DISPERSION FOR THE WAVE EQUATION INSIDE STRICTLY CONVEX DOMAINS II: THE GENERAL CASE

OANA IVANOVICI[1], RICHARD LASCAR[2], GILLES LEBEAU[3], AND FABRICE PLANCHON[4,*]

ABSTRACT. We consider the wave equation on a manifold (Ω, g) of dimension $d \geq 2$ with smooth strictly convex boundary $\partial\Omega \neq \emptyset$, with Dirichlet boundary conditions. We construct a sharp local in time parametrix and then proceed to obtain dispersion estimates: our fixed time decay rate for the Green function exhibits a $t^{1/4}$ loss with respect to the boundary less case. We precisely describe where and when these losses occur and relate them to swallowtail type singularities in the wave front set, proving that our decay is optimal. Moreover, we derive better than expected Strichartz estimates, balancing lossy long time estimates at a given incidence with short time ones with no loss: for $d = 3$, it heuristically means that, on average the decay loss is only $t^{1/6}$.

1. INTRODUCTION

Let us consider the wave equation on a smooth $d-$dimensional manifold (Ω, g), with $d \geq 2$, a strictly convex boundary $\partial\Omega$, and Δ_g its Laplace-Beltrami operator:

$$(1.1) \qquad \begin{cases} (\partial_t^2 - \Delta_g)u = 0, & \text{in } \Omega \\ u|_{t=0} = u_0, \ \partial_t u|_{t=0} = u_1, \ u|_{\partial\Omega} = 0, \end{cases}$$

On any smooth Riemannian manifold without boundary, one may construct an approximate solution, i.e. a parametrix, to any order by microlocal methods. In a suitable patch around $x_0 \in \Omega$ (within the radius of injectivity at x_0), such an approximate solution is a Fourier integral operator whose phase is a solution to the eikonal equation. That phase is non degenerate in a suitable way and one recovers pointwise decay estimates for the kernel of such parametrix similar to that for the flat case: let us denote by $e^{\pm it\sqrt{-\Delta_g}}$ the half-wave propagators on Ω with $\partial\Omega = \varnothing$, and $\varkappa \in C_0^\infty(]0, \infty[)$. Then, possibly only for (small) finite $|t|$, we have the so-called dispersion estimate,

$$(1.2) \qquad \|\varkappa(-h^2\Delta_g)e^{\pm it\sqrt{-\Delta_g}}\|_{L^1 \to L^\infty} \leq C(d)h^{-d}\min\{1, (h/|t|)^{\frac{d-1}{2}}\}.$$

Such fixed time decay estimates have been the key tool to obtain other families of estimates, from Strichartz to spectral projector estimates, all of which are of space-time type in (mixed) Lebesgue spaces, for data in Sobolev spaces. These in turn are invaluable tools for studying a large range of problems, from nonlinear waves to localization of eigenfunctions.

In the presence of a boundary, much less is known on the decay of the wave equation. In fact, before our recent work [9] on the wave equation on a model strictly convex domain, there were no known results on fixed time dispersion, even with lesser bounds than (1.2). Boundaries induce reflections, and the geometry of broken light rays can be quite complicated. These already

[1]SORBONNE UNIVERSITÉ, CNRS, LJLL, F-75005 PARIS, FRANCE

[2,3]UNIVERSITÉ CÔTE D'AZUR, CNRS, LJAD, FRANCE

[4]SORBONNE UNIVERSITÉ, CNRS, IMJ-PRG F-75005 PARIS, FRANCE

E-mail addresses: *corresponding author:* fabrice.planchon@sorbonne-universite.fr, oana.ivanovici@math.cnrs.fr, gilles.lebeau@univ-cotedazur.fr, richard.lascar@univ-cotedazur.fr.

Key words Dispersive estimates, wave equation, Dirichlet boundary condition.

cause difficulties in proving propagation of singularities results, and obtaining such results led to major developments [1, 13, 16, 17], along with constructions of suitable parametrices, see [4, 19, 20]. However, such parametrices, while efficient at proving that singularities travel along the (generalized) bi-characteristic flow, do not seem strong enough to obtain dispersion, at least in the presence of gliding rays and the more flexible microlocal energy arguments from [13, 16, 17] do not provide any information on the amplitude of the wave. Nevertheless, outside strictly convex obstacles, parametrices from [19, 20] were instrumental in matching results from $\mathbb{R}^d$: Strichartz estimates for the wave equation were obtained in [21], and dispersion estimates were finally proved to hold for $d = 3$ in [8]). For generic boundaries, some positive results for mixed space-time estimates ([22, 2] and references therein) have been obtained using the machinery developed for low regularity metrics ([23]): reflect the metric across the boundary and consider a boundary less manifold with a Lipschitz metric across an interface. These arguments require to work on very short time intervals, in order to consider only one reflection (and this, in turn, induces losses when summing time intervals). Counterexamples to the full set of Strichartz estimates inside a strictly convex domain were later constructed in [6, 7], by carefully propagating a cusp singularity along the boundary and across a large number of successive reflections, and these carefully crafted solutions provided hindsight for the parametrix construction on the model domain from [9].

Before stating our main result, let us define strict convexity: our boundary $\partial\Omega \neq \varnothing$ is said to be strictly (geodesically) convex if the induced second fundamental form on $\partial\Omega$ is positive definite. If Ω is actually a domain in $\mathbb{R}^d$ with the identity metric, this definition is equivalent to strict positivity of all principal curvatures at any point of the boundary, and Ω is a strictly convex domain (it admits a gauge function that is strictly convex.)

Theorem 1.1. *Let $\varkappa \in C_0^\infty(]0, +\infty[)$. There exist $C > 0$, $T_0 > 0$ and $a_0 > 0$ such that, uniformly in $a \in]0, a_0]$, $h \in (0, 1)$ and $t \in [-T_0, T_0]$, the solution u_a to (1.1) with $(u_0, u_1) = (\delta_a, 0)$, δ_a being any Dirac mass at distance a from $\partial\Omega$, is such that*

$$(1.3) \qquad \|\varkappa(-h^2\Delta_g)u_a(t, \cdot)\|_{L^\infty} \leq \frac{C}{h^d} \min\left\{1, \left(\frac{h}{|t|}\right)^{\frac{d-2}{2}+\frac{1}{4}}\right\}.$$

Remark 1.2. By finite speed of propagation for the wave equation, estimate (1.3) is local in time **and** space. Hence, compactness for Ω may be dropped if appropriate uniform assumptions are made on the metric.

The dispersion estimate (1.3) may be compared to (1.2): we notice a $1/4$ loss in the h/t exponent, which we may informally relate to the presence of caustics in arbitrarily small times if a is small. Moreover, one of the key features in Theorem 1.1 is that T_0 depends only on the geometry of $\partial\Omega$ and the metric g: (1.3) holds uniformly with respect to both the source point and its distance a to the boundary and the frequency $1/h$. In fact, say for $a = h^\nu$, $\nu > 0$, there are at most $1/\sqrt{a} = h^{-\nu/2}$ reflections, and caustics in between them, as we will see later; so in the large frequencies regime $h \to 0$, we have to deal with an increasingly large number of caustics, even to travel a small distance over a small time T_0. These caustics occur because optical rays are no longer diverging from each other in the normal direction, where less dispersion occurs when compared to the $\mathbb{R}^d$ case. In fact, we can track caustics and therefore Theorem 1.1 is optimal.

Theorem 1.3. *Let u_a be the solution to (1.1) with data $(u_0, u_1) = (\delta_a, 0)$. Let $h \in (0, 1)$ and $a \geq h^{1/3}$. There exist a constant $C > 0$, such that for all $\vartheta \in \mathbb{S}^{d-2}$, there exist a finite sequence $(t_n, x_n, y_n)_n$, $1 \leq n \leq a^{-1/2}$ with $d(x_n, \partial\Omega) \sim a$, $y_n/|y_n| \sim \vartheta$, such that*

$$h^{-d}(h/t_n)^{\frac{d-2}{2}}n^{-1/4}a^{\frac{1}{8}}h^{1/4} \sim a^{\frac{1}{4}}h^{-d}(h/t_n)^{\frac{d-2}{2}+\frac{1}{4}} \leq C|\varkappa(-h^2\Delta_g)u_a(t_n, x_n, y_n)|.$$

As a byproduct, we get that even for $t \in]0, T_0]$ with T_0 small, the 1/4 loss is unavoidable for a comparatively small to T_0 and independent of h. Later this optimal loss will be related to swallowtail type singularities in the wave front set of u_a.

Remark 1.4. There is nothing specific about the cosine part of the wave propagator in Theorem 1.1 and 1.3. Both hold equally true if one replaces $\varkappa(-h^2\Delta_g)u_a(t,x)$ by the half-wave propagators $\varkappa(hD_t)e^{\pm it\sqrt{-\Delta_g}}\delta_a$ with $\varkappa \in C_0^\infty(\mathbb{R}^*)$.

As a consequence of more elaborate estimates that lead to (1.3), we obtain improved Strichartz estimates when compared to those that routinely follow from fixed time dispersion.

Theorem 1.5. *Let $d \geq 3$ and u be a solution of (1.1) on a manifold Ω with strictly convex boundary. Then there exist T such that for all $\varepsilon > 0$, there exists $C_{T,\varepsilon}$ such that*

$$(1.4) \qquad \|u\|_{L^q(0,T)L^r(\Omega)} \leq C_{T,\varepsilon}\big(\|u_0\|_{\dot{H}^\beta(\Omega)} + \|u_1\|_{\dot{H}^{\beta-1}(\Omega)}\big),$$

where $\beta = d/2 - 1/q - d/r$ (scaling condition) and (d,q,r) such that $q \geq 2$ ($q \neq 2$ for $d = 3$),

$$\frac{1}{q} \leq \Big(\frac{d-1}{2} - \gamma(d)\Big)\Big(\frac{1}{2} - \frac{1}{r}\Big), \quad \text{with } \gamma(d) = \frac{1}{4} - \frac{1}{4d} + \varepsilon = \frac{1}{6} + \frac{1}{4}\Big(\frac{1}{3} - \frac{1}{d}\Big) + \varepsilon.$$

In dimension $d = 2$ the known range of admissible indices for which sharp Strichartz are already known to hold is in fact slightly larger, see [2] where $\gamma(2) = 1/6$ (which we may recover with our argument). Especially noteworthy is $d = 3$, for which we get $\gamma(3) = 1/6 + \varepsilon$: such a loss corresponds heuristically to a fixed time dispersion (1.3) where the 1/4 loss would be replaced by a 1/6 loss. In dimensions $d \geq 3$, Theorem 1.5 improves the known range of indices for which Strichartz estimates hold, and it does so in a uniform way with respect to dimension, in contrast to [2], where $\gamma(3) = 2/3$ and $\gamma(d) = (d-3)/2$ for $d \geq 4$. The results in [2] however apply to any domain or manifold with non-empty boundary.

In the negative direction, counterexamples from [6, 7] prove that $\gamma(d) \geq 1/12$, for $d = 2, 3, 4$. In other recent works [12], [11], on the model domain, both positive and negative results for $d = 2$ are pushed further. Estimates (1.4) are proved to hold with $\gamma(2) = 1/9$; improvements on counterexamples yield $\gamma(2) \geq 1/10$. These results extend beyond the model case for $d = 2$, and provide similar improvements in higher dimensions; these extensions, for the general case, will be addressed elsewhere, as they require significant new developments that are out of scope here.

In the present work, we mainly focus on constructing a sharp parametrix for the wave equation (1.1), providing optimal bounds on the amplitude of the wave, including at a discrete set of caustics of swallowtail type that increase to arbitrarily large numbers when the source gets closer and closer to the boundary. While a natural outcome of this parametrix is optimal dispersion bounds, we believe that such a sharp parametrix will prove useful for a broad range of applications beyond the study of dispersive effects and localization of eigenfunctions, including sharp quantitative versions of propagation of singularities results that are of importance in control theory.

We conclude this introduction with a brief overview of the content in the next sections.

- The second section is devoted to building our parametrix for the wave propagator, which is the key tool to prove Theorems 1.1 and 1.3. While one may think of [9] as inspirational, its inner knowledge is by no way a prerequisite and the present construction differs significantly for several reasons we briefly outline: unlike in the model case, we lack an explicit spectral representation. We therefore need to construct quasi-modes, and for this we rely on a parametrix for the Helmholtz equation (see [18] which relies crucially on [15]). Using the Airy-Poisson formula that we introduced in [10], we then obtain a parametrix, both as a "spectral" sum and its

counterpart after Poisson summation. One obvious benefit from this approach is that the Dirichlet boundary condition holds easily, unlike in [9]. Moreover, the Poisson sum turns out to coincide with the carefully constructed sum of reflected waves in [9], as each term has essentially the same phase (in the model case). The present construction is therefore a sophisticated version of the method of images, which was our inspiration for constructing suitably matching incoming and outgoing waves in between consecutive reflections in [9] (in turn drawing upon [6]). An additional benefit is that our parametrix holds for any a and h: we extend the reflected waves construction to the range $h^{2/3} < a < h^{4/7}$ (a crucial tool in further improvements alluded to earlier, see [12]). The range $0 < a \leq h^{2/3}$ requires to properly define gallery modes from the quasi-modes and prove that their decay properties are uniform with respect to their discrete parameter, at least in a range useful for our purpose. To our knowledge, these gallery modes had never been defined in such a uniform way in the general case before now; then, one has to carefully construct the initial data by decomposing over the gallery modes, a delicate issue that was notably absent from the model case.

- The third section deals with dispersion estimates for reflected waves. There the analysis of the oscillatory integrals follows [9] in spirit but it departs from it on several counts. We can no longer reduce the higher dimensional case to $d = 2$ by rotational invariance (i.e., the underlying model case is no longer isotropic). For $a < h^{4/7}$, we need to estimate both the size of each wave and their overlap, which is no longer bounded: we observe that after a very large number of reflections, waves start to exhibit dispersion along the tangential variable. We therefore obtain bounds that are sharper and cover an extended region when compared to [9].

- In the fourth section, for $a \leq h^{2/3}$, we use a mix of dispersion estimates on each gallery mode, the spectral sum, and Poisson summation on the worst terms to obtain a sharper decay than in [9], thereby proving that the worst decay (with a $1/4$ loss) really only happens when $h^{1/3} < a < 1$, whereas a lesser $1/6$ loss is seen below $h^{1/3}$, essentially due to cusp propagating and accumulating.

- The fourth section deals with Strichartz estimates and how to derive Theorem 1.5, taking advantage of the previously introduced decomposition with respect to angles of incidence, following [12], combined with short time Strichartz estimates (similar to those from [2]).

- Finally, the appendix provides hindsight on how to obtain the key properties (and required uniformity, in a suitable sense) of the generating function associated to the equivalence of glancing hypersurfaces ([15]) in our setting.

In the remaining of the paper, $A \lesssim B$ means that there exists a constant C such that $A \leq CB$ and this constant may change from line to line but is independent of all parameters. It will be explicit when (very occasionally) needed. Similarly, $A \sim B$ means both $A \lesssim B$ and $B \lesssim A$.

2. A parametrix construction

By finite speed of propagation, we may work locally near the boundary and chose boundary normal coordinates (x, y) on Ω, with $x > 0$ on Ω, $y \in \mathbb{R}^{d-1}$ such that $\partial\Omega = \{(0, y) : y \in \mathbb{R}^{d-1}\}$ (these coordinates may be interpreted as Fermi coordinates relative to the hypersurface that is the boundary); local coordinates on $\Omega \times \mathbb{R}_t$ are then (x, y, t). Local coordinates on the base induce local coordinates on the cotangent bundle, namely $(x, y, t, \xi, \eta, \tau)$ on $T^*(\Omega \times \mathbb{R}_t)$. The corresponding local coordinates on the boundary are (y, t, η, τ). In this coordinates (and up to conjugation by a non vanishing smooth factor $e_g(x, y)$), the Laplacian Δ_g can be written as ([5, III, Appendix C])

$$\Delta = e_g^{-1} \Delta_g e_g = \partial_x^2 + R(x, y, \partial_y).$$

We assume that the boundary is everywhere strictly (geodesically) convex: for every point $(0, y_0) \in \partial\Omega$ and every $(0, y_0, 0, \eta_0) \in T^*\Omega$ with $\eta_0 \neq 0$,

$$\{\xi^2 + R(x, y, \eta), x\}(0, y_0, 0, \eta_0) = 0\,,$$

$$\{\{\xi^2 + R(x, y, \eta), x\}, \xi^2 + R(x, y, \eta)\}(0, y_0, 0, \eta_0) = 2\partial_x R(0, y_0, \eta_0) > 0,$$

where $\{., .\}$ denotes the Poisson bracket (see [5, III, 24.3]). We assume (without loss of generality) that $y_0 = 0$, hence $\kappa_0 = (0, 0, 0, \eta_0)$. On the boundary and for $(0, y)$ near $(0, 0)$, the metric reads $\xi^2 + \sum_{j,k=1}^{d-1} R^{j,k}(0, y)\eta_j\eta_k$; using again [5, III, Appendix C], we assume moreover that $(R_{j,k}(0, 0))_{j,k}$ is the identity matrix, and define

$$(2.1) \quad R_0(y, \partial_y) := R(0, y, \partial_y) = \sum_j \partial_{y_j}^2 + O(|y|)\,, \quad R_1(y, \partial_y) := \partial_x R(0, y, \partial_y) = \sum_{j,k} R_1^{j,k}(y)\partial_{y_j}\partial_{y_k}\,.$$

Recall that strict convexity for $\partial\Omega$ is equivalent to R_1 being elliptic (the associated quadratic form is positive definite). Define our model Laplacian Δ_M and (Fourier) multipliers q, τ_q

$$(2.2) \quad \Delta_M = \partial_x^2 + \sum_j \partial_{y_j}^2 + x\sum_{j,k} R_1^{j,k}(0)\partial_{y_j}\partial_{y_k}\,, \quad q(\eta) = \sum_{j,k} R_1^{j,k}(0)\eta_j\eta_k\,, \quad \tau_q(\omega, \eta) = \sqrt{|\eta|^2 + \omega q(\eta)^{\frac{2}{3}}}\,.$$

Later we will use various functions of variables $(x, y, \eta, \omega, \sigma)$ (where some variables may be omitted depending on context and both new variables $\omega, \sigma \in \mathbb{R}$) that will be defined in a conic neighborhood of the set

$$(2.3) \quad N_0 = \{x = 0\,, \ y = 0\,, \ \omega = 0\,, \ \sigma = 0\,, \ \eta \in \mathbb{R}^{d-1} \setminus \{0\}\}\,.$$

Such a function f is said to be homogeneous of degree k if

$$f(x, y, \lambda\eta, \lambda^{2/3}\omega, \lambda^{1/3}\sigma) = \lambda^k f(x, y, \eta, \omega, \sigma)\,.$$

Definition 2.1. A symbol $a(x, y, \eta, \omega, \sigma)$ is of order m and type $((1, 2/3, 1/3), 0)$ if

$$\forall\beta = (\beta_0, \beta_1, \beta_2, \beta_3) \quad \exists C_\beta \quad |\partial_{(x,y)}^{\beta_0}\partial_\eta^{\beta_1}\partial_\omega^{\beta_2}\partial_\sigma^{\beta_3}a(x, y, \eta, \omega, \sigma)| \leq C_\beta(1 + |\eta|)^{m-|\beta_1|-\frac{2}{3}|\beta_2|-\frac{1}{3}|\beta_3|}\,.$$

We now recall the Airy function, defined for $z \in \mathbb{R}$ as the oscillatory integral

$$(2.4) \quad Ai(-z) = \int e^{i(\frac{\sigma^3}{3} - \sigma z)}\, d\sigma\,.$$

The choice of σ as an integration variable is consistent with our later use of oscillatory integrals with related phases and with symbols within the class we just defined; and ω may be chosen as a zero of the Airy function.

Constructing a parametrix near glancing or gliding rays has a long history, starting with Andersson-Melrose [1] and Eskin [4]. We also refer to Melrose and Taylor ([18] and references therein) and Zworski [24] for the exterior case. We now state an important theorem for our purposes. To our knowledge, this result is stated (for glancing rays) in [24] and a proof is available in [18].

Theorem 2.1. *[Melrose-Taylor, Zworski] Let $\tau_q(\omega, \eta)$ be defined in (2.2). There exist a neighborhood U of $(x, y, \eta, \omega) = (0, 0, 1, 0)$, phase functions $\psi(x, y, \eta, \omega)$ and $\zeta(x, y, \eta, \omega)$, symbols $p_0(x, y, \eta, \omega)$ and $p_1(x, y, \eta, \omega)$ and a function $e_0(x, y, \eta, \omega)$ such that*

- *the function ψ is homogeneous of degree 1, $(\nabla_y(\partial_{\eta_j}\psi))_{j=1,\cdots,d-1}$ are linearly independent;*
- *the function ζ is homogeneous of degree 2/3, and*

$$\zeta = \omega - xq(\eta)^{1/3}e_0(x, y, \eta/|\eta|, \omega/q(\eta)^{1/3})\,,$$

i.e. e_0 is homogeneous of degree 0;

- *the symbols p_0, p_1 (which do not depend on σ) and $p_0 + \sigma p_1$ are of order 0 and type $((1, 2/3, 1/3), 0)$;*
- *the phase functions ψ and ζ are solutions to the following eikonal equations*

$$(2.5) \qquad <\nabla_{(x,y)}\psi, \nabla_{(x,y)}\psi> + \zeta <\nabla_{(x,y)}\zeta, \nabla_{(x,y)}\zeta> = \tau_q^2(\omega, \eta), \quad <\nabla_{(x,y)}\psi, \nabla_{(x,y)}\zeta> = 0.$$

  Here $< .,. >$ is the symmetric bilinear form obtained by polarization of the principal symbol $\xi^2 + R(x, y, \eta)$ of the operator Δ (which is a second order homogeneous polynomial).
- *Define the function $G(x, y, \eta, \omega)$ to be*

$$(2.6) \qquad G(x, y, \eta, \omega) = e^{i\psi(\cdot)}(p_0(\cdot)Ai(-\zeta(\cdot)) + ip_1(\cdot)q^{-1/6}(\eta)Ai'(-\zeta(\cdot)));$$

  Then the following equation holds in U,

$$(2.7) \qquad -\Delta G = \tau_q^2 G + O_{C^\infty}(\tau_q^{-\infty}),$$

  with p_0, e_0 elliptic symbols, $e_0 > 0$ near any $(0, 0, \eta, 0)$ with $\eta \in \mathbb{R}^{d-1} \setminus \{0\}$ and $p_1 = 0$ on $\{x = 0\}$. We call G a quasimode in U.

Remark 2.2. Constructing an asymptotic solution to equation (2.7) with ansatz (2.6) is a classical result in geometrical optics. However, that such a solution can be constructed with $\zeta|_{x=0} = \omega$ independent of (y, η) is delicate and is a key point of the result. Moreover, that the construction can be done such that the symbol p_1 in front of Ai' in (2.6) vanishes on the boundary $\{x = 0\}$ is not obvious and proved in [18, Paragraph. 4.4, formula 4.4.6 and paragraph 7.1].

Remark 2.3. Near glancing rays, the same theorem holds true with $Ai(e^{\pm i\pi/3}\cdot)$ instead of $Ai(\cdot)$. As $Ai(e^{\pm i\pi/3}s)$ does not vanish for real values of s, one may define outgoing and incoming parametrix for the wave operator with given Dirichlet data on the boundary. Near gliding rays, which is our case, the Airy function may vanish and the same methodology no longer applies. In [4], Eskin deals with this difficulty by a conjugation of the wave operator by $e^{-\zeta t}$, replacing τ by $\tau - i\zeta$, therefore avoiding zeros of the Airy function. While one may then prove propagation of singularities, it is unknown (and unlikely) to be enough for dispersive estimates near gliding rays.

Let us now briefly review how to prove Theorem 2.1. First, observe that Melrose's classification Theorem for glancing hypersurfaces (see [15]) applies, in the non-homogeneous setting, locally near any point in the set Σ_0, defined as

$$(2.8) \qquad \Sigma_0 = \{(X_M, Y_M, \Xi, \Theta) : X_M = 0, Y_M = 0, \Xi = 0, |\Theta| = 1\}.$$

Therefore, there exists a canonical transform $\boldsymbol{\chi}_M$ such that, near Σ_0

$$(2.9) \qquad \boldsymbol{\chi}_M(\{X_M = 0\}) = \{x = 0\}, \quad \boldsymbol{\chi}_M(\{\Xi^2 + |\Theta|^2 + X_M q(\Theta) = 1\}) = \{\xi^2 + R(x, y, \eta) = 1\}.$$

The crucial fact that such a canonical transformation $\boldsymbol{\chi}_M$ may actually be defined in a neighborhood of Σ_0 then follows from the transversality of the Hamiltonian flow with respect to Σ_0. The following proposition will be essential for us.

Proposition 2.4. *The generating function for $\boldsymbol{\chi}_M$ may be written as $\varphi_\Gamma(x, y, \Xi, \Theta) = x\Xi + y\Theta + \Gamma(x, y, \Xi, \Theta)$, where $\Gamma(0, y, \Xi, \Theta)$ is independent of Ξ (as $\boldsymbol{\chi}_M(\{X_M = 0\}) = \{x = 0\}$) and*

$$(2.10) \qquad \Gamma(x, y, \Xi, \Theta) = B_\Gamma(y, \Theta) + xA_\Gamma(x, y, \Xi, \Theta).$$

The transformation $\boldsymbol{\chi}_M$ is such that $\boldsymbol{\chi}_M(\partial_\Xi \varphi_\Gamma, \nabla_\Theta \varphi_\Gamma, \Xi, \Theta) = (x, y, \partial_x \varphi_\Gamma, \nabla_y \varphi_\Gamma)$, and therefore generated by the following relations:

$$(2.11) \quad \begin{cases} X_M = x + x\frac{\partial A_\Gamma}{\partial \Xi}(x, y, \Xi, \Theta), \; Y_M = y + \frac{\partial B_\Gamma}{\partial \Theta}(y, \Theta) + x\frac{\partial A_\Gamma}{\partial \Theta}(x, y, \Xi, \Theta) \\ \xi = \Xi + A_\Gamma(x, y, \Xi, \Theta) + x\frac{\partial A_\Gamma}{\partial x}(x, y, \Xi, \Theta), \; \eta = \Theta + \frac{\partial B_\Gamma}{\partial y}(y, \Theta) + x\frac{\partial A_\Gamma}{\partial y}(x, y, \Xi, \Theta). \end{cases}$$

There exists an elliptic symbol $p(x, y, \eta, \omega, \sigma)$ of order 0 and type $((1, 2/3, 1/3), 0)$ with support near N_0 (from (2.3)) and

$$(2.12) \quad G(x, y, \eta, \omega) := \frac{1}{2\pi} e^{-i\tau_q B_\Gamma(0, \eta/\tau_q)} \int e^{i(y \cdot \eta + \frac{s^3}{3} + s(xq^{\frac{1}{3}}(\eta) - \omega) + \tau_q \Gamma(x, y, sq^{\frac{1}{3}}(\eta)/\tau_q, \eta/\tau_q))} p(x, y, \eta, \omega, s) ds$$

such that Theorem 2.1 holds with this G.

Proof. We postpone to the appendix how to obtain the generating function Γ and the $B_\Gamma(y, \Theta)$ and $A_\Gamma(x, y, \Xi, \Theta)$ terms. The function $\Gamma(x, y, \Xi, \Theta)$ is not unique: the group of canonical transformations under which the model $\{X_M = 0, \Xi^2 + |\Theta|^2 + X_M q(\Theta) = 1\}$ is invariant is non trivial and includes any symplectic transformation $(X_M, Y_M, \Xi, \Theta) \to (X_M, Y_M + h'(\Theta), \Xi, \Theta)$, where h is any function defined near the set $\{|\Theta| = 1\}$. Thus we may replace $\Gamma(x, y, \Xi, \Theta)$ by $\Gamma(x, y, \Xi, \Theta) + h(\Theta)$. We therefore assume that $B_\Gamma(0, \Theta) = 0$, which is equivalent to $\Gamma(0, 0, 0, \Theta) = 0$. This explains the factor $e^{-i\tau_q B_\Gamma(0, \eta/\tau_q)}$ in (2.12). Let us now verify that there exists a symbol $p(x, y, \eta, \omega, \sigma)$ such that G defined by (2.12) is such that $(-\Delta - \tau_q{}^2)G = O(\tau_q{}^{-\infty})$ near $x = 0, y = 0$.

We will work microlocally near the set Σ_0, defined in (2.8), in the semiclassical setting with $0 < h < 1$ as small parameter. Set $P = -h^2\Delta - 1$, $p_2(x, y, \xi, \eta) = \xi^2 + R(x, y, \eta) - 1$, $P_M = h^2(D_{X_M}^2 + D_{Y_M}^2 + Xq(D_{Y_M})) - 1$, and $p_{M,2}(X_M, Y_M, \Xi, \Theta) = \Xi^2 + |\Theta|^2 + X_M q(\Theta) - 1$, where $D_{X_M} = \frac{1}{i}\partial_{x_M}$, $D_{Y_M} = \frac{1}{i}\nabla_{Y_M}$. Let $W \subset \tilde{W}$ be small neighborhoods of Σ_0. Let $\chi(X_M, Y_M, \Xi, \Theta) \in C_0^\infty(\tilde{W})$ such that $\chi = 1$ in a neighborhood of $\overline{W}$ and such that $\chi|_{X_M=0}$ is independent of Ξ. Let $\mathcal{G}_h$ be the following semiclassical Fourier integral operator

$$\mathcal{G}_h(F)(x, y) := \frac{1}{(2\pi h)^d} \int e^{\frac{i}{h}(\varphi_\Gamma(x, y, \Xi, \Theta) - X\Xi - Y\Theta)} \chi(X, Y, \Xi, \Theta) F(X, Y) dX dY d\Xi d\Theta,$$

where φ_Γ has been introduced in Proposition 2.4. Then for any semiclassical operator Q_M such that $Q_M = \sum_{n \geq 0}(-ih)^n Q_{M,n}(X_M, Y_M, hD_{X_M}, hD_{Y_M})$ defined on $\tilde{W}$, there exists a semiclassical operator $Q = \sum_{n \geq 0}(-ih)^n Q_n(x, y, hD_x, hD_y)$ defined on $\boldsymbol{\chi}_M(\tilde{W})$ and unique on $\boldsymbol{\chi}_M(W)$ such that one has

$$WF_h\Big((Q\mathcal{G}_h - \mathcal{G}_h Q_M)(F)\Big) \cap \boldsymbol{\chi}_M(W) = \emptyset,$$

where WF_h denotes the semiclassical wavefront set (see [25]) and F denotes any function such that $WF_h(F) \subset W$. Moreover, $Q_0(\boldsymbol{\chi}_M(X_M, Y_M, \Xi, \Theta)) = Q_{M,0}(X_M, Y_M, \Xi, \Theta)$ for $(X_M, Y_M, \Xi, \Theta) \in W$. Taking $Q_M = P_M$ and using (2.9), Q has simple characteristics on the set $p_2(x, y, \xi, \eta) = 0$ near $\boldsymbol{\chi}_M(\Sigma_0)$. Thus, if $W \subset \tilde{W}$ are small enough, there exists a function $l(x, y, \xi, \eta) \in C_0^\infty(\boldsymbol{\chi}_M(\tilde{W}))$, which is elliptic on $\boldsymbol{\chi}_M(W)$, such that $l(x, y, \xi, \eta)p_2(x, y, \xi, \eta) = Q_0(x, y, \xi, \eta) = p_{M,2} \circ \boldsymbol{\chi}_M^{-1}(x, y, \xi, \eta)$ in a neighborhood of $\overline{\boldsymbol{\chi}_M(W)}$. Set $L = l(x, y, hD_x, hD_y)$, then there exists R_M with $R_M = \sum_{n \geq 0}(-ih)^n R_{M,n}(X_M, Y_M, hD_{X_M}, hD_{Y_M})$ such that

$$(2.13) \qquad WF_h\Big((LP\mathcal{G}_h - \mathcal{G}_h(P_M - ihR_M))(F)\Big) \cap \boldsymbol{\chi}_M(W) = \emptyset.$$

We now exhibit a suitable $F = F_{\theta, p_q, h}(X_M, Y_M)$ as an oscillatory integral with symbol p_q to be chosen later and $\theta \in \mathbb{R}^{d-1}$ with $|\theta|$ close to 1. Define a function

$$\alpha_q(\theta) = \frac{1 - |\theta|^2}{q^{2/3}(\theta)},$$

let p_q be such that $p_q(X_M, Y_M, s, \theta, h^{-1}) = \sum_{n \geq 0}(-ih)^n p_{q,n}(X_M, Y_M, s, \theta)$, compactly supported near $X = 0$, $Y = 0$, $s = 0$, with $p_{q,n}$ homogeneous of order 0 and $|\partial^\beta p_{q,n}| \leq C_\beta$, and set

$$F_{\theta,p_q,h}(X_M, Y_M) := \frac{1}{2\pi h^{1/3}} \int e^{\frac{i}{h}(Y \cdot \theta + s^3/3 + s(X_M q^{1/3}(\theta) - \alpha_q(\theta)))} p_q(X_M, Y_M, s, \theta, h^{-1}) \, ds \,,$$

where $Y_M \cdot \theta$ denotes the scalar product in $\mathbb{R}^{d-1}$. Define the model Lagrangian submanifold $\Lambda_{M,\theta}$

$$\Lambda_{M,\theta} := \left\{ (X_M, Y_M, \Xi, \Theta = \theta) : \exists s \in \mathbb{R} \text{ such that } X_M = q(\theta)^{-1/3}(\alpha_q(\theta) - s^2), \Xi = sq(\theta)^{1/3} \right\},$$

then one has $WF_h(F_{\theta,p_q,h}) \subset \Lambda_{M,\theta}$ and $\Lambda_{M,\theta}$ is contained in the characteristic set of P_M, defined by $p_{M,2}(X_M, Y_M, \Xi, \Theta) = 0$, which is the same as the characteristic set of $P_M - ihR_M$ as ihR_M is a lower order term. Let $V \subset W$ be a small neighborhood of $p_{q,0}$. By solving transport equations, we can select the symbol p_q to be elliptic on $X_M = 0, Y_M = 0, s = 0, |\theta| = 1$, such that for all $\theta \in \mathbb{R}^{d-1}$ with $|\theta|$ close to 1 one has

$$WF_h\left((P_M - ihR_M)(F_{\theta,p_q,h^{-1}})\right) \cap V = \emptyset, \quad WF_h(F_{\theta,p_q,h}) \subset W.$$

We now set $G_{\theta,p_q,h}(x,y) := \mathcal{G}_h(F_{\theta,p_q,h})(x,y)$. Using (2.13) and the ellipticity of L on $\chi_M(V)$ yields

$$(2.14) \qquad WF_h\left(P(G_{\theta,p_q,h})\right) \cap \chi_M(V) = \emptyset, \quad WF_h(G_{\theta,p_q,h}) \subset \chi_M(WF_h(F_{\theta,p_q,h})) \subset \chi_M(W).$$

Moreover, we may write $G_{\theta,p_q,h}(x,y)$ as follows

$$\frac{1}{(2\pi h)^d} \frac{1}{2\pi h^{1/3}} \int e^{\frac{i}{h}\Phi_\theta(x,y,X_M,Y_M,\Xi,\Theta,s)} \chi(X_M, Y_M, \Xi, \Theta) p_q(X_M, Y_M, s, \theta, h^{-1}) \, dX_M dY_M d\Xi d\Theta ds \,,$$

where the phase is defined as

$$\Phi_\theta(x,y,X_M,Y_M,\Xi,\Theta,s) = \varphi_\Gamma(x,y,\Xi,\Theta) - X_M \Xi - Y_M \cdot \Theta + Y_M \cdot \theta + s^3/3 + s(X_M q^{1/3}(\theta) - \alpha_q(\theta)) \,,$$

and we recall that $\varphi_\Gamma(x,y,\Xi,\Theta) = x\Xi + y \cdot \Theta + \Gamma(x,y,\Xi,\Theta)$ and we required $\Gamma(0,0,0,\Theta) = 0$. Since at $(x,y) = (0,0)$ we have $\varphi_\Gamma(0,0,0,\Theta) = 0$ ($\Gamma|_{x=0}$ is independent of Ξ), $\Phi_\theta(0,0,X_M,Y_M,\Xi,\Theta,s)$ is explicit and we easily check that it has an unique non degenerate stationary point in the variables X_M, Y_M, Ξ, Θ at $X_{M,c} = 0, Y_{M,c} = 0, \Xi_c = sq^{1/3}(\theta), \Theta_c = \theta$. Therefore, for (x,y) close to $(0,0)$, the phase function Φ_θ also has a unique non degenerate critical point in these variables, such that $\Xi_c = sq^{1/3}(\theta)$ and $\Theta_c = \theta$; the critical value of the phase Φ_θ, that we denote $\phi_\theta(x,y,s)$ is given by

$$\phi_\theta(x,y,s) = y \cdot \theta + \Gamma(x,y,sq^{1/3}(\theta),\theta) + s^3/3 + s(xq^{1/3}(\theta) - \alpha_q(\theta)).$$

Using (2.10), we have for the stationary point $X_{M,c} = x(1 + \partial_\Xi A_\Gamma)$; stationary phase provides a symbol $\tilde{p}_q$ such that $\tilde{p}_q = \sum_{n \geq 0}(-ih)^n \tilde{p}_{q,n}(x,y,s,\theta)$, elliptic on $x = 0, y = 0, s = 0, |\theta| = 1$ and

$$G_{\theta,p_q,h}(x,y) = \frac{1}{2\pi h^{1/3}} \int e^{\frac{i}{h}\phi_\theta(x,y,s)} \tilde{p}_q(x,y,s,\theta,h^{-1}) ds \,.$$

Notice that $WF_h(G_{\theta,p_q,h}) \subset \chi_M(\Lambda_{M,\theta})$, $\chi_M^{-1}(\{x = 0\}) = \{X = 0\}$ and $A_\Gamma(0,0,0,\theta) = 0$ (and also $\nabla_y B_\Gamma(0,\theta) = 0$ for $|\theta| = 1$). From (2.11) we thus get $\{|\theta| = 1$ and $(0,0,\xi,\eta) \in WF_h(G_{\theta,p_q,h})\}$ if and only if $\{\xi = 0$ and $\eta = \theta\}$. Together with (2.14) which gives $WF_h(P(G_{\theta,p_q,h})) \cap \chi_M(V) = \emptyset$, we proved that there exists a small neighborhood U of $(x,y) = (0,0)$ such that for all $|\theta| \sim 1$, one has $WF_h(P(G_{\theta,p_q,h})) \cap \{(x,y) \in U\} = \emptyset$.

Taking $\theta = h\eta$ and $h^{2/3}\omega = \alpha_q(\theta)$, we obtain by direct computation that $1/h = \tau_q(\eta,\omega)$, and (2.7) holds. Rescaling the variable $s \to \tau_q^{-1/3}s$, $G(x,y,\eta,\omega) = G_{\theta,p_q,\tau_q}(x,y)$ is given by the formula (2.12) where $p(x,y,\eta,\omega,s) = \tilde{p}_q(x,y,s/\tau_q^{1/3},\eta/\tau_q,\tau_q)$ and is a symbol or order 0 and type

$((1, 2/3, 1/3), 0)$. Finally, integration with respect to the variable s in (2.12) yields G of the form (2.6) near $x = 0$ as $\Gamma|_{x=0}$ is independent of s. $\square$

Let $a_0 > 0$ be small and $a \in (0, a_0]$; denote by $\mathcal{G}(t, x, y, a)$ the Green function for the wave equation with Dirichlet boundary condition, and $\delta_{(a,0)} := \delta_{x=a,y=0}$ the source point,

$$(2.15) \qquad (\partial_t^2 - \Delta)\mathcal{G} = 0, \text{ for } x > 0, \ \mathcal{G}\big|_{x=0} = 0, \mathcal{G}|_{t=0} = \delta_{(a,0)} \text{ and } \partial_t \mathcal{G}|_{t=0} = 0.$$

We will frequently need smooth cut-off functions $\varkappa \geq 0$ in $C_0^\infty(\mathbb{R}^m)$ with $m = 1$ or with $m = d-1$. For $m = 1$, $\varkappa$ will be such that $\varkappa = 1$ near 1, $\varkappa = 0$ outside a small neighborhood of 1, and for $m = d-1$, $\varkappa$ will be radial and such that $\varkappa = 1$ near $\mathbb{S}^{m-1}$, $\varkappa = 0$ outside a small neighborhood of $\mathbb{S}^{m-1}$. We will abuse notations and retain $\varkappa$ as a generic notation, irrespective of the value of m (which will be clear from context) as well as the size of the (small) support of $\varkappa$, which we assume from now on to be smaller than $0 < \epsilon_0 < 1/100$.

Definition 2.5. Let $h \in (0, 1)$. A function $\mathcal{P}_{h,a}(t, x, y)$ is a parametrix for (2.15) if and only if there exists $a_0 > 0$, $r > 0$ and a neighborhood V of $(t, y) = (0, 0)$ such that for all α one has

$$\sup_{0 < a \leq a_0} \sup_{0 < x \leq r} \sup_{(t,y) \in V} \left| \partial_{t,x,y}^\alpha (\varkappa(hD_t)\varkappa(hD_y)(\mathcal{P}_{h,a} - \mathcal{G}(\cdot, a))) \right| \in O(h^\infty).$$

Remark 2.6. We have $\mathcal{G} = \cos(t\sqrt{|\Delta|})$, but we will work with the half wave propagator $e^{\pm it\sqrt{|\Delta|}}$, from which we may obtain $\mathcal{G}$ and $\partial_t \mathcal{G}$. The operator $\varkappa(hD_t)$ is really a spectral localization with respect to Δ, if applied to a solution to the wave equation. The operator $\varkappa(hD_y)$ further restricts this localization to spatial frequencies whose dominant part is tangential: the general heuristic is that waves propagating along the boundary are the most dangerous ones, whereas other waves are transverse and can be handled by simpler arguments (with a finite number of reflections). While $\varkappa(hD_y)$ does not commute with Δ (unlike in the model case), the support of η in phase space will not significantly move over a finite time interval as a consequence of the Melrose-Sjöstrand propagation of singularities theorem. Therefore, up to $O_{C^\infty}(h^\infty)$ terms, we may insert $\varkappa(hD_y)$ operators before and after the propagator.

