the remaining consecutive strings of dots. A simple counting argument shows that the total number of ways of doing this (including the case where *all* dots are deleted, which corresponds to the trivial solution $\phi_{(0)} = 0$) is given by the number, $p(N)$, of (unrestricted) partitions of N.

The $SU(4)$ case is sufficient to illustrate this procedure. There are 5 partitions of 4, and they correspond to the following solutions for $\phi_{(0)}$:

$$\begin{array}{ll}
o - o - o & \phi_{(0)} = \frac{1}{\sqrt{2}}\left(\sqrt{3}E_1 + 2E_2 + \sqrt{3}E_3\right) \\
o - o \quad \times & \phi_{(0)} = E_1 + E_2 \\
o \quad \times \quad o & \phi_{(0)} = \frac{1}{\sqrt{2}}E_1 + \frac{1}{\sqrt{2}}E_3 \\
o \quad \times \quad \times & \phi_{(0)} = \frac{1}{\sqrt{2}}E_1 \\
\times \quad \times \quad \times & \phi_{(0)} = 0
\end{array} \tag{30}$$

Thus we have a simple constructive procedure, and a correspondingly simple labelling notation, for finding all $p(N)$ gauge inequivalent solutions $\phi_{(0)}$ to the algebraic embedding condition (15). Recall that each such $\phi_{(0)}$ characterizes a distinct minimum of the potential V, as well as a class of zero energy solutions to the selfduality equations (13).

Since each vacuum solution $\phi_{(0)}$ corresponds to an embedding of $SU(2)$ into $SU(N)$, an alternative shorthand for labelling the different vacua consists of listing the block diagonal spin content of the $SU(2)$ Cartan subagebra element $[\phi_{(0)}, \phi_{(0)}^\dagger] \sim J_3$. For example, consider the matter fields ϕ taking values in the $N \times N$ defining representation. Then, for each vacuum solution, $[\phi_{(0)}, \phi_{(0)}^\dagger]$ takes the $N \times N$ diagonal sub-blocked form:

$$[\phi_{(0)}, \phi_{(0)}^\dagger] = \begin{pmatrix} j_1 & & & & & & & & & \\ & \ddots & & & & & & & & \\ & & -j_1 & & & & & & & \\ & & & j_2 & & & & & & \\ & & & & \ddots & & & & & \\ & & & & & -j_2 & & & & \\ & & & & & & \ddots & & & \\ & & & & & & & j_M & & \\ & & & & & & & & \ddots & \\ & & & & & & & & & -j_M \end{pmatrix} \tag{31}$$

Each spin j sub-block has dimension $2j+1$, and so it is therefore natural to associate this particular $\phi_{(0)}$ with the following partition of N :

$$N = (2j_1 + 1) + (2j_2 + 1) + \ldots + (2j_M + 1) \tag{32}$$

For example, the $SU(4)$ solutions listed in (30) may be labelled by the partitions 4, $3+1$, $2+2$, $2+1+1$, and $1+1+1+1$, respectively.

4 Vacuum Mass Spectra

Having classified all possible gauge inequivalent vacua of the potential V, we now determine the spectrum of massive excitations in each vacuum. In the abelian model [1, 3] there is only one nontrivial vacuum, and a consequence of the particular 6^{th} order self-dual form of the potential was that in this broken vacuum the massive gauge excitation and the remaining real massive scalar field were degenerate in mass.[3] In the nonabelian models considered here the situation is considerably more complicated, due to the presence of many fields and also due to the many different gauge inequivalent vacua. Nevertheless, we shall see that an analogous mass degeneracy pattern exists.

[3]This degeneracy of the gauge and scalar masses in the broken vacuum is also true of the $2+1$ dimensional Abelian Higgs model [2].

The scalar masses are determined by expanding the shifted potential $V(\phi+\phi_{(0)})$ to quadratic order in the field ϕ:

$$V(\phi+\phi_{(0)}) \quad = \quad \frac{v^4}{4\kappa^2}\mathrm{tr}\left(|[[\phi_{(0)},\phi^\dagger],\phi_{(0)}]+[[\phi,\phi_{(0)}^\dagger],\phi_{(0)}]+[[\phi_{(0)},\phi_{(0)}^\dagger],\phi]-\phi|^2\right) \qquad (33)$$

With the fields normalized appropriately, the masses are then given by the square roots of the eigenvalues of the $2(N^2-1)\times 2(N^2-1)$ mass matrix in (33). Since V is a 6^{th} order potential[4], diagonalizing this scalar field mass matrix is considerably more complicated than for the conventional ϕ^4 Higgs model.