Rescale $\omega = \frac{\alpha}{h^{2/3}}$, $\eta = \frac{\theta}{h}$, $s = \frac{\sigma}{h^{1/3}}$ in (2.12) (defining G), hence $\tau_q(\alpha, \theta) = h\tau_q(\omega, \eta)$. Let also $\chi^\flat \in C^\infty(\mathbb{R})$ such that $\chi^\flat = 1$ on $(-\infty, 1]$ and $\chi^\flat = 0$ on $[2, \infty)$, and $\chi^\sharp = 1 - \chi^\flat$. We let

$$(2.16) \quad \Phi(x, y, \theta, \alpha, \sigma) = y \cdot \theta + \sigma^3/3 + \sigma(xq^{1/3}(\theta) - \alpha) + \tau_q(\alpha, \theta)\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q(\alpha, \theta), \theta/\tau_q(\alpha, \theta)),$$

and as our change of variables $s = \frac{\sigma}{h^{1/3}}$ provides a factor $h^{-1/3}$, we set

$$(2.17) \qquad p_h(x, y, \theta, \alpha, \sigma) := h^{-1/3} p(x, y, \theta/h, \alpha/h^{2/3}, \sigma/h^{1/3})\varkappa(\theta)\varkappa(\tau_q(\alpha, \theta))\chi^\sharp(\alpha/h^{2/3}),$$

where the relevance of all cut-off functions will reveal itself later on. We get

$$(2.18) \quad \chi^\sharp(\alpha/h^{2/3})\varkappa(\theta)\varkappa(\tau_q(\alpha, \theta))G(x, y, \theta/h, \alpha/h^{2/3}) = \frac{1}{2\pi}\int e^{\frac{i}{h}\Phi(x,y,\theta,\alpha,\sigma)} p_h(x, y, \theta, \alpha, \sigma)d\sigma.$$

We now define an operator acting on smooth $f(y', \rho)$, with $\hat{f}$ its Fourier transform in all variables,

$$(2.19) \qquad J(f)(x, y) = \int G(x, y, \eta, \omega)\chi^\sharp(\omega)q(\eta)^{1/6}\varkappa(h\eta)\varkappa(h\tau_q(\omega, \eta))\hat{f}(\eta, \omega/h^{1/3})d\eta d\omega.$$

After rescaling and substitution of (2.18) in (2.19),

$$J(f)(x, y) = \frac{1}{2\pi h^d}\int e^{\frac{i}{h}(\Phi(x,y,\theta,\alpha,\sigma) - y'\cdot\theta - \varrho\alpha)} p_h(x, y, \theta, \alpha, \sigma)q(\theta)^{1/6} f(y', \varrho) \, dy'd\varrho d\theta d\alpha d\sigma.$$

Lemma 2.7. *The operator J is well defined from tempered distributions $\mathcal{S}'_{y',\varrho}$ into smooth functions of (x, y) near $(0, 0)$. In the semiclassical setting with h as small parameter, J is a semi-classical Fourier integral operator associated to a canonical transform $\boldsymbol{\chi}_J$, defined near the set $\{y' = 0, \varrho = 0, |\theta| = 1, \alpha = 0\}$ and such that $\boldsymbol{\chi}_J(y' = 0, \varrho = 0, |\theta| = 1, \alpha = 0) = \{y = 0, x = 0, |\theta| = 1, \xi = 0\}$. Moreover, J is elliptic on this set and, microlocally near this set, an intertwining relation holds,*

$$-h^2 \Delta J(f) = J(\tau_q^2(hD_\varrho, hD_{y'})f) + O(h^\infty).$$

As the symbol p_h is smooth and compactly supported in (θ, α, σ), J is easily extended to $\mathcal{S}'_{y',\varrho}$. The Lemma then follows from Theorem 2.1 (p_0 is elliptic and xp_1 vanishes on $\partial\Omega$.)

Remark 2.8. When $\Gamma = 0$ (the model case), this canonical transform is given explicitly:

$$\boldsymbol{\chi}_J(y', \varrho, \theta, \alpha) = (y, x, \theta, \xi)$$

where

$$y = y' + \varrho(\alpha - \varrho^2)\nabla q(\theta)/(3q(\theta)), \quad x = (\alpha - \varrho^2)/q^{1/3}(\theta), \quad \xi = -\varrho q^{1/3}(\theta).$$

2.1. Some useful results on Airy functions.

We now digress and present a variation on the Poisson summation formula, the "Airy-Poisson summation formula". For $z \in \mathbb{C}$ we set

(2.20) $\quad A_\pm(z) = e^{\mp i\pi/3} Ai(e^{\mp i\pi/3}z)$, then $Ai(-z) = A_+(z) + A_-(z)$ and $\overline{A_+(z)} = A_-(\overline{z})$.

The next two Lemmas are proved in [11, Lemmas 1 and 3] :

Lemma 2.9. *Define, for $\omega \in \mathbb{R}$, the function $L(\omega) = \pi + i \log \frac{A_-(\omega)}{A_+(\omega)}$: L is an analytic, real valued, strictly increasing function with $L(0) = \pi/3$, $\lim_{\omega \to -\infty} L(\omega) = 0$, and, for $\omega \geq 1$,*

(2.21) $\quad L(\omega) = \frac{4}{3}\omega^{\frac{3}{2}} + \frac{\pi}{2} - B_L(\omega^{\frac{3}{2}}), \quad B_L(u) = \sum_{k=1}^{\infty} b_k u^{-k}, \quad (b_k)_k \in \mathbb{R}, \ b_1 > 0.$

Finally, let $\{-\omega_k\}_{k \geq 1}$ denote the zeros of the Airy function in decreasing order,

(2.22) $\quad L(\omega_k) = 2\pi k \ and \ L'(\omega_k) = 2\pi \int_0^\infty Ai^2(x - \omega_k)\, dx.$

Lemma 2.10. *Let $\mathbb{N}^* = \mathbb{N} \setminus \{0\}$. In $\mathcal{D}'(\mathbb{R}_\omega)$, one has*

(2.23) $\quad \sum_{N \in \mathbb{Z}} e^{-iNL(\omega)} = 2\pi \sum_{k \in \mathbb{N}^*} \frac{1}{L'(\omega_k)}\delta(\omega - \omega_k).$

Let us define, for $\omega \in \mathbb{R}$, and without loss of generality, an arbitrary choice of $+$ sign for the time propagator $\exp(it\tau_q(\omega, \eta))$,

(2.24) $\quad K_\omega(f)(t, x, y) = \int e^{it\tau_q(\omega,\eta)} G(x, y, \eta, \omega)\chi^\sharp(\omega)q^{1/6}(\eta)\varkappa(h\eta)\varkappa(h\tau_q(\omega, \eta))\hat{f}(\eta, \frac{\omega}{h^{1/3}})\, d\eta.$

Due to both cut-off in ω as well as that in η, $K_\omega(f)$ is supported in $1 \leq \omega \leq \epsilon_0 h^{-2/3}$ and so is $R(t, x, y, \omega, a, h) := ((\partial_t^2 - \Delta)K_\omega(f))(t, x, y)$. By design of G, using (2.7), we have moreover that, for small r_0 and a_0 and for all (large) $M \in \mathbb{N}$,

(2.25) $\quad \sup_{|a|<a_0} \sup_{|(t,x,y)|<r_0} \sup_\omega \left|\nabla_{t,x,y,\omega}^\alpha R\right| \leq C_{M,\alpha} h^M.$

Moreover, at $x = 0$, we have $K_{\omega_k}(f)(t, 0, y) = 0$ as $G(0, y, \eta, \omega_k) = 0$ (recall $\zeta(x, y, \eta, \omega)|_{x=0} = \omega$ and (2.6)). In other words, $K_\omega(f)(t, x, y)$ is a solution to the wave equation, up to $O(h^\infty)$; and when $\omega = \omega_k$, it satisfies the Dirichlet boundary condition.

To get a sense of perspective, let us remark that, in the model case, $\Gamma = 0$ and then (up to normalization) $G_M(x, y, \eta, \omega) = \int_\eta \exp(iy \cdot \eta) Ai(xq^{1/3}(\eta) - \omega) \, d\eta$; for $\omega = \omega_k$, G_M is a so-called gallery mode, and $K_\omega(f)$ is an exact solution to the half-wave equation, satisfying the Dirichlet boundary condition if $\omega = \omega_k$, but f should not be considered as its data: if one picks f such that, on the model, $J(f)$ is a Dirac at $(x = a, y = 0)$, then $f = \int_\eta \exp(-iy \cdot \eta) Ai(aq^{1/3}(\eta) - \omega) \, d\eta$ and then integrating over ω recovers $\delta_{x=a,y=0}$ by a standard identity on Airy functions. For this f, the integral over ω of $K_\omega(f)(t, x, y)$ is then just an half-wave solution with no boundary condition. In [9] such a solution is then iterated by reflecting it on the boundary; here, the Airy-Poisson formula would, on the model, directly provide a sum of waves (the sum over N) that may later be identified as analogue of the reflected waves from [9], while the spectral sum (over k) provides the boundary condition and a direct way to decompose the Dirac data.

We now revert to the general case, where we follow the model case strategy we just sketched, but replace gallery modes by $G(x, y, \eta, \omega)$. Recall we defined $J(f)(x, y)$ in (2.19) and we may rewrite $J(f)(x, y) = \int_{\mathbb{R}} K_\omega(f)(0, x, y) \, d\omega$. Let $\eta = \frac{\theta}{h}$, $q^{\frac{1}{6}}(\eta) = h^{-\frac{1}{3}} q^{\frac{1}{6}}(\theta)$ and $\alpha = h^{\frac{2}{3}}\omega$, then (with elliptic symbol p_h defined in (2.17))

(2.26)

$$K_\omega(f)(t, x, y) = \frac{h^{\frac{2}{3}-d}}{2\pi} \int e^{\frac{i}{h}(t\tau_q(\alpha,\theta)+\Phi(x,y,\theta,\alpha,\sigma)-y'\cdot\theta-\varrho\alpha)} p_h(x, y, \theta, \alpha, \sigma) q^{\frac{1}{6}}(\theta) f(y', \varrho) \, dy' d\varrho d\theta d\sigma \, .$$

As we will see later, both cut-off functions $\varkappa$ in $K_\omega(f)$ relate to localization operators from Definition 2.5. Moreover, $K_\omega(f)$ is a suitable test function in ω (smooth and compactly supported in ω). Using (2.23),

(2.27)
$$\langle \sum_{N\in\mathbb{Z}} e^{-iNL(\omega)}, K_\omega(f)(t, x, y)\rangle_\omega = 2\pi \sum_{k\in\mathbb{N}^*} \frac{1}{L'(\omega_k)} K_{\omega_k}(f)(t, x, y) \, .$$

The $N = 0$ term in the sum over N is $J(f)$. Moreover, at $x = 0$ the RHS vanishes, as the sum over k is finite and each term vanishes as we just observed, and this finite sum (on the RHS) satisfies the wave equation, up to $O(h^\infty)$ terms, due to (2.25).

These remarks will later be of crucial importance to verify that Definition 2.5 will hold for the parametrix we shall introduce in the next sections, up to finding a suitable function f that will recover the data at $t = 0$ in (2.27). This remaining step is far from trivial, unlike in the model case (see [11]), for which we know explicitly the spectral resolution of the Laplacian and can therefore expand a Dirac mass over the eigenmodes, as alluded to earlier.

One may expect that it should be enough to consider initial data (at time 0) $\chi_0(hD_x)\varkappa(hD_y)\delta_{(a,0)}$, for $\varkappa$ supported near $\mathbb{S}^{d-1}$ and $\chi_0 \in C_0^\infty$ supported near 0. Indeed, classical geometric optics arguments provide a parametrix for data $(1 - \chi_0(hD_x))\varkappa(hD_y)\delta_{(a,0)}$: due to the cut-off $(1 - \chi_0(hD_x))$, singularities are transverse to the boundary at $x = 0$ (there is at most one reflection). However,

(2.28) $$\chi_0(hD_x)\varkappa(hD_y)\delta_{(a,0)} = \int e^{i((x-a)\xi+y\cdot\eta)}\chi_0(h\xi)\varkappa(h\eta) d\xi d\eta = \frac{1}{h^d}\widehat{\chi}_0\left(\frac{x-a}{h}\right)\widehat{\varkappa}\left(\frac{y}{h}\right) \, .$$

Therefore, at $x = 0$, this data will be $O(h^\infty)$ only if we assume that $a \geq h^{1-\epsilon}$ for some $\epsilon > 0$. For smaller a, in the case of the Friedlander model operator Δ_M, we can take advantage of the known, explicit, spectral resolution of $-\Delta_M$ in order to consider an initial data $\chi_0(-h^2\Delta_M)\varkappa(hD_y)\delta_{(a,0)}$ that can be further expressed as a sum of eigenfunctions that vanish on the boundary. By contrast, in the general case, we only have quasimodes and this is a source of significant difficulties for these very small a. Nevertheless, we will decompose the parametrix construction according to the respective values of a and $h^{2/3}$, with an overlap between the two regimes where any construction

holds. In subsection 2.2, for $a \gg h^{2/3}$, we will use (2.28) as a data, and mainly proceed with the sum over N in our Airy-Poisson formula (2.27). In subsection 2.3, dealing with $a \lesssim h^{2/3}$, stationary phase methods in this sum over N break down (although one could push them down to $a \gg h$, matching the heuristic above, but with no obvious benefit) in addition to the problem of defining a suitable initial data. We will solve the data issue in subsection 2.3.1 by choosing the model initial data $\chi_0(-h^2\Delta_M)\varkappa(hD_y)\delta_{(a,0)}$. In some sense, in the very narrow strip where it is located, the spectral localizations with respect to Δ_M or Δ are close enough that gallery modes are good substitutes to the quasimodes in defining said data. One then proceeds with a parametrix construction where such data is, again, split according to the values of k in the spectral sum defining it: either k is large enough and we recover a large parameter and can proceed as in the previous regime, or we have the relatively small value of k for which we proceed with the spectral sum, proving in subsections 2.3.2 and 2.3.3 that terms appearing in that expansion are close enough to the model gallery modes and therefore retain enough of their properties to provide a parametrix. This part of the construction is quite delicate and obviously absent in the model case, while of independent interest as far as uniform estimates on quasimodes are concerned as these will be proved in the range $k \ll h^{-1/4}$, exceeding by far what we need in our construction.

2.2. Parametrix construction for $a \geq h^{\frac{2}{3}-\epsilon}$, $0 < \epsilon < 2/3$.
An initial data (2.28) is $O(h^\infty)$ on the boundary for any χ_0, compactly supported near 0. Let $\chi_0 \in C_0^\infty(-2\epsilon_0, 2\epsilon_0)$ with $\chi_0\big|_{[-\epsilon_0,\epsilon_0]} = 1$.

Lemma 2.11. *Let $a_0 > 0$, $r_0 > 0$ be small enough. For all $a \in [h^{\frac{2}{3}-\epsilon}, a_0]$, there exists a smooth function $g_{h,a}$ such that $\varkappa(hD_{y'})g_{h,a} = g_{h,a}$ and*

$$(2.29) \qquad J(g_{h,a})(x,y) = \chi_0(hD_x)\varkappa(hD_y)\delta_{(a,0)} + O_{C^\infty(|(x,y)|\leq r_0)}(h^\infty),$$

where the remainder is $O_{C^\infty(|(x,y)|\leq r_0)}(h^\infty)$ uniformly in a.

The lemma follows from the aforementioned fact that J is an elliptic Fourier integral operator, however, we compute $g_{h,a}$ explicitely:

Lemma 2.12. *There exists a smooth phase function $\psi_a(\varrho, \theta')$ and a symbol $r_h(\varrho, \theta')$ of order $1/3$, with support near $\varrho = 0$, $|\theta'| = 1$, of the form $r_h(\varrho, \theta') = h^{1/3}\sum_{k\geq 0} r_{h,k}h^k$ with $r_{h,0}(0, \theta') \neq 0$ for $|\theta'| = 1$, such that*

$$(2.30) \qquad \psi_a(\varrho, \theta') = \varrho^3/3 + a(\varrho q^{1/3}(\theta') + O(\varrho^3)) + O(a^2)$$

and the function $g_{h,a}$, defined as

$$(2.31) \qquad g_{h,a}(y', \varrho) = h^{-d}\int e^{\frac{i}{h}(\psi_a(\varrho,\theta')+y'\cdot\theta')}r_h(\varrho, \theta')q^{-1/6}(\theta')\, d\theta',$$

solves (2.29). Moreover, ψ_a is the critical value of the phase $\varrho\alpha - \Phi(a, 0, \theta', \alpha, s)$ at critical points in (α, s).

Proof. We may invert microlocally the operator J from (2.19) by setting

$$J^{-1}(F)(y', \varrho) = h^{-d-1}\int e^{\frac{i}{h}(-\Phi(x,y,\theta',\alpha',s)+y'\cdot\theta'+\varrho\alpha')}q_h(x, y, \theta', \alpha', s)q^{-1/6}(\theta')F(x,y)dxdyd\theta'd\alpha'ds,$$

where $q_h(x, y, \theta', \alpha', s)$ is a symbol of order $1/3$, $q_h = h^{1/3} \sum_{k \geq 0} h^k q_{h,k}$, with support near $\{x = 0, y = 0, \alpha' = 0, s = 0\}$ and elliptic on this set: we aim at proving that $J^{-1} \circ J(f) = f$ modulo $O(h^\infty)$:

$$J^{-1} \circ J(f) = h^{-d-1} \int e^{\frac{i}{h}(-\Phi(x,y,\theta',\alpha',s)+y'\cdot\theta'+\varrho\alpha')} q_h(x, y, \theta', \alpha', s) q^{-1/6}(\theta')$$

$$\times \frac{1}{2\pi h^d} e^{\frac{i}{h}\Phi(x,y,\theta,\alpha,\sigma)} p_h(x, y, \theta, \alpha, \sigma) q^{1/6}(\theta) \hat{f}(\theta/h, \alpha/h) d\theta d\alpha d\sigma dx dy d\theta' d\alpha' ds.$$

We now apply stationary phase in variables $(\sigma, s, x, y, \theta', \alpha')$: one checks that critical points are non-degenerate, such that $\theta'_c = \theta$, $\alpha'_c = \alpha$, and stationary phase provides a factor $h^{d-1+1+1} = h^{d+1}$ (one factor h^{d-1} from $dy d\theta'$, one factor h from $dx ds$ and one factor h from $d\sigma d\alpha'$). The critical value of the phase is $y'\theta + \varrho\alpha$ and we obtain (modulo $O(h^\infty)$)

$$J^{-1} \circ J(f)(y', \varrho) = h^{-d-1} \times \frac{h^{d+1}}{2\pi h^d} \int e^{\frac{i}{h}(y'\cdot\theta+\varrho\alpha)} \tilde{q}_h(\theta, \alpha) \hat{f}(\theta/h, \alpha/h) d\theta d\alpha,$$

where $\tilde{q}_h$ is obtained from the product $q_h(x, y, \theta', \alpha', s) q^{-1/6}(\theta') p_h(x, y, \theta, \alpha, \sigma) q^{1/6}(\theta)$ after stationary phase; asking $\tilde{q}_h = 1$ for θ such that $|\theta| \sim 1$ and ϱ near 0 allows to chose q_h; since $p_h = h^{-1/3} p(x, y, \theta/h, \alpha/h^{2/3}, \sigma/h^{1/3}) \chi^\sharp(\alpha/h^{2/3}) \varkappa(\theta) \varkappa(\tau_q(\alpha, \theta))$, we obtain q_h as announced. Define

$$(2.32) \qquad \tilde{g}_{h,a} := J^{-1}(\chi_0(hD_x)\varkappa(hD_y)\delta_{(a,0)}).$$

Then, using the second line in (2.28),

$$(2.33) \quad \tilde{g}_{h,a}(y', \varrho) = \frac{1}{h^d} \int e^{\frac{i}{h}y'\cdot\theta'} F_{h,a}(\varrho, \theta') d\theta', \quad F_{h,a}(\varrho, \theta') = \frac{1}{h^{d+1}} \int e^{\frac{i}{h}(-\Phi(x,y,\theta',\alpha,s)+\varrho\alpha+(x-a)\sigma+y\cdot\theta)}$$

$$\times q_h(x, y, \theta', \alpha, s) q^{-\frac{1}{6}}(\theta') \chi_0(\sigma)\varkappa(\theta) d\sigma d\theta dx dy d\alpha ds.$$

We apply stationary phase to $F_{h,a}$ with respect to variables (x, σ, y, θ): (non-degenerate) critical points are $x = a$, $y = 0$, $\sigma_c = \partial_x \Phi(a, 0, \theta', \alpha, s)$ and $\theta_c = \partial_y \Phi(a, 0, \theta', \alpha, s)$. The resulting symbol $\tilde{q}_h(\theta', \alpha, s)$ is of order $1/3$, with support near $\{|\theta'| = 1, \alpha = 0, s = 0\}$ and elliptic on this set, and

$$(2.34) \qquad F_{h,a}(\varrho, \theta') = h^{-1} \int e^{\frac{i}{h}(-\Phi(a,0,\theta',\alpha,s)+\varrho\alpha)} \tilde{q}_h(\theta', \alpha, s) q^{-1/6}(\theta') d\alpha ds.$$

Here α is bounded, as $\tau_q(\alpha, \theta') \in \text{supp}\,\varkappa$; indeed, $\alpha = h^{2/3}\omega$ and we assumed $|\omega| \leq \epsilon_0 h^{-2/3}$; therefore $\alpha \leq \epsilon_0$ on the support of the symbol p_h, as well as on the support of q_h and also $\tilde{q}_h$. The phase of $F_{h,a}$ is stationary in α for $s + \varrho + O(a) = 0$ and in s for $s^2 + O(a) \sim \alpha \leq \epsilon_0$ (as we will see below using the explicit form of Φ) and $a \leq a_0 < 1$ is small enough, therefore $s^2 \lesssim \epsilon_0 + a_0$ (otherwise non stationary phase in s provides an $O(h^\infty)$ contribution.) Hence there exists a cut-off $\chi(\varrho) \in C_0^\infty((-2r, 2r))$, equal to 1 on $[-r, r]$ for $r \sim \sqrt{\epsilon_0} + a_0$, such that $(1-\chi(\varrho))F_{h,a}(\varrho, \theta') = O(h^\infty)$ in $\mathcal{S}_\varrho$, uniformly in θ' near $|\theta'| = 1$.

We now apply stationary phase in (2.34) with respect to α and s: with (ϱ, θ', a) as parameters and for $\varrho \in (-2r, 2r)$ and $|\theta'|$ close to 1, let (α_c, s_c) denote the critical points of the phase, define $\psi_a(\varrho, \theta') = \varrho\alpha_c - \Phi(a, 0, \theta', \alpha_c, s_c)$, where (ϱ, a) are small parameters in $(-2r, 2r) \times (0, a_0)$. In order for ψ_a to be (2.30), we need more information on the phase Φ. Recall from (2.16)

$$\Phi(x, y, \theta, \alpha, \sigma) = y\theta + \sigma^3/3 + \sigma(xq^{1/3}(\theta) - \alpha) + \tau_q(\alpha, \theta)\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q(\alpha, \theta), \theta/\tau_q(\alpha, \theta)),$$

and from the Appendix, Proposition 6.5, $\Gamma(x, y, \Xi, \Theta) = B_\Gamma(y, \Theta) + x A_\Gamma(x, y, \Xi, \Theta)$, where $B_\Gamma(0, \Theta) = 0$ and $A_\Gamma(x, y, \Xi, \Theta) = \Xi\ell(y, \Theta/|\Theta|) + \mu(y, \Theta/|\Theta|)(\Xi^2 + |\Theta|^2 - 1) + \mathcal{H}_{j\geq 3}$, where ℓ, μ are smooth functions such that $\ell(0, \omega) = 0$, and $\mathcal{H}_{j\geq k}$ denotes any function which is an expansion of the form

$\sum_{j\geq k} f_j$ with f_j homogeneous of order j with respect to weights on variable (x,y,Ξ,Θ): $x:2$, $(|\Theta|-1):2$, $\Xi:1$. Therefore, the phase of $F_{h,a}$ reads

$$(2.35) \quad -\Phi(a,0,\theta',\alpha,s)+\varrho\alpha = -s^3/3 - s(aq^{1/3}(\theta')-\alpha)+\varrho\alpha$$
$$- a\tau_q(\alpha,\theta')A_\Gamma(a,0,sq^{1/3}(\theta')/\tau_q(\alpha,\theta'),\theta'/\tau_q(\alpha,\theta'))\,.$$

Setting $\Xi := sq^{1/3}(\theta')/\tau_q(\alpha,\theta')$ and $\Theta := \theta'/\tau_q(\alpha,\theta')$ and using that $\ell(0,\Theta/|\Theta|)=0$,

$$A_\Gamma(a,0,\Xi,\Theta)\big|_{\Xi=sq^{1/3}(\theta')/\tau_q(\alpha,\theta'),\Theta=\theta'/\tau_q(\alpha,\theta')} = \mu(0,\theta'/|\theta'|)\frac{(s^2-\alpha)q^{2/3}(\theta')}{\tau_q^{\,2}(\alpha,\theta')} + \mathcal{H}_{j\geq 3},$$

where $\mathcal{H}_{j\geq 3}$ contains terms with factors as, s^3 and $s\alpha$; as $A_\Gamma(a,\cdots)$ comes with a factor a, cancelling the derivative of the phase (2.35) with respect to s yields an equation for s_c,

$$-s^2 - (aq^{1/3}(\theta')-\alpha) - a\tau_q(\alpha,\theta')\Big(2s\mu(0,\theta'/|\theta'|)q^{2/3}(\theta')/\tau_q^{\,2}(\alpha,\theta') + O(a,s^2,\alpha)\Big) = 0\,,$$
$$s_c^2(1+O(a)) + aq^{1/3}(\theta')(1+2s_cq^{1/3}(\theta')\mu/\tau_q) = \alpha_c(1+O(a))\,,$$

while cancelling the derivative of (2.35) with respect to α yields $\varrho + s_c(1+O(a))=0$. Therefore, $s_c = -\varrho(1+O(a))$ and $\alpha_c = \varrho^2(1+O(a))+aq^{1/3}(\theta')(1+O(\varrho))$. We now compute the critical value of the phase (2.35) at s_c,α_c: let $\phi(a,\varrho,\theta') := -\Phi(a,0,\theta',\alpha_c,s_c)+\varrho\alpha_c$ and write the Taylor expansion of ϕ near $a=0$. We have $\phi(a,\varrho,\theta') = \phi(0,\varrho,\theta') + a\partial_a\phi(0,\varrho,\theta') + O(a^2)$, where $\alpha_{c|a=0} = \varrho^2$ and $s_{c|a=0} = -\varrho$, and then, with $(\cdots)=(0,0,\theta')$,

$$\phi(0,\varrho,\theta') = -\Phi(\cdots,\alpha_{c|a=0},s_{c|a=0})+\varrho\alpha_{c|a=0} = \big(-(\frac{s^3}{3}+s\alpha)+\varrho\alpha\big)\big|_{s=-\varrho,\alpha=\varrho^2} = \frac{\varrho^3}{3}\,,$$

$$(2.36) \qquad \partial_a\phi(0,\varrho,\theta') = -\partial_a\Phi(\cdots,\varrho^2,-\varrho) - \partial_a s_{c|a=0}\partial_s\Phi(\cdots,\varrho^2,-\varrho)$$
$$+ \partial_a\alpha_{c|a=0}(\varrho-\partial_\alpha\Phi(\cdots,\varrho^2,-\varrho))\,.$$

As s_c,α_c are critical points for $-\Phi(\cdot,\alpha,s)+\varrho\alpha$, the last two terms in (2.36) vanish. We have

$$\partial_a\phi(0,\varrho,\theta') = -\partial_a\Phi(a,0,\theta',\alpha,s)|_{(0,0,\theta',\varrho^2,-\varrho)} = \varrho q^{1/3}(\theta') - \tau_q(\varrho^2,\theta')A_\Gamma(0,0,-\varrho q^{1/3}(\theta')/\tau_q,\theta'/\tau_q)\,.$$

For $(\Xi,\Theta) = (-\varrho q^{1/3}(\theta')/\tau_q,\theta'/\tau_q)$, with $\tau_q = \tau_q(\varrho^2,\theta')$, the term homogeneous of degree 1 of $A_\Gamma(0,0,\Xi,\Theta)$ is $\Xi\ell(0,\Theta)=0$ and the term homogeneous of degree 2 equals $\mu(0,\Theta)(\Xi^2+|\Theta|^2-1) = 0$ as $(\Xi^2+|\Theta|^2-1) = \frac{(s^2-\alpha)}{\tau_q^2}|_{s=-\varrho,\alpha=\varrho^2} = 0$. The terms homogeneous of higher order $j\geq 3$ of $A_\Gamma(0,0,\Xi,\Theta)$ are powers of $\Xi^j \sim \varrho^j$ (as we can replace $|\Theta|^2-1$ by Ξ^2), hence $A_\Gamma(0,0,\Xi,\Theta) = O(\varrho^3)$. Therefore, stationary phase in s,α yields, for some new symbol $r_h(\varrho,\theta')$ of order $1/3$,

$$\tilde{g}_{h,a}(y',\varrho) = h^{-d}\int e^{\frac{i}{h}(\psi_a(\varrho,\theta')+y'\cdot\theta')}r_h(\varrho,\theta')q^{-1/6}(\theta')\,d\theta' + O_{C^\infty}(h^\infty)\,,$$

where we set $\psi_a(\varrho,\theta') := \phi(a,\varrho,\theta')$ that is indeed the required (2.31). One has $WF_h(\tilde{g}_{h,a}) \subset \{(y',\varrho,\theta',\alpha'), y' = -\nabla_{\theta'}\psi_a(\varrho,\theta'), \alpha' = \partial_\varrho\psi_a(\varrho,\theta')\}$. Using (2.30) and (2.32), we now set $g_{h,a}(y',\varrho) := \chi(\varrho)\tilde{g}_{h,a}(y',\varrho)$, such that (2.29) holds and this completes the proof of Lemma 2.12. $\square$

Definition 2.13. Let $g_{h,a}$ be defined in (2.31): using (2.23), we define $\mathcal{P}_{h,a}$ equivalently as

$$(2.37) \qquad \mathcal{P}_{h,a}(t,x,y) = \langle\sum_{N\in\mathbb{Z}}e^{-iNL(\omega)}, K_\omega(g_{h,a})(t,x,y)\rangle_\omega$$

$$(2.38) \qquad \mathcal{P}_{h,a}(t,x,y) = 2\pi\sum_{k\in\mathbb{N}^*}\frac{1}{L'(\omega_k)}K_{\omega_k}(g_{h,a})(t,x,y)\,.$$

We are abusing notation here: one should consider $\mathcal{P}^{\pm}$ depending on the sign on t and then obtain $2\mathcal{P}$ from Definition 2.5 as $\mathcal{P}^{+} + \mathcal{P}^{-}$. Considering $\mathcal{P}^{+}$ is enough by time symetry and we therefore drop the $+$.

We now recall (see the discussion after (2.24)) that, using both localizations in η and $\tau_q(\omega, \eta)$, (2.38) may be reduced to a finite sum over $k \lesssim h^{-1}$: support considerations on K_ω (as a function of ω) provide $|\omega| \leq \epsilon_0 h^{-2/3}$; after Airy-Poisson summation, this translates into $|\omega_k| \leq \epsilon_0 h^{-2/3}$. The zeroes $\{-\omega_k\}_{k \geq 1}$ of the Airy function have asymptotic $\omega_k \sim (3\pi k/2)^{2/3}$. We therefore introduce a cut-off in the sum over k, $\chi^{\flat}_{\epsilon_0}(h^{2/3}\omega_k) := \chi^{\flat}(h^{2/3}\omega_k/\epsilon_0)$. Then, (2.38) may be rewritten as a finite sum,

$$(2.39) \qquad \mathcal{P}_{h,a}(t,x,y) := 2\pi \sum_{k \in \mathbb{N}^*} \frac{\chi^{\flat}_{\epsilon_0}(h^{2/3}\omega_k)}{L'(\omega_k)} K_{\omega_k}(g_{h,a})(t,x,y) \,.$$

From $\omega_1 \geq 2.33$, we remark that the cut-off function $\chi^{\sharp}(\omega)$ that was introduced in the definition (2.24) of K_ω is no longer needed when restricting ω to the set $\{\omega_k\}_{k \in \mathbb{N}^*}$. But it will help on the other sum (2.37), in estimating how many N's contribute significantly. Again with (2.23), we also have

$$(2.40) \qquad \mathcal{P}_{h,a}(t,x,y) = \langle \sum_{N \in \mathbb{Z}} e^{-iNL(\omega)}, \chi^{\flat}(h^{2/3}\omega/\epsilon_0) K_\omega(g_{h,a})(t,x,y) \rangle_\omega \,.$$

The sum $\sum_{N \in \mathbb{Z}}$ converges in $\mathcal{D}'_\omega$ and $\chi^{\flat}(h^{2/3}\omega/\epsilon_0)K_\omega(g_{h,a})(t,x,y)$ is smooth in (t,x,y) in a neighborhood W of $(0,0,0)$ and smooth and compactly supported in ω. For the moment we use $g_{h,a}$ as expressed from (2.33) and (2.34) (integral over α, s, ϱ). We can however replace the cut-off $\chi(\varrho)$, introduced in the proof of Lemma 2.12, by $\chi(s)$; as $s_c = -\varrho(1 + O(a))$, defining $g_{h,a}$ without the factor $\chi(\varrho)$ but with $\chi(s)$ inside the integral provides the same contribution modulo $O(h^\infty)$ but allows to immediately obtain $\hat{g}_{h,a}(\theta/h, \alpha/h)$, which is useful in the formula for $K_\omega(g_{h,a})$:

$$\hat{g}_{h,a}(\theta/h, \alpha/h) = h^{-1}q^{-1/6}(\theta) \int e^{-\frac{i}{h}\Phi(a,0,\theta,\alpha,s)} \chi(s)\tilde{q}_h(\theta, \alpha, s)ds,$$

and substitution in (2.26) yields

$$(2.41) \quad K_\omega(g_{h,a})(t,x,y) = \frac{h^{2/3}}{2\pi h^{d+1}} \int e^{\frac{i}{h}(t\tau_q(h^{2/3}\omega,\theta)+\Phi(x,y,\theta,h^{2/3}\omega,\sigma)-\Phi(a,0,\theta,h^{2/3}\omega,s))}$$

$$\times p_h(x,y,\theta,h^{2/3}\omega,\sigma)\chi(s)\tilde{q}_h(\theta,h^{2/3}\omega,s)dsd\theta d\sigma \,.$$

We set $\mathcal{P}_{h,a}(t,x,y) = \sum_{N \in \mathbb{Z}} V_N(t,x,y)$, where V_N is defined as

$$(2.42) \qquad V_N(t,x,y) := \int e^{-iNL(\omega)}\chi^{\flat}(h^{2/3}\omega/\epsilon_0)K_\omega(g_{h,a})(t,x,y)d\omega$$

$$= \frac{1}{2\pi h^{d+1}} \int e^{\frac{i}{h}(t\tau_q(\alpha,\theta)+\Phi(x,y,\theta,\alpha,\sigma)-\Phi(a,0,\theta,\alpha,s)-NhL(h^{-2/3}\alpha))}$$

$$\times \chi^{\flat}(\alpha/\epsilon_0)\chi(s)p_h(x,y,\theta,\alpha,\sigma)\tilde{q}_h(\theta,\alpha,s)ds\,d\theta d\sigma d\alpha \,.$$

The symbol $\chi^{\flat}(\alpha/\epsilon_0)\chi(s)p_h\tilde{q}_h$ of V_N is the same for every N, is of order 0 and is given by an asymptotic expansion with small parameter h and main term equal to 1 (indeed, since $\tilde{q}_h$ has been obtained by inverting J, whose symbol is p_h). Note that we do not have a finite sum over N: convergence should be understood in the distributional sense. The cut-off in α is redundant but we will leave it there to emphasize compact support in α. In the forthcoming Lemma 2.16, we prove that for a generic function f_h replacing $g_{h,a}$ the sum over N converges and is $O(h^\infty)$ for $N > h^{-1/3}$,

provided f_h is of moderate growth with respect to h. Practically, f_h is an oscillatory integral with an Airy type phase and with a smooth rapidly decaying or compactly supported symbol and as such, is of moderate growth.

As such, we may indeed replace $g_{h,a}$ modulo $O(h^\infty)$ as it will only concern a (large but) finite number of terms. Our main result in this section is the following proposition:

Proposition 2.14. *Let $a \in (h^{\frac{2}{3}-\epsilon}, a_0]$ with small $a_0, \epsilon > 0$.*

- *For $|t| \lesssim 1$, $\mathcal{P}_{h,a}$ is essentially a finite sum in N at any given time,*

$$(2.43) \qquad \mathcal{P}_{h,a}(t,x,y) = \sum_{|N| \lesssim |t| a^{-1/2}} V_N(t,x,y) + O_{C^\infty}(h^\infty).$$

  Moreover we can introduce a cut-off $\chi^\sharp(4\alpha/a)$ in the definition of V_N without changing its main contribution modulo $O(h^\infty)$ terms.
- *At $t = 0$, we have $\mathcal{P}_{h,a}(0,x,y) = \chi_0(hD_x)\varkappa(hD_y)\delta_{(a,0)} + O_{C^\infty}(h^\infty)$.*
- *$\mathcal{P}_{h,a}$ is a parametrix in the sense of the Definition 2.5.*

Remark 2.15. The cut off $\chi^\sharp(h^{-2/3}\alpha)$ from K_ω restricts to $1 \leq h^{-2/3}\alpha$. The last statement in the first part of Proposition 2.14 translates into the contribution of the integrals defining V_N being irrelevant for small values $\alpha \leq a/2$: for $a > h^{2/3-\epsilon}$ we can further restrict to $\alpha > a/2$. This follows right away from the expression of $G(x,y,\theta/h,\omega)$ appearing in the definition of $K_\omega(g_{h,a})$ (recall (2.24)): using (2.6), G reads as a sum of Airy functions computed at $-\zeta = x|\eta|^{2/3}e_0(x,y,\eta,\omega) - \omega$ with an elliptic e_0, close to 1. These Airy functions are exponentially decreasing for $-\zeta > 0$; hence, if $a > h^{2/3-\epsilon}$, $\eta = \theta/h$ with $|\theta| \sim 1$ and $\omega = h^{-2/3}\alpha$, we must have $a \lesssim \alpha$ since otherwise the contribution from G is $O(h^\infty)$. Note that $h^{2/3-\epsilon} \lesssim \alpha$ is required to perform stationary phase arguments; for $\alpha \leq h^{2/3-\epsilon}$ (to be dealt with if $a \leq h^{2/3-\epsilon}$!), rescaling no longer provides a large parameter.

Proof. We start with the easiest part: from Theorem 2.1 $G(0,y,\eta,\omega_k) = e^{i\psi(0,y,\eta,\omega_k)}p_0 Ai(-\omega_k)$, which immediately yields $G(0,y,\eta,\omega_k) = 0$. Therefore, from (2.39) being a finite sum, we get $\mathcal{P}_{h,a}(t,x,y)_{|\partial\Omega} = 0$, which is to say, the Dirichlet boundary condition holds for $\mathcal{P}_{h,a}$. From (2.23), we get that the distribution $\sum_{N\in\mathbb{Z}} e^{-iNL(\omega)} \in \mathcal{S}'(\mathbb{R})$. Moreover, from upcoming Lemma 2.16, for $|N| > h^{-1/3}$, the sum is $O(h^\infty)$ irrespective of $g_{h,a}$. As such, we are reduced to a finite number of N's, and from (2.37) and (2.25), it follows that, taking W smaller if needed, and uniformly in $a < a_0$, one has $(\partial_t^2 - \Delta)\mathcal{P}_{h,a} \in O_{C^\infty(W)}(h^\infty)$, not only for $x > 0$ but in the full neighborhood W of $(0,0,0,1)$. Both statements on $\mathcal{P}_{h,a}$ are independent on the particular choice of the function $g_{h,a}$ such that (2.29) holds. It remains to check that, with our choice of $g_{h,a}$ given in (2.29), $\mathcal{P}_{h,a}(0,x,y)$ is the right initial value, which turns out to be the most difficult part of the proof. We first prove that the sum over N is (large but) finite and that at $t = 0$, in the sum over N in (2.43), all the oscillatory integrals $V_N(0,x,y)$ for $|N| \geq 1$ provide a $O(h^\infty)$ contribution, while $V_0(0,x,y) = J(g_{h,a})(x,y)$ which, by design, is our initial data. The fact that the number of N is finite will allow to deduce that $\mathcal{P}_{h,a}(0,x,y) = V_0(0,x,y) = J(g_{h,a})(x,y) = \chi_0(hD_x)\varkappa(hD_y)\delta_{(a,0)}$ and conclude.

Lemma 2.16. *Let f_h be a smooth function of (y',ϱ), with compact support in ϱ and of moderate growth in h, and $K_\omega(f_h)$ be defined by (2.26). Then*

$$(2.44) \qquad \langle \sum_{|N| \gtrsim h^{-1/3}} e^{-iNL(\omega)}, \chi^\flat(h^{2/3}\omega/\epsilon_0)K_\omega(f_h)(t,x,y)\rangle_\omega = O(h^\infty).$$

Proof. We consider the sum over all N: all phases in the sum are linear in t and N and given by

$$(2.45) \qquad t\tau_q(\alpha,\theta) + \Phi(x,y,\theta,\alpha,\sigma) - NhL(\alpha h^{-2/3}) - y'\theta - \varrho\alpha,$$

with large parameter $1/h$ as a factor; it follows from (2.21) that

$$NL(\alpha h^{-2/3}) = N\frac{\pi}{2} + \frac{1}{h}\Big(\frac{4}{3}N\alpha^{3/2} - NhB_L(\alpha^{3/2}/h)\Big).$$

Integration variables are σ, α, θ and also y' and ϱ; only stationary points with respect to α and σ will be required for the sum in N to be finite. Critical points in α are such that

$$(2.46) \qquad t\partial_\alpha\tau_q(\alpha,\theta) + \partial_\alpha\Phi(x,y,\theta,\alpha,\sigma) = \varrho + 2N\alpha^{1/2}\Big(1 - \frac{3}{4}B'_L(\alpha^{3/2}/h)\Big),$$

while those with respect to σ are such that $\partial_\sigma\Phi(x,y,\theta,\alpha,\sigma) = 0$. We used $1 \leq h^{-2/3}\alpha$ to expand $L(h^{-2/3}\alpha)$ with (2.21). Recall that

$$\Phi(x,y,\theta,\alpha,\sigma) = y\cdot\theta + \sigma^3/3 + \sigma(xq^{1/3}(\theta) - \alpha) + \tau_q(\alpha,\theta)\Gamma(x,y,\sigma q^{1/3}(\theta)/\tau_q(\alpha,\theta),\theta/\tau_q(\alpha,\theta)),$$

where $\Gamma(x,y,\Xi,\Theta) = B_\Gamma(y,\Theta) + xA_\Gamma(x,y,\Xi,\Theta)$ from Proposition 6.5 in the Appendix. In addition to properties of A_Γ and B_Γ listed in Proposition 6.5, we will use Lemma 6.4. We start with computing derivatives of $\tau_q(\alpha,\theta)\Gamma(x,y,\sigma sq^{1/3}(\theta)/\tau_q,\theta/\tau_q)$ which depends on α only through τ_q. Take $\Theta = \theta/\tau_q(\alpha,\theta)$, $\vartheta = \theta/|\theta|$, then

$$\tau_q B_\Gamma(y,\theta/\tau_q) = \tau_q\big(B_0(y,\vartheta) + (|\theta|/\tau_q - 1)B_2(y,\vartheta) + ... + (|\theta|/\tau_q - 1)^j B_{2j}(y,\vartheta) + ...\big)$$

and, writing $\rho = |\theta|/\tau_q$, $\theta/\tau_q = \rho\vartheta$,

$$(2.47)\quad \partial_\alpha\Big(\tau_q B_\Gamma(y,\theta/\tau_q)\Big) = \partial_\alpha\tau_q\partial_w\Big(wB_\Gamma(y,\theta/w)\Big)_{\big|w=\tau_q}$$

$$= \partial_\alpha\tau_q\Big(B_\Gamma(y,\theta/\tau_q) - \frac{|\theta|}{\tau_q}\partial_\rho B_\Gamma(y,\rho\vartheta)_{\big|\rho=|\theta|/\tau_q}\Big)$$

$$= \partial_\alpha\tau_q\Big[B_0(y,\vartheta) - B_2(y,\vartheta) - \Big(\frac{|\theta|^2}{\tau_q^2} - 1\Big)B_4(y,\vartheta) + \mathcal{H}_{j\geq4}\Big].$$

$$(2.48)\quad \partial_\alpha\Big(\tau_q A_\Gamma(x,y,sq^{1/3}(\theta)/\tau_q,\theta/\tau_q)\Big) = \partial_\alpha\tau_q\partial_w\Big(wA_\Gamma(x,y,sq^{1/3}/w,\theta/w)\Big)_{\big|w=\tau_q}$$

$$= \partial_\alpha\tau_q\Big(A_\Gamma(x,y,sq^{1/3}/\tau_q,\theta/\tau_q) - sq^{1/3}(\theta)\ell(y,\vartheta)/\tau_q$$

$$- 2\mu(y,\vartheta)(s^2q^{2/3}(\theta)/\tau_q^2 + |\theta|^2/\tau_q^2) + \mathcal{H}_{j\geq3}\Big)$$

$$= \partial_\alpha\tau_q\Big[-\mu(y,\vartheta)\Big(s^2q^{2/3}(\theta)/\tau_q^2 + |\theta|^2/\tau_q^2 + 1\Big) + \mathcal{H}_{j\geq3}\Big],$$

where in the second to last line we used that the terms in $\mathcal{H}_{j\geq3}$ are powers of $x, sq^{1/3}(\theta)/w, \theta/w$ and therefore $w\partial_w\mathcal{H}_{j\geq3|w=\tau_q} = \mathcal{H}_{j\geq3}$. We have $\partial_\alpha\tau_q(\alpha,\theta) = \frac{q^{2/3}(\theta)}{2\tau_q(\alpha,\theta)}$ and using (2.47), (2.48) (which comes with a factor $x \in \mathcal{H}_{j\geq2}$), (2.46) becomes

$$(2.49)\quad \partial_\alpha\tau_q\Big[t + B_0(y,\vartheta) - B_2(y,\vartheta) - \Big(\frac{|\theta|^2}{\tau_q^2} - 1\Big)B_4(y,\vartheta) + \mathcal{H}_{j\geq4}\Big] + \sigma = \varrho$$

$$+ 2N\alpha^{1/2}\Big(1 - \frac{3}{4}B'_L(\alpha^{3/2}/h)\Big).$$

Using now (2.48) and that $B_\Gamma(y, \theta/\tau_q)$ does not depend on σ, $\partial_\sigma \Phi(\cdots) = 0$ becomes

$$(2.50) \qquad \sigma^2 + xq^{1/3}(\theta) - \alpha + \tau_q(\alpha, \theta)x\partial_\sigma A_\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q(\alpha, \theta), \theta/\tau_q(\alpha, \theta)) = 0.$$

Recalling (6.29), let $\Xi = \frac{\sigma q^{1/3}(\theta)}{\tau_q(\alpha, \theta)}$ and $\Theta = \frac{\theta}{\tau_q(\alpha, \theta)}$. Taking a derivative with respect to σ of A_Γ always provides a factor $q^{1/3}(\theta)/\tau_q$: $A_\Gamma(x, y, \Xi, \Theta)$ depends on σ only through $\Xi = \frac{\sigma q^{1/3}(\theta)}{\tau_q(\alpha, \theta)}$, hence

$$\partial_\sigma A_\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q, \theta/\tau_q) = \partial_\sigma \Xi \times \partial_\Xi A_\Gamma(x, y, \Xi, \Theta),$$

and $\partial_\Xi A_\Gamma(x, y, \Xi, \Theta) = \ell(y, \Theta/|\Theta|) + 2\Xi\mu(y, \vartheta) + \mathcal{H}_{j\geq 2}$. This yields

$$(2.51) \qquad \partial_\sigma A_\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q, \theta/\tau_q) = \frac{q^{1/3}(\theta)}{\tau_q(\alpha, \theta)}\Big(\ell(y, \vartheta) + 2\frac{\sigma q^{1/3}(\theta)}{\tau_q(\alpha, \theta)}\mu(y, \vartheta) + \mathcal{H}_{j\geq 2}\Big),$$

where $\mathcal{H}_{j\geq 2}$ contains weights x, $\frac{\sigma^2 q^{2/3}(\theta)}{\tau_q^2}$ and $\frac{|\theta|}{\tau_q} - 1 = \frac{|\theta|^2 - \tau_q^2}{\tau_q(|\theta| + \tau_q)} = -\frac{\alpha q^{2/3}(\theta)}{\tau_q(|\theta| + \tau_q)}$. Using (2.50) and (2.51),

$$(2.52) \qquad \sigma^2 + xq^{1/3}(\theta)\Big(1 + \ell(y, \vartheta) + 2\frac{\sigma q^{1/3}(\theta)}{\tau_q(\alpha, \theta)}\mu(y, \vartheta) + \mathcal{H}_{j\geq 2}\Big) = \alpha.$$

Recall that ℓ was chosen after (6.26) and depends on the curvature at the boundary near $y = 0$:

$$1 + \ell(y, \vartheta) = \Big(R_1(y, \vartheta + \nabla_y B_0(y, \vartheta))/q(\vartheta)\Big)^{1/3}, \quad R_1(0, \vartheta) = q(\vartheta),$$

where from (6.25) we have $\vartheta + \nabla_y B_0(y, \vartheta) = \vartheta(1 + O(y))$. By finite speed of propagation of the wave flow, for bounded time $|t|$ we must have $|y|$ bounded (see Lemma 3.2); hence, there exists $T_0 < 1$ sufficiently small such that if $|t| \leq T_0$ then

$$(2.53) \qquad \Big(R_1(y, \vartheta + \nabla_y B_0(y, \vartheta))/q(\vartheta)\Big)^{1/3} > 1/2.$$

Recall that $\alpha \leq \epsilon_0$, small. We have $B_0 = O(|y|^2)$, $B_2 = O(|y|^2)$, $B_{2j} = O(|y|)$, $\forall j \geq 2$ and $\frac{|\theta|}{\tau_q} - 1 = \frac{|\theta|^2 - \tau_q^2}{\tau_q(|\theta| + \tau_q)} = -\frac{\alpha q^{2/3}(\theta)}{\tau_q(|\theta| + \tau_q)} = O(\alpha)$, therefore the coefficient of $\partial_\alpha \tau_q$ in (2.47) is like

$$B_0(y, \vartheta) - B_2(y, \vartheta) + O(\alpha y) = y(O(y) + O(\alpha)).$$

For $|t| \leq T_0$, taking T_0 smaller if necessary, we assume $|B_0(y, \vartheta) - B_2(y, \vartheta)| = O(y^2) \leq |t|/8$. As $y \lesssim t$ by finite speed of propagation, taking ϵ_0 smaller if necessary, we assume $O(\alpha y) \leq |t|/8$ for $\alpha \leq \epsilon_0$. Therefore, the contribution from $\partial_\alpha(\tau_q B_\Gamma(y, \theta/\tau_q))/\partial_\alpha \tau_q$ in (2.49) is $O(|t|/4)$ and the coefficient of $\partial_\alpha \tau_q$ in (2.49) behaves like $t \leq T_0 < 1$. As ϱ is bounded (f has compact support in ϱ), it remains to compare σ and $2N\alpha^{1/2}$ in (2.49). Going back to (2.52), using that $0 \leq x < 1$ and that the terms in $\mathcal{H}_{j\geq 2}$ come with the factors σ^2, x, α, it follows that

$$(\sigma + x\mu q^{2/3}(\theta)/\tau_q)^2 + xq^{1/3}(1 + \ell + \mathcal{H}_{j\geq 2}) = \alpha + x^2\mu^2 q^{4/3}(\theta)/\tau_q^2,$$

which implies that σ^2 is bounded; hence for $|\sigma| > C$ for some constant C, repeated integrations by parts in σ provide a contribution $O(h^\infty)$ in every integral in the sum in N in (2.44). We obtain that the phase functions in (2.44) may be stationary in α only for

$$(2.54) \qquad 2|N|\alpha^{1/2} \leq 2(\frac{c}{2}|t| + |\varrho| + |\sigma|), \quad c := \sup_{|\theta|\sim 1, \alpha \leq \epsilon_0} \frac{q^{2/3}(\theta)}{\tau_q(\alpha, \theta)},$$

where, as $h^{2/3 - \epsilon} \lesssim \alpha$, we used $|B_L'(\alpha^{3/2}/h)| = \frac{h^2}{\alpha^3}\Big(b_1 + \sum_{j\geq 2} jb_j(\frac{h}{\alpha^{3/2}})^{j-1}\Big) \leq h^\epsilon$. As the righthand side from (2.54) is bounded, phase functions in (2.44) are stationary in α only for $|N| \leq \tilde{C}\alpha^{-1/2}$

for some constant $\tilde{C} := \frac{c}{2}T_0 + 1 + C$. Again, as $h^{2/3-\epsilon} \lesssim \alpha$ (which is crucial here), we get for N's that may provide non-trivial contributions $|N| \lesssim \alpha^{-1/2} \lesssim h^{-1/3+\epsilon/2}$. For $|N|$ larger than $h^{-1/3}$ we perform non stationary phase with respect to α: each integration by parts provides a factor h and "loses" a factor $h^{-2/3}$ corresponding to taking a derivative on p_h which has been defined in (2.17) in terms of $p(x, y, \theta/h, \alpha/h, \sigma/h^{1/3})$, together with a negative power of the derivative of the phase of $K_\omega(f)$ with respect to α, which depends on N (through the term $2\sqrt{\alpha}N$). Therefore, if $|N| \geq h^{-1/3}$, we get, after $M \geq 1$ integrations by parts,

$$\left| \langle \sum_{|N| \geq h^{-\frac{1}{3}}} e^{-iNL(\omega)}, \chi^\flat(h^{\frac{2}{3}}\omega/\epsilon_0)K_\omega(f)(t, x, y) \rangle_\omega \right| \leq C_M \sum_{|N| \geq h^{-\frac{1}{3}}} \left(\frac{h^{1-\frac{2}{3}}}{\sqrt{\alpha}N} \right)^M \leq C_M h^{M\frac{\epsilon}{2}}.$$

as the sum in N is bounded for $M \geq 2$, and therefore this provides a contribution $O(h^\infty)$. We just proved that, for any smooth f, the sum over N (2.44) is essentially finite over $|N| \lesssim h^{-1/3}$. $\square$

In the following we introduce $g_{h,a}$ provided by Lemma 2.12 in this finite sum (for $|N| \lesssim h^{-1/3}$) and prove that for $a > h^{2/3-\epsilon}$, $N > |t|a^{-1/2}$, V_N provides an $O_{C^\infty}(h^\infty)$ contribution. Let $K_\omega(g_{h,a})$ be given by (2.41). The phase function of $V_N(t, x, y)$ defined in (2.42) is

$$t\tau_q(\alpha, \theta) + \Phi(x, y, \theta, \alpha, \sigma) - \Phi(a, 0, \theta, \alpha, s) - NhL(\alpha h^{-2/3}),$$

with large parameter $1/h$ in front. This phase function is stationary with respect to α if

$$(2.55) \quad \partial_\alpha \tau_q \left[t + B_0(y, \vartheta) - B_2(y, \vartheta) - \left(\frac{|\theta|^2}{\tau_q^2} - 1 \right) B_4(y, \vartheta) + \mathcal{H}_{j \geq 4} \right]$$

$$+ \sigma - s = 2N\alpha^{1/2} \left(1 - \frac{3}{4}B'_L(\alpha^{3/2}/h) \right).$$

Using now (2.48), the phase is stationary in σ and s when

$$\sigma^2 + xq^{1/3}(\theta) - \alpha + \tau_q(\alpha, \theta)x\partial_\sigma A_\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q(\alpha, \theta), \theta/\tau_q(\alpha, \theta)) = 0,$$

$$s^2 + aq^{1/3}(\theta) - \alpha + \tau_q(\alpha, \theta)a\partial_s A_\Gamma(a, 0, sq^{1/3}(\theta)/\tau_q(\alpha, \theta), \theta/\tau_q(\alpha, \theta)) = 0.$$

Using (2.51) for $\partial_\sigma A_\Gamma$, we obtain as in (2.52)

$$(2.56) \quad \sigma^2 + xq^{1/3}(\theta)\left(1 + \ell(y, \vartheta) + 2\frac{\sigma q^{1/3}(\theta)}{\tau_q(\alpha, \theta)}\mu(y, \vartheta) + \mathcal{H}_{j \geq 2} \right) = \alpha,$$

$$s^2 + aq^{1/3}(\theta)\left(1 + 2\frac{sq^{1/3}(\theta)}{\tau_q(\alpha, \theta)}\mu(0, \vartheta) + \mathcal{H}_{j \geq 2} \right) = \alpha.$$

Both $a \leq a_0$ and $\alpha \leq \epsilon_0$ being small, the second equation in (2.56) yields

$$\left(s(1 + O(a) + aq^{2/3}(\theta)/\tau_q \right)^2 + aq^{1/3}(\theta)(1 + O(a)) \sim \alpha,$$

and therefore $|s| = \sqrt{\alpha} + O(a)$. Moreover, we must have $a \lesssim \alpha$, otherwise non stationary phase in s provides an $O(h^\infty)$ contribution. Therefore we introduce a cut-off $(1 - \chi)(4\alpha/a)$ supported for $\alpha > a/4$ in the symbol of V_N without changing its contribution modulo $O(h^\infty)$ (see Remark 2.15). For T_0 sufficiently small such that (2.53) to hold, the first equation in (2.56) yields,

$$\left(\sigma(1 + O(x)) + x\mu q^{2/3}(\theta)/\tau_q \right)^2 + xq^{1/3}(1 + \ell(y, \vartheta) + O(x)) \sim \alpha.$$

Note that x remains small (comparable to α), otherwise non stationary phase with respect to σ will provide an $O(h^\infty)$ contribution. We also obtain $|\sigma| = \sqrt{\alpha} + O(x)$. Moreover, both σ and s

are small and $x \geq 0$, so that (2.56) implies that $|\sigma|, |s| \leq \sqrt{\alpha}$ for the phase of V_N to be stationary in σ, s; for $|\sigma|, |s| \geq 2\sqrt{\alpha}$ we can apply the non-stationary phase theorem, and from (2.55), for T_0 sufficiently small such that (2.53) to hold, the phase of $V_N(t, x, y)$ is stationary in α only for

$$(2.57) \qquad 2|N|\sqrt{\alpha} \leq |t| + |\sigma| + |s| \leq |t| + 2\sqrt{\alpha}.$$

We used again here that for T_0 sufficiently small and by finite speed of propagation, the coefficient of $\partial_\alpha \tau_q$ in the lefthand side of (2.55) is $t + y(O(y) + O(a)) \sim t$ and that $\partial_\alpha \tau_q \sim 1/2$. Moreover, for values $2|N|\sqrt{\alpha} \geq 2|t| + 4\sqrt{\alpha}$, non stationary phase in α provides an $O(h^\infty)$ contribution from all V_N with $|N| \geq 2 + |t|/\sqrt{\alpha}$. As both $a \lesssim \alpha$ and the number of V_N is (large but) finite ($|N| \leq h^{-1/3}$), non-trivial contributions in $\mathcal{P}_{h,a}(t, x, y)$ may only be provided by the sum over $|N| \lesssim |t|/\sqrt{\alpha} \lesssim |t|/\sqrt{a}$. We have thus proved the second point in Proposition 2.14.

Finally, we turn to the data: by design, $V_0(0, x, y) = \chi_0(hD_x)\varkappa(hD_y)\delta_{(a,0)} + O_{C^\infty}(h^\infty)$, and we are left with proving that, for $0 < |N| \lesssim h^{-1/3}$, $V_N(0, x, y) \in O_{C^\infty}(h^\infty)$. In this part we consider $g_{h,a}$ as provided by Lemma 2.12, of the form (2.31) with phase function $\psi_a(\varrho, \theta') = \varrho^3/3 + a(\varrho q^{1/3}(\theta') + O(\varrho^3)) + O(a^2)$. Then $K_\omega(g_{h,a})$ is of the form (2.26) with f replaced by $g_{h,a}$, that we re-write here

$$K_\omega(g_{h,a})(t, x, y) = \frac{h^{2/3}}{2\pi h^d} \int e^{\frac{i}{h}(t\tau_q(h^{2/3}\omega,\theta) + \Phi(x,y,\theta,h^{2/3}\omega,\sigma) - y'\cdot\theta - \varrho h^{2/3}\omega)} p_h(x, y, \theta, h^{2/3}\omega, \sigma) q^{1/6}(\theta)$$

$$\times h^{-d} e^{\frac{i}{h}(\psi_a(\varrho,\theta') + y'\cdot\theta')} r_h(\varrho, \theta') q^{-1/6}(\theta') \, d\theta' \, dy' \, d\varrho \, d\theta \, d\sigma.$$

For all N, V_N is an oscillatory integral, that we rewrite, using (2.42) and Proposition 2.14,

$$V_N(t, x, y) = h^{-2/3} \int e^{-iNL(h^{-2/3}\alpha)} \chi^\flat(\alpha/\epsilon_0) \chi^\sharp(4\alpha/a) K_{h^{-2/3}\alpha}(g_{h,a})(t, x, y) d\alpha$$

$$= \frac{1}{2\pi h^d} \int e^{-iNL(h^{-2/3}\alpha)} \chi^\flat(\alpha/\epsilon_0) \chi^\sharp(4\alpha/a) e^{\frac{i}{h}(t\tau_q(\alpha,\theta) + \Phi(x,y,\theta,\alpha,\sigma) - y'\cdot\theta - \varrho\alpha)}$$

$$\times p_h(x, y, \theta, \alpha, \sigma) q^{1/6}(\theta) h^{-d} e^{\frac{i}{h}(\psi_a(\varrho,\theta') + y'\cdot\theta')} r_h(\varrho, \theta') q^{-1/6}(\theta') \, d\theta' \, dy' \, d\varrho \, d\theta \, d\sigma.$$

We can write $q_h := \chi^\flat(\alpha/\epsilon_0)\chi^\sharp(4\alpha/a)p_h r_h$, which is an elliptic symbol of order 0; indeed, recall that p_h comes with a factor $h^{-1/3}$ while r_h comes with a factor $h^{1/3}$. At $t = 0$, the stationary points of V_N with respect to $\alpha, s, \varrho, y', \theta', \theta$ are solutions to the following equations

$$(2.58) \qquad \begin{cases} \partial_\alpha \Phi(x, y, \theta, \alpha, \sigma) = \varrho + 2N\alpha^{1/2}\left(1 - \frac{3}{4}B'_L(\alpha^{3/2}/h)\right), \\ \sigma^2 + xq^{1/3}(\theta) + \tau_q(\alpha, \theta)x\partial_\sigma A_\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q(\alpha, \theta), \theta/\tau_q(\alpha, \theta)) = \alpha, \\ \partial_\varrho \psi_a(\varrho, \theta') = \alpha, \\ \theta' = \theta, \quad \nabla_{\theta'}\psi_a(\varrho, \theta') = y', \quad \nabla_\theta \Phi(x, y, \theta, \alpha, \sigma) = y'. \end{cases}$$

From the first two equations we get (2.57) (at $t = 0$), which allows to conclude that if $|N| \geq 2$ the phase is non-stationary in α. Hence we are left with $|N| = 1$. The equations from the last line in (2.58) give $\nabla_\theta \Phi(x, y, \theta, \alpha, \sigma) = \nabla_\theta \psi_a(\varrho, \theta)$ and therefore

$$(2.59) \qquad a\varrho\nabla_\theta(q^{1/3}(\theta)) + O(a\varrho^2, a^2) = y + \sigma x\nabla_\theta(q^{1/3}(\theta)) + \nabla_\theta(\tau_q(\alpha, \theta)\Gamma).$$

For $\tau_q = \tau_q(\alpha, \theta)$ we have $\nabla_\theta \tau_q = (\theta + \alpha\frac{\nabla_\theta(q^{1/3}(\theta))}{3q^{1/3}(\theta)})/\tau_q$ and

$$\nabla_\theta\left(\tau_q\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q, \theta/\tau_q)\right) = (\nabla_\theta \tau_q)\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q, \theta/\tau_q)$$

$$+ \tau_q\left(\nabla_\theta(B_\Gamma(y, \theta/\tau_q)) + x\nabla_\theta(A_\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q, \theta/\tau_q))\right).$$

From $B_0 = O(|y|^2)$, $B_2 = O(|y|^2)$, $B_{2j} = O(|y|)$ we get

$$\partial_\theta(\tau_q(\alpha,\theta)\Gamma) = \partial_\theta \tau_q \Big(O(y^3) + O(y^2\alpha) + O(y\alpha^2) + x(O(\sigma) + O(\alpha)) \Big) + \tau_q(O(y\alpha) + xO(\sigma))$$
$$= O(y^3) + O(y\alpha) + O(x\sigma).$$

Combining this with (2.59) yields $y = O(a\varrho) + O(x\sigma) + O(a^2)$, at the stationary phase points y', θ', θ of V_N. Rescale $x = aX$, $\alpha = aq^{1/3}(\theta)\mathcal{A}$, $\sigma = \sqrt{a}q(\theta)^{1/6}\mathcal{S}$ and $\varrho = \sqrt{a}q(\theta)^{1/6}\mathcal{T}$; from the first three equations of (2.58) (and $\theta' = \theta$) we get

$$\mathcal{S} + \mathcal{T} + 2\mathcal{A}^{\frac{1}{2}} = O(a^{1/2}), \quad \mathcal{S}^2 + (1+\ell)X + O(a^{1/2}) = \mathcal{A} = \mathcal{T}^2 + 1 + O(a^{1/2}).$$

As $a \leq a_0$ is small (taking a_0 even smaller if necessary), $\mathcal{T}^2 \leq \mathcal{A} - \frac{1}{2}$; then, using that $\ell = O(y) = O(a^{3/2})$, there exists $\varepsilon > 0$ such that for $X \geq -\varepsilon$, $\mathcal{S}^2 \leq \mathcal{A} + \frac{1}{10}$. For these values, the first equation cannot hold and non stationary phase in α allows to conclude. Again, that we can integrate by parts in $V_{\pm 1}(0, x, y)$ relies on $a \geq h^{2/3-\epsilon}$. $\qquad \square$

2.3. Parametrix construction for $a < h^{2/3-\epsilon}$, $0 < \epsilon < 1/12$.

In formulas (2.37) and (2.38), $G(\cdots, \omega)$ can be written in terms of the Airy function $Ai(-\zeta)$ and its derivative, as stated in Theorem 2.1. Using the explicit form of ζ, for $\zeta < 0$ (hence for $\omega \leq a/(4h^{2/3})$), these Airy factors in G are exponentially decreasing, so the main contribution comes from values $a \lesssim 4\omega h^{2/3}$. For $a \gtrsim h^{2/3-\epsilon}$, this implies $\omega \gtrsim h^{-\epsilon}$ and one may perform stationary phase arguments in the integrals from (2.37), as long as we pick any $\epsilon > 0$. In this section, a is much smaller and very different issues arise compared to the previous one; we need a different way to construct a suitable $g_{h,a}$ to recover the initial data. The explicit upper bound on ϵ will be of use in this section and later required in Section 4; there is quite an overlap between both parametrix constructions for $h^{2/3} < a < h^{2/3-1/12}$, but we made not attempt at enlarging it; the reader is advised to think ϵ to be really small.

Besides the lack of a large parameter, which forces us to work with (2.38), the regime $a < h$ has its own difficulties: even deciding how the initial data should be chosen in order the Dirichlet condition to be satisfied on the boundary becomes a non trivial issue. Indeed, (2.28) as initial data provides a non-trivial contribution on the boundary. Taking $\varkappa(-h^2\Delta)\varkappa(hD_y)\delta_{(a,0)}$ would be a natural choice: it may be expanded on eigenfunctions of the Laplace operator on the compact set Ω, but we know very little on them. Instead, we use the spectral theory for the model Laplace operator (2.2) in order to expand $\varkappa(-h^2\Delta_M)\varkappa(hD_y)\delta_{(a,0)}$ in terms of the eigenfunctions of $-\Delta_M$, as they have been used extensively in our previous work [9], [12]. This will turn out to be sufficient for our later purposes. We now recall some properties of $-\Delta_M$.

2.3.1. *Spectral theory for* $-\Delta_M$: *the initial data in terms of model gallery modes.* Let $-\Delta_M$ be the Friedlander model operator introduced in (2.2), and recall q is a positive definite quadratic form. Taking the Fourier transform in the y variable, the operator $-\Delta_M$ becomes $-\partial_x^2 + |\eta|^2 + xq(\eta)$. For $\eta \neq 0$, this operator is a positive self-adjoint operator on $L^2(\mathbb{R}_+)$, with compact resolvent. The next Lemma is proved in [11] (with $q(\eta) = |\eta|^2$, but only using $q(\eta) \neq 0$):

Lemma 2.17. *(see* [11, *Lemma 2]) There exists an Hilbert basis of* $L^2(\mathbb{R}_+)$ *where* $\{e_k(x,\eta)\}_{k\geq 0}$ *are eigenfunctions of* $-\partial_x^2 + |\eta|^2 + xq(\eta)$, *with eigenvalues* $\lambda_k(\eta) = |\eta|^2 + \omega_k q(\eta)^{2/3} = \tau_q^2(\omega_k, \eta)$. *These eigenfunctions are translated and rescaled Airy functions:*

$$(2.60) \qquad e_k(x,\eta) = \frac{\sqrt{2\pi}q(\eta)^{1/6}}{\sqrt{L'(\omega_k)}} Ai\Big(xq(\eta)^{1/3} - \omega_k\Big),$$

where $L'(\omega_k)$ *(from* (2.22)*) normalizes* $\|e_k(.,\eta)\|_{L^2(\mathbb{R}_+)} = 1$.

For $a > 0$, the Dirac distribution $\delta_{x=a}$ on $\mathbb{R}_+$ may be decomposed as $\delta_{x=a} = \sum_{k \geq 1} e_k(x, \eta) e_k(a, \eta)$. Eigenfunctions of $-\partial_x^2 + xq(\eta)$ are $e_k(x, \eta)$ with eigenvalue $\lambda_k(\eta) - |\eta|^2 = \tau_q^2(\omega_k, \eta) - |\eta|^2 = \omega_k q^{2/3}(\eta)$, and for cut-offs $\varkappa$, χ_0 to be chosen, the following spectral decomposition holds

$$(2.61) \quad \chi_0(-h^2\partial_x^2 + xq(h\partial_y))\varkappa(hD_y)\delta_{(a,0)} = \sum_{k \geq 1} \int e^{iy\cdot\eta}\chi_0\big(h^{\frac{2}{3}}\omega_k q^{\frac{2}{3}}(h\eta)\big)\varkappa(h\eta)e_k(x, \eta)e_k(a, \eta)d\eta.$$

With $\chi_0 \in C_0^\infty(-2\epsilon_0, 2\epsilon_0)$, $\chi_0 = 1$ on $[-\epsilon_0, \epsilon_0]$, (2.61) is a finite sum with $O(\epsilon_0/h)$ terms. Setting $\theta = h\eta$, from support considerations, taking the support of χ_0 smaller if necessary, we can assume that $\chi^\flat(\alpha/\epsilon_0)\chi_0(\alpha q^{2/3}(\theta))\varkappa(\theta) = \chi_0(\alpha q^{2/3}(\theta))\varkappa(\theta)$, where $\alpha = h^{2/3}\omega$ and $\chi^\flat(h^{2/3}\omega/\epsilon_0)$ is the cut-off introduced in (2.39) (which restricts the support of K_ω to values $\omega \leq \epsilon_0/h^{2/3}$).

For $a \leq h^{2/3-\epsilon}$, the easiest way to define an initial data is to chose the lefthand side term in (2.61), which does vanish on the boundary ($e_k(0, \eta) = 0$ for every $k \geq 1$). Using both (2.40) and (2.39), we are left to obtain a smooth function $g_{h,a}$ such that for $\mathcal{P}_{h,a}$ as in (2.39) to have

$$(2.62) \qquad \mathcal{P}_{h,a}(0, x, y) = \chi_0(-h^2\partial_x^2 + xq(h\partial_y))\varkappa(hD_y)\delta_{(a,0)} + O_{C^\infty}(h^\infty).$$

We proceed as follows : for a given K_ϵ such that $h^{-\epsilon} \ll h^{-2\epsilon} \lesssim K_\epsilon \lesssim h^{-1/4+\epsilon} \ll h^{-1/4}$, define $\mathcal{E}_M$ and split the sum over k in (2.61),

$$(2.63)\, \mathcal{E}_M(x, y, a, \omega_k) := \int e^{iy\cdot\eta}\chi_0(h^{2/3}\omega_k q^{2/3}(h\eta))\varkappa(h\eta)e_k(x, \eta)e_k(a, \eta)d\eta$$

$$= \frac{2\pi}{L'(\omega_k)}\int e^{iy\cdot\eta}\chi_0(h^{2/3}\omega_k q^{2/3}(h\eta))\varkappa(h\eta)q^{1/3}(\eta)Ai(xq^{1/3}(\eta) - \omega_k)Ai(aq^{1/3}(\eta) - \omega_k)d\eta.$$

$$(2.64) \quad \sum_{k \geq 1}(\cdots) = \sum_{k \geq 1}\chi^\flat\Big(\frac{\omega_k}{\omega_{K_\epsilon}}\Big)\mathcal{E}_M(x, y, a, \omega_k) + \sum_{k \geq 1}\chi^\flat(h^{2/3}\omega_k/\epsilon_0)\chi^\sharp\Big(\frac{\omega_k}{\omega_{K_\epsilon}}\Big)\mathcal{E}_M(x, y, a, \omega_k).$$

Proposition 2.18. *For all $a \in (0, h^{2/3-\epsilon})$ there exists a smooth function $g_{h,a,1}$ such that*

$$(2.65) \qquad \sum_{k \geq 1}\frac{2\pi}{L'(\omega_k)}\chi^\flat(h^{2/3}\omega_k/\epsilon_0)K_{\omega_k}(g_{h,a,1})(0, x, y) = \sum_{k \geq 1}\chi^\flat\Big(\frac{\omega_k}{\omega_{K_\epsilon}}\Big)\mathcal{E}_M(x, y, a, \omega_k) + O(h^\infty).$$

The proof of Proposition 2.18 is postponed to Section 2.3.3, as it requires arguments from section 2.3.2; we will introduce the cutoff $\chi^\flat(\omega_k/\omega_{K_\epsilon})$ in the (LHS) term of (2.65) as well, due to how $g_{h,a,1}$ is obtained.

Proposition 2.19. *For all $a \in (0, h^{2/3-\epsilon})$, there exists a smooth function $g_{h,a,2}$ such that*

$$(2.66) \qquad \langle\sum_{N \in \mathbb{Z}} e^{-iNL(\omega)}, \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp(h^{2\epsilon}\omega)K_\omega(g_{h,a,2})(0, x, y)\rangle_\omega$$

$$= \sum_{k \geq 1}\chi^\sharp\Big(\frac{\omega_k}{\omega_{K_\epsilon}}\Big)\chi^\flat(h^{2/3}\omega_k/\epsilon_0)\mathcal{E}_M(x, y, a, \omega_k) + O(h^\infty),$$

$$(2.67) \qquad \langle\sum_{N \in \mathbb{Z}} e^{-iNL(\omega)}, \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp(\omega)\chi^\flat(h^{2\epsilon}\omega)K_\omega(g_{h,a,2})(0, x, y)\rangle_\omega = O(h^\infty).$$

We introduced a new cut-off $\chi^\sharp(h^{2\epsilon}\omega)$ in (2.66) and removed $\chi^\sharp(\omega)$ (which is supported for $\omega \geq 2$ and identically 1 on the support of $\chi^\sharp(h^{2\epsilon}\omega)$). The proof of Proposition 2.19 will be provided in Section 2.3.4. The sum of (2.66) and (2.67) yields the second term in (2.64). Finally, we have

Proposition 2.20. *For all $a \in (0, h^{2/3-\epsilon})$, let $g_{h,a} := g_{h,a,1} + g_{h,a,2}$, then $\mathcal{P}_{h,a}$, defined in Definition 2.13 is a parametrix in the sense of Definition 2.5 and (2.62) holds.*

Proposition 2.20 follows easily from Propositions 2.18 and 2.19 using $K_\omega(g_{h,a,1} + g_{h,a,2}) = K_\omega(g_{h,a,1}) + K_\omega(g_{h,a,2})$ together with (2.38).

Remark 2.21. For later purposes (dispersion for small a, Section 4), taking $K_\epsilon \sim h^{-2\epsilon}$ would be enough. However, in the next subsection, we aim at obtaining gallery modes for k as large as possible, which turns out to be up to $K_\epsilon \sim h^{-1/4+\epsilon}$. This is of independent interest and will prove useful to deal with the Schrödinger operator as well as generalize [12] from the model case to the general case; both will be adressed elsewhere.

2.3.2. *Pseudo-differential calculus; construction of gallery modes.*

Definition 2.22. Let G be defined in (2.12). We set

$$(2.68) \qquad e(x,y,\eta,\omega) = \frac{\sqrt{2\pi}q(\eta)^{1/6}}{\sqrt{L'(\omega)}}e^{-iy\cdot\eta}G(x,y,\eta,\omega).$$

Replacing $G(\cdots)$ with $G_M(\cdots) := e^{iy\cdot\eta}Ai(xq(\eta)^{1/3}-\omega)$ in (2.68) yields (2.60) instead of $e(x,y,\eta,\omega)$.

Definition 2.23. Let $\varkappa$ be like in Definition (2.5). For $g \in L^2(\mathbb{R}^{d-1})$, we define

$$(2.69) \qquad F_{\omega_k}(g)(x,y) := \frac{1}{(2\pi)^{d-1}}\int e^{i(y-y')\cdot\eta}e(x,y,\eta,\omega_k)(\varkappa(hD_{y'})g)(y')d\eta dy'.$$

Definition 2.24. Let $\tilde{\varkappa} \in C_0^\infty(\mathbb{R}^{d-1})$ be such that $\tilde{\varkappa} = 1$ on the support of $\varkappa$ and vanishing outside a neighborhood of $\mathbb{S}^{d-2}$. Let also $\chi \in C_0^\infty$ be a smooth cutoff supported in the ball of center 0 and radius 1/16 of $\mathbb{R}^{d-1}$. For $f \in L^2(\mathbb{R}^{d-1})$ we define an operator $\mathfrak{L}$ as

$$(2.70) \qquad \mathfrak{L}(f)(y) := \int e^{i(y-y')\cdot\eta-i|\eta|B_0(y',\eta/|\eta|)}\tilde{\varkappa}(h\eta)\chi(y')f(y')dy'd\eta,$$

where B_0 is the first term in the development of B_Γ in (6.8) and is homogeneous of degree 0.

To define $g_{h,a,1}$ and $g_{h,a,2}$, we need to "invert" F_{ω_k}, which requires estimating derivatives of $e(x,y,\eta,\omega_k)$ with respect to (y,η): in the Friedlander model case the corresponding mode $e_k(x,\eta)$ from (2.60) does not depend on y, but here, deriving with respect to y yields a large factor due to τB_0 in the phase $\tau\Gamma(x,y,\sigma q^{1/3}(\eta)/\tau,\eta/\tau)$. In order to get rid of the homogeneous term of degree zero $|\eta|B_0(y,\eta/|/\eta|)$ in the phase function of $e(x,y,\eta,\omega_k)$, we set $\tilde{F}_{\omega_k} := F_{\omega_k} \circ \mathfrak{L}$ to obtain a new operator whose phase function does not include the contribution $|\eta|B_0(y,\eta/|\eta|)$; we need to prove that, at least for $k \ll h^{-1/4}$, these operators can be inverted and this will be our main result in this section:

Proposition 2.25. *Let $0 < \epsilon_1 < 1/4$ be small. The operators $\tilde{F}_{\omega_k}^* \circ \tilde{F}_{\omega_k} : L^2(\mathbb{R}^{d-1}) \to L^2(\mathbb{R}^{d-1})$ are pseudo-differential operators that are uniformly elliptic with respect to $1 \le k \le h^{-1/4+\epsilon_1}$.*

Note that we would like to construct quasi-modes for as many k as possible in the next section, and therefore ϵ_1 should really be seen as very small. We momentarily postpone the proof of Proposition 2.25 to perform some preliminary steps.

We compute $F_{\omega_k} \circ \mathfrak{L}(f)$ for $f \in L^2(\mathbb{R}^{d-1})$, setting $\tilde{e}(x,y,\eta,\omega_k) = e^{-i|\eta|B_0(y,\eta/|\eta|)}e(x,y,\eta,\omega_k)$:

$$F_{\omega_k} \circ \mathfrak{L}(f)(x,y) = \frac{1}{(2\pi)^{d-1}}\int_\eta e^{iy\cdot\eta}e(x,y,\eta,\omega_k)\varkappa(h\eta)\widehat{\mathfrak{L}(f)}(\eta)d\eta$$

$$= \frac{1}{(2\pi)^{d-1}}\int_{y'}\int_\eta e^{i(y-y')\cdot\eta+i|\eta|(B_0(y,\eta/|\eta|)-B_0(y',\eta/|\eta|))}\tilde{e}(x,y,\eta,\omega_k)\varkappa(h\eta)\chi(y')f(y')d\eta dy'.$$

Remark 2.26. As y' is small on the support of $\chi(y')$, there exists a cut-off $\tilde{\chi}$, supported in a ball of radius $1/8$ in $\mathbb{R}^{d-1}$, such that $\tilde{\chi} = 1$ on the support of χ and

$$F_{\omega_k} \circ \mathfrak{L}(f)(x,y) = \tilde{\chi}(y) F_{\omega_k} \circ \mathfrak{L}(f)(x,y) + O(h^\infty).$$

When it will be necessary to emphasize the small size of the support in y of the operators we will work with, we will add the cutoff $\tilde{\chi}(y)$.