In the unbroken vacuum, with $\phi_{(0)}=0$, there are N^2-1 complex scalar fields, each with mass

$$m=\frac{v^2}{2\kappa} \qquad (34)$$

In one of the broken vacua, where $\phi_{(0)}\neq 0$, some of these $2(N^2-1)$ massive scalar degrees of freedom are converted to massive gauge degrees of freedom. The gauge masses are determined by expanding $\mathrm{tr}\left(\left(D_\mu\left(\phi+\phi_{(0)}\right)\right)^\dagger\left(D_\mu\left(\phi+\phi_{(0)}\right)\right)\right)$ and extracting the piece quadratic in the gauge field A:

$$v^2\ \mathrm{tr}\left([A_\mu,\phi_{(0)}]^\dagger[A^\mu,\phi_{(0)}]\right) \qquad (35)$$

Since the Lagrange density (1) for this model only contains a Chern-Simons term for the gauge fields, and no Yang-Mills term, the gauge field masses are generated by the Chern-Simons-Higgs mechanism [11, 18], which is different from the conventional Higgs mechanism. Because the Chern-Simons term is *first order* in spacetime derivatives, a quadratic term $v^2A_\mu A^\mu$ coming from one algebraic component of (35) leads to a gauge mode of mass $\sim v^2$ (and not $\sim v$ as would be the case in the conventional Higgs mechanism). Thus, the gauge masses are determined by finding the eigenvalues (*not* the square roots of the eigenvalues) of the $(N^2-1)\times(N^2-1)$ mass matrix in (35).

This procedure of finding the eigenvalues of the scalar and gauge mass matrices, must be performed for each of the $p(N)$ gauge inequivalent minima $\phi_{(0)}$ of V. The results for $SU(3)$, $SU(4)$ and $SU(5)$ are presented in Tables 1, 2 and 3. The masses for the two nontrivial vacua in $SU(3)$ are in agreement with the results of [10].

A number of interesting observations can be made at this point, based on the evaluation of these mass spectra for the various vacua in $SU(N)$ for N up to 10.

(i) All masses, both gauge and scalar, are integer or half-odd-integer multiples of the fundamental mass scale $m=v^2/2\kappa$. The fact that all the scalar masses are proportional to m is clear from the form of the potential V in (2). The fact that the gauge masses

[4]Note that a 6^{th} order potential is renormalizable in three dimensional spacetime.

are multiples of the *same* mass scale depends on the fact that the Chern-Simons coupling parameter κ has been included in the overall normalization of the potential in (2). This is a direct consequence of the self-duality of the model.

(ii) In each vacuum, the masses of the real scalar excitations are equal to the masses of the real gauge excitations, whereas this is not true of the complex scalar and gauge fields[5]. Indeed, in some vacua the *number* of complex scalar degrees of freedom and complex gauge degrees of freedom is not even the same. This will be discussed further in the next section.

(iii) In each vacuum, each mass appears at least twice, and always an even number of times. For the complex fields this is a triviality, but for the real fields this is only true as a consequence of the feature mentioned in (ii). This pairing of the masses is a reflection of the $N = 2$ supersymmetry of the relativistic self-dual Chern-Simons systems [4].

(iv) While the distribution of masses between gauge and scalar modes is different in the different vacua, the total number of degrees of freedom is, in each case, equal to $2(N^2 - 1)$, as in the unbroken phase.

The most complicated, and most interesting, of the nontrivial vacua is the "maximal embedding" case, with $\phi_{(0)}$ given by (26). For this vacuum, the gauge and scalar mass spectra have additional features of note. First, this "maximal embedding" also corresponds to "maximal symmetry breaking", in the sense that in this vacuum all $N^2 - 1$ gauge degrees of freedom acquire a mass. The original $2(N^2 - 1)$ massive scalar modes divide equally between the scalar and gauge fields. Moreover, the mass spectrum reveals an intriguing and intricate pattern, as shown in Table 4. It is interesting to note that for the $SU(N)$ maximal symmetry breaking vacuum, the entire scalar mass spectrum is *almost* degenerate with the gauge mass spectrum : there is just *one* single complex component for which the masses differ!