Setting $\tilde{F}_{\omega_k} = F_{\omega_k} \circ \mathfrak{L}$, we compute the adjoint operator $\tilde{F}^*_{\omega_k}$ using that for every $f \in L^2(\mathbb{R}^{d-1})$ and $\mathcal{E} \in L^2(\Omega)$ we have $< \tilde{F}_{\omega_k}(f), \mathcal{E} >_{L^2(\Omega)} = < f, \tilde{F}^*_{\omega_k}(\mathcal{E}) >_{L^2(\mathbb{R}^{d-1})}$. This yields

$$< \tilde{F}_{\omega_k}(f), \mathcal{E} >_{L^2(\Omega)} = \int_\Omega \left(\frac{1}{(2\pi)^{d-1}} \int_z \int_\eta e^{i(y-z)\cdot\eta} e(x,y,\eta,\omega_k) \varkappa(h\eta) \mathfrak{L}(f)(z) d\eta dz \right) \overline{\mathcal{E}}(x,y) dx dy$$

$$= \int_{y'} f(y') \left(\frac{1}{(2\pi)^{d-1}} \int_\Omega \int_\eta e^{i(y'-y)\cdot\eta + i|\eta|(B_0(y',\eta/|\eta|) - B_0(y,\eta/|\eta|))} \overline{\tilde{e}(x,y,\eta,\omega_k)} \varkappa(h\eta) \chi(y') \mathcal{E}(x,y) d\eta dx dy \right) dy'$$

$$= \int_z f(z) \overline{\tilde{F}^*_{\omega_k}(\mathcal{E})}(z) dz = < f, \tilde{F}^*_{\omega_k}(\mathcal{E}) >_{L^2(\mathbb{R}^{d-1})},$$

which yields $\tilde{F}^*_{\omega_k}(\mathcal{E})$. We can now explicitly compute $\tilde{F}^*_{\omega_k} \circ \tilde{F}_{\omega_k}(f)(z)$ for $f \in L^2(\mathbb{R}^{d-1})$ and $z \in \mathbb{R}^{d-1}$:

$$\tilde{F}^*_{\omega_k} \circ \tilde{F}_{\omega_k}(f)(z) = \frac{\chi(z)}{(2\pi)^{d-1}} \int_\Omega \int_\eta e^{i(z-y)\cdot\eta + i|\eta|(B_0(z,\eta/|\eta|) - B_0(y,\eta/|\eta|))} \varkappa(h\eta) \overline{\tilde{e}(x,y,\eta,\omega_k)} \tilde{F}_{\omega_k}(f)(x,y) d\eta dx dy$$

$$(2.71) \qquad = \int_{z'} M_k(z,z') f(z') dz' + O(h^\infty),$$

where we have introduced the cut-off $\tilde{\chi}(y)$ and have set

$$M_k(z,z') = \frac{1}{(2\pi)^{d-1}} \int_{y,\eta,\eta'} e^{i\left((z-y)\cdot\eta + |\eta|(B_0(z,\eta/|\eta|) - B_0(y,\eta/|\eta|))\right) - i\left((z'-y)\cdot\eta' + |\eta'|(B_0(z',\eta'/|\eta'|) - B_0(y,\eta'/|\eta'|))\right)}$$

$$\times \tilde{\chi}(y) \chi(z) \chi(z') \varkappa(h\eta) \varkappa(h\eta') \int_0^\infty \overline{\tilde{e}(x,y,\eta,\omega_k)} \tilde{e}(x,y,\eta',\omega_k) dx d\eta' d\eta dy.$$

We let $v = v(\eta,z,y) := \eta + |\eta| \int_0^1 \nabla_y B_0(\varpi y + (1-\varpi)z, \eta/|\eta|) d\varpi$ (resp. $v' = v(\eta',z',y)$) such that

$$(z-y)v = (z-y)\cdot\eta + |\eta|(B_0(z,\eta/|\eta|) - B_0(y,\eta/|\eta|)),$$
$$(z'-y)v' = (z'-y)\cdot\eta' + |\eta'|(B_0(z',\eta'/|\eta'|) - B_0(y,\eta'/|\eta'|)).$$

$\nabla_\eta v = \mathbb{I}_{d-1} + O(|\nabla_y B_0|)$ being close to the identity matrix $\mathbb{I}_{d-1}$ for small y, z, $|\nabla_v \eta| = 1 + O(|\nabla_y B_0|)$ and, in the same way $|\nabla_{v'} \eta'| = 1 + O(|\nabla_y B_0|)$. Denoting $\eta(v,z,y) = v + |v|O(y,z)$ the inverse function, we have $\eta = \eta(v,z,y)$ and $\eta' = \eta(v',z',y)$, respectively, and therefore

$$(2.72) \quad M_k(z,z') = \frac{1}{(2\pi)^{d-1}} \int_{y,v,v'} e^{i(z-y)\cdot v - i(z'-y)\cdot v'} |\nabla_v \eta| |\nabla_{v'} \eta'| \times \tilde{\chi}(y) \chi(z) \chi(z')$$

$$\times \varkappa(h\eta(v,z,y)) \varkappa(h\eta(v',z',y)) \int_0^\infty \overline{\tilde{e}(x,y,\eta(v,z,y),\omega_k)} \tilde{e}(x,y,\eta(v',z',y),\omega_k) dx dv' dv dy.$$

Taking $y = z' + w$, $v' = v - \zeta$ in (2.72), yields

$$(2.73) \qquad M_k(z, z') = \int_v e^{i(z-z')\cdot v} m_k(z, z', v) dv ,$$

$$m_k(z, z', v) = \frac{1}{(2\pi)^{d-1}} \int_\zeta \int_w \int_0^\infty e^{-iw\cdot\zeta} |\nabla_v \eta|(v, z, z' + w) |\nabla_{v'} \eta|(v - \zeta, z', z' + w)$$
$$\times \varkappa(h\eta(v, z, z' + w)) \varkappa(h\eta'(v - \zeta, z', z' + w)) \tilde{\chi}(z' + w) \chi(z) \chi(z')$$
$$\times \overline{\tilde{e}(x, z' + w, \eta(v, z, z' + w), \omega_k)} \tilde{e}(x, z' + w, \eta(v - \zeta, z', z' + w), \omega_k) dx dw d\zeta .$$

On the support of the two cut-offs $\varkappa$, $h(v + O(z, z' + w)) \in [1/2, 2]$, $h(v - \zeta + O(z', z' + w)) \in [1/2, 2]$, and on the support of $\chi(z)\chi(z'+w)\tilde{\chi}(z'+w)$, $|z, z'| < 1/16$, $|z' + w| < 1/8$; then, set $v = \frac{\theta}{h}$, $\zeta = \frac{\varrho}{h}$ where $\theta \in [1/4, 5/4]$ and $|\varrho| \leq 2$. In (2.73) we may replace $m_k(z, z', v)$ by $m_k(z', z', v) =: \tilde{m}_k(z', v)$ without changing the integral modulo $O(v^{-\infty}) = O(h^\infty)$. In the new variables (and modulo $O(h^\infty)$ terms), the symbol of (2.73) becomes

$$\tilde{m}_k(z', \theta/h) = \frac{1}{(2\pi h)^{d-1}} \int_\varrho \int_w e^{-\frac{i}{h} w\cdot\varrho} a_k((z', \theta); (w, \varrho); h) dw d\varrho ,$$

where

$$(2.74) \quad a_k((z', \theta); (w, \varrho); h) := |\nabla_v \eta|(\theta/h, z', z' + w) |\nabla_{v'} \eta|((\theta - \varrho)/h, z', z' + w)$$
$$\times \varkappa(h\eta(\theta/h, z', z' + w)) \varkappa(h\eta((\theta - \varrho)/h, z', z' - w)) \tilde{\chi}(z' + w) \chi(z) \chi(z')$$
$$\times \int_0^\infty \overline{\tilde{e}(x, z' + w, \eta(\theta/h, z, z' + w), \omega_k)} \tilde{e}(x, z' + w, \eta((\theta - \varrho)/h, z', z' + w), \omega_k) dx .$$

Define

$$\mathcal{S}_\gamma := \{a \in C^\infty \text{ such that } |\partial_w^{\beta_1} \partial_\varrho^{\beta_2} a(w, \varrho, h)| \leq C_\beta h^{-\gamma(|\beta_1| + |\beta_2|)}\} .$$

We now prove

Proposition 2.27. *Let $0 < \epsilon_1 < 1/4$ be small. The symbols $a_k((z', \theta); (w, \varrho); h)$ are in the class $\mathcal{S}_{\frac{1}{2} - \frac{2}{3}\epsilon_1}$, uniformly with respect to $1 \leq k \leq h^{-\frac{1}{4} + \epsilon_1}$.*

Proof. We check that there exists $\epsilon_1 > 0$ such that for every $|\beta| \geq 1$,

$$(2.75) \qquad h^{|\beta|} \left| \partial_w^\beta \partial_\varrho^\beta a_k(z', \theta; w, \varrho; h)|_{w=0, \varrho=0} \right| \lesssim h^{\frac{4}{3}\epsilon_1|\beta|} .$$

This easily follows from following lemma whose proof is postponed to the Appendix:

Lemma 2.28. *Uniformly for $h^{2/3}\omega_k \ll 1$, we have:*

$$(2.76) \qquad \|\partial_{(y, \theta)}^\beta \tilde{e}(., y, \theta/h, \omega_k)\|_{L^2(x \geq 0)} \lesssim (\omega_k/h^{1/3})^{|\beta|} .$$

Using (2.74), (2.76), and $\omega_k \sim k^{2/3}$, we get that for every $|\beta| \geq 1$, as $k \leq h^{-1/4+\epsilon_1}$.

$$h^{|\beta|} \left| \partial_w^\beta \partial_\varrho^\beta a_k((z', \theta); (w, \varrho); h)|_{w=0, \varrho=0} \right| \lesssim C_\beta h^{|\beta|} (\omega_k/h^{1/3})^{2|\beta|} \lesssim C_\beta (h^{4\epsilon_1/3})^{|\beta|} ,$$

which proves the Lemma and completes the proof of Proposition 2.27. □

We now turn our attention to $\tilde{m}_k$ and recall the classical expansion:

Lemma 2.29. *Let* $a(w, \varrho, h) : \mathbb{R}^{2n} \to \mathbb{C}$ *be* C^∞ *with* $a \in \mathcal{S}_{\frac{1}{2}-\epsilon_1}$, *with small* $\epsilon_1 > 0$. *Then*

$$\frac{1}{(2\pi h)^n} \int_\varrho \int_w e^{-\frac{i}{h} w \cdot \varrho} a(w, \varrho, h) dw d\varrho = \sum_\beta \frac{h^{|\beta|}}{i^{|\beta|}|\beta|!} \partial_w^\beta \partial_\varrho^\beta a(w, \varrho, h) \Big|_{(w,\varrho)=(0,0)}.$$

Using Lemma 2.29 for $n = d - 1$ with a_k defined in (2.74) and Proposition 2.27, we get that $\tilde{m}_k(z', \theta/h)$ may be written as

$$(2.77) \qquad \tilde{m}_k(z', \theta/h) = \sum_\beta \frac{h^{|\beta|}}{i^{|\beta|}|\beta|!} \partial_w^\beta \partial_\varrho^\beta a_k((z', \theta); (w, \varrho); h)|_{w=0, \varrho=0}.$$

We may now return to the proof of our main Proposition.

Proof. (of Proposition 2.25) From (2.71) and (2.73),

$$\tilde{F}_{\omega_k}^* \circ \tilde{F}_{\omega_k}(f)(z) = \int e^{\frac{i}{h}(z-z') \cdot \theta} \tilde{m}_k(z', \theta/h) f(z') dz',$$

where the symbol $\tilde{m}_k(z', \theta/h)$ is given by (2.77) and where, for every $k \leq h^{-1/4-\epsilon_1}$, (2.75) holds true. Moreover, $\tilde{m}_k$ is elliptic. Indeed, from (2.74), it follows that

$$a_k((z', \theta); (0,0); h) = \sigma(z', \theta/h; 0, 0) \|\tilde{e}(., z', \theta/h, \omega_k)\|_{L^2(x>0)}^2,$$

where for $\upsilon = \theta/h$ and $\zeta = \varrho/h$ like before, we define

$$(2.78) \quad \sigma((z', \upsilon); (w, \zeta)) = |\nabla_\upsilon \eta|(\upsilon, z', z' + w)|\nabla_{\upsilon'} \eta|(\upsilon - \zeta, z', z' + w)$$
$$\times \varkappa(h\eta(\upsilon, z', z' + w))\varkappa(h\eta(\upsilon - \zeta, z', z' - w))\tilde{\chi}(z' + w)\chi(z)\chi(z').$$

Using Lemma 6.1 in the Appendix, the $e(., y, \eta, \omega_k)$ are almost L^2-normalized in $x > 0$ and $\|\tilde{e}(\cdot, y, \eta, \omega_k)\|_{L^2(x>0)} \sim 1$. On the other hand the symbol σ defined in (2.78) is elliptic, therefore a_k is elliptic and $a_k((z', \theta); (0,0); h) \sim 1$ for θ close to 1; using (2.75), $\tilde{m}_k$ is therefore elliptic; $\tilde{F}_{\omega_k}^* \circ \tilde{F}_{\omega_k}$ are pseudo-differential operators, uniformly elliptic for $k \leq h^{-1/4+\epsilon_1}$. $\qquad \square$

2.3.3. *Construction of quasi-modes* $k \ll h^{-1/4}$ *(proof of Proposition 2.18).* It suffices to construct a smooth function $g_{h,a,1}$ such that, for all k such that $\chi_\flat(\omega_k/\omega_{K_\epsilon}) \neq 0$,

$$\frac{2\pi}{L'(\omega_k)} K_{\omega_k}(g_{h,a,1})(0, x, y) = \mathcal{E}_M(x, y, a, \omega_k) + O(h^\infty).$$

Indeed, for a function $g(y', \varrho)$, let $\overset{2\wedge}{g}(y', \alpha/h)$ be its Fourier transform w.r.t. ϱ at α/h. Using the definition of $K_\omega(g)$, we only need $\overset{2\wedge}{g}_{h,a,1}(., \omega/h^{1/3})$ for $\omega \in \{\omega_k, 1 \leq k \ll h^{-1/4}\}$.

Lemma 2.30. *Let* $h^{-2\epsilon} \lesssim K_\epsilon \lesssim h^{-1/4+\epsilon}$, $\epsilon > 0$ *small. For* $1 \leq k \leq 4K_\epsilon$, *define*

$$(2.79) \qquad f_{\omega_k}(y) := \left(\tilde{F}_{\omega_k}^* \circ \tilde{F}_{\omega_k}\right)^{-1} \left(\tilde{F}_{\omega_k}^*(\mathcal{E}_M(., \omega_k))\right)(y),$$

then define $g_{h,a,1}$ *with:* $\forall k$ *such that* $\chi^\flat(\omega_k/\omega_{K_\epsilon}) \neq 0$,

$$(2.80) \qquad \overset{2\wedge}{g}_{h,a,1}(., \omega_k/h^{1/3}) := \frac{\sqrt{L'(\omega_k)}}{\sqrt{2\pi}} \mathfrak{L}(f_{\omega_k}),$$

and $\forall k$ *such that* $\chi^\flat(\omega_k/\omega_{K_\epsilon}) = 0$, $\overset{2\wedge}{g}_{h,a,1}(., \omega_k/h^{1/3}) := 0$. *Then Proposition 2.18 holds for* $g_{h,a,1}$.

Proof. As $k \leq 4K_\epsilon \lesssim h^{-1/4+\epsilon} \ll h^{-1/4}$, we can use Proposition 2.25 and define f_{ω_k} by (2.79). For such f_{ω_k} and $\chi_\flat(\omega_k/\omega_{K_\epsilon}) \neq 0$, we now define $\overset{2\wedge}{g}_{h,a,1}(y, \omega_k/h^{1/3})$ by (2.80) (and zero for larger k's).

By construction, on the support of $\chi_\flat(\omega_k/\omega_{K_\epsilon})$, we have $F_{\omega_k}(\mathfrak{L}(f_{\omega_k}))(x,y) = \tilde{F}_{\omega_k}(f_{\omega_k})(x,y) = \mathcal{E}_M(x,y,a,\omega_k)$, as we chose f_{ω_k} such that $\tilde{F}^*_{\omega_k} \circ \tilde{F}_{\omega_k}(f_{\omega_k}) = \tilde{F}^*_{\omega_k}(\mathcal{E}_M(., \omega_k))$. In turn, we have

$$F_{\omega_k}\Big(\frac{2\pi}{\sqrt{L'(\omega_k)}}\overset{2\wedge}{g}_{h,a,1}(., \omega_k/h^{1/3})\Big)(x,y) = \mathcal{E}_M(x,y,a,\omega_k) + O(h^\infty),$$

(inverting $\tilde{F}^*_{\omega_k}$); using (2.24) and (2.69), $\frac{2\pi}{L'(\omega_k)} K_{\omega_k}(g_{h,a,1})(0,x,y) = \mathcal{E}_M(x,y,a,\omega_k) + O(h^\infty)$. $\qquad\square$

We obtain the explicit form of $\overset{2\wedge}{g}_{h,a,1}$ as a corollary of Lemma 2.30:

Corollary 2.31. *We keep the notations from the proof of Lemma 2.30. Let*

$$(2.81) \qquad I_a(\eta, \omega_k) := \int_{x,y} e^{-iy\cdot\eta}\overline{e(x,y,\eta,\omega_k)}\mathcal{E}_M(x,y,a,\omega_k)dxdy.$$

For $1 \leq k \leq K_\epsilon$, $g_{h,a,1}$ (from Lemma 2.30) may be rewritten

$$\overset{2\wedge}{g}_{h,a,1}(y', \omega_k/h^{1/3}) = \frac{\sqrt{L'(\omega_k)}}{\sqrt{2\pi}} \int e^{iy'\cdot\eta}\varkappa(h\eta)r(\eta, \omega_k)I_a(\eta, \omega_k)d\eta,$$

where $r(., \omega_k)$ is an elliptic symbol of order 0 and main contribution $1/\tilde{m}_k(y, \eta + |\eta|\partial_y B_0(y, \eta/|\eta|))$ with $\tilde{m}_k$ defined in (2.77).

Proof. We compute explicitly

$$\tilde{F}^*_{\omega_k}(\mathcal{E}_M)(z) = \frac{1}{(2\pi)^{d-1}} \int e^{i(z\cdot\eta+|\eta|B_0(z,\eta/|\eta|))}\varkappa(h\eta) \int_{x,y} e^{-iy\cdot\eta}\overline{e(x,y,\eta,\omega_k)}\mathcal{E}_M(x,y,a,\omega_k)dxdyd\eta.$$

Moreover, using (2.71) and 2.73, there exists an elliptic symbol $\tilde{r}_k(y', \upsilon)$ of order 0 with main contribution $1/\tilde{m}_k$ and $\tilde{\varkappa}$ supported for $\upsilon \sim \frac{1}{h}$ and equal to 1 on the support of $\varkappa$ such that $(\tilde{F}^*_{\omega_k} \circ \tilde{F}_{\omega_k})^{-1}(F)(y') = \int e^{i(y'-z)\cdot\eta}\tilde{\varkappa}(h\eta)\tilde{r}_k(y', \eta)F(z)dzd\eta$. Taking $F = \tilde{F}^*_{\omega_k}(\mathcal{E}_M)$ yields

$$f_{\omega_k}(y') = \frac{1}{(2\pi)^{d-1}} \int e^{i(y'-z)\cdot\eta}\tilde{r}_k(y',\eta)\varkappa(h\eta') \int e^{i(z\cdot\eta'+|\eta'|B_0(z,\eta'/|\eta'|))}I_a(\eta', \omega_k)d\eta'dzd\eta.$$

Applying stationary phase with respect to z, η, critical points are $z = y'$, $\eta = \eta' + |\eta'|\partial_z B_0(z, \eta'/|\eta'|)|_{z=y'}$ and

$$f_{\omega_k}(y') = \frac{1}{(2\pi)^{d-1}} \int e^{i(y'\cdot\eta+|\eta|B_0(y',\eta/|\eta|))}r_k(y',\eta)I_a(\eta, \omega_k)d\eta$$

where the new symbol $r_k(y', \eta)$ is obtained from $\tilde{r}_k$ and has main contribution $\tilde{r}_k(y', \eta) = 1/\tilde{m}_k$ and where I remains unchanged since depended only on η' and not on x, η. We therefore get

$$\mathfrak{L}(f_{\omega_k})(y) = \frac{1}{(2\pi)^{d-1}} \int e^{i(y-y')\cdot\tilde{\eta}-|\tilde{\eta}|B_0(y',\tilde{\eta}/|\tilde{\eta}|)}\varkappa(h\tilde{\eta}) \int e^{i(y'\cdot\eta+|\eta|B_0(y',\eta/|\eta|))}r_k(y',\eta)I_a(\eta, \omega_k)d\eta dy'd\tilde{\eta}$$

and integrating in $y', \tilde{\eta}$ give $y' = y$, $\tilde{\eta} = \eta$ achieves the proof. $\qquad\square$

2.3.4. *Proof of Proposition 2.19.* Our goal is to obtain $g_{h,a,2}$ such that (2.66) holds. Observe that the sum in the second line of (2.66), involving $\mathcal{E}_M$ for large ω_k, is, using (2.23),

$$(2.82) \quad \sum_{k \geq 1} \chi^\sharp\left(\frac{\omega_k}{\omega_{K_\epsilon}}\right) \chi^\flat(h^{2/3}\omega_k/\epsilon_0)\mathcal{E}_M(x,y,a,\omega_k)$$

$$= \langle \sum_{N \in \mathbb{Z}} e^{-iNL(\omega)}, \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp\left(\frac{\omega}{\omega_{K_\epsilon}}\right)\frac{L'(\omega)}{2\pi}\mathcal{E}_M(x,y,a,\omega))\rangle_\omega,$$

and non-stationary phase (with respect to ω) easily applies for all $|N| \geq 2$ in the second line of (2.82), providing an $O(h^\infty)$ contribution (this is just the model case). Therefore, we are left to obtain $g_{h,a,2}$ such that

$$(2.83) \quad \int \chi^\flat(h^{2/3}\omega)\chi^\sharp(h^{2\epsilon}\omega)\left(1 + \sum_\pm e^{\pm iL(\omega)}\right)K_\omega(g_{h,a,2})(0,x,y)d\omega$$

$$+ \sum_{|N| \geq 2} \int e^{-iNL(\omega)}\chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp(h^\epsilon\omega)K_\omega(g_{h,a,2})(0,x,y)d\omega$$

$$= \int \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp\left(\frac{\omega}{\omega_{K_\epsilon}}\right)\left(1 + \sum_\pm e^{\pm iL(\omega)}\right)\frac{L'(\omega)}{2\pi}\mathcal{E}_M(x,y,a,\omega)d\omega + O(h^\infty).$$

Let us analyze the last line of (2.83), corresponding to the sum over model gallery modes. Here $\mathcal{E}_M$ is a product of two Airy functions $e^{iy\cdot\eta}\mathrm{Ai}(-\zeta_M(x,\eta,\omega))\mathrm{Ai}(-\zeta_M(a,\eta,\omega))$, where the phases $\zeta_M = \omega - xq^{1/3}(\eta)$ and $\psi_M(y,\eta) = y \cdot \eta$ are such that (2.5) holds with $<.,.>$ replaced by the scalar product obtained by polarization of the principal symbol $\xi^2 + |\eta|^2 + xq(\eta)$ of the model Laplace operator Δ_M. Using the definition of L in Lemma 2.9, $1 + \sum_\pm e^{\pm iL(\omega)} = 1 - \left(\frac{A_+}{A_-}\right)(\omega) - \left(\frac{A_-}{A_+}\right)(\omega)$. As $h^{2/3}\omega_{K_\epsilon} \sim (hK_\epsilon)^{2/3} \gtrsim (h^{1-2\epsilon})^{2/3} \gg h^{2/3-\epsilon}$, it follows that $h^{2/3}\omega$ is much larger than a on the support of the symbol of the integral in the last line of (2.83), and we can use (2.20) to write $\mathrm{Ai}(-\zeta_M(a,\eta,\omega)) = \sum_\pm A_\pm(\zeta_M(a,\eta,\omega))$. The phase of $\mathcal{E}_M$ is now

$$(2.84) \quad y \cdot \eta + \xi^3/3 + \xi(xq^{1/3}(\eta) - \omega) \pm \frac{2}{3}(\omega - aq^{1/3}(\eta))^{3/2}.$$

Lemma 2.32. *In the integral defining $\mathcal{E}_M(.,\omega)$, the usual stationary phase in ξ applies. Moreover, for the phase corresponding to $N = 0$ in the second line of (2.82), we have*

$$\phi_{M,\pm,\mp}(x,y,\eta,\omega) := y \cdot \eta \pm \frac{2}{3}\left((\omega - xq^{1/3}(\eta))^{3/2} - (\omega - aq^{1/3}(\eta))^{3/2}\right).$$

In the same way, the phases corresponding to $N = \pm 1$ in the second line of (2.82) are

$$\phi_{M,\pm,\pm}(x,y,\eta,\omega) \mp \frac{4}{3}\omega^{3/2} := y \cdot \eta \pm \frac{2}{3}\left((\omega - xq^{1/3}(\eta))^{3/2} + (\omega - aq^{1/3}(\eta))^{3/2} - 2\omega^{3/2}\right).$$

Moreover, for $x > 2h^{2/3-\epsilon}$, the integral in the second line of (2.83) is $O(h^\infty)$.

Proof. Let $N = 0$. The derivative with respect to ω of the phase (2.84) of $\mathcal{E}_M$ vanishes when $\xi = \pm\sqrt{\omega - aq^{1/3}(\eta)}$. As $a \leq h^{2/3-\epsilon}$ and $\omega \geq \omega_{K_\epsilon} \gtrsim h^{-2\epsilon} \gg a/h^{2/3}$, we introduce a cut-off in ξ localizing for $|\xi| \in [\frac{1}{2}\sqrt{\omega}, 2\sqrt{\omega}]$ without changing the contribution of the integral corresponding to $N = 0$ modulo $O(h^\infty)$ (for $|\xi| \leq \frac{1}{2}\sqrt{\omega}$ the phase (2.84) is non-stationary in ω). The second

derivative of (2.84) with respect to ξ is $2\xi \sim \pm\sqrt{\omega}$ and for ξ such that $|\xi| \sim \sqrt{\omega} \geq \sqrt{\omega_{K_\epsilon}} \gg h^{-\epsilon}$ on the support of the symbol, stationary phase yields the following phase for $N = 0$,

$$\phi_{M,\pm,\pm}(x,y,\eta,\omega) := y \cdot \eta \pm \frac{2}{3}(\omega - xq(\eta)^{1/3})^{3/2} \pm \frac{2}{3}(\omega - aq(\eta)^{1/3})^{3/2}.$$

We further notice that the phases $\phi_{M,+,+}$ and $\phi_{M,-,-}$ are non-stationary in ω: indeed, $\partial_\omega \phi_{M,+,+} \sim 2\sqrt{\omega}$ and we get $O(h^\infty)$ by integrations by parts. Moreover, taking switching signs, the derivative with respect to ω becomes

$$(2.85) \qquad \sqrt{\omega - xq(\eta)^{1/3}} - \sqrt{\omega - aq(\eta)^{1/3}} \sim \frac{(a-x)}{2\sqrt{\omega}}q(\eta)^{1/3},$$

and from (2.85), for $x - a \geq 2h^{2/3-\epsilon} - a \geq h^{2/3-\epsilon}$ and $\omega_{K_\epsilon} \lesssim \omega \leq \epsilon_0 h^{-2/3}$, integrations by parts in ω provide a $O(h^\infty)$ contribution in the integral corresponding to $N = 0$ in the second line of (2.82). In fact, with $\mathcal{E}_M$ as in (2.63), the symbol of the integral in the second line of (2.82) depends on ω only through $\chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp(\omega/\omega_k)$, and therefore, in order to integrate by parts in ω, it remains to check that, for some $\tilde\epsilon > 0$, we have

$$(2.86) \qquad \frac{2\sqrt{\omega}}{(x-a)q(\eta)^{1/3}}\partial_\omega\left(\chi^\sharp\left(\frac{\omega}{\omega_{K_\epsilon}}\right)\right) \ll h^{\tilde\epsilon}.$$

As $\chi^\sharp(\omega/\omega_{K_\epsilon})$ is constant everywhere but for $\omega \sim \omega_{K_\epsilon}$, (2.86) vanishes everywhere but for $\omega \sim \omega_{K_\epsilon}$; from $\eta \sim \frac{1}{h}$ and $\omega_{K_\epsilon} \sim K_\epsilon^{2/3} \gg h^{-4\epsilon/3}$, the lefthand side in (2.86) is at most $\frac{h^{2/3}}{h^{2/3-\epsilon}\sqrt{\omega_{K_\epsilon}}}$; we obtain (2.86) with $\tilde\epsilon = 5\epsilon/3$ and we can integrate by parts infinitely many times.

For $N = \pm 1$ we proceed in a similar manner. Let for instance $N = 1$, then the phase of $e^{-iL(\omega)}\mathcal{E}_M$ is just $(2.84) - \frac{4}{3}\omega^{3/2}$; it is stationary with respect to ω for $\xi = 2\sqrt{\omega} - \sqrt{\omega - aq^{1/3}(\eta)}$. As $\omega \geq \omega_{K_\epsilon} \gtrsim h^{-4\epsilon/3} \gg a/h^{2/3}$ we again introduce a cut-off, supported for $|\xi| \in [\frac{1}{2}\sqrt{\omega}, 2\sqrt{\omega}]$ without changing the contribution of the integral modulo $O(h^\infty)$. Stationary phase then applies in ξ and provides the phase function $\phi_{M,\pm,\pm} - \frac{4}{3}\omega^{3/2}$. We easily see that $\phi_{M,\pm,\mp}$ and $\phi_{M,-,-}$ are non-stationary with respect to ω and provide a $O(h^\infty)$ contribution. We are left with $\phi_{M,+,+} - \frac{4}{3}\omega^{3/2}$ whose derivative with respect to ω is

$$\sqrt{\omega - xq(\eta)^{1/3}} + \sqrt{\omega - aq(\eta)^{1/3}} - 2\sqrt{\omega} \sim -(x+a)q(\eta)^{1/3}/(2\sqrt{\omega}).$$

From $a \lesssim h^{2/3-\epsilon}$ we obtain that, for $x \geq 2h^{2/3-\epsilon}$, the phase $\phi_{M,+,+} - \frac{4}{3}\omega^{3/2}$ is non-stationary in ω and yields an $O(h^\infty)$ contribution. The exact same line of reasoning applies to $N = -1$. $\qquad\square$

Remark 2.33. For $|x,a| \leq \frac{h^{2/3}}{\sqrt{\omega_{K_\epsilon}}}$, we cannot get an $O(h^\infty)$ contribution for $N = \pm 1$: (2.86) does not hold anymore even though for $x \geq 0$ the derivative with respect to ω does not vanish; for such small values of a we cannot perform integrations by parts. Specifically, as $K_\epsilon \ll h^{-1/4}$, we are to deal with this case for all $|x,a| \leq h^{2/3+1/12} = h^{3/4}(\ll \frac{h^{2/3}}{\sqrt{\omega_{K_\epsilon}}})$.

Putting all this together, the integral in the last line in (2.83) reads as

$$\int \chi^\flat(h^{2/3}\omega/\varepsilon_0)\chi^\sharp\left(\frac{\omega}{\omega_{K_\epsilon}}\right)\left(1 + \sum_\pm e^{\pm iL(\omega)}\right)\frac{L'(\omega)}{2\pi}\mathcal{E}_M(x,y,a,\omega)d\omega = E_{M,+}(x,y,a) + E_{M,-}(x,y,a),$$

where we have set, modulo $O(h^\infty)$,

$$(2.87) \quad E_{M,+}(x,y,a) := \int \chi^\flat(h^{2/3}\omega/\varepsilon_0)\chi^\sharp(\omega/\omega_{K_\epsilon}) \int q(\eta)^{1/6}\varkappa(h\eta)\chi_0(h^{2/3}\omega q^{2/3}(h\eta))$$

$$\times e^{iy\cdot\eta}\Big(A_+(\zeta_M(x,\eta,\omega)) - \Big(\frac{A_+}{A_-}\Big)(\omega)A_-(\zeta_M(x,\eta,\omega))\Big)A_-(\zeta_M(a,\eta,\omega))d\eta d\omega,$$

$$(2.88) \quad E_{M,-}(x,y,a) := \int \chi^\flat(h^{2/3}\omega/\varepsilon_0)\chi^\sharp(\omega/\omega_{K_\epsilon}) \int q(\eta)^{1/6}\varkappa(h\eta)\chi_0(h^{2/3}\omega q^{2/3}(h\eta))$$

$$\times e^{iy\cdot\eta}\Big(A_-(\zeta_M(x,\eta,\omega)) - \Big(\frac{A_-}{A_+}\Big)(\omega)A_+(\zeta_M(x,\eta,\omega))\Big)A_+(\zeta_M(a,\eta,\omega))d\eta d\omega,$$

where the phase functions of the Airy terms in the second line of (2.87) are $\phi_{M,+,-}$ and $\phi_{M,-,-} + L(\omega)$, while the phase functions of the Airy terms in the second line of (2.88) are $\phi_{M,-,+}$ and $\phi_{M,+,+} - L(\omega)$. Moreover, for $x > 2h^{2/3-\epsilon}$, $E_{M,\pm}(x,y,a) = O(h^\infty)$ and $E_{M,\pm}(0,y,a) = 0$. This means that we can introduce a smooth cut-off $\chi_1(x/h^{2/3-\epsilon})$ with $\chi_1 \in C_0^\infty$ equal to 1 on $[-1,1]$ and equal to 0 for $x \geq 2h^{2/3-\epsilon}$ such that $E_{M,\pm}(x,y,a) = \chi_1(x/h^{2/3-\epsilon})E_{M,\pm}(x,y,a) + O(h^\infty)$, and therefore we need to construct $g_{h,a,2}$ such that (2.83) holds with the last line replaced by $\chi_1(x/h^{2/3-\epsilon})(E_{M,+}(x,y,a) + E_{M,-}(x,y,a))$ (instead of $E_{M,+}(x,y,a) + E_{M,-}(x,y,a)$).

We now go back to (2.83): the symbol of its left hand side has support in $\omega \gtrsim h^{-2\epsilon}$, while the right hand side is essentially supported for $x, a \lesssim h^{2/3-\epsilon}$. For such values of x and ω we have $\zeta(x,y,\eta,\omega) = \omega - x|\eta|^{2/3}e_0(x,y,\eta/|\eta|,\omega_k/|\eta|^{2/3}) \geq \omega/2$; using (2.6) we write

$$G(x,y,\eta,\omega) = e^{i\psi}\sum_\pm \Big(p_0 A_\pm(\zeta) + i|\eta|^{-1/3}p_1 A'_\pm(\zeta)\Big) =: G_\pm(x,y,\eta,\omega).$$

Proposition 2.34. *There exists smooth functions $g_{h,a,2,\pm}$ such that, with $g_{h,a,2} := \sum_\pm g_{h,a,2,\pm}$,*

$$(2.89) \quad \int \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp(h^{2\epsilon}\omega)\Big(G_+(x,y,\eta,\omega) - \Big(\frac{A_+}{A_-}\Big)(\omega)G_-(x,y,\eta,\omega)\Big)$$

$$q(\eta)^{1/6}\varkappa(h\eta)\varkappa(h\tau_q(\omega,\eta))\hat{g}_{h,a,2,-}(\eta,\omega/h^{1/3})d\eta d\omega = E_{M,+}(x,y,a) + O(h^\infty),$$

$$(2.90) \quad \int \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp(h^{2\epsilon}\omega)\Big(G_-(x,y,\eta,\omega) - \Big(\frac{A_-}{A_+}\Big)(\omega)G_+(x,y,\eta,\omega)\Big)$$

$$\times q(\eta)^{1/6}\varkappa(h\eta)\varkappa(h\tau_q(\omega,\eta))\hat{g}_{h,a,2,+}(\eta,\omega/h^{1/3})d\eta d\omega = E_{M,-}(x,y,a) + O(h^\infty).$$

$$(2.91) \quad \int \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp(h^{2\epsilon}\omega)\Big(G_+(x,y,\eta,\omega) - \Big(\frac{A_+}{A_-}\Big)(\omega)G_-(x,y,\eta,\omega)\Big)$$

$$q(\eta)^{1/6}\varkappa(h\eta)\varkappa(h\tau_q(\omega,\eta))\hat{g}_{h,a,2,+}(\eta,\omega/h^{1/3})d\eta d\omega = O(h^\infty),$$

$$(2.92) \quad \int \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp(h^{2\epsilon}\omega)\Big(G_-(x,y,\eta,\omega) - \Big(\frac{A_-}{A_+}\Big)(\omega)G_+(x,y,\eta,\omega)\Big)$$

$$\times q(\eta)^{1/6}\varkappa(h\eta)\varkappa(h\tau_q(\omega,\eta))\hat{g}_{h,a,2,-}(\eta,\omega/h^{1/3})d\eta d\omega = O(h^\infty),$$

$$(2.93) \quad \sum_{|N|\geq 2}\int e^{-iNL(\omega)}\int \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp(h^{2\epsilon}\omega)K_\omega(g_{h,a,2})(0,x,y)d\omega = O(h^\infty).$$

Proof. Proving that both operators in the first lines of (2.90) and (2.89) are invertible is sufficient: once we define $g_{h,a,2,\pm}$ it will be clear that (2.91), (2.92) and (2.93) hold (using non-stationary phase arguments). In fact, we notice from (2.89) that $g_{h,a,2,-}$ has to have a phase function whose derivative with respect to ω should equal $2\sqrt{\omega} + O(a)$, since otherwise the phase of (2.89) is non-stationary in ω. Introducing such a function $\hat{g}_{h,a,2,-}$ in the integral in the first line of (2.92) yields a phase function (for (2.92)) whose derivative with respect to ω behaves like $4\sqrt{\omega}$ and since ω is large on the support of the symbol this allows to perform repeated integrations by part with respect to ω to obtain a $O(h^\infty)$ contribution. In the same way we prove (2.93), since for $|N| \geq 2$ all the phase functions will be non-stationary in ω and after each integration by parts we obtain a factor $(N\sqrt{\omega})^{-1}$, which will allow to sum up over N to conclude. We are reduced to proving that we can define $g_{h,a,2,-}$ satisfying (2.89) (solving (2.90) follows in exactly the same way).

Proposition 2.35. *Let* $\tilde{J}_+ := J_+ + R_+$, *with*

$$J_+(f)(x,y) = \int \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp)(h^{2\epsilon}\omega)G_+(x,y,\eta,\omega)\chi_1(x/h^{2/3-\epsilon})$$

$$\times q(\eta)^{1/6}\varkappa(h\eta)\varkappa(h\tau_q(\omega,\eta))e^{i(y'\cdot\eta+\varrho\omega/h^{1/3})}f(y',\varrho)d\eta d\omega dy'd\varrho.$$

$$R_+(f)(x,y) = \int \chi^\flat(h^{2/3}\omega/\epsilon_0)\chi^\sharp)(h^{2\epsilon}\omega)\Big(\frac{A_+}{A_-}\Big)(\omega)G_-(x,y,\eta,\omega)\chi_1(x/h^{2/3-\epsilon})$$

$$\times q(\eta)^{1/6}\varkappa(h\eta)\varkappa(h\tau_q(\omega,\eta))e^{i(y'\cdot\eta+\varrho\omega/h^{1/3})}f(y',\varrho)d\eta d\omega dy'd\varrho.$$

The operator $\tilde{J}_+$ *is well defined from* $\mathcal{S}'_{y',\varrho}$ *into the space of functions of* (x,y) *near* $(0,0)$, *and with* h *as small parameter,* J_+ *is an elliptic semi-classical Fourier integral operator. Moreover,* $\|J_+^{-1} \circ R_+\|_{\mathcal{L}(L^2)} = O(h^\infty)$, *hence* $\tilde{J}_+$ *is invertible and* $\tilde{J}_+^{-1} = \Big(I + J_+^{-1} \circ R_+\Big)^{-1} \circ J_+^{-1}$.

If we now chose $g_{h,a,2,-}(y',\varrho) := \tilde{J}_+^{-1}(E_{M,+})$, this achieves the proof of Proposition 2.34. $\square$

Proof. (of Proposition 2.35) The operator J_+ is easily elliptic and invertible with phase function $\psi + \frac{2}{3}\zeta^{3/2}$ with ψ and ζ defined in Theorem 2.1. The phase function of R_+ is $\psi - \frac{2}{3}\zeta^{3/2} + \frac{4}{3}\omega^{3/2}$. Therefore the phase function of $J_+^{-1} \circ R_+$ is given by

$$(2.94) \qquad -\psi(x,y,\eta,\omega) - \frac{2}{3}\zeta^{3/2}(x,y,\eta,\omega) + \psi(x,y,\tilde{\eta},\tilde{\omega}) - \frac{2}{3}\zeta^{3/2}(x,y,\tilde{\eta},\tilde{\omega}) + \frac{4}{3}\tilde{\omega}^{3/2},$$

where x,y are now integration variables. The derivative of (2.94) with respect to x is

$$-\partial_x\zeta(x,y,\eta,\omega)\sqrt{\zeta(x,y,\eta,\omega)} - \partial_x\zeta(x,y,\tilde{\eta},\tilde{\omega})\sqrt{\zeta(x,y,\tilde{\eta},\tilde{\omega})} - \partial_x\psi(x,y,\eta,\omega) + \partial_x\psi(x,y,\tilde{\eta},\tilde{\omega}),$$

where $|\eta|,|\tilde{\eta}| \sim 1/h$, $\omega,\tilde{\omega} \geq h^{-2\epsilon}$ and $xq^{1/3}(\eta) \leq h^{2/3-\epsilon-2/3} = h^{-\epsilon}$ on the support of the symbol. As $\zeta = \omega - x|\eta|^{2/3}e_0(x,y,\eta/|\eta|,\omega/|\eta|^{2/3})$ with e_0 elliptic and close to 1, the derivatives of the two terms involving ζ in (2.94) are such that

$$\Big|\partial_x\zeta(x,y,\eta,\omega)\sqrt{\zeta(x,y,\eta,\omega)} + \partial_x\zeta(x,y,\tilde{\eta},\tilde{\omega})\sqrt{\zeta(x,y,\tilde{\eta},\tilde{\omega})}\Big| \gtrsim \frac{1}{2}(\sqrt{\omega}|\eta|^{2/3} + \sqrt{\tilde{\omega}}|\tilde{\eta}|^{2/3}).$$

On the other hand, using Corollary 6.10 and (6.31) in particular, the derivative with respect to x of ψ is of the form

$$\partial_x\psi(x,y,\eta,\omega) = \mu(y,\eta/|\eta|)q^{2/3}(\eta)/\tau_q(\omega,\eta)\Big(\omega + 2xq^{1/3}(1 + \ell(y,\eta/|\eta|)) + \mathcal{H}_{j\geq4}\Big),$$

and therefore

$$\left| \partial_x \psi(x, y, \eta, \omega) - \partial_x \psi(x, y, \tilde{\eta}, \tilde{\omega}) \right| \lesssim \omega |\eta|^{1/3} + \tilde{\omega} |\tilde{\eta}|^{1/3},$$

where we have used that $q^{2/3}(\eta)/\tau_q(\omega, \eta) = |\eta|^{4/3} q^{2/3}(\eta/|\eta|)/(|\eta|\tau_q(\omega/|\eta|^{2/3}, \eta/|\eta|)) \sim |\eta|^{1/3}$. On the support of the symbol $\chi(h^{2/3}\omega)\chi(h^{2/3}\tilde{\omega})$ we have $|\omega|, |\tilde{\omega}| \leq \epsilon_0 h^{-2/3}$, which means that the main contribution of the derivative of (2.94) comes from the terms involving ζ and behaves like $\simeq (\sqrt{\omega}|\eta|^{2/3} + \sqrt{\tilde{\omega}}|\tilde{\eta}|^{2/3})$, as for $|\eta|, |\tilde{\eta}| \simeq 1/h$ we have $\sqrt{\omega}|\eta|^{2/3} \gg \omega|\eta|^{1/3}$ ($\omega h^{2/3} \ll 1$). To perform non stationary phase and obtain an $O(h^\infty)$ contribution, we check that taking one derivative with respect to x of the symbol provides a factor $O(h^\epsilon)$. Indeed, $h^{2/3}\partial_x(\chi(x/h^{2/3-\epsilon}))/\sqrt{\omega} \sim h^\epsilon/\sqrt{\omega} \leq h^{\frac{5}{3}\epsilon}$, which completes the proof. $\square$

To complete the proof of Proposition 2.19, it remains to prove that, for $g_{h,a,2,-} := \tilde{J}_+^{-1}(E_{M,+})$, (2.67) holds: but then in (2.67) one obtains a vanishing symbol as $\chi^\flat(h^{2\epsilon}\omega)\chi^\sharp(h^{2\epsilon}\omega) = 0$. $\square$

3. DISPERSION ESTIMATES WHEN $a \geq h^{2/3-\epsilon}$

Here again one should think of ϵ as being very small: we may set $0 < \epsilon < 1/12$ to be consistent with the parametrix construction we just did in the opposite regime (subsection 2.3), to have an overlap between both regimes where we get dispersion estimates by different arguments. We now use the parametrix as a sum over N to obtain the following dispersion estimates, restricting to positive times for the sake of simplicity.