5 Mass Matrices for Real Fields

The masses of the real fields exhibit special simple properties, which we discuss in this section. As mentioned above, in each vacuum $\phi_{(0)}$ the *number* of real scalar modes is equal to the number of real gauge modes. Furthermore, the two mass spectra coincide exactly, and are all *integer* multiples of the mass scale m in (34). The real gauge fields come from the diagonal algebraic components H_a, while the real scalar fields come from the simple root step operator components E_a. Indeed, the real scalar fields correspond to those fields shifted by the symmetry breaking minimum field $\phi_{(0)}$, which is decomposed in terms of the simple root step operators as in (22). This means that the *number* of real scalars in a given vacuum $\phi_{(0)}$ is given by the number of nonzero coefficients $\phi^a_{(0)}$ in the decomposition (22). This can be seen explicitly for $SU(3)$, $SU(4)$ and $SU(5)$ in the Tables 1, 2 and 3. This also serves as an

[5]By 'complex' gauge fields we simply mean those fields which naturally appear as complex combinations of the (nonhermitean) step operator generators.

easy count of the number of real gauge masses. This also means that to determine the mass matrix for the real gauge fields we can expand A_μ in terms of the Cartan subalgebra elements H_a (the other, off-diagonal, algebraic components do not mix with these ones at quadratic order). In fact, in order to normalize the gauge fields correctly, it is more convenient to expand the A_μ in another Cartan subalgebra basis, h_a, for which the traces are orthonormal (in contrast to the traces (20) in the Chevalley basis which involve the Cartan matrix) :

$$\mathrm{tr}\,(h_a h_b) = \delta_{a\,b} \tag{36}$$

Such basis elements, h_a, are related to the Chevalley basis elements, H_a, by

$$h_a = \sum_{b=1}^{r} \omega_a^{(b)}\; H_b \tag{37}$$

where $\vec{\omega}^{(b)}$ is the b^{th} fundamental weight of the algebra [16], satisfying

$$\sum_{b=1}^{r} \omega_a^{(b)} \alpha_c^{(b)} = \delta_{ac} \tag{38}$$

where $\vec{\alpha}^{(b)}$ is the b^{th} simple root. For $SU(N)$ we can be more explicit:

$$h_a = \frac{1}{\sqrt{a(a+1)}} \sum_{b=1}^{a} b\, H_b \tag{39}$$

The orthogonality relation (38) means that the correspondence can be inverted to give

$$H_a = \sum_{b=1}^{r} \alpha_b^{(a)}\; h_b \tag{40}$$

The fundamental weights $\vec{\omega}^{(b)}$ and simple roots $\vec{\alpha}^{(b)}$ are also related by

$$\vec{\alpha}^{(a)} = \sum_{b=1}^{r} K_{ba}\, \vec{\omega}^{(b)} \tag{41}$$

These new basis elements have the following commutation relations with the simple root step operators:

$$[h_a, E_b] = \alpha_a^{(b)} E_b \tag{42}$$

Given the traces in (36) and the commutation relations (42), it is now a simple matter to expand the quadratic gauge field term (35) to find the following mass matrix:

$$\mathcal{M}_{ab}^{(\mathrm{gauge})} = 2\; m\; \sum_{c=1}^{r} |\phi_{(0)}^{c}|^2\, \alpha_a^{(c)}\, \alpha_b^{(c)} \tag{43}$$

where m is the fundamental mass scale in (34). For the maximal embedding vacuum (26) in $SU(N)$ this leads to a mass matrix

$$\mathcal{M}_{ab}^{(\mathrm{gauge})} = m \sum_{c=1}^{N-1} c(N-c)\, \alpha_a^{(c)}\, \alpha_b^{(c)} \tag{44}$$

This matrix has eigenvalues

$$2,\ 6,\ 12,\ 20,\ \ldots\ ,\ N(N-1) \tag{45}$$

in multiples of m. For any vacuum $\phi_{(0)}$ other than the maximal symmetry breaking one, the mass matrix for the real gauge fields decomposes into smaller matrices of the same form, according to the particular partition of the original $SU(N)$ Dynkin diagram, as described in Section 3.

The real scalar field mass matrix can be computed by expanding the ϕ field appearing in (33) in terms of the positive root step operators. With such a decomposition for ϕ, the quadratic term (33) simplifies considerably to give a mass (squared) matrix

$$\mathcal{M}_{ab}^{(\mathrm{scalar})} = 4\, m^2\, \phi_{(0)}^a \phi_{(0)}^b \sum_{c=1}^{r} |\phi_{(0)}^c|^2\, K_{ac}\, K_{bc} \tag{46}$$

where K is the Cartan matrix (19). For the $SU(N)$ maximal symmetry breaking vacuum (26) this mass matrix is

$$\mathcal{M}_{ab}^{(\mathrm{scalar})} = m^2 \sqrt{a\, b\, (N-a)\, (N-b)} \sum_{c=1}^{N-1} c\, (N-c)\, K_{ac}\, K_{bc} \tag{47}$$

which has eigenvalues

$$(2)^2,\ (6)^2,\ (12)^2,\ (20)^2,\ \ldots\ ,\ (N(N-1))^2 \tag{48}$$

in units of m^2. It is interesting to note that the eigenvalues in (48) are the squares of the eigenvalues (45) of $\mathcal{M}^{(\mathrm{gauge})}$, even though $\mathcal{M}^{(\mathrm{scalar})}$ is *not* the square of the matrix $\mathcal{M}^{(\mathrm{gauge})}$ in this basis. Nevertheless, as the real scalar masses are given by the square roots of the eigenvalues in (48), we see that the real scalar masses do indeed coincide with the real gauge masses, a consequence of the $N = 2$ supersymmetry of the theory.

6 Conclusion

In this paper we have analyzed the vacuum structure of the $SU(N)$ self-dual Chern-Simons-Higgs systems with adjoint coupling. Finding the locations of the potential minima (which are degenerate) is equivalent to a classic algebraic embedding problem. A simple explicit construction is given for enumerating and evaluating the gauge inequivalent minima for any gauge group. For $SU(N)$, the number of gauge inequivalent minima is equal to $p(N)$, the number of partitions of N. In the nontrivial vacua, the Chern-Simons-Higgs mechanism generates masses for some of the algebraic components of the gauge field. Both the number of and the actual mass values of these gauge excitations depend on which vacuum is being considered. We have analyzed the resulting mass spectra, for both the gauge and scalar fields, and identified a number of interesting symmetry properties of these spectra. There is clearly a very rich structure present in these spectra, some of which can be understood in terms of the self-duality, and the associated $N = 2$ supersymmetry, of these systems.

The picture is by no means as complete as for the corresponding *nonrelativistic* nonabelian self-dual Chern-Simons-matter systems, where a classification of all finite charge solutions is known [14], due to a deep relationship between the nonrelativistic self-duality equations and integrable models in two dimensions. Ideally, one would like to discover more about explicit solutions of the relativistic self-duality equations (12,13). Some properties of these equations and their possible solutions for $SU(2)$ and $SU(3)$ have been discussed in [7, 10]. An analysis of the integrability of the *abelian* relativistic models [19] suggests that the relativistic nonabelian self-duality equations are not completely integrable in general. However, it would be very interesting to learn if they may be integrable in certain special cases, as was found for the abelian theories [19]. Such information about nonzero energy solutions to the self-duality equations would shed some light on the quantization of this model and the quantum role of the intricate vacuum structure.

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vacuum $\phi_{(0)}$	gauge masses					
	real fields		complex fields			
$\frac{1}{\sqrt{2}}E_1$	2		1/2	1/2	1	
E_1+E_2	2	6	1	2	5	

vacuum $\phi_{(0)}$	scalar masses					
	real fields		complex fields			
$\frac{1}{\sqrt{2}}E_1$	2		1	3/2	3/2	2
E_1+E_2	2	6	2	3	5	

Table 1: $SU(3)$ vacuum mass spectra, in units of the fundamental mass scale $\frac{v^2}{2\kappa}$, for the inequivalent nontrivial minima $\phi_{(0)}$ of the potential V. Notice that for each vacuum the *total* number of massive degrees of freedom is equal to $2(N^2-1)=16$, although the distribution between gauge and scalar fields is vacuum dependent.

vacuum $\phi_{(0)}$	gauge masses								
	real fields			complex fields					
$\frac{1}{\sqrt{2}}E_1$	2			1/2	1/2	1/2	1/2	1	
$\frac{1}{\sqrt{2}}E_1 + \frac{1}{\sqrt{2}}E_3$	2	2		1	1	1	1	2	
$E_1 + E_2$	2	6		1	1	1	2	2	5
$\frac{1}{\sqrt{2}}\left(\sqrt{3}E_1 + 2E_2 + \sqrt{3}E_3\right)$	2	6	12	1	2	3	5	8	11

vacuum $\phi_{(0)}$	scalar masses											
	real fields			complex fields								
$\frac{1}{\sqrt{2}}E_1$	2			1	1	1	1	3/2	3/2	3/2	3/2	2
$\frac{1}{\sqrt{2}}E_1 + \frac{1}{\sqrt{2}}E_3$	2	2		1	1	1	2	2	2	2	2	
$E_1 + E_2$	2	6		1	2	2	2	2	3	5		
$\frac{1}{\sqrt{2}}\left(\sqrt{3}E_1 + 2E_2 + \sqrt{3}E_3\right)$	2	6	12	2	3	4	5	8	11			