Theorem 3.1. *There exist a_0, c, C, ϵ such that for all $|(t, x, y, h, a)| < a_0$, one has*

- *for $t \leq c\sqrt{a}$,*

$$(3.1) \qquad |\mathcal{P}_{h,a}(t, x, y)| \leq Ch^{-d} \min\left(1, (h/t)^{\frac{d-1}{2}}\right);$$

- *For $a \geq h^{2/3-\epsilon}$, $t > c\sqrt{a}$,*

$$(3.2) \qquad |\mathcal{P}_{h,a}(t, x, y)| \leq Ch^{-d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\left((\max(a,x))^{\frac{1}{4}}\left(\frac{h}{t}\right)^{\frac{1}{4}} + h^{\frac{1}{3}}\right);$$

- *For $a \leq h^{1/3+\epsilon}$,*

$$(3.3) \qquad |\mathcal{P}_{h,a}(t, x, y)| \leq Ch^{-d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}+\frac{1}{3}}.$$

The first estimate, (3.1), is just the (short time) dispersion for a free wave. On this timescale, the wave has at most one reflection and singularities have not appeared yet. One should point out that (a suitable version of) such dispersion is already proved in [2], for a more general boundary.

The second estimate, (3.2), is proved using the parametrix as a sum over reflected waves. The first term is due to swallowtail singularities (and always larger than the corresponding factor in the free dispersion) and the second term is due to the presence of cusps appearing after each swallowtail singularity, between two consecutive reflections; notice that here we use the parametrix construction in an extended region $a \geq h^{2/3-\epsilon}$ (when compared to the previous parametrix construction in [9], where it was obtained as a superposition of waves only for $a \geq h^{4/7-\epsilon}$).

The third estimate, (3.3), will be proved using the parametrix as a sum over quasi-modes, and we postpone its proof to the last section, where we deal with decay of such quasi-modes.

We start with a lemma which allows to deal with the parametrix "behind the wave front".

Lemma 3.2. *There exist c_0 and T_0 such that , with $\mathcal{B} = \{0 \leq x \leq a, |y| \leq c_0 t, 0 < h \leq t\}$,*

$$\forall t \in [0, T_0], \qquad \sup_{x,y,t \in \mathcal{B}} |\mathcal{P}_{h,a}(t, x, y)| \leq C h^{-d} O((h/t)^\infty).$$

Proof. The lemma follows from classical propagation of singularities: for $t \leq a/C_1$ with a large C_1, we apply Melrose-Sjöstrand's theorem in the interior of Ω. Now, let T_0 be small enough and consider a given $s \in [h, T_0]$. Rescale with (t', x', y') as new variables and $\hbar$ a new parameter: $t = st'$, $x = sx'$, $y = sy'$, $\hbar = h/s$, let $v_s(t', x', y') = v(st', sx', sy')$ for any function v, then

$$(\Box v)_s = \frac{1}{s^2} \Box_s v_s \text{ with } \Box_s = -\partial_{t'}^2 + \partial_{x'}^2 + s^2 R(sx', sy', s^{-1} D_{y'}).$$

Set $b = a/s$, $0 < b \leq c_1$ and $\mathcal{P}_{b,s,\hbar} = s^d (\mathcal{P}_{a,h})_s$; then one may apply the Melrose-Sjöstrand theorem to $\mathcal{P}_{b,s,\hbar}$ to obtain $\mathcal{P}_{b,s,\hbar} \in O(\hbar^\infty)$ for $0 \leq x' \leq b$, $|y'| \leq c_0$ and $t' = 1$, when $s \leq T_0$ and T_0 is small enough. In fact, $\mathcal{P}_{b,s,\hbar}$ is a parametrix for $\Box_s$, and $\Box_s$ is smooth in s and for s small enough, $\Box_s$ is close to the usual wave operator $-\partial_{t'}^2 + \partial_{x'}^2 + \Delta_{y'}$ (recall that we picked boundary normal coordinates, and $R_0(y, \eta) = |\eta|^2 + O(|y|)$). If we denote by $\mathcal{G}_{b,s}$ the Green function of $\Box_s$,

$$\Box_s \mathcal{G}_{b,s} = 0 \text{ in } \Omega, \quad \mathcal{G}_{b,s} = 0 \text{ on } \partial\Omega,$$

with $\mathcal{G}_{b,s}|_{t'=0} = \delta_{(b,0)}$ and $\partial_{t'} \mathcal{G}_{b,s}|_{t'=0} = 0$, then we have $\mathcal{P}_{b,s,\hbar} = \varkappa(\hbar D_{t'}) \mathcal{Q}(sx', sy', \hbar D_{y'}) \mathcal{G}_{b,s}$ and its wave front set is described by the Melrose-Sjöstrand theorem. To complete our proof, we need to check the following property of an optical ray $\sigma \in [0,1] \to \tau_s(\sigma)$: if it starts at $\sigma = 0$ from $(b, 0, \xi_0, \eta_0)$ with $\xi_0^2 + R(sb, 0; \eta_0) = 1$ and if $|\eta_0| \geq c_1 > 0$ for some constant c_1, then, if $\tau_s(\sigma) = (x'(\sigma), y'(\sigma), \xi'(\sigma), \eta'(\sigma))$, there exists c_0 such that $|y'(1)| \geq c_0 > 0$. Note that $(y', \eta')(\sigma)$ are solutions to the Hamilton-Jacobi equations $\partial_\sigma y' = \partial_{\eta'} R_s$, and $\partial_\sigma \eta' = -\partial_{y'} R_s$, with $y'(0) = b$, $\eta'(0) = \eta_0$, $R_s(x', y', \eta') = s^2 R(sx', xy', \eta'/s)$. For some $c > 0$ small, $|y'(\sigma)| \geq 2c_1 \sigma - cs\sigma$ for all $\sigma \in [0, 1]$, and if T_0 is small enough, we get the lower bound $|y'(1)| \geq c_0 > 0$ for some $c_0 > 0$. $\square$

3.1. Number of waves that contribute in $\mathcal{P}_{h,a}$.

We further localize our parametrix $\mathcal{P}_{h,a}$: let $\phi \in C_0^\infty(\mathbb{R})$ be even, $\phi = 1$ on $[-1, 1]$ and $\phi = 0$ outside $(-3/2, 3/2)$, set $\chi_1 = \phi - \phi(2\cdot)$, then define (see (2.24)), for any dyadic γ (i.e. $1/\gamma \in 2^{\mathbb{N}}$) such that $\gamma \leq \epsilon_0$,
(3.4)
$$K_{\omega,\gamma}(f)(t, x, y) := \int e^{it\tau_q(\omega,\eta)} G(x, y, \eta, \omega) \chi^\sharp(\omega) q^{1/6}(\eta) \varkappa(h\eta) \varkappa(h\tau_q(\omega,\eta)) \chi_1\left(\frac{\omega}{\gamma |\eta|^{2/3}}\right) \hat{f}\left(\eta, \frac{\omega}{h^{1/3}}\right) d\eta$$

as well as $\mathcal{P}_{h,a,\gamma}$ by replacing K_ω with $K_{\omega,\gamma}$ in (2.39). We just reduced the sum over k to k's such that $k \sim \gamma^{3/2}/h$, where $h^{2/3-\epsilon} \lesssim \gamma \leq \epsilon_0$. Then, $\mathcal{P}_{h,a}(t, x, y) = \sum_\gamma \mathcal{P}_{h,a,\gamma}(t, x, y)$, where the sum is intended to be dyadic γ's with $\gamma < \epsilon_0$. For $\gamma \ll a$, the corresponding $\mathcal{P}_{h,a,\gamma}$ is irrelevant, as the phase of $g_{h,a}$ is non-stationary. Using (2.40) or (2.23),

(3.5)
$$\mathcal{P}_{h,a,\gamma}(t, x, y) = \langle \sum_{N \in \mathbb{Z}} e^{-iNL(\omega)}, \chi^\flat(h^{2/3}\omega/\epsilon_0) K_{\omega,\gamma}(g_{h,a})(t, x, y) \rangle_\omega.$$

We will deal successively with $\gamma \sim a$ and $4a \leq \gamma < 1$. Heuristically, the first case contains "tangent" initial directions (worst scenario); the second case contains all "almost transverse" directions and

provide less important contributions (even after summing up in γ). We write

$$(3.6) \quad V_{N,\gamma}(t,x,y) = \int e^{-iNL(\omega)} \chi^\flat(h^{2/3}\omega/\epsilon_0) K_{\omega,\gamma}(g_{h,a})(t,x,y)d\omega$$

$$= \frac{1}{2\pi h^{d+1}} \int e^{\frac{i}{h}(t\tau_q(\alpha,\theta)+\Phi(x,y,\theta,\alpha,\sigma)-\Phi(a,0,\theta,\alpha,s)-NhL(h^{-2/3}\alpha))}\chi_1(\alpha/(|\theta|^{2/3}\gamma))$$

$$\times \chi^\flat(\alpha/\epsilon_0)\chi^\sharp(\alpha/h^{2/3})\chi(s)p_h(x,y,\theta,\alpha,\sigma)\tilde{q}_h(\theta,\alpha,s)ds\,d\theta d\sigma d\alpha\,,$$

$$(3.7) \quad \mathcal{P}_{h,a,\gamma}(t,x,y) = \sum_{N\in\mathbb{Z}} V_{N,\gamma}(t,x,y)\,,$$

where the symbol of $V_{N,\gamma}$ (the same for every N) is of order 0.

Lemma 3.3. *At fixed $|t| \lesssim T_0$, the significant contributions in the sum (3.7) defining $\mathcal{P}_{h,a,\gamma}$ come from $|N| \lesssim |t|/\sqrt{\gamma}$:*

$$\sum_{|N|\geq 4|t|/\sqrt{\gamma}} V_{N,\gamma}(t,x,y) = O(h^\infty)\,.$$

The proof of the lemma reproduces that of Proposition 2.14; the maximum number of integrals that provide non-trivial contributions in the sum over N is $1/\sqrt{a}$ when $\gamma \sim a$. Observe that Proposition 2.14 tells us that for $t = 0$, only the $N = 0$ term may contribute.

At fixed $t \geq C\gamma^{1/2}$, we can further bound the cardinal of those N that contribute significantly among the $C|t|\gamma^{-1/2}$ which are left. We introduce a few notations before stating a sharp bound of the number of waves that can overlap when $t/\sqrt{\gamma}$ is large : let $\mathcal{N}(x,y,t)$ be the set of N with significant contributions (e.g., we have a stationary point for the phase in all variables),

$$(3.8) \qquad \mathcal{N}(t,x,y) = \{N \in \mathbb{Z}, (\exists)(\sigma,s,\alpha,\theta) \text{ such that } \nabla_{(\sigma,s,\alpha,\theta)}\Phi_{N,a,\gamma} = 0\},$$

where $\Phi_{N,a,\gamma}$, the phase function of $V_{N,\gamma}$ for the large parameter $1/h$, is defined as follows

$$(3.9) \qquad \Phi_{N,a,\gamma} := t\tau_q(\alpha,\theta) + \Phi(x,y,\theta,\alpha,\sigma) - \Phi(a,0,\theta,\alpha,s) - \frac{4}{3}N\alpha^{3/2} + NhB_L(\alpha^{3/2}/h).$$

Let (t,x,y) such that $\mathcal{N}(t,x,y) \neq \emptyset$ and assume without loss of generality that $t > 0$ and $t/\sqrt{\gamma}$ is large. As we shall see below, at a critical point (σ,s,α,θ) of $\Phi_{N,a,\gamma}$,

$$\frac{y + \nabla B_0(y,\vartheta)}{|y + \nabla B_0(y,\vartheta)|} = -\vartheta + O(\alpha), \qquad \frac{|y + \nabla B_0(y,\vartheta)|}{t} = 1 + O(\alpha),$$

and on the support of the symbol χ_1, $\alpha \sim \gamma \ll 1$. Therefore, as $B_0(y,\omega) = O(|y|^2)$ and $\nabla B_0(y,\omega) = O(|y|^2)$, if $\mathcal{N}(t,x,y) \neq \emptyset$ then we must have $\frac{|y|}{t} = 1 + O(|y|)$.

Consider first $d > 2$. Let c satisfy $4\gamma < c < \frac{1}{16}$ and such that $1-c \leq \frac{|y|}{t} \leq 1+c$ (notice that such c does exist since otherwise $\mathcal{N}(t,x,y) = \emptyset$). As $B_2(y,\vartheta) = O(|y|^2)$, $|y| \lesssim T_0$ with T_0 sufficiently small, there exists $\tilde{\vartheta} = \tilde{\vartheta}(t,y) \in \mathbb{R}^{d-1} \setminus \{0\}$ such that, with $\not\nabla_\theta = \nabla_\theta - \frac{\theta}{|\theta|}(\frac{\theta}{|\theta|} \cdot \nabla)$,

$$(3.10) \qquad \tilde{\vartheta} = -\frac{y + \nabla B_0(y,\tilde{\vartheta})}{|y + \nabla B_0(y,\tilde{\vartheta})|} - 2\Big(\frac{|y + \nabla B_0(y,\tilde{\vartheta})|}{t} - 1\Big)\Big[\frac{\not\nabla q(\tilde{\vartheta})}{q(\tilde{\vartheta})} - \frac{3\not\nabla B_2(y,\tilde{\vartheta})}{2t(1 - B_2(y,\tilde{\vartheta})/t)}\Big],$$

as a fixed point of a continuous map from a ball to a ball. In fact, as $B_0(y,\theta)$ (resp. $B_2(y,\theta)$) is homogeneous of degree 1 (resp. 0) in θ, so is ∇B_0 (of degree 0), hence the right hand side in (3.10) does not depend on $|\tilde{\vartheta}|$. Uniqueness follows by taking differences and using smallness of B_0. Moreover, for sufficiently small c and T_0 we have $|\tilde{\vartheta}| \in [\frac{1}{2}, \frac{3}{2}]$. As we shall see below (in (3.39) from

Lemma [3.10]), $\tilde{\vartheta}(t,,y)$ is an approximation (modulo $O(\gamma^{3/2}/t)$ terms) of the critical point ϑ_θ of the phase $\Phi_{N,a,\gamma}$. Let (t',x',y') be such that $\mathcal{N}(t',x',y') \neq \emptyset$, then $1-c \leq \frac{|y'|}{t'} \leq 1+c$ and there exists $\tilde{\vartheta}(t',y')$ solution to [(3.10)] with (t,y) replaced by (t',y'). We now define a cylinder $\mathcal{C}_\gamma(t,x,y)$ as the set of (t',x',y') such that

$$(3.11) \qquad |(t' - B_2(y',\tilde{\vartheta}(t',y'))) - (t - B_2(y,\tilde{\vartheta}(t,y)))| \leq r_0\sqrt{\gamma},$$

$$(3.12) \qquad |x'(1 + \ell(y,\tilde{\vartheta}(t',y'))) - x(1 + \ell(y,\tilde{\vartheta}(t,y)))| \leq r_0\gamma,$$

$$(3.13) \qquad ||y' + \nabla B_0(y',\tilde{\vartheta}(t',y'))| - t' - |y + \nabla B_0(y,\tilde{\vartheta}(t,y))| + t| \leq r_0\gamma^{3/2},$$

and let, with $\mathcal{N}$ defined in [(3.8)],

$$(3.14) \qquad \mathcal{N}^1_{d>2}(t,x,y) = \cup_{\mathcal{C}_\gamma(t,x,y)}\mathcal{N}(t',x',y').$$

For $d = 2$, replace $\nabla B_0(y,\tilde{\vartheta}(t,y))$ by $B_0(y)$, $B_2(y,\tilde{\vartheta}(t,y))$ by $B_2(y)$ and $\ell(y,\tilde{\vartheta}(t,y))$ by $\ell(y)$ to define $\mathcal{N}^1_{d=2}(t,x,y)$.

Proposition 3.4. *For any $d \geq 2$, the following optimal upper bound holds for $\mathcal{N}^1_d$ defined in [(3.14)]*

$$(3.15) \qquad \left|\mathcal{N}^1_d(t,x,y)\right| \lesssim O(1) + \gamma^{-1/2}|t|(\gamma^{3/2}/h)^{-2}.$$

Remark 3.5. A particular case of the two dimensional version of Proposition [3.4] has been proved in [9, Lemma 2.17, Lemma 2.18] in the case $a \gg h^{4/7}$ (and without the γ cut-off), where we proved $|\mathcal{N}^1_{d=2}(t,x,y)|$ was bounded by a constant and that $\mathcal{N}(t,x,y) \subset [1, t/(2\sqrt{a}) + N_0]$, with N_0 being an absolute constant. For $a \sim \gamma \gg h^{4/7}$ and $|t| \lesssim 1$, one easily sees that $|t|/(\gamma^{1/2}(\gamma^3/h^2)) = O(1)$. Indeed, $\gamma^{7/2}/h^2 \gg 1$, hence the right hand side term in [(3.15)] is $O(1)$. Therefore, if $\gamma \gg h^{4/7}$, we only get non-trivial contributions from an uniformly bounded number of waves at a fixed t. The case $\gamma \gg h^{2/3}$ was recently dealt with in [12] where Propositions [3.4] and [3.13] were proved in the 2D Friedlander model domain. Compared to [12], there are significant additional difficulties with angles in the higher dimensional case (even in the model situation !)

Proof. We first provide a proof in all dimensions $d \geq 2$ for the model case, with Δ_M: a parametrix of the wave equation reads as [(3.7)], where $V_{N,\gamma}$ has symbol $\chi_1(\alpha/(\gamma|\theta|^{2/3}))\chi^\flat(\alpha/\varepsilon_0)\chi^\sharp(\alpha/h^{2/3})\chi(s)$ and phase function $\Phi^M_{N,a,\gamma}$ given by

$$\Phi^M_{N,a,\gamma}(t,x,y,\sigma,s,\alpha,\theta) := t\tau_q(\alpha,\theta) + \Phi_M(x,y,\theta,\alpha,\sigma) - \Phi_M(a,0,\theta,\alpha,s) - \frac{4}{3}N\alpha^{3/2} + NhB_L(\alpha^{3/2}/h),$$

where $\Phi_M(x,y,\theta,\alpha,\sigma) = y\cdot\theta + \frac{\sigma^3}{3} + \sigma(xq^{1/3}(\theta) - \alpha)$; note that the only difference between $\Phi^M_{N,a,\gamma}$ and $\Phi_{N,a,\gamma}$ comes from the additional terms $\tau_q(\alpha,\theta)\Gamma(x,y,\frac{\sigma q^{1/3}(\theta)}{\tau_q(\alpha,\theta)},\frac{\theta}{\tau_q(\alpha,\theta)}) - \tau_q(\alpha,\theta)\Gamma(a,0,\frac{sq^{1/3}(\theta)}{\tau_q(\alpha,\theta)},\frac{\theta}{\tau_q(\alpha,\theta)})$ (difference between Φ_M and Φ). In the model case, without the additional phase function Γ, we rescale $x = \gamma X$, $t = \sqrt{\gamma}T$, $y = \sqrt{\gamma}Y$, and our cylinder $\mathcal{C}_\gamma(t,x,y)$ and $\mathcal{N}^1_{d\geq 2}$ simplify to $\mathcal{N}^{1,M}_d(T,X,Y) := \cup_{\mathcal{C}^M_\gamma(T,X,Y)}\mathcal{N}^M(T',X',Y')$ and

$$\mathcal{C}^M_\gamma(T,X,Y) = \{(T',X',Y') \; : \; |Y'-Y| \leq r_0, |X-X'| < r_0, ||Y'|-T'-|Y|+T| < r_0\gamma\}.$$

Note that $(X',Y',Y') \in \mathcal{C}^M_\gamma(T,X,Y)$ implies $||Y'|-|Y|| \leq r_0$, $|Y|\left|\frac{Y'}{|Y'|} - \frac{Y}{|Y|}\right| \leq 2r_0$ and $|T'-T| \leq r_0$. We also rescale $\sigma = \sqrt{\gamma}|\theta|^{1/3}\Sigma$, $\alpha = \gamma|\theta|^{2/3}A$, $s = \sqrt{\gamma}|\theta|^{1/3}S$ and we let $\lambda_\gamma = \frac{\gamma^{3/2}}{h}$. Define our new

phase to be (with large parameter $1/h$ replaced by λ_γ)

$$\Psi_{N,a,\gamma}^M(T,X,Y,\Sigma,S,A,\theta) = |\theta|\Big(\frac{Y\cdot\vartheta + T\sqrt{1+\gamma Aq^{2/3}(\vartheta)}}{\gamma} + \frac{\Sigma^3}{3} + \Sigma(Xq^{1/3}(\vartheta)-A)$$
$$- \frac{S^3}{3} - S(\frac{a}{\gamma}q^{1/3}(\vartheta)-A) - \frac{4}{3}NA^{3/2}\Big) + \frac{N}{\lambda_\gamma}B_L(|\theta|\lambda_\gamma A^{3/2})\,.$$

The phase $\Psi_{N,a,\gamma}^M$ defines a Lagrangian Λ_N^M, which is described by $\nabla_{A,S,\Sigma,\theta}\Psi_{N,a\gamma}^M = 0$:

(3.16)
$$\frac{Tq^{2/3}(\vartheta)}{2\sqrt{1+\gamma Aq^{2/3}(\vartheta)}} - (\Sigma-S) - 2NA^{1/2}(1-\frac{3}{4}B_L{}'(|\theta|\lambda_\gamma A^{3/2})) = 0,$$

(3.17)
$$\Sigma^2 + Xq^{1/3}(\vartheta) - A = 0, \quad S^2 + \frac{a}{\gamma}q^{1/3}(\vartheta) - A = 0,$$

(3.18)
$$Y + T\vartheta\sqrt{1+\gamma Aq^{2/3}(\vartheta)} + \gamma\Big(\frac{TAq^{1/3}(\vartheta)}{\sqrt{1+\gamma Aq^{2/3}(\vartheta)}} + (\Sigma X - S\frac{a}{\gamma})\Big)\frac{\nabla q(\vartheta)}{3q^{2/3}(\vartheta)}$$
$$+ \frac{2}{3}\gamma\vartheta(S^3-\Sigma^3) = \frac{4}{3}\gamma NA^{3/2}(1-\frac{3}{4}B_L{}'(|\theta|\lambda_\gamma A^{3/2}))\vartheta\,,$$

where we used the second and third equation for substitution in the last one. We recover [9, formula (2.13) to (2.15)] when $d=2$, with small adjustments due to our (more complicated) phase construction there. Assume, without loss of generality, that $T>0$, then eliminating N between (3.16) and (3.18),

(3.19)
$$\frac{Y}{T} = -\Big(1 + \frac{\gamma Aq^{2/3}(\vartheta)}{1+\sqrt{1+\gamma Aq^{2/3}(\vartheta)}} - \frac{\gamma Aq^{2/3}(\vartheta)}{3\sqrt{1+\gamma Aq^{2/3}(\vartheta)}} + \frac{2\gamma}{3T}(A(\Sigma-S)+(S^3-\Sigma^3))\Big)\vartheta$$
$$- \Big(\frac{\gamma Aq^{1/3}(\vartheta)}{\sqrt{1+\gamma Aq^{2/3}(\vartheta)}} + \frac{\gamma}{T}\Big((\Sigma X - S\frac{a}{\gamma})\Big)\frac{\nabla q(\vartheta)}{3q^{2/3}(\vartheta)}$$

Using that $\vartheta\cdot\nabla q(\vartheta)=0$, we compute $|Y|^2/T^2$ and expand its square root to get

(3.20)
$$\frac{|Y|}{T} - 1 = \frac{\gamma}{6}Aq^{2/3}(\vartheta) + \frac{2\gamma}{3T}(A(\Sigma-S)+S^3-\Sigma^3) + \gamma^2\mathcal{E}\,,$$

where $\mathcal{E} = \mathcal{E}(1/T,\Sigma X - S\frac{a}{\gamma},A,S,\Sigma,\vartheta)$ is a smooth function of its arguments (that may be computed explicitly, although irrelevant). We then compute

(3.21)
$$\frac{Y}{|Y|} = -\vartheta - \frac{\gamma A}{3}\Big(\frac{\nabla q(\vartheta)}{q^{1/3}(\vartheta)}\Big) + \frac{\gamma}{T}(\Sigma X - S\frac{a}{\gamma})\frac{\nabla q(\vartheta)}{3q^{2/3}(\vartheta)} + \gamma^2\vec{\mathcal{E}}\,,$$

where $\vec{\mathcal{E}} = \vec{\mathcal{E}}(1/T,\Sigma X - S\frac{a}{\gamma},A,S,\Sigma,\vartheta)$ is an explicit smooth, vector-valued function. Later we will use $O(\gamma^2)\lesssim O(\gamma/T)$, as $T\sqrt{\gamma}=O(1)$. We now estimate the distance between any two elements of $\mathcal{N}_d^{1,M}(T,X,Y)$. Pick (T,X,Y) with $T>0$, and let $N_j\in\mathcal{N}_d^{1,M}(T,X,Y)$, $j\in\{1,2\}$. There exist $(T_j,X_j,Y_j)\in\mathcal{C}_\gamma^M$ and there exist $(\theta_j=|\theta_j|\vartheta_j,A_j,\Sigma_j,S_j)$, $j\in\{1,2\}$, A_j close to 1, such that (3.16), (3.17), (3.19) hold. Taking the difference between (3.16) for $j=1,2$ and using (3.17)

$(\frac{(\Sigma_j - S_j)}{A_j^{1/2}} = O(1))$ and $\gamma T_j = O(\sqrt{\gamma} t_j) = O(\sqrt{\gamma})$,

$$(3.22) \quad N_1 - N_2 = \frac{3}{4}\left(N_1 B_L'(|\theta_1|\lambda_\gamma A_1^{3/2}) - N_2 B_L'(|\theta_2|\lambda_\gamma A_2^{3/2})\right) - \frac{\Sigma_1 - S_1}{2\sqrt{A_1}} + \frac{\Sigma_2 - S_2}{2\sqrt{A_2}}$$

$$+ \frac{T_1 q^{2/3}(\vartheta_1)}{4A_1^{1/2}\sqrt{1 + \gamma A_1 q^{2/3}(\vartheta_1)}} - \frac{T_2 q^{2/3}(\vartheta_2)}{4A_2^{1/2}\sqrt{1 + \gamma A_2 q^{2/3}(\vartheta_2)}}$$

$$= O\left(\frac{N_1 + N_2}{\lambda_\gamma^2}\right) + O(1) + \frac{T_1 q^{2/3}(\vartheta_1)}{4A_1^{1/2}} - \frac{T_2 q^{2/3}(\vartheta_2)}{4A_2^{1/2}}.$$

Remark 3.6. The first term in the first line of (3.22) behaves like $(|N_1| + |N_2|)/\lambda_\gamma^2$, using that $B'(\theta\lambda A^{3/2}) \sim -\frac{b_1}{\theta^2\lambda^2 A^3}$ and $\theta, A \sim 1$. We cannot take advantage of the difference, since each term $N_j B'(\cdot)$ corresponds to some θ_j, A_j (close to 1) and, although the difference between A_j turns out to be $O(1/T)$, we do not have any better information about the difference between $|\theta_j|$ than bounded by a small constant on the support of χ_1. Therefore the bound $(N_1 + N_2)/\lambda_\gamma^2$ for the terms involving B_L' in (3.22) is sharp. As $N_j \sim T_j$, and $|T_j - T| \leq 1$, this contribution is of order $|T|/\lambda_\gamma^2$.

We are reduced to proving that the following difference (from (3.22)) is $O(1)$. Write

$$(3.23) \quad \frac{T_1}{A_1^{1/2}} q^{2/3}(\vartheta_1) - \frac{T_2}{A_2^{1/2}} q^{2/3}(\vartheta_2) = \frac{(T_1 - T_2)}{A_1^{1/2}} q^{2/3}(\vartheta_1) + \frac{T_2}{\sqrt{A_1 q^{2/3}(\vartheta_1)}}(q(\vartheta_1) - q(\vartheta_2))$$

$$+ T_2 q(\vartheta_2)\left(\frac{1}{\sqrt{A_1 q^{2/3}(\vartheta_1)}} - \frac{1}{\sqrt{A_2 q^{2/3}(\vartheta_2)}}\right).$$

As $|T_1 - T_2| \leq |T_1 - T| + |T_2 - T| \leq 2r_0$, it remains to prove

Lemma 3.7. *Let* (T, X, Y) *be fixed, let* $(T_j, X_j, Y_j) \in \mathcal{C}_\gamma^M$ *and let* $(\theta_j, A_j, \Sigma_j, S_j)$, $j \in \{1, 2\}$ *with* A_j *close to* 1 *such that* (3.16), (3.17) *and* (3.19) *hold, then*

$$(3.24) \quad T|\vartheta_1 - \vartheta_2| \lesssim 1, \quad T|A_1 q^{2/3}(\vartheta_1) - A_2 q^{2/3}(\vartheta_2)| \lesssim 1.$$

Proof. When T is not too large, (3.24) immediately follows. We consider T sufficiently large. For $T_j, Y_j, A_j, \vartheta_j$, (3.20) and (3.21) hold. Taking difference between (3.21) for $j = 1$ and $j = 2$,

$$-\frac{Y_1}{|Y_1|} + \frac{Y_2}{|Y_2|} = (\vartheta_1 - \vartheta_2)\left(1 - \frac{2}{3}\gamma A_1 q^{2/3}(\vartheta_1)\right) - \frac{2}{3}\vartheta_2\gamma(A_1 q^{2/3}(\vartheta_1) - A_2 q^{2/3}(\vartheta_2))$$

$$+ \gamma(A_1 q^{2/3}(\vartheta_1) - A_2 q^{2/3}(\vartheta_2))\frac{\nabla q(\vartheta_2)}{q(\vartheta_2)} + \gamma A_1 q^{2/3}(\vartheta_1)\left(\frac{\nabla q(\vartheta_1)}{q(\vartheta_1)} - \frac{\nabla q(\vartheta_2)}{q(\vartheta_2)}\right) + O\left(\frac{\gamma}{T}\right),$$

where we used that $O(\frac{\gamma}{T_1}) = O(\frac{\gamma}{T_2}) = O(\frac{\gamma}{T})$. As A_j's stay close to 1 and $\vartheta_j \in \mathbb{S}^{d-2}$, we get

$$(\vartheta_1 - \vartheta_2)(1 + O(\gamma)) = \left(-\frac{Y_1}{|Y_1|} + \frac{Y_2}{|Y_2|}\right) + O(\gamma).$$

Using $\left|-\frac{Y_1}{|Y_1|} + \frac{Y_2}{|Y_2|}\right| = \left|-\frac{y_1}{|y_1|} + \frac{y_2}{|y_2|}\right| \leq 2\frac{r_0}{|Y|}$ and $T|\vartheta_1 - \vartheta_2| \lesssim t(2r_0 + O(\sqrt{\gamma})) \lesssim 2r_0$, we obtain the first inequality in (3.24). Taking now the difference between (3.20) for $j = 1, 2$,

$$6\left(\frac{|Y_1|}{T_1} - \frac{|Y_2|}{T_2}\right) = \gamma(A_1 q^{2/3}(\vartheta_1) - A_2 q^{2/3}(\vartheta_2)) + O\left(\frac{\gamma}{T}\right),$$

and using $|(|Y_j| - T_j) - (|Y| - T)| \leq r_0\gamma$, $|T_1 - T_2| \leq 2r_0$ and (3.20) with $j = 2$ yields

$$T|A_1 q^{2/3}(\vartheta_1) - A_2 q^{2/3}(\vartheta_2)| \leq \frac{6T}{\gamma}\Big|\frac{|Y_1|}{T_1} - \frac{|Y_2|}{T_2}\Big| + O(1)$$

$$\leq \frac{6T}{\gamma}\frac{|(|Y_1| - T_1) - (|Y_2| - T_2)|}{T_1} + \frac{6T}{\gamma}\Big(\frac{|Y_2|}{T_2} - 1\Big)\frac{(T_2 - T_1)}{T_1} + O(1)$$

$$\leq 12r_0\frac{T}{T_1} + 2r_0\frac{T}{T_1}\Big(A_2 q^{2/3}(\vartheta_2) + O(\frac{1}{T_2})\Big) + O(1) = O(1),$$

where, again, on the support of the symbol χ_1, A_j is close to 1. The proof of Lemma 3.7 is complete and, combined with (3.23) and (3.22), this yields (3.15) for the model operator Δ_M. $\qquad\square$

We now proceed with the general case, following the same steps as above. We first deal with the most complicated situation $d > 2$. We also rescale variables as follows $\sigma = \sqrt{\gamma}|\theta|^{1/3}\Sigma$, $\alpha = \gamma|\theta|^{2/3}A$, $s = \sqrt{\gamma}|\theta|^{1/3}S$ and let $\lambda_\gamma = \frac{\gamma^{3/2}}{h}$. With $\vartheta = \theta/|\theta|$ we get $q(\theta) = |\theta|^2 q(\vartheta)$, $\tau_q(\alpha, \theta) = \tau_q(\gamma A|\theta|^{2/3}, \theta) = |\theta|\sqrt{1 + \gamma A q^{2/3}(\vartheta)} =: |\theta|\tau_q(\gamma A, \vartheta)$. We retain space-time variables (t, x, y) as our phase function is no longer homogeneous in y. Recall that

$$\Phi(x, y, \theta, \alpha, \sigma) = \Phi_M(x, y, \theta, \alpha, \sigma) + \tau_q(\alpha, \theta)(B_\Gamma(y, \theta/\tau_q) + xA_\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q, \theta/\tau_q)),$$

where $\tau_q = \tau_q(\alpha, \theta) = \sqrt{|\theta|^2 + \alpha q^{2/3}(\theta)}$. Let

$$\tilde{\Phi}_{M,\gamma}(x, y, \vartheta, A, \Sigma) := |\theta|^{-1}\Phi_M(x, y, \theta, \gamma|\theta|^{2/3}A, \sqrt{\gamma}|\theta|^{1/3}\Sigma)$$

$$= y \cdot \vartheta + \gamma^{3/2}\Big(\frac{\Sigma^3}{3} + \Sigma(\frac{x}{\gamma}q^{1/3}(\vartheta) - A)\Big).$$

Set $\tilde{\Phi}_\gamma(x, y, \vartheta, A, \Sigma) := |\theta|^{-1}\Phi(x, y, \theta, \gamma|\theta|^{2/3}A, \sqrt{\gamma}|\theta|^{1/3}\Sigma)$, then, using the homogeneity in $|\theta|$,

$$(3.25) \qquad \tilde{\Phi}_\gamma(x, y, \vartheta, A, \Sigma) := \tilde{\Phi}_{M,\gamma}(x, y, \vartheta, A, \Sigma) + \tau_q(\gamma A, \vartheta)\Big[B_\Gamma(y, \vartheta/\tau_q(\gamma A, \vartheta))$$

$$+ xA_\Gamma\Big(x, y, \frac{\sqrt{\gamma}\Sigma q^{1/3}(\vartheta)}{\tau_q(\gamma A, \vartheta)}, \frac{\vartheta}{\tau_q(\gamma A, \vartheta)}\Big)\Big],$$

$$\tilde{\Phi}_\gamma(a, 0, \vartheta, A, S) := \tilde{\Phi}_{M,\gamma}(a, 0, \vartheta, A, S) + \tau_q(\gamma A, \vartheta)aA_\Gamma\Big(a, 0, \frac{\sqrt{\gamma}S q^{1/3}(\vartheta)}{\tau_q(\gamma A, \vartheta)}, \frac{\vartheta}{\tau_q(\gamma A, \vartheta)}\Big),$$

where in the last line we used that $B_\Gamma(0, \vartheta/\tau_q) = 0$ and where in the new variables we have

$$(3.26) \qquad \tau_q(\gamma A, \vartheta)B_\Gamma(y, \frac{\vartheta}{\tau_q(\gamma A, \vartheta)}) = B_0(y, \vartheta) + (1 - \tau_q(\gamma A, \vartheta))B_2(y, \vartheta)$$

$$+ \tau_q(\gamma A, \vartheta)\sum_{j\geq 2}(\frac{1}{\tau_q(\gamma A, \vartheta)} - 1)^j B_{2j}(y, \vartheta),$$

$$(3.27) \qquad \tau_q(\gamma A, \vartheta)A_\Gamma\Big(x, y, \frac{\sqrt{\gamma}\Sigma q^{1/3}(\vartheta)}{\tau_q(\gamma A, \vartheta)}, \frac{\vartheta}{\tau_q(\gamma A, \vartheta)}\Big) = (\sqrt{\gamma}q^{1/3}(\vartheta))\Sigma\ell(y, \vartheta) + \mathcal{H}_{j\geq 3}$$

$$+ \frac{\gamma q^{2/3}(\vartheta)}{\tau_q(\gamma A, \vartheta)}(\Sigma^2 - A)\mu(y, \vartheta),$$

where we used (6.29); homogeneous terms of order j in $\mathcal{H}_{j\geq 3}$ have weights $\sqrt{\gamma}^j$. We also set

$$(3.28) \qquad \tilde{\Phi}_{N,a,\gamma}(t, x, y, \Sigma, S, A, \theta) := \Phi_{N,a,\gamma}(t, x, y, \sqrt{\gamma}|\theta|^{1/3}\Sigma, \sqrt{\gamma}|\theta|^{1/3}S, \gamma|\theta|^{2/3}A, \theta),$$

and from (3.9),

$$(3.29) \quad \tilde{\Phi}_{N,a,\gamma}(t,x,y,\Sigma,S,A,\theta) = |\theta|\Big(t\sqrt{1+\gamma Aq^{2/3}(\vartheta)} + \tilde{\Phi}_\gamma(x,y,\vartheta,A,\Sigma) - \tilde{\Phi}_\gamma(a,0,\vartheta,A,S)\Big)$$
$$- \frac{4}{3}\gamma^{3/2}|\theta|NA^{3/2} + NhB_L((|\theta|\lambda_\gamma A^{3/2}).$$

The phase function $\tilde{\Phi}_{N,a,\gamma}$ defines a Lagrangian Λ_N and, as in the model case, we obtain a parametrization of $\pi_N(\Lambda_N)$ by $(d+1)$ parameters $(\rho = |\theta|, \vartheta, \Sigma, A)$ as follows

(3.30)
$$\begin{cases} \frac{t\gamma q^{2/3}(\vartheta)}{2\sqrt{1+\gamma Aq^{2/3}(\vartheta)}} + \partial_A\Big(\tilde{\Phi}_\gamma(x,y,\vartheta,A,\Sigma) - \tilde{\Phi}_\gamma(a,0,\vartheta,A,S)\Big) = 2N\gamma^{3/2}A^{1/2}(1 - \frac{3}{4}B_L'(\rho\lambda_\gamma A^{3/2})), \\ \Sigma^2 + \frac{x}{\gamma}\Big(q^{1/3}(\vartheta) + \frac{\tau_q(\gamma A,\vartheta)}{\sqrt{\gamma}}\partial_\Sigma\Big(A_\Gamma(x,y,\frac{\sqrt{\gamma}\Sigma q^{1/3}(\vartheta)}{\tau_q(\gamma A,\vartheta)},\frac{\vartheta}{\tau_q(\gamma A,\vartheta)})\Big)\Big) = A, \\ S^2 + \frac{a}{\gamma}\Big(q^{1/3}(\vartheta) + \frac{\tau_q(\gamma A,\vartheta)}{\sqrt{\gamma}}\partial_S\Big(A_\Gamma(a,0,\frac{\sqrt{\gamma}Sq^{1/3}(\vartheta)}{\tau_q(\gamma A,\vartheta)},\frac{\vartheta}{\tau_q(\gamma A,\vartheta)})\Big)\Big) = A, \\ t\sqrt{1+\gamma Aq^{2/3}(\vartheta)} + \tilde{\Phi}_\gamma(x,y,\vartheta,A,\Sigma) - \tilde{\Phi}_\gamma(a,0,\vartheta,A,S) = \frac{4}{3}\gamma^{3/2}NA^{3/2}(1 - \frac{3}{4}B_L'(\rho\lambda_\gamma A^{3/2})), \\ \partial_{\vartheta_j}\Big(t\sqrt{1+\gamma Aq^{2/3}(\vartheta)} + \tilde{\Phi}_\gamma(x,y,\vartheta,A,\Sigma) - \tilde{\Phi}_\gamma(a,0,\vartheta,A,S)\Big) = 0, \quad \forall 1 \le j \le d-2, \end{cases}$$

where $\vartheta = (\vartheta_1,...,\vartheta_{d-1}) \in \mathbb{S}^{d-2}$ and, as $t \ne 0$ implies $|y| \ne 0$, we assumed that $y_{d-1} \ne 0$; then $\vartheta_{d-1} = \pm\sqrt{1 - \sum_{j=1}^{d-2}\vartheta_j^2}$. The last line in the system (3.30) reads as follows

$$(3.31) \qquad \partial_{\vartheta_j}\tilde{\Phi}_{0,a,\gamma} - \partial_{\vartheta_{d-1}}\tilde{\Phi}_{0,a,\gamma}\frac{\vartheta_j}{\vartheta_{d-1}} = 0, \quad j \in \{1,...,d-2\}.$$

With $\tilde{\Sigma} := \frac{\sqrt{\gamma}\Sigma q^{1/3}(\vartheta)}{\tau_q(\gamma A,\vartheta)}$, $\frac{d\tilde{\Sigma}}{d\Sigma} = \frac{\sqrt{\gamma}q^{1/3}(\vartheta)}{\tau_q(\gamma A,\vartheta)}$ the second equation in the system (3.30) reads as

$$\Sigma^2 + \frac{x}{\gamma}q^{1/3}(\vartheta)\Big(1 + \partial_{\tilde{\Sigma}}\Big(A_\Gamma(x,y,\tilde{\Sigma},\frac{\vartheta}{\tau_q(\gamma A,\vartheta)})\Big)\Big) = A.$$

We further compute, with $\tau_q = \tau_q(\gamma A,\vartheta) = \sqrt{1+\gamma Aq^{2/3}(\vartheta)}$,

$$(3.32) \qquad \partial_A\tilde{\Phi}_\gamma(x,y,\vartheta,A,\Sigma) = \partial_A\tau_q\Big(\sum_{k\ge 2}\partial_{\tau_q}\Big(\frac{(1-\tau_q)^k}{\tau_q^{k-1}}\Big)B_{2k}(y,\vartheta) - B_2(y,\vartheta)\Big)$$
$$+ \gamma^{3/2}\Big(-\Sigma + \frac{\partial_A\tau_q}{\sqrt{\gamma}}\frac{x}{\gamma}\partial_{\tau_q}\Big(\tau_q A_\Gamma(x,y,\sqrt{\gamma}\Sigma q^{1/3}(\vartheta)/\tau_q,\vartheta/\tau_q)\Big)\Big),$$

where the last term in the second line is small and behaves like $xO(\sqrt{\gamma})$; indeed, it follows from (3.27) that $\partial_{\tau_q}(\tau_q A_\Gamma) = -\gamma q^{2/3}(\vartheta)\frac{(\Sigma^2-A)}{\tau_q^2}\mu(y,\vartheta) + \mathcal{H}_{j\ge 3}$, hence its main contribution is $O(\gamma)$ (here we have also used the second and the third equations in (3.30) which imply that $\Sigma^2, S^2 \le 2A$ where the phase may be stationary in Σ, S). Using that for $k \ge 2$, all terms in B_Γ come with factors $O(\gamma^2)$ and $\partial_A\tau_q = \gamma q^{2/3}(\vartheta)/(2\tau_q)$, the main contribution in the first equation in (3.30) reads

$$\frac{q^{2/3}(\vartheta)}{2\sqrt{1+\gamma Aq^{2/3}(\vartheta)}}\frac{(t-B_2(y,\vartheta))}{\sqrt{\gamma}} - \Sigma + S + O(\sqrt{\gamma}x) + O(\gamma^{3/2}|y|) = 2NA^{1/2}(1 - \frac{3}{4}B_L'(\rho\lambda_\gamma A^{3/2})).$$

We now turn to critical points with respect to ϑ and deal with the last equation in (3.30).