Table 2: $SU(4)$ vacuum mass spectra, in units of the fundamental mass scale $\frac{v^2}{2\kappa}$, for the inequivalent nontrivial minima $\phi_{(0)}$ of the potential V. Notice that for each vacuum the *total* number of massive degrees of freedom is equal to $2(N^2-1) = 30$, although the distribution between gauge and scalar fields is vacuum dependent.

vacuum $\phi_{(0)}$	gauge masses	
	real fields	complex fields
$\frac{1}{\sqrt{2}}E_1$	2	1/2 1/2 1/2 1/2 1/2 1/2 1
$\frac{1}{\sqrt{2}}E_1 + \frac{1}{\sqrt{2}}E_3$	2 2	1/2 1/2 1/2 1/2 1 1 1 1 2
$E_1 + E_2$	2 6	1 1 1 1 1 2 2 2 5
$\frac{1}{\sqrt{2}}E_1 + E_3 + E_4$	2 2 6	1/2 1/2 1 1 3/2 3/2 2 5 7/2 7/2
$\frac{1}{\sqrt{2}}\left(\sqrt{3}E_1 + 2E_2 + \sqrt{3}E_3\right)$	2 6 12	1 3/2 3/2 2 3 7/2 7/2 5 8 11
$\sqrt{2}E_1 + \sqrt{3}(E_2 + E_3) + \sqrt{2}E_4$	2 6 12 20	1 2 3 4 5 8 11 11 16 19

vacuum $\phi_{(0)}$	scalar masses	
	real fields	complex fields
$\frac{1}{\sqrt{2}}E_1$	2	1 1 1 1 1 1 1 1 1 3/2 3/2 3/2 3/2 3/2 3/2 2
$\frac{1}{\sqrt{2}}E_1 + \frac{1}{\sqrt{2}}E_3$	2 2	1 1 1 1 3/2 3/2 3/2 3/2 2 2 2 2 2
$E_1 + E_2$	2 6	1 1 1 1 2 2 2 2 2 2 2 3 5
$\frac{1}{\sqrt{2}}E_1 + E_3 + E_4$	2 2 6	1 3/2 3/2 2 2 5/2 5/2 7/2 7/2 3 5
$\frac{1}{\sqrt{2}}\left(\sqrt{3}E_1 + 2E_2 + \sqrt{3}E_3\right)$	2 6 12	1 2 5/2 5/2 3 7/2 7/2 4 5 8 11
$\sqrt{2}E_1 + \sqrt{3}(E_2 + E_3) + \sqrt{2}E_4$	2 6 12 20	2 3 4 5 5 8 11 11 16 19

Table 3: $SU(5)$ vacuum mass spectra, in units of the fundamental mass scale $\frac{v^2}{2\kappa}$, for the inequivalent nontrivial minima $\phi_{(0)}$ of the potential V. Notice that for each vacuum the *total* number of massive degrees of freedom is equal to $2(N^2-1) = 48$, although the distribution between gauge and scalar fields is vacuum dependent.

gauge masses							
real fields	complex fields						
2	1	2	3	4	5	...	N-1
6	5	8	11	14	...	3N-4	
12	11	16	21	...	5N-9		
20	19	26	...	7N-16			
30	29	...	9N-25				
$\vdots$	$\vdots$						
N(N-1)	N(N-1)-1						

scalar masses							
real fields	complex fields						
2	N	2	3	4	5	...	N-1
6	5	8	11	14	...	3N-4	
12	11	16	21	...	5N-9		
20	19	26	...	7N-16			
30	29	...	9N-25				
$\vdots$	$\vdots$						
N(N-1)	N(N-1)-1						

Table 4: $SU(N)$ mass spectrum, in units of the fundamental mass scale $\frac{v^2}{2\kappa}$, for the maximal symmetry breaking vacuum, for which $\phi_{(0)}$ is given by (26). Notice that the gauge mass spectrum and the scalar mass are *almost* degenerate - they differ in just one complex field component.