Lemma 3.8. *The last equation in the system* (3.30) *has two solutions, denoted* $\vartheta_{\pm}$, *such that*

$$(3.33) \quad \pm \vartheta_{\pm} = \frac{y + \nabla B_0(y, \vartheta_{\pm})}{|y + \nabla B_0(y, \vartheta_{\pm})|} - \frac{\gamma A q^{2/3}(\vartheta_{\pm})}{|y + \nabla B_0(y, \vartheta_{\pm})|}\left[\frac{\slashed{\nabla} q(\vartheta_{\pm})}{3q(\vartheta_{\pm})}\left(t - B_2(y, \vartheta_{\pm})\right) - \frac{1}{2}\slashed{\nabla} B_2(y, \vartheta_{\pm})\right]$$

$$- \frac{\gamma^{3/2}}{|y + \nabla B_0(y, \vartheta_{\pm})|}\left[\frac{\slashed{\nabla} q(\vartheta_{\pm})}{3q^{2/3}(\vartheta_{\pm})}\left(\frac{x}{\gamma}\Sigma(1 + \ell(y, \vartheta_{\pm})) - \frac{a}{\gamma}S\right) + \frac{x}{\gamma}\Sigma q^{1/3}(\vartheta_{\pm})\slashed{\nabla}(\ell(y, \vartheta_{\pm}))\right] + \frac{\gamma^2}{|y|}\mathcal{E}_{\pm},$$

with $\mathcal{E}_{\pm}(t, x, y, \frac{t}{|y+\nabla B_0|}, \frac{\sqrt{\gamma}}{t}, \vartheta)$ *smooth functions. Moreover,* $\left(\frac{y+\nabla B_0(y,\vartheta_{\pm})}{|y+\nabla B_0(y,\vartheta_{\pm})|}\right) \cdot \vartheta_{\pm} = \pm 1 + \frac{\gamma^2}{|y+\nabla B_0|}\mathcal{E}_{\pm}.$

Proof. Using (3.31) with $\vartheta_{d-1}^2 = 1 - \sum_{j=1}^{d-2}\vartheta_j^2$, $\left(\sum_{j=1}^{d-2}|\partial_{\vartheta_j}\tilde{\Phi}_{0,a,\gamma}|^2\right)\vartheta_{d-1}^2 = (1 - \vartheta_{d-1}^2)|\partial_{\vartheta_{d-1}}\tilde{\Phi}_{0,a,\gamma}|^2$. We eventually obtain

$$(3.34) \quad \vartheta_{d-1} = \pm\frac{\partial_{\vartheta_{d-1}}\tilde{\Phi}_{0,a,\gamma}}{|(\sum_{j=1}^{d-1}|\partial_{\vartheta_j}\tilde{\Phi}_{0,a,\gamma}|^2|^{1/2}}, \quad \vartheta_j = \vartheta_{d-1}\frac{\partial_{\vartheta_j}\tilde{\Phi}_{0,a,\gamma}}{\partial_{\vartheta_{d-1}}\tilde{\Phi}_{0,a,\gamma}} = \pm\frac{\partial_{\vartheta_j}\tilde{\Phi}_{0,a,\gamma}}{|\sum_{j=1}^{d-1}|\partial_{\vartheta_j}\tilde{\Phi}_{0,a,\gamma}|^2|^{1/2}}.$$

For each sign $\pm$ there exists an unique solution as the maps from $\mathbb{S}^{d-2}$ to $\mathbb{S}^{d-2}$: $\vartheta \to \pm\frac{\nabla_{\vartheta}\tilde{\Phi}_{0,a,\gamma}}{|\nabla_{\vartheta}\tilde{\Phi}_{0,a,\gamma}|}$ each have a unique fixed point for small $|y|, t$ (as $B_0 = O(|y|^2)$, $B_2 = O(|y|^2)$ and γ is small).

In the following we approximate the equation satisfied by ϑ up to $O(\gamma^2)$ terms. This will turn out to be useful later on, in the proof of Proposition 3.13. Using (3.29) with $N = 0$ together with (3.25),(3.26) and (3.27), we write

$$\tilde{\Phi}_{0,a,\gamma}(t, x, y, \Sigma, S, A, \rho\vartheta) = \rho\Big(t\tau_q + \tilde{\Phi}_{M,\gamma}(x, y, \vartheta, A, \Sigma) - \tilde{\Phi}_{M,\gamma}(a, 0, \vartheta, A, S)$$

$$+ B_0(y, \vartheta) + (1 - \tau_q)B_2(y, \vartheta) + \gamma^{3/2}\frac{x}{\gamma}q^{1/3}(\vartheta)\Sigma\ell(y, \vartheta) + \gamma^2\tilde{\Gamma}_{\gamma}(x, a, y, \vartheta, \sqrt{\gamma}\Sigma, \sqrt{\gamma}S, A)\Big),$$

where we have set, for $\tau_q = \tau_q(\gamma A, \vartheta)$,
$$(3.35)$$
$$\tilde{\Gamma}_{\gamma} := \gamma^{-2}\Big(\tau_q B_{\Gamma}(y, \vartheta/\tau_q) - B_0(y, \vartheta) - (1 - \tau_q)B_2(y, \vartheta) + x(\tau_q A_{\Gamma} - \sqrt{\gamma}xq^{1/3}(\vartheta)\ell(y, \vartheta)\Sigma) - a\tau_q A_{\Gamma}S\Big).$$

From (3.26) and (3.27), $\tilde{\Gamma}_{\gamma}$ is a smooth bounded function and

$$\tilde{\Gamma}_{\gamma} = \frac{(1 - \tau_q)^2}{\gamma^2\tau_q}\Big(B_4(y, \vartheta) + \sum_{j\geq 3}(\frac{1}{\tau_q} - 1)^{j-2}B_{2j}(y, \vartheta)\Big) + \frac{x}{\gamma}\Big(q^{2/3}(\vartheta)(\Sigma^2 - A)\mu(y, \vartheta) + \gamma^{-1}\mathcal{H}_{j\geq 3}\Big)$$

$$- \frac{a}{\gamma}\Big(q^{2/3}(\vartheta)(S^2 - A)\mu(0, \vartheta) + \gamma^{-1}\mathcal{H}_{j\geq 3}\Big),$$

where homogeneous terms of order j in $\gamma^{-1}\mathcal{H}_{j\geq 3}$ come from A_{Γ} and have factors $\gamma^{j/2-1}$, $j \geq 3$ and where $(1 - \tau_q)^2/\gamma^2 = A^2q^{4/3}(\vartheta)/(1 + \tau_q)^2$ is bounded. We compute explicitly

$$(3.36) \quad \frac{1}{\rho}\partial_{\vartheta_j}\tilde{\Phi}_{0,a,\gamma} = y_j + \partial_j B_0(y, \vartheta) + \frac{\gamma A}{2\tau_q}q^{2/3}(\vartheta)\left[\frac{2\partial_j q(\vartheta)}{3q(\vartheta)}(t - B_2(y, \vartheta)) - \frac{2}{1 + \tau_q}\partial_j B_2(y, \vartheta)\right]$$

$$+ \gamma^{3/2}\left[\frac{\partial_j q(\vartheta)}{3q^{2/3}(\vartheta)}\left(\frac{x}{\gamma}\Sigma(1 + \ell(y, \vartheta)) - \frac{a}{\gamma}S\right) + \frac{x}{\gamma}\Sigma q^{1/3}(\vartheta)\partial_j\ell(y, \vartheta)\right] + \gamma^2\partial_{\vartheta_j}\tilde{\Gamma}_{\gamma}.$$

With $\nabla F = (\partial_{\vartheta_1} F, ..., \partial_{\vartheta_{d-1}} F)$, let $\Omega_1 = \frac{A}{2\tau_q} q^{2/3}(\vartheta) \left(\frac{2\nabla q(\vartheta)}{3q(\vartheta)} (1 - \frac{B_2(y,\vartheta)}{t}) - \frac{2}{1+\tau_q} \frac{\nabla B_2(y,\vartheta)}{t} \right)$, $\Omega_2 = \frac{\nabla q(\vartheta)}{3q^{2/3}(\vartheta)} \left(\frac{x}{\gamma} \Sigma(1 + \ell(y,\vartheta)) - \frac{a}{\gamma} S \right) + \frac{x}{\gamma} \Sigma q^{1/3}(\vartheta) \nabla \ell(y,\vartheta)$, then, using (3.36) we compute,

$$(3.37) \quad \sum_{j=1}^{d-1} |\partial_{\vartheta_j} \tilde{\Phi}_{0,a,\gamma}|^2 = \left| y + \nabla B_0 + \gamma t \Omega_1 + \gamma^{3/2} \Omega_2 + \gamma^2 \nabla \tilde{\Gamma}_\gamma \right|^2 = |y + \nabla B_0|^2 \Big[1 + \frac{2\gamma t}{|y + \nabla B_0|} \times$$

$$\Omega_1 \cdot \left(\frac{y + \nabla B_0}{|y + \nabla B_0|} \right) + \frac{2\gamma^{3/2}}{|y + \nabla B_0|} \Omega_2 \cdot \left(\frac{y + \nabla B_0}{|y + \nabla B_0|} \right) + \frac{\gamma^2}{|y + \nabla B_0|} \left(\mathcal{E}_1 + \frac{t^2 \mathcal{E}_2 + t\sqrt{\gamma} \mathcal{E}_3 + \gamma \mathcal{E}_4}{|y + \nabla B_0|} \right) \Big],$$

where we have set $\mathcal{E}_1 := 2 \left(\frac{y + \nabla B_0}{|y + \nabla B_0|} \right) \cdot \nabla \tilde{\Gamma}_\gamma$, $\mathcal{E}_2 := |\Omega_1|^2$, $\mathcal{E}_3 = 2\Omega_1 \cdot (\Omega_2 + \sqrt{\gamma} \nabla \tilde{\Gamma}_\gamma)$, $\mathcal{E}_4 = |\Omega_2|^2 + \sqrt{\gamma} \Omega_2 \cdot (\nabla \tilde{\Gamma}_\gamma) + \gamma |\nabla \tilde{\Gamma}_\gamma|^2$. As $\frac{\gamma}{|y + \nabla B_0|} = \sqrt{\gamma} \frac{\sqrt{\gamma}}{t} \frac{t}{|y + \nabla B_0|}$ is small when $t/\sqrt{\gamma}$ is sufficiently large, the coefficient of $\mathcal{E}_3$ in (3.37) is bounded by $O(\sqrt{\gamma})$. All the three terms $\mathcal{E}_j$ are smooth, bounded functions and we relabel their sum as follows

$$\mathcal{E}_1 + \frac{t^2 \mathcal{E}_2 + t\sqrt{\gamma} \mathcal{E}_3}{|y + \nabla B_0|} + \frac{\gamma}{|y + \nabla B_0|} \mathcal{E}_4 = \mathcal{E}(t, x, y, \frac{t}{|y + \nabla B_0|}, \frac{\sqrt{\gamma}}{t}, \vartheta).$$

We eventually find

$$\left(\sum_{j=1}^{d-1} |\partial_{\vartheta_j} \tilde{\Phi}_{0,a,\gamma}|^2 \right)^{1/2} = |y + \nabla B_0| \Big(1 + \frac{\gamma t}{|y + \nabla B_0|} < \Omega_1, \frac{y + \nabla B_0}{|y + \nabla B_0|} >$$

$$+ \frac{\gamma^{3/2}}{|y + \nabla B_0|} < \Omega_2, \frac{y + \nabla B_0}{|y + \nabla B_0|} > + \frac{\gamma^2}{|y + \nabla B_0|} \tilde{\mathcal{E}} \Big),$$

where $\tilde{\mathcal{E}}$ is a smooth function of $(t, x, y, \frac{t}{|y + \nabla B_0|}, \frac{\sqrt{\gamma}}{t}, \vartheta)$ obtained by taking the square root in (3.37) and using the asymptotic expansion of $\sqrt{1 + \kappa} = 1 + \kappa/2 + O(\kappa^2)$. From (3.34) we have

$$\pm \vartheta = \frac{y + \nabla B_0 + \gamma t \Omega_1 + \gamma^{3/2} \Omega_2 + \gamma^2 \nabla \tilde{\Gamma}_\gamma}{|y + \nabla B_0| \left(1 + \frac{\gamma t}{|y + \nabla B_0|} \Omega_1 \cdot \left(\frac{y + \nabla B_0}{|y + \nabla B_0|} \right) + \frac{\gamma^{3/2}}{|y + \nabla B_0|} \Omega_2 \cdot \left(\frac{y + \nabla B_0}{|y + \nabla B_0|} \right) + \frac{\gamma^2}{|y + \nabla B_0|} \tilde{\mathcal{R}} \right)}.$$

Setting $\tilde{\Omega}_j = \frac{\Omega_j}{|y + \nabla B_0|}$ and using $(1 + \kappa)^{-1} = 1 - \kappa + O(\kappa^2)$, we obtain, for another smooth and bounded function $\breve{\mathcal{E}}(t, x, y, \frac{t}{|y + \nabla B_0|}, \frac{\sqrt{\gamma}}{t}, \vartheta)$,

$$\pm \vartheta = \Big\{ \frac{y + \nabla B_0}{|y + \nabla B_0|} + \gamma t \tilde{\Omega}_1 + \gamma^{3/2} \tilde{\Omega}_2 + \gamma^2 \nabla \tilde{\Gamma}_\gamma / |y + \nabla B_0| \Big\}$$

$$\times \Big\{ 1 - \gamma t \tilde{\Omega}_1 \cdot \left(\frac{y + \nabla B_0}{|y + \nabla B_0|} \right) - \gamma^{3/2} \tilde{\Omega}_2 \cdot \left(\frac{y + \nabla B_0}{|y + \nabla B_0|} \right) + \frac{\gamma^2}{|y + \nabla B_0|} \breve{R} \Big\}$$

$$= \frac{y + \nabla B_0}{|y + \nabla B_0|} + \frac{\gamma t}{|y + \nabla B_0|} \Big(\Omega_1 - \frac{y + \nabla B_0}{|y + \nabla B_0|} \Omega_1 \cdot \left(\frac{y + \nabla B_0}{|y + \nabla B_0|} \right) \Big)$$

$$+ \frac{\gamma^{3/2}}{|y + \nabla B_0|} \Big(\Omega_2 - \frac{y + \nabla B_0}{|y + \nabla B_0|} \Omega_2 \cdot \left(\frac{y + \nabla B_0}{|y + \nabla B_0|} \right) \Big) + \frac{\gamma^2}{|y + \nabla B_0|} \mathcal{E},$$

and replacing $\frac{y + \nabla B_0}{|y + \nabla B_0|}$ in the last two lines by $\pm \vartheta - \gamma t \tilde{\Omega}_1 - \gamma^{3/2} \tilde{\Omega}_2 - \gamma^2 \nabla \tilde{\Gamma}_\gamma / |y + \nabla B_0|$ yields (3.33) where $\mathcal{E}_\pm$ are smooth, bounded functions of $(t, x, y, \frac{t}{|y + \nabla B_0|}, \frac{\sqrt{\gamma}}{t}, \vartheta)$. $\qquad \square$

Remark 3.9. For $t > 0$, considering the critical point satisfying (3.33) with $+$ sign in the left hand side term produces $O(h^\infty)$ contributions, as the critical value of the phase is not stationary in ρ. Therefore in the following for $t > 0$ we pick $\vartheta_c(t, x, y, \Sigma, S, A) := \vartheta_-(t, x, y, \Sigma, S, A)$ such that (3.33) holds (and for $t < 0$ one should take $\vartheta_c := \vartheta_+$).

In Lemma 3.8 we only considered critical points of $\tilde{\Phi}_{N,a,\gamma}$ with respect to ϑ. In the next Lemma we deal with critical points with respect to A, ρ and ϑ of $\tilde{\Phi}_{N,a,\gamma}$, where $\rho = |\theta|$, $\theta = \rho\vartheta$.

Lemma 3.10. *Let $t > 0$. If $\nabla_{A,\rho,\vartheta}\tilde{\Phi}_{N,a,\gamma} = 0$ then there exists a smooth, bounded function $\tilde{\mathcal{E}}$ depending on $(t, x, y, \frac{t}{|y+\nabla B_0|}, \frac{\sqrt{\gamma}}{t}, \vartheta)$ such that for the critical point with respect to ϑ, that we denote $\vartheta_c^\sharp$ (and for which (3.33) holds with ϑ_-)*

$$(3.38) \quad \frac{|y + \nabla B_0(y, \vartheta_c^\sharp)|}{t} = 1 + \frac{1}{6}\gamma A q^{2/3}(\vartheta_c^\sharp)\Big(1 - \frac{B_2(y, \vartheta_c^\sharp)}{t}\Big) + \frac{\gamma^{3/2}}{t}\Big(\frac{2}{3}A(\Sigma - S)$$
$$+ \frac{\Sigma^3}{3} + \Sigma(\frac{x}{\gamma}q^{1/3}(\vartheta_c^\sharp)(1 + \ell(y, \vartheta_c^\sharp)) - A) - \frac{S^3}{3} - S(\frac{a}{\gamma}q^{1/3}(\vartheta_c^\sharp) - A)\Big) + \frac{\gamma^2}{t}\tilde{\mathcal{E}}.$$

Moreover, the critical point $\vartheta_c^\sharp$ is such that

$$(3.39) \quad -\vartheta_c^\sharp = \frac{y + \nabla B_0(y, \vartheta_c^\sharp)}{|y + \nabla B_0(y, \vartheta_c^\sharp)|} - 2\Big(\frac{|y + \nabla B_0(y, \vartheta_c^\sharp)|}{t} - 1\Big)\Big[\frac{\nabla q(\vartheta_c^\sharp)}{q(\vartheta_c^\sharp)} - \frac{3\nabla B_2(y, \vartheta_c^\sharp)}{2t(1 - B_2(y, \vartheta_c^\sharp)/t)}\Big] + O(\frac{\gamma^{\frac{3}{2}}}{t}).$$

Proof. From Lemma 3.8, (3.33) holds (critical point w.r.t. ϑ). Let $\partial_A\tilde{\Phi}_{N,a,\gamma} = \partial_\rho\tilde{\Phi}_{N,a,\gamma} = 0$ (first and second to last equations in (3.30)): substitution between the two equations yields

$$t\sqrt{1 + \gamma A q^{2/3}(\vartheta)} + \tilde{\Phi}_\gamma(x, y, \vartheta, A, \Sigma) - \tilde{\Phi}_\gamma(a, 0, \vartheta, A, S)$$
$$= \frac{2}{3}A\Big(\frac{t\gamma q^{2/3}(\vartheta)}{2\sqrt{1 + \gamma A q^{2/3}(\vartheta)}} + \partial_A\Big(\tilde{\Phi}_\gamma(x, y, \vartheta, A, \Sigma) - \tilde{\Phi}_\gamma(a, 0, \vartheta, A, S)\Big)\Big),$$

and using (3.25) and (3.32) we further obtain, with $\tau_q = \tau_q(\gamma A, \vartheta)$, $\partial_A\tau_q = \gamma q^{2/3}(\vartheta)/(2\tau_q)$,

$$(3.40) \quad t\tau_q + y \cdot \vartheta + \gamma^{3/2}\Big(\frac{\Sigma^3}{3} + \Sigma(\frac{x}{\gamma}q^{1/3}(\vartheta) - A) - \frac{S^3}{3} - S(\frac{a}{\gamma}q^{1/3}(\vartheta) - A)\Big) + B_0(y, \vartheta)$$

$$+(1-\tau_q)B_2(y, \vartheta) + \gamma^{3/2}\frac{x}{\gamma}q^{1/3}(\vartheta)\Sigma\ell(y, \vartheta) + \gamma^2\tilde{\Gamma}_\gamma = \frac{2}{3}\frac{\gamma A q^{2/3}}{2\tau_q}\Big[t - B_2(y, \vartheta) + \sum_{k \geq 2}\partial_{\tau_q}\Big(\frac{(1 - \tau_q)^k}{\tau_q^{k-1}}\Big)B_{2k}(y, \vartheta)$$

$$+x\partial_{\tau_q}\Big(\tau_q A_\Gamma(x, y, \sqrt{\gamma}\Sigma q^{1/3}(\vartheta)/\tau_q, \vartheta/\tau_q)\Big) - a\partial_{\tau_q}\Big(\tau_q A_\Gamma(a, 0, \sqrt{\gamma}Sq^{1/3}(\vartheta)/\tau_q, \vartheta/\tau_q)\Big)\Big] - \frac{2}{3}\gamma^{3/2}(\Sigma - S)A$$

with $\tilde{\Gamma}_\gamma$ defined in (3.35). As, from (3.27), we obtain $\partial_{\tau_q}(\tau_q A_\Gamma) = -\gamma q^{2/3}(\vartheta)(\Sigma^2 - A)\mu(y, \vartheta)/\tau_q^2 + \mathcal{H}_{j \geq 3}$, we define a smooth, bounded function $\breve{\Gamma}_\gamma$

$$\breve{\Gamma}_\gamma = \frac{1}{3}\frac{A q^{1/3}(\vartheta)}{\tau_q}\Big[\sum_{k \geq 2}\partial_{\tau_q}\Big(\frac{(1 - \tau_q)^k}{\gamma\tau_q^{k-1}}\Big)B_{2k}(y, \vartheta)$$
$$+ \frac{x}{\gamma}\partial_{\tau_q}\Big(\tau_q A_\Gamma(x, y, \sqrt{\gamma}\Sigma q^{1/3}(\vartheta)/\tau_q, \vartheta/\tau_q)\Big) - \frac{a}{\gamma}\partial_{\tau_q}\Big(\tau_q A_\Gamma(a, 0, \sqrt{\gamma}Sq^{1/3}(\vartheta)/\tau_q, \vartheta/\tau_q)\Big)\Big].$$

Note that $\check{\Gamma}_\gamma = \frac{Aq(\vartheta)}{3\tau_q^2}(\frac{2A}{(1+\tau_q)} - x(\Sigma^2 - A)\mu(y,\vartheta) + a(S^2 - A)\gamma(0,\vartheta) + \gamma^{-1}\mathcal{H}_{j\geq3})$, where terms in $\mathcal{H}_{j\geq3}$ come with wights $\gamma^{j/2}$ and $j \geq 3$. Equation (3.40) becomes

$$t + y \cdot \vartheta + B_0(y,\vartheta) = -\gamma Aq^{2/3}(\vartheta)(t - B_2(y,\vartheta))(\frac{1}{1+\tau_q} - \frac{1}{3\tau_q}) - \gamma^2(\tilde{\Gamma}_\gamma - \check{\Gamma}_\gamma)$$

$$- \gamma^{3/2}\Big(\frac{\Sigma^3}{3} + \Sigma(\frac{x}{\gamma}q^{1/3}(\vartheta)(1 + \ell(y,\vartheta)) - A) - \frac{S^3}{3} - S(\frac{a}{\gamma}q^{1/3}(\vartheta) - A) + \frac{2}{3}(\Sigma - S)A\Big).$$

As $\tau_q^2 - 1 = \gamma Aq^{2/3}/(1 + \tau_q)$ and $(\frac{1}{1+\tau_q} - \frac{1}{3\tau_q}) = \frac{1}{6} + \frac{1}{12}\frac{(\tau_q^2-1)(1-(\tau_q^2-1)/4)}{\tau_q^2(1+(\tau_q^2-1)/2)}$, we move the part of the coefficient of $(t - B_2(y,\vartheta))$ with factor γ^2 into the next term. Recall $B_0(y,\vartheta) = \vartheta \cdot \nabla B_0(y,\vartheta)$, which eventually yields

$$(y + \nabla B_0(y,\vartheta)) \cdot \vartheta + t = -\gamma Aq^{2/3}(\vartheta)(t - B_2(y,\vartheta))\Big(\frac{1}{6} + \frac{1}{12}\frac{(\tau_q^2-1)(1-(\tau_q^2-1)/4)}{\tau_q^2(1+(\tau_q^2-1)/2)}\Big)$$

$$-\gamma^{3/2}\Big(\frac{\Sigma^3}{3} + \Sigma(\frac{x}{\gamma}q^{1/3}(\vartheta)(1 + \ell(y,\vartheta)) - A) - \frac{S^3}{3} - S(\frac{a}{\gamma}q^{1/3}(\vartheta) - A) + \frac{2}{3}(\Sigma - S)A\Big) + \gamma^2(\tilde{\Gamma}_\gamma - \check{\Gamma}_\gamma).$$

As ϑ is itself a critical point, we obtain, using the last statement of Lemma 3.8 and Remark 3.9, that (3.33) holds with $-$ sign on the left, and that $\vartheta \cdot (y + \nabla B_0(y,\vartheta)) = -|y + \nabla B_0| + \gamma^2\mathcal{E}_-$. As a result we obtain (3.38) with $\tilde{\mathcal{E}} = \tilde{\mathcal{E}}(t,x,y,\frac{t}{|y+\nabla B_0|},\frac{\sqrt{\gamma}}{t},\vartheta)$, defined as follows

$$\tilde{\mathcal{E}} := \tilde{\Gamma}_\gamma - \check{\Gamma}_\gamma + \mathcal{E}_- + \frac{1}{12}\frac{A^2q^{4/3}(\vartheta v)}{1+\tau_q}\frac{(1-(\tau_q^2-1)/4)}{\tau_q^2(1+(\tau_q^2-1)/2)}(t - B_2(y,\vartheta)),$$

which completes the proof of Lemma 3.10. $\qquad\square$

Pick (t,x,y) with $t > 0$ and $\mathcal{N}(t,x,y) \neq \varnothing$. As already noticed, we must have $||y|/t - 1| \leq c$ (since otherwise $\mathcal{N}(t,x,y) = \emptyset$ by (3.38)). Let now $N_j \in \mathcal{N}_d^1(t,x,y)$, $j \in \{1,2\}$. Then there exist (t_j,x_j,y_j) such that $N_j \in \mathcal{N}(t_j,x_j,y_j)$ and $(t_j,x_j,y_j) \in \mathcal{C}_\gamma(t,x,y)$; there exist $(\theta_j,A_j,\Sigma_j,S_j)$, $j \in \{1,2\}$, A_j close to 1 such that (3.30) holds with $(t,x,y,\theta,A,\Sigma,S)$ replaced by $(t_j,x_j,y_j,\theta_j,A_j,\Sigma_j,S_j)$. Then we have, with $\vartheta_j = \theta_j/|\theta_j|$,

$$(3.41) \quad 2N_j(1 - \frac{3}{4}B_L{}'(|\theta_j|\lambda_\gamma A_j^{3/2})) = \frac{q^{2/3}(\vartheta_j)}{2A_j^{1/2}\sqrt{1+\gamma A_jq^{2/3}(\vartheta_j)}}\frac{1}{\sqrt{\gamma}}\Big(t_j - B_2(y_j,\vartheta_j)\Big) + O(1),$$

$$(3.42) \quad \Sigma_j^2 + \frac{x_j}{\gamma}q^{1/3}(\vartheta_j)\Big(1 + \ell(y_j,\vartheta_j) + \mathcal{H}_{j\geq1}\Big) = A_j, S_j^2 + \frac{a}{\gamma}q^{1/3}(\vartheta_j)(1 + \mathcal{H}_{j\geq1}) = A_j,$$

$$(3.43) \quad \frac{y_j + \nabla B_0(y_j,\vartheta_j)}{t_j} = -\vartheta_j + \gamma A_jq^{2/3}(\vartheta_j)\Big[(1 - \frac{B_2(y_j,\vartheta_j)}{t_j})(\frac{1}{2}\vartheta_j - \frac{\nabla q(\vartheta_j)}{3q(\vartheta_j)})$$

$$+ \frac{1}{2t_j}\nabla B_2(y_j,\vartheta_j)\Big] + O(\frac{\gamma^{3/2}}{t_j}).$$

Taking the difference between (3.41) for $j = 1,2$ yields

$$(3.44) \quad N_1 - N_2 = \frac{3}{4}\Big(N_1B_L'(|\theta_1|\lambda_\gamma A_1^{3/2}) - N_2B_L'(|\theta_2|\lambda_\gamma A_2^{3/2})\Big) + O(1)$$

$$+ \frac{(t_1 - B_2(y_1,\vartheta_1))q^{2/3}(\vartheta_1)}{4\sqrt{\gamma}A_1^{1/2}\sqrt{1+\gamma A_1q^{2/3}(\vartheta_1)}} - \frac{(t_2 - B_2(y_2,\vartheta_2))q^{2/3}(\vartheta_2)}{4\sqrt{\gamma}A_2^{1/2}\sqrt{1+\gamma A_2q^{2/3}(\vartheta_2)}}.$$

Using Remark 3.6, the first term in the RHS of (3.44) is $O\left(\frac{N_1+N_2}{\lambda_\gamma^2}\right)$. We are reduced to proving

$$\left|\frac{(t_1-B_2(y_1,\vartheta_1))q(\vartheta_1)}{\sqrt{A_1 q^{2/3}(\vartheta_1)}} - \frac{(t_2-B_2(y_2,\vartheta_2))q(\vartheta_2)}{\sqrt{A_2 q^{2/3}(\vartheta_2)}}\right| = O(\sqrt{\gamma}).$$

This will follow from the next lemma :

Lemma 3.11. *Let (t,x,y) be fixed, let (t_j,x_j,y_j) satisfy (3.12), (3.13) and let $(\theta_j = |\theta_j|\vartheta_j, A_j, \Sigma_j, S_j)$, $j \in \{1,2\}$ with A_j close to 1 such that (3.41), (3.42) and (3.43) hold true, then*

$$(3.45) \qquad t|\vartheta_1-\vartheta_2| \lesssim \sqrt{\gamma}, \quad t|A_1 q^{2/3}(\vartheta_1) - A_2 q^{2/3}(\vartheta_2)| \lesssim \sqrt{\gamma}.$$

Proof. If $\frac{t}{\sqrt{\gamma}}$ is bounded then both inequalities in (3.45) follow immediately. Suppose $\frac{t}{\sqrt{\gamma}}$ is sufficiently large. From (3.43) and (3.33) we get (as in (3.38))

$$(3.46) \qquad \frac{|y_j+\nabla B_0(y_j,\vartheta_j)|-t_j}{t_j-B_2(y_j,\vartheta_j)} = \frac{1}{6}\gamma A_j q^{2/3}(\vartheta_j) + O(\frac{\gamma^{3/2}}{t_j}).$$

$$(3.47) \qquad \frac{y_j+\nabla B_0(y_j,\vartheta_j)}{|y_j+\nabla B_0(y_j,\vartheta_j)|} + \vartheta_j = -\gamma A_j q^{2/3}(\vartheta_j)\left[\frac{\nabla q(\vartheta_j)}{3q(\vartheta_j)}\left(1-\frac{B_2(y_j,\vartheta_j)}{t_j}\right)\right.$$
$$\left. -\frac{1}{2t_j}\nabla B_2(y_j,\vartheta_j)\right] + O(\frac{\gamma^{3/2}}{t_j}).$$

Taking the difference between (3.47) for $j=1$ and $j=2$ yields

$$|\vartheta_1-\vartheta_2| = \left|\frac{y_1+\nabla B_0(y_1,\vartheta_1)}{|y_1+\nabla B_0(y_1,\vartheta_1)|} - \frac{y_2+\nabla B_0(y_2,\vartheta_2)}{|y_2+\nabla B_0(y_2,\vartheta_2)|}\right| + O(\gamma).$$

As $\nabla B_0(y,\vartheta) = O(|y|^2)$, $\left|\frac{y_1}{|y_1|} - \frac{y_2}{|y_2|}\right| \leq \frac{2r_0\sqrt{\gamma}}{|y|}$ from (3.12) and $\frac{t}{|y|} \sim 1$, the first inequality in (3.45) holds true. We proceed with the second one, which is, as for the model, more delicate to handle.

Lemma 3.12. *Let $\tilde{\vartheta}_j = \tilde{\vartheta}_j(t_j,y_j)$ be the solution to (3.48) below, then $|\vartheta_j - \tilde{\vartheta}_j| \lesssim \frac{\gamma^{3/2}}{t_j}$.*

$$(3.48) \quad \tilde{\vartheta}_j + \frac{y_j+\nabla B_0(y_j,\tilde{\vartheta}_j)}{|y_j+\nabla B_0(y_j,\tilde{\vartheta}_j)|} = -2\left(\frac{|y_j+\nabla B_0(y_j,\tilde{\vartheta}_j)|}{t_j}-1\right)\left[\frac{\nabla q(\tilde{\vartheta}_j)}{q(\tilde{\vartheta}_j)} - \frac{3\nabla B_2(y_j,\tilde{\vartheta}_j)}{2t_j(1-\frac{B_2(y_j,\tilde{\vartheta}_j)}{t_j})}\right].$$

Proof. The coefficient of γA_j in (3.46) does not vanish as $B_2(y_j,\vartheta_j) = O(|y_j|^2)$ and $|y_j|/t_j \sim 1$; we replace γA_j in (3.47) by its first approximation given in (3.46). We obtain

$$(3.49) \quad \vartheta_j = -\frac{y_j+\nabla B_0(y_j,\vartheta_j)}{|y_j+\nabla B_0(y_j,\vartheta_j)|}$$
$$-2\left(\frac{|y_j+\nabla B_0(y_j,\vartheta_j)|}{t_j}-1\right)\left[\frac{\nabla q(\vartheta_j)}{q(\vartheta_j)} - \frac{3\nabla B_2(y_j,\vartheta_j)}{2t_j(1-\frac{B_2(y_j,\vartheta_j)}{t_j})}\right] + O(\frac{\gamma^{3/2}}{t_j}).$$

Taking the difference between (3.49) and (3.48), using $\nabla B_2(y_j,\vartheta) = O(|y_j|^2)$ and smallness of $\left|\frac{|y_j+\nabla B_0(y_j,\tilde{\vartheta}_j)|}{t_j}-1\right|$, completes the proof of Lemma 3.12. $\qquad\square$

Using (3.46), in order to achieve the proof of Lemma 3.11 we are reduced to proving that

$$\frac{t}{\gamma^{3/2}}\left|\frac{|y_1+\nabla B_0(y_1,\vartheta_1)|-t_1}{t_1-B_2(y_1,\vartheta_1)} - \frac{|y_2+\nabla B_0(y_2,\vartheta_2)|-t_2}{t_2-B_2(y_2,\vartheta_2)}\right| = O(1).$$

Set $\tilde{y}_j = y_j + \nabla B_0(y_j, \vartheta_j)$ and write

$$\left| \frac{|\tilde{y}_1| - t_1}{t_1 - B_2(y_1, \vartheta_1)} - \frac{|\tilde{y}_2| - t_2}{t_2 - B_2(y_2, \vartheta_2)} \right| \leq \left| \frac{|\tilde{y}_1| - t_1 + t_2 - |\tilde{y}_2|}{t_1 - B_2(y_1, \vartheta_1)} \right.$$
$$\left. - (t_2 - |\tilde{y}_2|) \left| \frac{1}{t_1 - B_2(y_1, \vartheta_1)} - \frac{1}{t_2 - B_2(y_2, \vartheta_2)} \right| \right|$$
$$\leq \left| \frac{|y_1 + \nabla B_0(y_1, \tilde{\vartheta}_1)| - t_1 + t_2 - |y_2 + \nabla B_0(y_2, \tilde{\vartheta}_2)|}{t_1 - B_2(y_1, \vartheta_1)} \right| + \left| \frac{|\tilde{y}_1| - |y_1 + \nabla B_0(y_1, \tilde{\vartheta}_1)|}{t_1 - B_2(y_1, \vartheta_1)} \right|$$
$$+ \left| \frac{|\tilde{y}_2| - |y_2 + \nabla B_0(y_2, \tilde{\vartheta}_2)|}{t_1 - B_2(y_1, \vartheta_1)} \right| + \left| \frac{|\tilde{y}_2| - t_2}{t_2 - B_2(y_2, \vartheta_2)} \right| \left| \frac{t_2 - B_2(y_2, \vartheta_2)}{t_1 - B_2(y_1, \vartheta_1)} - 1 \right|$$

and using (3.11) and (3.13), Lemma 3.12 and (3.46) for $j = 2$ yields

$$\frac{t}{\gamma^{3/2}} \left| \frac{|\tilde{y}_1| - t_1}{t_1 - B_2(y_1, \vartheta_1)} - \frac{|\tilde{y}_2| - t_2}{t_2 - B_2(y_2, \vartheta_2)} \right| \leq \frac{t}{\gamma^{3/2}} \times \frac{r_0 \gamma^{3/2}}{t_1}$$
$$+ \frac{t}{\gamma^{3/2}} \times \left(O(\frac{|y_1|^2}{t_1}) |\vartheta_1 - \tilde{\vartheta}_1| + O(\frac{|y_2|^2}{t_1}) |\vartheta_2 - \tilde{\vartheta}_2| \right)$$
$$+ \frac{t}{\gamma^{3/2}} \times O(\gamma) \times \left(\frac{|t_2 - B_2(y_2, \tilde{\vartheta}_2) - t_1 + B_2(y_1, \tilde{\vartheta}_1)|}{t_1 - B_2(y_1, \vartheta_1)} + O(\frac{|y_1|^2}{t_1}) |\vartheta_1 - \tilde{\vartheta}_1| + O(\frac{|y_2|^2}{t_1}) |\vartheta_2 - \tilde{\vartheta}_2| \right) = O(1).$$

For $d = 2$ the proof is much simpler (no angle ϑ). In this case Lemma 3.11 reduces to obtaining suitable estimates for $|A_1 - A_2|$; this follows from the $2D$ equivalent of (3.46) which reads

$$\frac{|y_j + B_0(y_j)|}{t_j} = 1 + \frac{1}{6} \gamma A_j \left(1 - \frac{B_2(y_j)}{t_j} \right) + O(\gamma^{3/2}/t),$$

and we are done with Lemma 3.11. $\qquad \square$

This completes the proof of Proposition 3.4 in the general case. $\qquad \square$

Proposition 3.13. *Let (t, x, y) such that $\mathcal{N}(t, x, y) \neq \emptyset$. Then,*

$$\sum_{N \notin \mathcal{N}_d^1(t,x,y), |N| \lesssim \frac{1}{\sqrt{\gamma}}} V_{N,\gamma}(t, x, y) = O(h^\infty).$$

Proof. We, again, consider first the model case: in rescaled variables T, X, Y, the Lagrangian Λ_N^M is defined by (3.16), (3.17), (3.18). We start with integration in ϑ : $\Psi_{N,a,\gamma}^M = \Psi_{0,a,\gamma}^M - \frac{4}{3} |\theta| N A^{3/2} + \frac{N}{\lambda_\gamma} B_L((|\theta|\lambda_\gamma A^{3/2})$ and the last two terms do not depend on ϑ; $\nabla_\vartheta^2 \Psi_{N,a,\gamma}^M = \nabla_\vartheta^2 \Psi_{0,a,\gamma}^M = \frac{|Y|}{\gamma} \times O(1) = \frac{|y|}{\gamma^{3/2}} \times O(1)$. Using Lemma 3.2, we have $|y| > c_0|t|$, hence critical points with respect to ϑ are non-degenerate and stationary phase applies (providing a factor $(\lambda_\gamma \frac{|y|}{\gamma^{3/2}})^{-(d-2)/2} \lesssim (\frac{h}{t})^{(d-2)/2})$. Let $t > 0$, then using Lemma 3.8 with $\Gamma = 0$ (hence $B_0 = B_2 = l = 0$), there exists a smooth, bounded function $\mathcal{E}_{M,-}$ such that the critical point (associated to the $-$ sign in (3.33)) is

$$(3.50) \qquad \vartheta_c = -\vartheta_Y - \frac{\gamma A}{3} \frac{T}{|Y|} \frac{\nabla q(-\vartheta_Y)}{q^{1/3}(-\vartheta_Y)} + \frac{\gamma}{|Y|} (\Sigma X - S \frac{a}{\gamma}) \frac{\nabla q(-\vartheta_Y)}{3q^{2/3}(-\vartheta_Y)} + \frac{\gamma^{3/2}}{|Y|} \mathcal{E}_{M,-},$$

where we set $\vartheta_Y = Y/|Y|$. The critical point (up to its sign) is unique, and from Lemma 3.10 (in the model case) we see that we recover (3.21). Let $\rho := |\theta|$ and denote $\tilde{\Psi}_{N,a,\gamma}^M(T, X, Y, \Sigma, S, A, \rho) :=$

$\Psi^M_{N,a,\gamma}(T,X,Y,\Sigma,S,A,\rho\vartheta_c)$ the critical value of the phase at ϑ_c given above, then

$$(3.51) \quad \tilde\Psi^M_{N,a,\gamma}(T,X,Y,\Sigma,S,A,\rho) = \rho\Big(\frac{Y\cdot\vartheta_c + T\sqrt{1+\gamma A q^{2/3}(\vartheta_c)}}{\gamma} + \frac{\Sigma^3}{3} + \Sigma(Xq^{1/3}(\vartheta_c)-A)$$
$$- \frac{S^3}{3} - S(\frac{a}{\gamma}q^{1/3}(\vartheta_c)-A) - \frac{4}{3}NA^{3/2}\Big) + \frac{N}{\lambda_\gamma}B_L(\rho\lambda_\gamma A^{3/2})\,.$$

Define sets of integers related to stationary points of this phase:

$$\mathcal{N}^M_{\tilde\Psi}(T,X,Y) := \{N\in\mathbb{Z}, \exists(\Sigma,S,A,\rho) \text{ such that } \nabla_{(\Sigma,S,A,\rho)}\tilde\Psi^M_{N,a,\gamma}(T,X,Y,\Sigma,S,A,\rho)=0\}\,,$$
$$\mathcal{N}^{1,M}_{\tilde\Psi}(T,X,Y) := \cup_{\mathcal{C}^M_\gamma(T,X,Y)}\mathcal{N}^M_{\tilde\Psi}(T',X',Y')\,.$$

One sees that $\mathcal{N}^M_{\tilde\Psi}(T,X,Y) = \mathcal{N}^M(t,x,y)$, which implies $\mathcal{N}^{1,M}_{\tilde\Psi}(T,X,Y) = \mathcal{N}^{1,M}_d(t,x,y)$, where $(T,X,Y)=(\frac{t}{\sqrt\gamma},\frac{x}{\gamma},\frac{y}{\sqrt\gamma})$: indeed, if N is such that there exists a critical point (Σ,S,A,ρ) for $\tilde\Psi^M_{N,a,\gamma}$ at (T,X,Y), then $(\Sigma,S,A,\rho,\vartheta_c(T,X,Y,\Sigma,S,A,a,\gamma))$ is a critical point for $\Psi^M_{N,a,\gamma}$ and the converse also holds. We now need to prove that

$$\sum_{N\notin\mathcal{N}^{1,M}_{\tilde\Psi}(T,X,Y),|N|\lesssim\frac{1}{\sqrt\gamma}} V^M_{N,\gamma}(T,X,Y) = O(h^\infty),$$

where $V^M_{N,\gamma}(T,X,Y)$ had phase $\Psi^M_{N,a,\gamma}$ that became $\tilde\Psi^M_{N,a,\gamma}$ after the stationary phase in ϑ.

Let first $4a\lesssim\gamma$: stationary phase applies in S. Indeed, $\tilde\Psi^M_{N,a,\gamma}$ is stationary in S when $S^2 + \frac{a}{\gamma}q^{1/3}(\vartheta_c) = A$ and for $\frac{a}{\gamma}$ small enough there are two non-degenerate critical points $S_\pm$ (with main contributions $\pm\sqrt{A}$). We denote by $\tilde\Psi^{M,\pm}_{N,a,\gamma}$ the critical values of the phase $\tilde\Psi^M_{N,a,\gamma}$ at $S_\pm$. For $\varepsilon\in\{\pm\}$, we define

$$\mathcal{N}^{M,\varepsilon}_{\tilde\Psi}(T,X,Y) := \{N\in\mathbb{Z}\,:\, \exists(\Sigma,A,\rho) \text{ such that } \nabla_{(\Sigma,A,\rho)}\tilde\Psi^{M,\varepsilon}_{N,a,\gamma}(T,X,Y,\Sigma,A,\rho)=0\}\,,$$
$$\mathcal{N}^{1,M,\varepsilon}_{\tilde\Psi}(T,X,Y) := \cup_{\mathcal{C}^M_\gamma(T,X,Y)}\mathcal{N}^{M,\varepsilon}_{\tilde\Psi}(T',X',Y') \quad\text{and}\quad \cap_\pm\mathcal{N}^{M,\pm}_{\tilde\Psi}(T,X,Y) = \mathcal{N}^M_{\tilde\Psi}(T,X,Y)\,.$$

If $N\in\mathcal{N}^M_{\tilde\Psi}(T,X,Y)$, then there exists a critical point Σ_c,S_c,A_c,ρ_c for the phase $\tilde\Psi^M_{N,a,\gamma}$; S_c satisfies $S^2+\frac{a}{\gamma}q^{1/3}(\vartheta_c) = A$, hence $S_c\in\{S_\pm\}$. Therefore (Σ_c,A_c,ρ_c) is a critical point for $\tilde\Psi^M_{N,a,\gamma}(T,X,Y,\Sigma,S_\pm,A,\rho\vartheta_c|_{S_\pm}) = \tilde\Psi^\pm_{N,a,\gamma}$. Conversely, let $\Sigma_\pm,A_\pm,\rho_\pm$ be a critical point for $\tilde\Psi^{M,\pm}_{N,a,\gamma}$; for each sign $\varepsilon\in\{\pm\}$, let $S^\pm_\varepsilon$ denote both solutions to $S^2+\frac{a}{\gamma}q^{1/3}(\vartheta_c|_{A_\varepsilon,\Sigma_\varepsilon,S})=A_\varepsilon$, then $(\Sigma_\varepsilon,S^\pm_\varepsilon,A_\varepsilon,\rho_\varepsilon)$ are critical points for $\tilde\Psi^M_{N,a,\gamma}$ and both inclusions hold. Using $\cap_\pm\mathcal{N}^{M,\pm}_{\tilde\Psi}(T,X,Y) = \mathcal{N}^M_{\tilde\Psi}(T,X,Y)$ we obtain $\cap_\pm\mathcal{N}^{1,M,\pm}_{\tilde\Psi}(T,X,Y) = \mathcal{N}^{1,M}_{\tilde\Psi}(T,X,Y)$ and therefore $(\mathcal{N}^{1,M}_{\tilde\Psi}(T,X,Y))^c = \cup_\pm(\mathcal{N}^{1,M,\pm}_{\tilde\Psi}(T,X,Y))^c$, where $(\mathcal{N}^{1,M}_{\tilde\Psi})^c$ is the complement set. Hence, the proof of Proposition 3.13 for Δ_M (for $4a\lesssim\gamma$) rests on proving

$$\sum_{N\notin\cup_{\varepsilon\in\{\pm\}}\mathcal{N}^{1,M,\varepsilon}_{\tilde\Psi}(T,X,Y),|N|\lesssim\frac{1}{\sqrt\gamma}} V^{M,\varepsilon}_{N,\gamma}(T,X,Y) = O(h^\infty),$$

where now $V_{N,\gamma}^{M,\varepsilon}$ has phase function $\tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}$ and its symbol is obtained from the one of $V_{N,\gamma}^M$ after stationary phase in S. Using (3.51), we obtain, with ϑ_c satisfying (3.21)

$$\tilde{\Psi}_{N,a,\gamma}^{M,\pm}(T,X,Y,\Sigma,A,\rho) = \rho\Big(\frac{Y\cdot\vartheta_c + T\sqrt{1+\gamma Aq^{2/3}(\vartheta_c)}}{\gamma} + \frac{\Sigma^3}{3} + \Sigma(Xq^{1/3}(\vartheta_c) - A)$$
$$\mp \frac{2}{3}(\frac{a}{\gamma}q^{1/3}(\vartheta_c) - A)^{3/2} - \frac{4}{3}NA^{3/2}\Big) + \frac{N}{\lambda_\gamma}B_L(\rho\lambda_\gamma A^{3/2}).$$

Lemma 3.14. *There exists a uniform constant $c > 0$ such that, if $N \notin \cup_{\varepsilon\in\{\pm\}}\mathcal{N}_{\tilde{\Psi}}^{1,M,\varepsilon}(T,X,Y)$ then, for all (Σ,A,ρ) on the support of the symbol of $V_{N,\gamma}^{M,\pm}(T,X,Y)$,*

$$(3.52) \qquad\qquad |\nabla_{(\Sigma,A,\rho)}\tilde{\Psi}_{N,a,\gamma}^{\varepsilon}(T,X,Y,\Sigma,A,\rho)| \geq c.$$

Remark 3.15. The lemma allows to conclude the proof of Proposition 3.13 in the case $4a \lesssim \gamma$ as, using (3.52), we apply non stationary phase with respect to (Σ,A,ρ) and obtain a contribution $O(h^\infty)$ for each such $V_{N,a,\gamma}^{M,\varepsilon}$; the sum over N is finite (up to $|N| \lesssim \frac{1}{\sqrt{\gamma}}$) and we can sum up.

Before dealing with the proof of Lemma 3.14, we go back to (3.16), (3.17) and (3.18): we set $S = S_\pm(A,\vartheta)$ is such that $S^2 + \frac{a}{\gamma}q^{1/3}(\vartheta) = A$. For each $\varepsilon \in \{\pm\}$, we explicitly obtain the integral curves $(T_\varepsilon, X_\varepsilon, Y_\varepsilon)(A,\Sigma,\theta)$ depending on the parameters (A,Σ,θ),

$$(3.53) \quad \begin{cases} X_\varepsilon(A,\Sigma,\vartheta)q^{1/3}(\vartheta) = A - \Sigma^2, \quad S_\varepsilon = \varepsilon\sqrt{A - \frac{a}{\gamma}q^{1/3}(\vartheta)}, \\[2mm] T_\varepsilon(A,\Sigma,|\theta|,\vartheta)\frac{q^{2/3}(\vartheta)}{2\sqrt{1+\gamma Aq^{2/3}(\vartheta)}} = 2NA^{1/2}(1 - \frac{3}{4}B_L'(|\theta|\lambda_\gamma A^{3/2})) + (\Sigma - S_\varepsilon(A,\vartheta)), \\[2mm] Y_\varepsilon(A,\Sigma,|\theta|,\vartheta) = -\frac{T_\varepsilon(A,\Sigma,|\theta|,\vartheta)q^{2/3}(\vartheta)}{\sqrt{1+\gamma Aq^{2/3}(\vartheta)}}\Big(\frac{\vartheta}{q^{2/3}(\vartheta)}(1+\gamma Aq^{2/3}(\vartheta)) + \frac{\gamma A}{3}\frac{\nabla q(\vartheta)}{q(\vartheta)}\Big) \\[2mm] \qquad - \gamma\Big(\frac{\Sigma^3}{3} + \Sigma(X_\varepsilon q^{1/3}(\vartheta_c) - A) - \varepsilon\frac{2}{3}S_\varepsilon^3\Big)\vartheta + \frac{4}{3}\gamma NA^{3/2}(1 - \frac{3}{4}B_L'(|\theta|\lambda_\gamma A^{3/2}))\vartheta \\[2mm] \qquad\qquad - \gamma\Big(\Sigma X_\varepsilon(A,\Sigma,\vartheta) - S_\varepsilon\frac{a}{\gamma}\Big)q^{1/3}(\vartheta)\frac{\nabla q(\vartheta)}{3q(\vartheta)}. \end{cases} \quad v$$

Taking the scalar product with ϑ in the last equation and using that $\vartheta \cdot \nabla q(\vartheta) = 0$ yields

$$Y_\varepsilon \cdot \vartheta + T_\varepsilon\sqrt{1+\gamma Aq^{\frac{2}{3}}(\vartheta)} + \gamma\Big(\frac{\Sigma^3}{3} + \Sigma(X_\varepsilon q^{\frac{1}{3}}(\vartheta_c) - A) - \varepsilon\frac{2}{3}S_\varepsilon^3\Big) = \frac{4}{3}\gamma NA^{\frac{3}{2}}(1 - \frac{3}{4}B_L'(|\theta|\lambda_\gamma A^{\frac{3}{2}})),$$

which is nothing but the derivative of the phase with respect to $\rho = |\theta|$.

Proof. (of (3.52)) Fix (T,X,Y). Let $N \notin \cup_{\varepsilon\in\{\pm\}}\mathcal{N}_{\tilde{\Psi}}^{1,M,\varepsilon}(T,X,Y)$, hence $N \notin \cup_{\varepsilon\in\{\pm\}}\mathcal{N}_{\tilde{\Psi}}^{M,\varepsilon}(T',X',Y')$ for all $(T',X',Y') \in \mathcal{C}_\gamma^M(T,X,Y)$, which is equivalent to $\nabla_{(\Sigma,A,\rho)}\tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}(T',X',Y',\Sigma,A,\rho) \neq 0$ for all $(T',X',Y') \in \mathcal{C}_\gamma^M(T,X,Y)$. Pick a sign $\varepsilon \in \{\pm\}$, let $(T',X',Y') \in \mathcal{C}_\gamma^M(T,X,Y)$, and let (A,Σ,ρ) be a given point (on the support of the symbol of $V_{N,a,\gamma}^{M,\varepsilon}$); let $S_\varepsilon(A,\vartheta_c')$ and ϑ_c' be solutions to $S^2 + \frac{a}{\gamma}q^{1/3}(\vartheta_c') = A$ and (3.50), where in (3.50) we replace (T,X,Y) by (T',X',Y') and where the sign of S_ε is ε. We may compute $\nabla_{(A,\Sigma,\rho)}\tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}(T',X',Y',\Sigma,A,\rho)$, taking advantage of ϑ_c' and $S_\varepsilon(A,\vartheta_c')$ being stationary points for $\Psi_{N,a,\gamma}^M(\cdots)$ to cancel their derivatives. Let now

$\vartheta \in \mathbb{S}^{d-2}$, using (3.53) and $S_\varepsilon(A, \vartheta) = \varepsilon\sqrt{A - \frac{a}{\gamma}q^{1/3}(\vartheta)})$, we get

$$(3.54) \begin{cases} \partial_A \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}(T', X', Y', \cdot) = \dfrac{T'q^{2/3}(\vartheta'_c)}{2\sqrt{1+\gamma A q^{2/3}(\vartheta'_c)}} - \dfrac{T_\varepsilon q^{2/3}(\vartheta)}{2\sqrt{1+\gamma A q^{2/3}(\vartheta)}} + S_\varepsilon(A, \vartheta'_c) - S_\varepsilon(A, \vartheta), \\[2mm] \partial_\Sigma \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}(T', X', Y', \cdot) = X'q^{1/3}(\vartheta'_c) - X_\varepsilon q^{1/3}(\vartheta), \\[2mm] \partial_\rho \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}(T', X', Y', \cdot) = \gamma^{-1}\Big(Y' \cdot \vartheta'_c + T'\sqrt{1+\gamma A q^{\frac{2}{3}}(\vartheta'_c)} - Y_\varepsilon \cdot \vartheta - T_\varepsilon\sqrt{1+\gamma A q^{\frac{2}{3}}(\vartheta)}\Big) \\[2mm] \qquad\qquad\qquad\qquad + \Sigma\Big(X'q^{\frac{1}{3}}(\vartheta'_c) - X_\varepsilon q^{\frac{1}{3}}(\vartheta)\Big) - \dfrac{2}{3}\Big(S_\varepsilon^3(A, \vartheta'_c) - S_\varepsilon^3(A, \vartheta)\Big). \end{cases}$$

Recall that ϑ'_c is provided by (3.50) with (T, X, Y) replaced by (T', X', Y'); the same formula holds for ϑ with (T, X, Y) replaced by $(T_\varepsilon, X_\varepsilon, Y_\varepsilon)$. For such ϑ'_c (resp. ϑ) we have $Y' \cdot \vartheta'_c = -|Y'| + O(\gamma^2)$ (resp. $Y_\varepsilon \cdot \vartheta = -|Y_\varepsilon| + O(\gamma^2)$). Setting $\gamma Z' := -|Y'| + T'$ and $\gamma Z_\varepsilon := -|Y_\varepsilon| + T_\varepsilon$, $\partial_\rho \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}$ may be rewritten as

$$\partial_\rho \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}(T', X', Y', \cdots) = A\partial_A \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}(T', X', Y', \cdots) + \Sigma\partial_\Sigma \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}(T', X', Y', \cdots)$$
$$+ Z' - Z_\varepsilon - A(S_\varepsilon(A, \vartheta'_c) - S_\varepsilon(A, \vartheta)) - \Big(\frac{S_\varepsilon^3(A, \vartheta'_c)}{3} - \frac{S_\varepsilon^3(A, \vartheta)}{3}\Big) + O(\gamma),$$

where all the small terms $O(\gamma)$ come with differences $T'q^{2/3}(\vartheta'_c) - T_\varepsilon q^{2/3}(\vartheta)$, $q^{1/3}(\vartheta'_c) - q^{1/3}(\vartheta)$, $X'q^{1/3}(\vartheta'_c) - X_\varepsilon q^{1/3}(\vartheta)$ etc. Using (3.54) we bound the gradient of $\tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}$

$$(3.55) \quad (|\partial_A \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}| + |\partial_\Sigma \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}| + |\partial_\rho \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}|)(T', X', Y', \Sigma, A, \rho) \le 4(|T'q^{2/3}(\vartheta'_c) - T_\varepsilon q^{2/3}(\vartheta)|$$
$$+ |q^{1/3}(\vartheta'_c) - q^{1/3}(\vartheta)| + |X'q^{1/3}(\vartheta'_c) - X_\varepsilon q^{1/3}(\vartheta)| + |Z' - Z_\varepsilon|).$$

As $N \notin \mathcal{N}_{\tilde{\Psi}}^{1,M,\varepsilon}(T, X, Y)$ implies that $\nabla_{(\Sigma, A, \rho)} \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}(T', X', Y', \Sigma, A, \rho) \ne 0$ for all $(T', X', Y') \in \mathcal{C}_\gamma^M(T, X, Y)$, it follows that the right hand side of (3.55) doesn't vanish for any $(A, \Sigma, \rho, \vartheta)$. Hence, for all (A, Σ, ρ) on the support of the symbol and for every $\vartheta \in \mathcal{S}^{d-2}$,

$$(3.56) \qquad (T_\varepsilon(A, \Sigma, \rho, \vartheta), X_\varepsilon(A, \Sigma, \rho, \vartheta), Z_\varepsilon(A, \Sigma, \rho, \vartheta)) \notin B_{r_0}(T, X, Z), \quad \gamma Z = -|Y| + T.$$

In fact, if the last statement does not hold, then there exist $(A, \Sigma, \rho, \vartheta)$ such that

$$(|T - T_\varepsilon(A, \Sigma, \rho, \vartheta)| + |X - X_\varepsilon(A, \Sigma, \rho, \vartheta)| + |Z - Z_\varepsilon(A, \Sigma, \rho, \vartheta)| \le r_0.$$

Taking $(T', X', Y') = (T_\varepsilon, X_\varepsilon, Y_\varepsilon)(A, \Sigma, \rho, \vartheta)$, $\gamma Z' = -|Y'| + T'$, then $\vartheta'_c = \vartheta$ and therefore the right hand side term in (3.55) vanishes which contradicts $(T', X', Y') \in \mathcal{C}_\gamma^M(T, X, Y)$. As (3.56) holds true for all ϑ, it also holds for $\vartheta = \vartheta_c$ from (3.50) (corresponding to $(T, X, Y, \Sigma, A, \rho)$), and

$$|T - T_\varepsilon(A, \Sigma, \rho, \vartheta_c)| + |X - X_\varepsilon(A, \Sigma, \rho, \vartheta_c)| + |Z - Z_\varepsilon(A, \Sigma, \rho, \vartheta_c)| \ge r_0.$$

Moreover, at $(T, X, Y, \Sigma, A, \rho)$ and $\vartheta = \vartheta_c$ we obtain from (3.54)

$$|\partial_A \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}|(T, X, Y, \Sigma, A, \rho) = \frac{q^{2/3}(\vartheta_c)}{2\sqrt{1+\gamma A q^{2/3}(\vartheta_c)}}|T - T_\varepsilon(A, \Sigma, \rho, \vartheta_c)|,$$

$$|\partial_\Sigma \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}|(T, X, Y, \Sigma, A, \rho) = q^{1/3}(\vartheta_c)|X - X_\varepsilon|,$$

$$|\partial_\rho \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}|(T, X, Y, \Sigma, A, \rho) \ge |Z - Z_\varepsilon(A, \Sigma, \rho, \vartheta_c)| - \frac{A q^{2/3}(\vartheta_c)|T - T_\varepsilon|}{1 + \sqrt{1+\gamma A q^{2/3}(\vartheta_c)}} - |\Sigma|q^{1/3}(\vartheta_c)|X' - X_\varepsilon|,$$

and therefore there exists a uniform constant (depending only on q) such that

$$(|\partial_A \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}| + |\partial_\Sigma \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}| + |\partial_\rho \tilde{\Psi}_{N,a,\gamma}^{M,\varepsilon}|)(T, X, Y, \Sigma, A, \rho) \geq C(|T - T_\varepsilon(A, \Sigma, \rho, \vartheta_c)|$$
$$+ |X - X_\varepsilon(A, \Sigma, \rho, \vartheta_c)| + |Z' - Z_\varepsilon(A, \Sigma, \rho, \vartheta_c)|) \geq Cr_0.$$

which allows to conclude the proof of Lemma 3.14. □

Let now $a \sim \gamma$. We rescale variables as follows $s = \sqrt{a}|\theta|^{1/3}S$, $\sigma\sqrt{a}|\theta|^{1/3}\Sigma$, $\alpha = a|\theta|^{2/3}A$, let $\lambda = \lambda_a = \frac{a^{3/2}}{h}$ and $\Psi_{N,a,a}^M(T, X, Y, \Sigma, S, A, \theta) := \Phi_{N,a,a}^M(\sqrt{a}T, aX, \sqrt{a}Y, \sqrt{a}|\theta|^{1/3}\Sigma, \sqrt{a}|\theta|^{1/3}S, a|\theta|^{2/3}A)$. The saddle points of $\Psi_{N,a,a}^M$ satisfy $S^2 = A - q^{1/3}(\vartheta_c)$ and they undergo coalescence when $A = q^{1/3}(\vartheta_c)$. Let $\chi_0 \in C^\infty$ be a smooth cutoff, equal to 1 on $[0, \infty]$ and equal to 0 on $[-\infty, -2]$. Then $(\chi_0 Ai)(-(|\theta|\lambda)^{2/3}(A - q^{1/3}(\vartheta_c)))$ is a symbol of order $2/3$ supported for values $(|\theta|\lambda)^{2/3}(A - q^{1/3}(\vartheta_c)) \leq 2$ and $(1 - \chi_0)Ai(-(|\theta|\lambda)^{2/3}(A - q^{1/3}(\vartheta_c)))$ is supported for $A \geq q^{1/3}(\vartheta_c)$ and is equal to 1 on $(|\theta|\lambda)^{2/3}(A - q^{1/3}(\vartheta_c)) \geq 2$. Notice that on the support of $(1 - \chi_0)$ the Airy function can be written as a sum of two contributions $A_\pm((|\theta|\lambda)^{2/3}(A - q^{1/3}(\vartheta_c)))$ corresponding to the saddle points $S_\pm$. We split the symbol of V_N^M in two parts using $\chi_0 + (1 - \chi_0) = 1$ and notice that on the support of χ_0 the Airy function behaves as a symbol of order $2/3$. Therefore we can write each integral $V_{N,a}^M(t, x, y)$ as a sum $V_{N,a}^M(t, x, y) = \sum_{\varepsilon \in \{0, \pm\}} V_{N,a}^{M,\varepsilon}(t, x, y)$ where for $\varepsilon \in \{\pm\}$, $V_N^{M,\varepsilon}$ has phase $\tilde{\Psi}_{N,a,a}^{M,\varepsilon}(T, X, Y, \Sigma, A, \rho)$ while $V_{N,a}^{M,0}$ has phase function $\Psi_{N,a,a}(t, x, y, \Sigma, 0, A, \rho)$. We are left to prove that for ever $\varepsilon \in \{0, \pm\}$ the following holds

$$\sum_{N \notin \mathcal{N}_\Psi^{1,M,\varepsilon}(T,X,Y), |N| \lesssim 1/\sqrt{a}} V_{N,a}^{M,\varepsilon}(T, X, Y) = O(h^\infty).$$

For $\varepsilon \in \{\pm\}$ we act exactly like in the transverse case $4a \leq \gamma$ since on the support of $(1 - \chi_0)$ we obtain two distinct saddle points $S_\pm$. For $\varepsilon = 0$ we use the fact that we got rid of variable S hence $\partial_S \Psi_{N,a,a}(T, X, Y, \Sigma, 0, A, \rho) = 0$ and act again as in the previous case, completing the proof of Proposition 3.13 for the model Laplace Δ_M.

We can now proceed with the proof of Proposition 3.13 in the general situation. Recall that the phase of V_N is $\Phi_{N,a,\gamma}(t, x, y, \sigma, s, \alpha, \theta) = t\tau_q(\alpha, \theta) + \Phi(x, y, \theta, \alpha, \sigma) - \Phi(a, 0, \theta, \alpha, s) - NhL(\alpha^{3/2}/h)$ (see (3.9)). Rescaling $s = \sqrt{\gamma}|\theta|^{1/3}S$, $\alpha = \gamma|\theta|^{2/3}A$ we obtain a new phase $\tilde{\Phi}_{N,a,\gamma}(t, x, y, \Sigma, S, A, \theta) := \Phi_{N,a,\gamma}(t, x, y, \sigma, s, \alpha, \theta)$ whose saddle points $S_\pm$ are solutions to the third equation in (3.30) that we re-write here, using (2.51),

$$(3.57) \qquad S^2 + \frac{a}{\gamma}q^{1/3}(\vartheta)\Big(1 + \mathcal{H}_{j \geq 1}\Big) = A,$$

where terms in $\mathcal{H}_j$ come with factors $O(\gamma^{j/2})$. As this equation is independent of $|\theta|$, we recover that $S_\pm = S_\pm(a, \gamma, \vartheta, A)$ are independent of $|\theta|$ or N.

Let first $4a \lesssim \gamma$; as A is close to 1 on the support of the symbol, critical points $S_\pm$ are non-degenerate and have opposite signs. Stationary phase in S yields $\tilde{\Phi}_{N,a,\gamma} \in \{\tilde{\Phi}_{N,a,\gamma}^\varepsilon\}$, where for $\varepsilon \in \{\pm\}$ we set $\tilde{\Phi}_{N,a,\gamma}^\varepsilon(t, x, y, \Sigma, A, \theta) := \tilde{\Phi}_{N,a,\gamma}(t, x, y, \Sigma, S_\varepsilon, A, \theta)$. Assume $t > 0$. For $\varepsilon \in \{\pm\}$, let

$$\mathcal{N}^\varepsilon(t, x, y) := \{(N \in \mathbb{Z}, \exists(\Sigma, A, \theta) \text{ such that } \nabla_{(\Sigma, A, \theta)}\tilde{\Phi}_{N,a,\gamma}^\varepsilon(t, x, y, \Sigma, A, \theta) = 0\},$$

then $\mathcal{N}(t, x, y) = \cap_{\varepsilon = \pm}\mathcal{N}^\varepsilon(t, x, y)$. Indeed, if $N \in \mathcal{N}(t, x, y)$, then there exists a critical point $\Sigma_c, S_c, A_c, \theta_c$ for the phase $\tilde{\Phi}_{N,a,\gamma}$; S_c satisfies (3.57), hence $S_c \in \{S_\pm\}$. Therefore $(\Sigma_c, A_c, \theta_c)$ is a critical point for $\tilde{\Phi}_{N,a,\gamma}(t, x, y, \Sigma, S_\pm(a, \gamma, \vartheta, A), A, \theta) = \tilde{\Phi}_{N,a,\gamma}^\pm$. Conversely, if $\Sigma_\pm, A_\pm, \theta_\pm$ is a critical point for $\tilde{\Phi}_{N,a,\gamma}^\pm$, then $(\Sigma_\pm, S_\pm(a, \gamma, \vartheta_\pm, A_\pm), A_\pm, \theta_\pm)$ are critical points for $\tilde{\Phi}_{N,a,\gamma}$. For

$\varepsilon \in \{\pm\}$, let $\mathcal{N}_d^{1,\varepsilon}(t,x,y) = \cup_{\mathcal{C}_\gamma(t,x,y)} \mathcal{N}^\varepsilon(t',x',y')$, then we have $\mathcal{N}_d^1 = \cap_\pm \mathcal{N}_d^{1,\pm}$ and therefore, $(\mathcal{N}_1)^c = \cup_{\varepsilon \in \{\pm 1\}} (\mathcal{N}_d^{1,\varepsilon})^c$. Proposition 3.13 will follow from proving

$$(3.58) \qquad \forall \varepsilon \in \{\pm\}, \qquad \sum_{N \notin \cup_\pm \mathcal{N}_d^{1,\pm}(t,x,y), |N| \lesssim \frac{1}{\sqrt{\gamma}}} V_{N,\gamma}^\varepsilon(t,x,y) = O(h^\infty),$$

where $V_{N,\gamma}^\varepsilon(t,x,y)$ has phase function $\tilde{\Phi}_{N,a,\gamma}^\varepsilon$ and symbol obtained from the one of $V_{N,\gamma}$ after stationary phase in S. For every N the phase $\tilde{\Phi}_{N,a,\gamma}^\varepsilon$ has two critical points with respect to ϑ, which are non-degenerate. Indeed, $\tilde{\Phi}_{N,a,\gamma}^\varepsilon$ depends on N through $-NhL(|\theta|\lambda_\gamma A^{3/2})$ and therefore its dependence on ϑ comes only through $\tilde{\Phi}_{0,a,\gamma}^\varepsilon = \tilde{\Phi}_{0,a,\gamma}|_{S=S_\varepsilon}$. Moreover, $\nabla_\vartheta^2 \tilde{\Phi}_{0,a,\gamma} = |y|O(1)$, $(\nabla_\vartheta^2 \tilde{\Phi}_{0,a,\gamma})^{-1} = |y|^{-1}O(1)$ and using Lemma 3.2 it follows that $|y| > c_0|t|$ and in particular $y \neq 0$. Lemma 3.8 provide the explicit form of the critical point ϑ_c and for $t > 0$ we set $\vartheta_c(t,x,y,\Sigma,A) = \vartheta_-(t,x,y,\Sigma,S_\varepsilon,A)$. Let $\rho := |\theta|$ and set

$$\Psi_{N,a,\gamma}^\varepsilon(t,x,y,\Sigma,A,\rho) := \tilde{\Phi}_{0,a,\gamma}^\varepsilon(t,x,y,\Sigma,A,\rho\vartheta_c) - \frac{4}{3}N\rho\gamma^{3/2}A^{3/2} + NhB_L(\rho\lambda_\gamma A^{3/2}),$$

$$\mathcal{N}_{\Psi^\varepsilon}(t,x,y) := \{(N \in \mathbb{Z}, \exists(\Sigma,A,\rho) \text{ such that } \nabla_{(\Sigma,A,\rho)}\Psi_{N,a,\gamma}^\varepsilon(t,x,y,\Sigma,A,\rho) = 0\}.$$

Remark again that $\mathcal{N}_{\Psi^\varepsilon}(t,x,y) = \mathcal{N}^\varepsilon(t,x,y)$: indeed, if N is such that there exists a critical point (Σ,A,ρ) for $\Psi_{N,a,\gamma}^\varepsilon$ at (t,x,y), then $(\Sigma,A,\rho,\vartheta_c(t,x,y,\Sigma,A))$ is a critical point for $\tilde{\Phi}_{N,a,\gamma}^\varepsilon$ and the reverse statement holds as well. Let $\varepsilon \in \{\pm\}$ and $N \notin \mathcal{N}_d^{1,\varepsilon}$. Then $N \notin \mathcal{N}^\varepsilon(t',x',y')$ for all $(t',x',y') \in \mathcal{C}_\gamma(t,x,y)$ and therefore for all such (t',x',y') we have

$$\nabla_{(\Sigma,A,\theta)}\tilde{\Phi}_{N,a,\gamma}^\varepsilon(t',x',y',\Sigma,A,\theta) \neq 0, \quad \forall(A,\Sigma,\rho).$$

This translates into $N \notin \mathcal{N}_{\Psi^\varepsilon}(t',x',y')$ and $\nabla_{(\Sigma,A,\rho)}\Psi_{N,a,\gamma}^\varepsilon(t',x',y',\Sigma,A,\rho) \neq 0$ for all $(t',x',y') \in \mathcal{C}_\gamma(t,x,y)$ and all Σ,A,ρ on the support of the symbol of $V_{N,\gamma}^\varepsilon(t,x,y)$.

Lemma 3.16. *There exists a uniform constant $c > 0$ such that, for all $N \notin \cup_{\varepsilon \in \{\pm\}} \mathcal{N}_{\tilde{\Psi}}^{1,\varepsilon}(t,x,y)$,*

$$(3.59) \qquad |\nabla_{(\Sigma,A,\rho)}\Psi_{N,a,\gamma}^\varepsilon(t,x,y,\Sigma,A,\rho)| \geq c\gamma^{3/2}$$

for all (Σ,A,ρ) on the support of the symbol of $V_{N,\gamma}^\varepsilon$.

This lemma allows to perform non stationary phase in $V_{N,\gamma}^\varepsilon(t,x,y)$ with large parameter λ_γ and conclude that (3.58) holds true.

Remark 3.17. We need to define the integral curves of $\tilde{\Phi}_{N,a,\gamma}^\varepsilon$: recall that (3.30) allows to parametrize the Lagrangian Λ_N; the projection $\pi_N(\Lambda_N)$ can be parametrized by $(\rho,\vartheta,\Sigma,A)$, with $\rho \sim 1$, $\vartheta \in \mathbb{S}^{d-2}$, A,Σ,S on the support of the symbol of $V_{N,\gamma}$ and $S \in \{S_\pm(a,\gamma,\vartheta,A)\}$. We define the integral curves $\tilde{t}_\varepsilon = \tilde{t}_\varepsilon(\Sigma,A,\rho,\vartheta,a,\gamma)$, $\tilde{x}_\varepsilon = \tilde{x}_\varepsilon(\Sigma,A,\rho,\vartheta,a,\gamma)$, $\tilde{y}_\varepsilon = \tilde{y}_\varepsilon(\Sigma,A,\rho,\vartheta,a,\gamma)$ (as we did in Remark 3.14 in the model case) such that (3.30) holds at $(\tilde{t}_\varepsilon,\tilde{x}_\varepsilon,\tilde{y}_\varepsilon)$. Therefore, at a given point $(\rho,\vartheta,\Sigma,A)$, $(\tilde{t}_\varepsilon,\tilde{x}_\varepsilon,\tilde{y}_\varepsilon)$ is defined such that (3.30) holds with $(t,x,y) = (\tilde{t}_\varepsilon,\tilde{x}_\varepsilon,\tilde{y}_\varepsilon)$, $S = S_\varepsilon(a,\gamma,\vartheta,A)$ and $\theta = \rho\vartheta$. Using Lemma 3.8 with $\vartheta \in \{\vartheta_\pm\}$ and (t,x,y) replaced by $(\tilde{t}_\varepsilon,\tilde{x}_\varepsilon,\tilde{y}_\varepsilon)$, we obtain $(\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon,\vartheta))\vartheta = -|\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon,\vartheta)| + \gamma^2 O(\frac{|\tilde{y}_\varepsilon|}{\tilde{t}_\varepsilon})$.

Proof. Let (Σ,A,ρ) be a point on the support of the symbol of $V_{N,\gamma}^\varepsilon(t,x,y)$, let $\vartheta \in \mathbb{S}^{d-2}$ and define $S_\varepsilon(a,\gamma,\vartheta,A)$ to be the solution to the third equation in (3.30) (where ϑ is replaced by ϑ) with sign ε. Let $(\tilde{t}_\varepsilon,\tilde{x}_\varepsilon,\tilde{y}_\varepsilon) := (\tilde{t}_\varepsilon,\tilde{x}_\varepsilon,\tilde{y}_\varepsilon)(\Sigma,A,\rho,\vartheta,a,\gamma)$ denote the corresponding point on the integral curve (as in Remark 3.17); then, by construction of the integral curves,

$$\nabla_{(A,\Sigma,\rho)}\tilde{\Psi}_{N,a,\gamma}^\varepsilon(\tilde{t}_\varepsilon,\tilde{x}_\varepsilon,\tilde{y}_\varepsilon,\Sigma,A,\rho) = \nabla_{(A,\Sigma,\rho)}\tilde{\Phi}_{N,a,\gamma}^\varepsilon((\tilde{t}_\varepsilon,\tilde{x}_\varepsilon,\tilde{y}_\varepsilon)(\Sigma,\cdots,\gamma),\Sigma,A,\rho\vartheta) = 0 \in \mathbb{R}^3.$$

Using the explicit form of (3.30), this translates into

$$\frac{\gamma q^{2/3}(\vartheta)}{2\sqrt{1+\gamma A q^{2/3}(\vartheta)}}(\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon, \vartheta)) + \gamma^{\frac{3}{2}}(\Sigma - S_\varepsilon(a, \gamma, \vartheta, A)) = 2N\gamma^{\frac{3}{2}}A^{1/2}(1 - \frac{3}{4}B'_L(\rho\lambda_\gamma A^{\frac{3}{2}})) + O(\gamma^2),$$

$$\tilde{x}_\varepsilon(1 + \ell(\tilde{y}_\varepsilon, \vartheta) + O(\sqrt{\gamma})) = \gamma(A - \Sigma^2),$$

$$\left(((\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon, \vartheta))\vartheta + \tilde{t}_\varepsilon\right) - (1 - \tau_q(A, \vartheta))(\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon, \vartheta))$$

$$+ \gamma^{\frac{3}{2}}\left[\frac{\Sigma^3}{3} + \Sigma\left(\frac{\tilde{x}_\varepsilon}{\gamma}q^{1/3}(\vartheta)(1 + \ell(\tilde{y}_\varepsilon, \vartheta) + O(\sqrt{\gamma})) - A\right) - \frac{S_\varepsilon^3}{3} - S_\varepsilon\left(\frac{a}{\gamma}q^{1/3}(\vartheta)(1 + O(\sqrt{\gamma})) - A\right)\right]$$

$$= \frac{4}{3}\gamma^{\frac{3}{2}}NA^{\frac{3}{2}}(1 - \frac{3}{4}B'_L(\rho\lambda_\gamma A^{\frac{3}{2}})) + O(\gamma^2),$$

where the last equation is obtained using the second to last equation in the system (3.30), as satisfied by $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{y}_\varepsilon)(\Sigma, A, \rho, \vartheta, a, \gamma), \Sigma, S_\varepsilon(a, \gamma, \vartheta, A), A, \rho, \vartheta$. Let $(t', x', y') \in \mathcal{C}_\gamma(t, x, y)$ and let $\vartheta'_c := \vartheta_c(t', x', y', \Sigma, A)$ be the critical point of $\tilde{\Phi}^\varepsilon_{N,a,\gamma}(t', x', y', \Sigma, A, \rho\vartheta)$, then, for any $\vartheta \in \mathbb{S}^{d-2}$,

$$(3.60) \quad \partial_A\Psi^\varepsilon_{N,a,\gamma} = \frac{\gamma q^{2/3}(\vartheta'_c)}{2\sqrt{1+\gamma q^{2/3}(\vartheta'_c)}}(t' - B_2(y', \vartheta'_c))$$

$$- \frac{\gamma A q^{2/3}(\vartheta)}{2\sqrt{1+\gamma A q^{2/3}(\vartheta)}}(\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon, \vartheta)) - \gamma^{3/2}(S_\varepsilon(a, \gamma, \vartheta'_c, A) - S_\varepsilon(a, \gamma, \vartheta, A)) + O(\gamma^2),$$

$$(3.61) \quad \partial_\Sigma\Psi^\varepsilon_{N,a,\gamma} = \gamma^{\frac{3}{2}}\left(q^{1/3}(\vartheta'_c)\frac{x'}{\gamma}(1 + \ell(y', \vartheta'_c) + O(\gamma^{\frac{1}{2}})) - q^{1/3}(\vartheta)\frac{\tilde{x}_\varepsilon}{\gamma}(1 + \ell(\tilde{y}_\varepsilon, \vartheta) + O(\gamma^{\frac{1}{2}}))\right),$$

$$(3.62) \quad \partial_\rho\Psi^\varepsilon_{N,a,\gamma} = ((y' + \nabla B_0(y', \vartheta'_c)) \cdot \vartheta'_c + t') - ((\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon, \vartheta)) \cdot \vartheta + \tilde{t}_\varepsilon)$$

$$+ \frac{\gamma A q^{2/3}(\vartheta'_c)}{2\sqrt{1+\gamma A q^{2/3}(\vartheta'_c)}}(t' - B_2(y', \vartheta'_c)) - \frac{\gamma A q^{2/3}(\vartheta)}{2\sqrt{1+\gamma A q^{2/3}(\vartheta)}}(\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon, \vartheta))$$

$$+ \gamma^{3/2}\Sigma\left(\frac{x'}{\gamma}q^{1/3}(\vartheta'_c)(1 + \ell(y', \vartheta'_c) + O(\sqrt{\gamma})) - \frac{\tilde{x}_\varepsilon}{\gamma}q^{1/3}(\vartheta)(1 + \ell(\tilde{y}_\varepsilon, \vartheta) + O(\sqrt{\gamma}))\right)$$

$$- \frac{2}{3}\gamma^{3/2}\left(S_\varepsilon^3(a, \gamma, \vartheta'_c, A) - S_\varepsilon^3(a, \gamma, \vartheta, A)\right) + O(\gamma^2),$$

where in the last line we used $\frac{S_\varepsilon^3}{3} + S_\varepsilon\left(\frac{a}{\gamma}q^{1/3}(\vartheta)(1 + O(\sqrt{\gamma})) - A\right) = -\frac{2}{3}S_\varepsilon^{3/2} + O(\sqrt{\gamma})$.

Remark 3.18. All terms $O(\gamma^2)$ are smooth functions of differences: $\vartheta'_c - \vartheta$, $q^{2/3}(\vartheta'_c)(t' - B_2(y', \vartheta'_c)) - q^{2/3}(\vartheta)(\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon, \vartheta))$, $q^{1/3}(\vartheta'_c)\frac{x'}{\gamma} - q^{1/3}(\vartheta)\frac{\tilde{x}_\varepsilon}{\gamma}$ as well as $(f(y', \vartheta'_c) - f(\tilde{y}_\varepsilon, \vartheta))$ where f is either $B_2(y, \vartheta)$ or $\nabla B_2(y, \vartheta)$ and such that $f(y, \vartheta) = O(|y|^2)$ or $f \in \{B_{2k}(y, \vartheta), \nabla B_{2k}(y, \vartheta)\}$ is such that $f(y, \vartheta) = O(|y|)$ and coefficients $O(\gamma^{2+k})$ instead of $O(\gamma^2)$, or $f \in \{\alpha_j(y, \vartheta), \gamma_j(y, \vartheta)\}$ (where α_j, γ_j are coefficients of homogeneous terms of degree $2j$ in A_Γ) with coefficients $O(\gamma^{2+j/2})$.

We then obtain and upper bound for $\nabla_{A,\Sigma,\rho}\tilde{\Psi}^{\varepsilon}_{N,a,\gamma}$ (similar to (3.55))

$$\gamma^{-3/2}(|\partial_A\tilde{\Psi}^{\varepsilon}_{N,a,\gamma}| + |\partial_\Sigma\tilde{\Psi}^{\varepsilon}_{N,a,\gamma}| + |\partial_\rho\tilde{\Psi}^{\varepsilon}_{N,a,\gamma}|)(t',x',y',\Sigma,A,\rho)$$

$$\leq C\left(\left|q^{2/3}(\vartheta'_c)\frac{(t'-B_2(y',\vartheta'_c))}{\sqrt{\gamma}} - q^{2/3}(\vartheta)\frac{(\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon,\vartheta))}{\sqrt{\gamma}}\right|\right.$$

$$+ |q^{1/3}(\vartheta'_c) - q^{1/3}(\vartheta)|(1+O(\sqrt{\gamma})) + |q^{1/3}(\vartheta'_c)\frac{x'}{\gamma}(1+\ell(y',\vartheta'_c)) - q^{1/3}(\vartheta)\frac{\tilde{x}_\varepsilon}{\gamma}(1+\ell(\tilde{y}_\varepsilon,\vartheta))|$$

$$+ \left|\frac{(y'+\nabla B_0(y',\vartheta'_c))\cdot\vartheta'_c + t'}{\gamma^{3/2}} - \frac{(\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon,\vartheta))\cdot\vartheta + \tilde{t}_\varepsilon}{\gamma^{3/2}}\right| + O(\sqrt{\gamma})|y'-\tilde{y}_\varepsilon|\right),$$

where the term $|q^{1/3}(\vartheta'_c) - q^{1/3}(\vartheta)|$ comes from $\left(S_\varepsilon(a,\gamma,\vartheta'_c,A) - S_\varepsilon(a,\gamma,\vartheta,A)\right)$ and the term $O(\sqrt{\gamma})|y'-\tilde{y}_\varepsilon|$ comes from differences involving B_{2k}, α_j, γ_j. The constant $C > 0$ depends on q.

Lemma 3.19. *Let (t,x,y) be fixed, let $(\Sigma, A, \rho, \vartheta)$ belong to the support of the symbol of $V^{\varepsilon}_{N,\gamma}$, then $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{y}_\varepsilon)(\Sigma, A, \rho, \vartheta, a, \gamma) \notin \mathcal{C}_\gamma(t,x,y)$.*

Proof. If not, then taking $(t',x',y') := (\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{y}_\varepsilon)(\Sigma, A, \rho, \vartheta)$ we have $\vartheta'_c = \vartheta$, $(t',x',y') \in \mathcal{C}_\gamma$ and $\nabla_{A,\Sigma,\rho}\tilde{\Psi}^{\varepsilon}_{N,a,\gamma}(t',x',y',\Sigma,A,\rho) = 0$, which cannot happen for $N \notin \mathcal{N}_{\Psi^\varepsilon}(t',x',y')$. □

We now prove (3.59): from the previous lemma, for all A, Σ, ρ we have $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{y}_\varepsilon)(\Sigma, A, \rho, \vartheta_c, a, \gamma) \notin \mathcal{C}_\gamma(t,x,y)$, where $\vartheta_c = \vartheta_c(t,x,y,\Sigma,A)$ is the critical point of $\tilde{\Phi}^{\varepsilon}_{N,a,\gamma}(t,x,y,\Sigma,A,\rho\vartheta)$ given in Lemma 3.8 (with $\vartheta_c = \vartheta_-(t,x,y,\Sigma,A)$ for $t > 0$). This yields

$$(3.63) \quad \left|\frac{t-B_2(y,\tilde{\vartheta}(t,y))}{\sqrt{\gamma}} - \frac{\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon,\tilde{\vartheta}(\tilde{t}_\varepsilon,\tilde{y}_\varepsilon))}{\sqrt{\gamma}}\right| + |\frac{x}{\gamma}(1+\ell(y,\tilde{\vartheta}(t,y))) - \frac{\tilde{x}_\varepsilon}{\gamma}(1+\ell(\tilde{y}_\varepsilon,\tilde{\vartheta}(\tilde{t}_\varepsilon,\tilde{y}_\varepsilon)))|$$

$$+ \left|\frac{|y+\nabla B_0(y,\tilde{\vartheta}(t,y))| - t}{\gamma^{3/2}} - \frac{|\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon,\tilde{\vartheta}(\tilde{t}_\varepsilon,\tilde{y}_\varepsilon))| - \tilde{t}_\varepsilon}{\gamma^{3/2}}\right| > r_0.$$

As (3.60),(3.61) and (3.62) do hold for any $\vartheta \in \mathbb{S}^{d-2}$, taking $(t',x',y') = (t,x,y)$ and $\vartheta := \vartheta_c(= \vartheta'_c) = \vartheta_c(t,x,y,\Sigma,A)$ yields $S_\varepsilon(a,\gamma,\vartheta'_c,A) = S_\varepsilon(a,\gamma,\vartheta,A)$ and therefore, for all (A,Σ,ρ) on the support of $V^{\varepsilon}_{N,\gamma}(t,x,y)$ and $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{y}_\varepsilon) = (\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{y}_\varepsilon)(\Sigma, A, \rho, \vartheta_c)$ we have

$$\gamma^{-3/2}|\partial_A\tilde{\Psi}^{\varepsilon}_{N,a,\gamma}|(t,x,y,\Sigma,A,\rho) = \frac{1}{2}q^{2/3}(\vartheta_c)\left|\frac{(t-B_2(y,\vartheta_c))}{\sqrt{\gamma}} - \frac{(\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon,\vartheta_c))}{\sqrt{\gamma}}\right| + O(\sqrt{\gamma}),$$

$$\gamma^{-3/2}|\partial_\Sigma\tilde{\Psi}^{\varepsilon}_{N,a,\gamma}|(t,x,y,\Sigma,A,\rho) = q^{1/3}(\vartheta_c)|\frac{x}{\gamma}(1+\ell(y,\vartheta_c)) - \frac{\tilde{x}_\varepsilon}{\gamma}(1+\ell(\tilde{y}_\varepsilon,\vartheta_c))| + O(\sqrt{\gamma}),$$

$$\gamma^{-3/2}|\partial_\rho\tilde{\Psi}^{\varepsilon}_{N,a,\gamma}|(t,x,y,\Sigma,A,\rho) \geq \left|\frac{(y+\nabla B_0(y,\vartheta_c))\cdot\vartheta_c + t}{\gamma^{3/2}} - \frac{(\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon,\vartheta_c))\cdot\vartheta_c + \tilde{t}_\varepsilon}{\gamma^{3/2}}\right|$$

$$- \frac{1}{2}q^{2/3}(\vartheta_c)\left|\frac{(t-B_2(y,\vartheta_c))}{\sqrt{\gamma}} - \frac{(\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon,\vartheta_c))}{\sqrt{\gamma}}\right|$$

$$- q^{1/3}(\vartheta_c)|\frac{x}{\gamma}(1+\ell(y,\vartheta_c)) - \frac{\tilde{x}_\varepsilon}{\gamma}(1+\ell(\tilde{y}_\varepsilon,\vartheta_c))| - O(\sqrt{\gamma}),$$

and therefore there exists a uniform constant $C > 0$ (depending only on q) such that

$$(3.64) \quad C\gamma^{-3/2}(|\partial_A \tilde{\Psi}^\varepsilon_{N,a,\gamma}| + |\partial_\Sigma \tilde{\Psi}^\varepsilon_{N,a,\gamma}| + |\partial_\rho \tilde{\Psi}^\varepsilon_{N,a,\gamma}|)(t, x, y, \Sigma, A, \rho)$$

$$\geq \Big[\Big|\frac{(t - B_2(y, \vartheta_c))}{\sqrt{\gamma}} - \frac{(\tilde{t}_\varepsilon - B_2(\tilde{y}_\varepsilon, \vartheta_c))}{\sqrt{\gamma}}\Big| + \Big|\frac{x}{\gamma}(1 + \ell(y, \vartheta_c)) - \frac{\tilde{x}_\varepsilon}{\gamma}(1 + \ell(\tilde{y}_\varepsilon, \vartheta_c))\Big|$$

$$+ \Big|\frac{(y + \nabla B_0(y, \vartheta_c)) \cdot \vartheta_c + t)}{\gamma^{3/2}} - \frac{(\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon, \vartheta_c)) \cdot \vartheta_c + \tilde{t}_\varepsilon)}{\gamma^{3/2}}\Big| - O(\sqrt{\gamma})\Big].$$

As $(y + \nabla B_0(y, \vartheta_c)) \cdot \vartheta_c = -|y + \nabla B_0(y, \vartheta_c)| + O(\gamma^2)$ by Lemma 3.8, $(\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon, \vartheta_c)) \cdot \vartheta_c = -|\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon, \vartheta_c)| + O(\gamma^2)$ by construction and Lemma 3.8, the last line in (3.64) may be rewritten as

$$\Big|\frac{|y + \nabla B_0(y, \vartheta_c)| - t}{\gamma^{3/2}} - \frac{|\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon, \vartheta_c)| - \tilde{t}_\varepsilon}{\gamma^{3/2}}\Big| - O(\sqrt{\gamma}).$$

As $(\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{y}_\varepsilon) = (\tilde{t}_\varepsilon, \tilde{x}_\varepsilon, \tilde{y}_\varepsilon)(\Sigma, A, \rho, \vartheta_c)$ is the integral curve associated to $(\Sigma, A, \rho, \vartheta_c(t, x, y, \Sigma, A))$, it follows from the definition of $\tilde{\vartheta}(t, y)$ in (3.10) and Lemma 3.12 (applied to ϑ_c and $\tilde{\vartheta}(\tilde{t}_\varepsilon, \tilde{y}_\varepsilon)$) that

$$(3.65) \qquad\qquad |\tilde{\vartheta}(\tilde{t}_\varepsilon, \tilde{y}_\varepsilon) - \vartheta_c(t, x, y, \Sigma, A)| = O(\gamma^{3/2}/\tilde{t}_\varepsilon),$$

which further yields

$$\big|\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon, \vartheta_c)\big| - \big|\tilde{y}_\varepsilon + \nabla B_0(\tilde{y}_\varepsilon, \tilde{\vartheta}(\tilde{t}_\varepsilon, \tilde{y}_\varepsilon))\big| = O\Big(\frac{|\tilde{y}_\varepsilon|^2}{\tilde{t}_\varepsilon}\gamma^{3/2}\Big).$$

We now consider the difference between ϑ_c and $\tilde{\vartheta}(t, y)$: the assumption of Proposition 3.13 is $\mathcal{N}(t, x, y) \neq \emptyset$, hence there exists $N_0 \in \mathcal{N}(t, x, y)$ for which $\tilde{\Phi}_{N_0,a,\gamma}$ is stationary with respect to A, ρ, ϑ and from Lemma 3.10 it follows that (3.38) must hold. Then $\tilde{\vartheta}(t, y)$ is an approximation modulo $O(\gamma^{3/2}/t)$ of the critical point ϑ (when we consider all variables (A, ρ, ϑ)) satisfying (3.39). From the formulas (3.33) for ϑ_c and (3.39) for $\vartheta_c^\sharp$, we always have $|\vartheta_c - \vartheta_c^\sharp| \lesssim \gamma$, but this is not small enough to conclude. If $|\vartheta_c - \vartheta_c^\sharp| \lesssim \gamma^{3/2}/t$, then, using (3.63), (3.64), (3.65), we obtain

$$|(y + \nabla B_0(y, \vartheta_c)) \cdot \vartheta_c - (y + \nabla B_0(y, \tilde{\vartheta}(t, y))) \cdot \tilde{\vartheta}(t, y)| = O(|y|^2\gamma^{3/2}/t) = O(|y|\gamma^{3/2}).$$

Suppose that, with $\vartheta_c^\sharp$ given in (3.39) from Lemma 3.10, we have $\gamma \gtrsim |\vartheta_c - \vartheta_c^\sharp| \gtrsim \gamma^{3/2}/t$ such that

$$(3.66) \qquad\qquad |(y + \nabla B_0(y, \vartheta_c)) \cdot \vartheta_c - (y + \nabla B_0(y, \vartheta_c^\sharp)) \cdot \vartheta_c^\sharp| \geq r_0\gamma^{3/2}.$$

Let $C_q \geq 2$ be such that $\|q(\vartheta)\|_\infty \leq C_q$. If the first line in (3.63) is bounded from below by $r_0/(10C_q)$, then using $|\vartheta_c - \tilde{\vartheta}(t, y)| \leq |\vartheta_c - \vartheta_c^\sharp| + |\vartheta_c^\sharp - \tilde{\vartheta}(t, y)| \lesssim \gamma$, the second line in (3.64) is bounded from below by $r_0/(20C_q)$ (if $\sqrt{\gamma}$ is small compared to r_0) and we are done. Therefore we are reduced to considering the first line in (3.63) bounded from above by $r_0/(10C_q)$, which is the same as assuming that $|\partial_A \tilde{\Psi}^\varepsilon(t, x, y, \Sigma, A, \rho)| \leq \gamma^{3/2}r_0/10$ and $|\partial_\Sigma \tilde{\Psi}^\varepsilon_{N,a,\gamma}(t, x, y, \Sigma, A, \rho)| \leq \gamma^{3/2}r_0/10$. We prove that, if, moreover, (3.66) holds, then $\gamma^{-3/2}|\partial_\rho \tilde{\Psi}^\varepsilon_{N,a,\gamma}(t, x, y, \Sigma, A, \rho)| \geq r_0/5$. As $\gamma^{-3/2}|\partial_A \tilde{\Psi}^\varepsilon(t, x, y, \Sigma, A, \rho)| \leq \frac{r_0}{10}$, we find

$$\Big|\frac{q^{2/3}(\vartheta_c)}{2}\frac{(t - B_2(y, \vartheta_c))}{\sqrt{\gamma}} + (\Sigma - S_\varepsilon(a, \gamma, \vartheta_c, A)) + O(\sqrt{\gamma}) - 2NA^{1/2}(1 - \frac{3}{4}B'_L(\rho\lambda_\gamma A^{3/2}))\Big| \leq r_0/10.$$

Recall that

$$\gamma^{-3/2}\partial_\rho\tilde\Psi^\varepsilon_{N,a,\gamma} = \gamma^{-3/2}\Big((y+\nabla B_0(y,\vartheta_c))\cdot\vartheta_c + t\Big) - \frac{Aq^{2/3}(\vartheta_c)}{2}\frac{(t-B_2(y,\vartheta_c))}{\sqrt\gamma}$$
$$+ \Big(\Sigma^3/3 + \Sigma(\frac{x}{\gamma}q^{1/3}(\vartheta_c)(1+\ell(y,\vartheta_c)) + \frac{2}{3}S_\varepsilon(a,\gamma,\vartheta_c,A)\Big) - \frac{4}{3}NA^{3/2}(1-B'_L(\rho\lambda_\gamma A^{3/2})) + O(\sqrt\gamma)$$

and therefore, by substitution in the $NA^{3/2}$ term,

$$\Big|\gamma^{-3/2}\partial_\rho\tilde\Psi^\varepsilon_{N,a,\gamma} - \Big(\gamma^{-3/2}\Big((y+\nabla B_0(y,\vartheta_c))\cdot\vartheta_c + t\Big) - \frac{Aq^{2/3}(\vartheta_c)}{6}\frac{(t-B_2(y,\vartheta_c))}{\sqrt\gamma}$$
$$+ \Big(\Sigma^3/3 + \Sigma\frac{x}{\gamma}q^{1/3}(\vartheta_c)(1+\ell(y,\vartheta_c)) + \frac{2}{3}S_\varepsilon(a,\gamma,\vartheta_c,A) - \frac{2}{3}(\Sigma - S_\varepsilon(a,\gamma,\vartheta_c,A))\Big)\Big)\Big| \le r_0/10\,.$$

With $\vartheta_c^\sharp$ given by (3.39) in Lemma 3.10 (recall that it is independent of N), we have

$$\gamma^{-3/2}\Big((y+\nabla B_0(y,\vartheta_c^\sharp))\cdot\vartheta_c^\sharp + t\Big) - \frac{Aq^{2/3}(\vartheta_c^\sharp)}{6}\frac{(t-B_2(y,\vartheta_c^\sharp))}{\sqrt\gamma}$$
$$+ \Big(\Sigma^3/3 + \Sigma\frac{x}{\gamma}q^{1/3}(\vartheta_c^\sharp)(1+\ell(y,\vartheta_c)) + \frac{2}{3}S_\varepsilon(a,\gamma,\vartheta_c^\sharp,A) - \frac{2}{3}(\Sigma - S_\varepsilon(a,\gamma,\vartheta_c^\sharp,A))\Big) + O(\sqrt\gamma) = 0.$$

Taking the difference between the last two equations and using that $|\vartheta_c - \vartheta_c^\sharp| \lesssim \gamma$ and that differences always provide functions of $\vartheta_c - \vartheta_c^\sharp$, we get, using (3.66)

$$\gamma^{-3/2}|\partial_\rho\tilde\Psi^\varepsilon_{N,a,\gamma}| = \gamma^{-3/2}\Big|(y+\nabla B_0(y,\vartheta_c))\cdot\vartheta_c - (y+\nabla B_0(y,\vartheta_c^\sharp))\cdot\vartheta_c^\sharp\Big| - O(\sqrt\gamma) - r_0/10 \ge \frac{r_0}{5}\,.$$

Therefore we always have $\Big|\nabla_{(\Sigma,A,\rho)}\tilde\Psi^\varepsilon_{N,a,\gamma}\Big| \gtrsim r_0/5$, which completes the proof in the case $4a \lesssim \gamma$. $\qquad\square$

Let now $\gamma \sim a$. We write $\Phi_{N,a,a} := \Phi_{N,a,\gamma\sim a}$ and rescale variables as follows $s = \sqrt{a}|\theta|^{1/3}S$, $\sigma\sqrt{a}|\theta|^{1/3}\Sigma$, $\alpha = a|\theta|^{2/3}A$ and let $\lambda = \lambda_a = \frac{a^{3/2}}{h}$. We define

$$(3.67)\qquad \tilde\Phi_{N,a,a}(t,x,y,\Sigma,S,A,\theta) := \Phi_{N,a,a}(t,x,y,\sqrt{a}|\theta|^{1/3}\Sigma,\sqrt{a}|\theta|^{1/3}S,a|\theta|^{2/3}A,\theta).$$

Let $s_0(a,\theta,\alpha)$ be the unique solution to $\partial_s^2\Phi(a,0,\theta,\alpha,s) = 0$ and let $s_\pm(a,\theta,\alpha)$ be the critical points of $\Phi(a,0,\theta,\alpha)$ (see Lemma 6.9 from the Appendix for details on critical points $s_\pm$ of $\Phi(x,y,\theta,\alpha,\sigma)$), then after rescaling variables we obtain at most two critical points such that

$$S_\pm(a,\vartheta,A) - S_0(a,\vartheta,A) = \pm\sqrt{\frac{\zeta(a,0,\vartheta,aA)}{a}}(1 + O(\sqrt{\zeta(a,0,\vartheta,aA)})),$$

where $\zeta(a,0,\vartheta,aA)/a = A - e_0(a,0,\vartheta,aA)$ is the phase function introduced in Theorem 2.1, with e_0 elliptic and close to 1. In this case $\zeta(a,0,\vartheta,aA)/a$ is close to 0 and $S_\pm$ are real only for $\zeta(a,0,\vartheta,aA) \ge 0$. We now repeat the argument from the model case : let $\chi_0 \in C^\infty$ be a smooth cut-off, equal to 1 on $[0,\infty]$ and equal to 0 on $[-\infty,-2]$. Then $(\chi_0 Ai)(-(|\theta|\lambda)^{2/3}\zeta(a,0,\vartheta,aA))$ is a symbol of order 2/3 supported for values $(|\theta|\lambda)^{2/3}\zeta(a,0,\vartheta,aA) \le 2$ and $(1-\chi_0)Ai(-(|\theta|\lambda)^{2/3}\zeta(\cdots)))$ is supported on $A \ge e_0(a,0,\vartheta,aA)$ with value 1 on $(|\theta|\lambda)^{2/3}\zeta(a,0,\vartheta,aA) \ge 2$. On the support of $(1-\chi_0)$ the Airy function may be written as a sum of two contributions $A_\pm((|\theta|\lambda)^{2/3}\zeta(a,0,\vartheta,aA)))$ corresponding to critical points $S_\pm$. We split the symbol of V_N in two parts using $\chi_0 + (1-\chi_0) = 1$ and on the support of χ_0 the Airy function behaves as a symbol of order 2/3. Therefore we write each integral $V_{N,a}(t,x,y)$ as a sum $V_{N,a}(t,x,y) = \sum_{\varepsilon\in\{0,\pm\}} V^\varepsilon_{N,a}(t,x,y)$ where for $\varepsilon \in \{\pm\}$, V^ε_N has

phase $\tilde{\Phi}_{N,a,a}(t, x, y, \Sigma, S_\varepsilon, A, \theta)$ while $V_{N,a}^0$ has phase $\tilde{\Phi}_{N,a,a}(t, x, y, \Sigma, S_0(a, \vartheta, A), A, \theta)$. It remains to prove that, for each $\varepsilon \in \{0, \pm\}$,

$$\sum_{N \notin \mathcal{N}_d^1(t,x,y), |N| \lesssim 1/\sqrt{a}} V_{N,a}^\varepsilon(t, x, y) = O(h^\infty).$$

For $\varepsilon \in \{\pm\}$ we act exactly like in the transverse case $4a \leq \gamma$, as on the support of $(1 - \chi_0)$ we have two separate critical points $S_\pm$. For $\varepsilon = 0$ we use that we got rid of variable S hence $\partial_S \tilde{\Phi}_{N,a,a}(t, x, y, \Sigma, S_0(a, \vartheta, A), A, \theta)$ and act again as in the previous case, finally completing the proof of Proposition 3.13. □

In the last part of this section we prove Theorem 3.1 : writing $\mathcal{P}_{h,a}$ as the sum over γ, we evaluate each $\mathcal{P}_{h,a,\gamma}$ and then sum up in $a \lesssim \gamma \ll 1$. We deal separately with the cases $\gamma > 4a$ and $\gamma \sim a$ when use the notation $\mathcal{P}_{h,a,a}$ for $\mathcal{P}_{h,a,\gamma\sim a}$ and $V_{N,a}$ for $V_{N,\gamma\sim a}$. From (2.42) and (3.6) we have

$$V_N = \sum_{a \lesssim \gamma \ll 1} V_{N,\gamma} = V_{N,a} + \sum_{4a \leq \gamma} V_{N,\gamma}.$$

Remark 3.20. In (3.6) we set $\omega = h^{-2/3}\alpha$, and then $\alpha = \gamma A$, which yields $\omega = \lambda_\gamma^{2/3} A$, with $\lambda_\gamma = \gamma^{3/2}/h$ and $A \sim 1$ on the support of ψ. Since the "main" contribution in the parametrix (2.37) comes from values $\omega \sim \lambda^{2/3}$ with $\lambda = \lambda_a = \frac{a^{3/2}}{h}$ (or, in terms of $\mathcal{P}_{h,a}$ written as a sum (2.38) with $\omega_k \sim k^{2/3}$, the main contribution comes from $k \sim \lambda$), the part $\mathcal{P}_{h,a,a}$ will provide the "worst" case scenario.

Before stating estimates for $V_{N,\gamma}$, when $|N| \geq 1$ and $a \lesssim \gamma$, we recall that we have the free space dispersion for V_0, assuming that $t > h$ so that the dispersive effect takes over:

$$(3.68) \qquad |V_0(t, x, y)| \leq Ch^{-d} \left(\frac{h}{t}\right)^{\frac{d-1}{2}}.$$

Proposition 3.21. *Assume* $\epsilon > 0$, $a \in [h^{2/3-\epsilon}, a_0]$, *and* $h \in (0, 1)$. *Then there exists* $C(\epsilon, a_0)$ *such that, for* $t \gtrsim h$,

$$(3.69) \qquad |V_{\pm 1}(t, x, y)| \leq Ch^{-d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \left(\left(\frac{h}{t}\right)^{1/2} + a^{1/4} \left(\frac{h}{t}\right)^{1/4} + h^{\frac{1}{3}}\right).$$

Proposition 3.22. *Assume* $\epsilon > 0$, $a \in [h^{2/3-\epsilon}, a_0]$, $\gamma \geq 4a$, $h \in (0, 1)$ *and let* $\lambda_\gamma = \gamma^{3/2}/h$. *Then there exists* $C(\epsilon, a_0)$ *such that, for* $t \gtrsim \sqrt{\gamma}$,

$$(3.70) \qquad |\sum_{|N| \geq 2} V_{N,\gamma}(t, x, y)| \leq Ch^{-d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} h^{1/3} \left(\frac{\gamma^{1/4}}{t^{1/2}} + \frac{1}{\lambda_\gamma^{3/2}}\right).$$

Proposition 3.23. *Assume* $\epsilon > 0$, $a \in [h^{2/3-\epsilon}, a_0]$, $h \in (0, 1)$ *and let* $\lambda = a^{3/2}/h$. *Then there exists* $C(\epsilon, a_0)$ *such that for* $0 \leq x \leq 2a$,

$$(3.71) \qquad |\sum_{|N| \geq 2} V_{N,a}(t, x, y)| \leq Ch^{-d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \left(a^{1/4} \left(\frac{h}{t}\right)^{1/4} + \frac{h^{1/3}}{\lambda^{4/3}}\right).$$

Our main estimate, (3.2) from Theorem 3.1, follows at once from the previous propositions, in the regime $a \geq h^{2/3-\epsilon}$. Recall that for $a \lesssim \gamma$, $V_{N,\gamma}$ was defined in (3.6) as

$$V_{N,\gamma}(t,x,y) = \frac{1}{2\pi h^{d+1}} \int e^{\frac{i}{h}(t\tau_q(\alpha,\theta)+\Phi(x,y,\theta,\alpha,\sigma)-\Phi(a,0,\theta,\alpha,s)-NhL(h^{-2/3}\alpha))}\psi(\alpha/(|\theta|^{2/3}\gamma))$$

$$\times \chi^\flat(\alpha/\epsilon_0)\chi^\sharp(\alpha/h^{2/3})\chi(s)p_h(x,y,\theta,\alpha,\sigma)\tilde{q}_h(\theta,\alpha,s)ds\,d\theta d\sigma d\alpha\,,$$

where the symbol of $V_{N,\gamma}$ (the same for every N) is of order 0. Let $|N| \geq 1$ and $a \lesssim \gamma$, and set $\lambda_\gamma = \frac{\gamma^{3/2}}{h}$. We rescale variables as follows

- If $\gamma \sim a$ we set $s = \sqrt{a}|\theta|^{1/3}S$, $\sigma\sqrt{a}|\theta|^{1/3}\Sigma$, $\alpha = a|\theta|^{2/3}A$ and let $\lambda = \frac{a^{3/2}}{h}$;
- If $4a \leq \gamma$ we set $s = \sqrt{\gamma}|\theta|^{1/3}S$, $\sigma\sqrt{\gamma}|\theta|^{1/3}\Sigma$, $\alpha = \gamma|\theta|^{2/3}A$.

We let $\tilde{\Phi}_{N,a,a}(t,x,y,\Sigma,S,A,\theta)$ be given by (3.67) when $\gamma \sim a$ and $\tilde{\Phi}_{N,a,\gamma}(t,x,y,\Sigma,S,A,\theta)$ be given by (3.28) when $4a \leq \gamma$. We have (including the case where γ is replaced by a)

$$\tilde{\Phi}_{N,a,\gamma}(t,x,y,\Sigma,S,A,\theta) = \tilde{\Phi}_{0,a,\gamma}(t,x,y,\Sigma,S,A,\theta) - \frac{4}{3}N\gamma^{3/2}|\theta|A^{3/2} + NhB_L(|\theta|\lambda_\gamma A^{3/2}).$$

The phase $\tilde{\Phi}_{0,a,\gamma}(t,x,y,\Sigma,S,A,\theta)$, with $\theta = \rho\vartheta$, $|\vartheta| = 1$, has two critical points $\vartheta_\pm$ and they are non-degenerate. Indeed (see the proof of Proposition 3.4), $\nabla^2_\vartheta\tilde{\Phi}_{0,a,\gamma} = |y|O(1)$ for any $a \lesssim \gamma$. From Lemma 3.2 we have $|y| \geq c_0|t|$ and stationary phase in $\vartheta \in \mathbb{S}^{d-2}$ yields a decay factor

(3.72) $$\left(\frac{h}{|y|}\right)^{\frac{d-2}{2}} \leq C\left(\frac{h}{|t|}\right)^{\frac{d-2}{2}}, \quad \text{for } |y| \geq c_0|t|.$$

Recall that the critical points (in ϑ) of $\Phi_{N,a,\gamma}$ are (3.33) (Lemma 3.8). According to Remark 3.9, if we fix a sign for t, only one of these critical points provide non-trivial contributions. Let $t > 0$ and denote $\vartheta_c = \vartheta_-(t,x,y,\Sigma,S,A)$ the critical point in ϑ, and let

$$\Psi_{N,a,\gamma}(t,x,y,\Sigma,S,A,\rho) = \tilde{\Phi}_{N,a,\gamma}(t,x,y,\Sigma,S,A,\rho\vartheta_c)\,.$$

Recall from Lemma 3.8 that ϑ_c doesn't depend on N, neither on $\rho = |\theta|$ (since $\tilde{\Phi}_{0,a,\gamma}(t,x,y,\Sigma,S,A,\theta)$ is linear in $|\theta|$).

Lemma 3.24. *The critical points ϑ_c are such that*

(3.73) $$\partial_\Sigma\vartheta_c = O(\gamma^{3/2}/t), \quad \partial_S\vartheta_c = O(\gamma^{3/2}/t).$$

Proof. We evaluate its derivatives in Σ and S of ϑ_c using the equation (3.33) from Lemma 3.8. $\square$

Remark 3.25. For $|N| \leq \lambda_\gamma$, the factor $e^{iNB_L(\rho\lambda_\gamma A^{3/2})}$ does not oscillate. Indeed $NB_L(\rho\lambda_\gamma A^{3/2}) \sim N/\lambda_\gamma \lesssim 1$, and moving this factor to the symbol, the phase becomes linear in ρ. On the other hand, as soon as $N > \lambda_\gamma$ we can take advantage of the stationary phase in ρ, as we shall see below. Applying the stationary phase in ρ turns out to be of particular interest for $N \geq \lambda_\gamma^2$, since in this case for a given t such that $t/\sqrt{\gamma} \geq \lambda_\gamma^2$, the cardinal of the set $\mathcal{N}_1(t,x,y)$ is not uniformly bounded anymore but starts to increase like $\frac{t}{\sqrt{\gamma}\lambda_\gamma^2}$.

3.2. **The tangent part $\gamma \sim a$.** Let first $a \sim \gamma$ and set $\lambda = \lambda_a = a^{3/2}/h$. We apply stationary phase to $\Psi_{N,a,a}$ with respect to A.

Lemma 3.26. *The equation $\partial_A\Psi_{N,a,a} = 0$ has at most one solution on the support of the symbol $\psi(A)$, that we denote A_c. Then A_c is a non-degenerate critical point and $\partial_A^2\Psi_{N,a,a}|_{A=A_c} \sim Na^{3/2}$.*

Proof. The phase is stationary when $\partial_A \Psi_{N,a,a} = \partial_A \tilde{\Phi}_{N,a,a}|_{\vartheta=\vartheta_c} = 0$. The equation $\partial_A \tilde{\Phi}_{N,a,a} = 0$ is the first in (3.30) and we easily see that for any $N \neq 0$ there exists a solution $A_c^{1/2} \sim \frac{t}{4N\sqrt{a}}$. For $N \sim t/(4\sqrt{a})$ we have $A_c \sim 1$. Moreover,

$$\partial_A^2 \Psi_{N,a,\gamma} = \partial_A^2 \tilde{\Phi}_{N,a,\gamma}|_{\vartheta=\vartheta_c} + \partial_A \vartheta_c \partial_A \nabla_\vartheta \tilde{\Phi}_{N,a,\gamma}|_{\vartheta=\vartheta_c} \,.$$

The derivative with respect to A of the first term in the left hand side of (3.30) is $t\, O(a^2)$. Using (3.32), $\partial_A^2(\Phi(x,y,\theta,a|\theta|^{2/3}A, \sqrt{a}|\theta|^{1/3}\Sigma) - \Phi(a,0,\theta,a|\theta|^{2/3}A, \sqrt{a}|\theta|^{1/3}S)) = |y|O(a^2)$. As the derivative of the coefficient of N in (3.30) is close to $Na^{3/2}$ and $|N| \sim |t|/4\sqrt{a}$, $|t| \sim |y|$, the main contribution of $\partial_A^2 \tilde{\Phi}_{N,a,a}|_{\vartheta=\vartheta_c}$ is also $Na^{3/2}$. On the other hand,

$$(3.74) \qquad \partial_A \nabla_\vartheta \tilde{\Phi}_{N,a,a}|_{\vartheta=\vartheta_c} = \partial_A \nabla_\vartheta \tilde{\Phi}_{0,a,a}|_{\vartheta=\vartheta_c} \sim t\, O(a).$$

Moreover, taking the derivative with respect to A of $\nabla_\vartheta \tilde{\Phi}_{0,a,a}|_{\vartheta=\vartheta_c} = 0$ gives $\partial_A \vartheta_c \nabla_\vartheta^2 \tilde{\Phi}_{0,a,a} = -\partial_A \nabla_\vartheta \tilde{\Phi}_{0,a,a}$; using $\left(\nabla_\vartheta^2 \tilde{\Phi}_{0,a,a}\right)^{-1} = |y|^{-1}O(1)$, $|y| \geq c_0|t|$ and (3.74) eventually yields

$$\partial_A \vartheta_c \partial_A \nabla_\vartheta \tilde{\Phi}_{0,a,a}|_{\vartheta=\vartheta_c} \sim \left|\partial_A \nabla_\vartheta \tilde{\Phi}_{0,a,a}\right|^2 \times |y|^{-1}O(1) \sim \frac{t^2}{|y|}O(a^2) \lesssim \frac{|t|}{c_0}O(a^2) \,.$$

Therefore the main contribution of $\partial_A^2 \Psi_{N,a,a}$ is $Na^{3/2}$. $\qquad \square$

Hence we may apply stationary phase in A and get another decay factor

$$(3.75) \qquad \frac{1}{\sqrt{a^{3/2}|N|}} \times \frac{1}{\sqrt{1/h}} = \frac{1}{(\lambda|N|)^{1/2}} \,.$$

Lemma 3.27. *Assume $N\sqrt{a} \lesssim 1$. The critical point A_c is*

$$(3.76) \qquad A_c = \mathbb{A}_0^2 - 2\mathbb{A}_0\mathbb{A}_1 + \left(\frac{\Sigma}{2N} - \frac{S}{2N}\right)^2 + O\left(a(\frac{\Sigma}{N}, \frac{S}{N})^2\right) + \frac{f_0}{N\lambda^2} \,,$$

$$(3.77) \qquad \mathbb{A}_0 = \frac{q^{2/3}(\vartheta_c)|_{\Sigma=S=0}}{4N\sqrt{a}}(t + E_0(y,a)), \quad \mathbb{A}_1 = \frac{\Sigma}{2N}(1 - xE_1) - \frac{S}{2N}(1 - aE_2) \,,$$

and f_0 is an asymptotic expansion in λ^{-1}, $E_0 = O(|y|^2, a|y|)$, $E_{1,2} = O(1)$, $E_{0,1,2}$ are independent of Σ and S.

Proof. At the critical point A_c, (3.30) holds (replacing ϑ with ϑ_c and γ with a). Using (3.32),

$$(3.78) \quad A^{1/2}(1 - \frac{3}{4}B_L'(\rho\lambda A^{3/2})) = \frac{q^{2/3}(\vartheta_c)\Big(t - B_2(y,\vartheta_c) + \sum_{k\geq 2}\partial_\tau\Big(\frac{(1-\tau)^k}{\tau^{k-1}}\Big)B_{2k}(y,\vartheta_c)\Big)}{4N\sqrt{a}\sqrt{1 + aAq^{2/3}(\vartheta_c)}}\Big|_{\tau=\tau_q(aA,\vartheta_c)}$$

$$+ \frac{(-\Sigma + S)}{2N} + \frac{aq^{2/3}(\vartheta_c)}{2N\sqrt{1 + aAq^{2/3}(\vartheta_c)}}\left[\frac{x}{a^{3/2}}\partial_\tau\Big(\tau A_\Gamma(x,y,\sqrt{a}\Sigma q^{1/3}(\vartheta_c)/\tau, \vartheta_c/\tau)\Big)\Big|_{\tau=\tau_q(aA,\vartheta_c)}\right.$$

$$\left. - \frac{a}{a^{3/2}}\partial_\tau\Big(\tau A_\Gamma(a,0,\sqrt{a}S q^{1/3}(\vartheta_c)/\tau, \vartheta_c/\tau)\Big)\Big|_{\tau=\tau_q(aA,\vartheta_c)}\right] \,,$$

where we recall that $B_{2k}(y,\vartheta_c) = O(|y|)$ for all $k \geq 2$ and $1 - \tau_q(aA, \vartheta_c) = \frac{aAq^{2/3}(\vartheta_c)}{(1 + \tau_q(aA,\vartheta_c))}$.

Remark 3.28. When $|N| \lesssim \lambda$, according to Remark 3.25 we get rid of the factor $(1 - \frac{3}{4}B_L'(\rho\lambda A^{3/2}))$ in the LHS of (3.78), without which the equation satisfied by A_c is independent of ρ, λ.

At $\Sigma = S = 0$, we get for $A_c^{1/2}|_{\Sigma=S=0}$

$$(3.79) \quad A^{\frac{1}{2}}(1 - \frac{3}{4}B_L'(\rho\lambda A^{\frac{3}{2}})) = \frac{q^{2/3}(\vartheta_c)/(4N\sqrt{a})}{\sqrt{1 + aAq^{\frac{2}{3}}(\vartheta_c)}}\left[t - B_2(y,\vartheta_c) + \sum_{k\geq 2}\partial_\tau\left(\frac{(1-\tau)^k}{\tau^{k-1}}\right)B_{2k}(y,\vartheta_c)\right.$$
$$\left. + 2a\left(\frac{x}{a}\partial_\tau\left(\tau A_\Gamma(x,y,0,\vartheta_c/\tau)\right) - \frac{a}{a}\partial_\tau\left(\tau A_\Gamma(a,0,0,\vartheta_c/\tau)\right)\right)\right]\Big|_{\tau=\tau_q(aA,\vartheta_c),\Sigma=S=0}.$$

Using that $\partial_\tau(\tau A_\Gamma) = O(1)$ and the expansion (2.21) of B_L, by the implicit function theorem applied to (3.79), we obtain that $A_c^{1/2}|_{\Sigma=S=0}$ is of the form $\mathbb{A}_0$ given in (3.77). Taking the derivative of (3.78) with respect to Σ yields

$$\partial_\Sigma(A_c^{1/2})\left(1 - \frac{3}{4}B_L'(z) - \frac{9}{4}zB_L''(z)\right)|_{z=\rho\lambda A^{3/2}} = \frac{q^{2/3}(\vartheta_c)}{4N\sqrt{a}\sqrt{1 + aA_cq^{2/3}(\vartheta_c)}}\left[\partial_\Sigma\vartheta_c\cdot\left(-\nabla B_2(y,\vartheta_c)\right)\right.$$
$$\left. + \sum_{k\geq 2}\partial_\tau\left(\frac{(1-\tau)^k}{\tau^{k-1}}\right)\nabla B_{2k}(y,\vartheta_c)\right) + (\partial_\Sigma A_c\partial_A\tau + \partial_\Sigma\vartheta_c\cdot\nabla_\vartheta\tau)\sum_{k\geq 2}\partial_{\tau,\tau}^2\left(\frac{(1-\tau)^k}{\tau^{k-1}}\right)B_{2k}(y,\vartheta_c)\right]\Big|_{\tau=\tau_q(aA_c,\vartheta_c)}$$
$$+ \partial_\Sigma\left(\frac{q^{2/3}(\vartheta_c)}{\sqrt{1 + aA_cq^{2/3}(\vartheta_c)}}\right)\frac{\left(t - B_2(y,\vartheta_c) + \sum_{k\geq 2}\partial_\tau\left(\frac{(1-\tau)^k}{\tau^{k-1}}\right)B_{2k}(y,\vartheta_c)\right)}{4N\sqrt{a}}\Big|_{\tau=\tau_q(aA_c,\vartheta_c)}$$
$$- \frac{1}{2N} + \frac{\sqrt{a}}{2N}\partial_\Sigma\left(\frac{q^{2/3}(\vartheta_c)}{\sqrt{1 + aA_cq^{2/3}(\vartheta_c)}}\right)\frac{x}{a}\partial_\tau\left(\tau A_\Gamma(x,y,\sqrt{a}\Sigma q^{1/3}(\vartheta_c)/\tau,\vartheta_c/\tau)\right)|_{\tau=\tau_q(aA,\vartheta_c)}$$
$$+ \frac{\sqrt{a}}{2N}\frac{q^{2/3}(\vartheta_c)}{\sqrt{1 + aA_cq^{2/3}(\vartheta_c)}}\frac{x}{a}\partial_\Sigma\left[\partial_\tau\left(\tau A_\Gamma(x,y,\sqrt{a}\Sigma q^{1/3}(\vartheta_c)/\tau,\vartheta_c/\tau)\right)|_{\tau=\tau_q(aA,\vartheta_c)}\right],$$

where $\partial_A\tau_q = aq^{2/3}(\vartheta)/(2\tau_q)$, $B_{2k}(y,\vartheta_c) = O(|y|)$, $\nabla_\vartheta\tau_q = aA\nabla(q^{2/3}(\vartheta))/(2\tau_q)$ and $\partial_\Sigma(\partial_\tau(\tau A_\Gamma)) = O(a)$. At $\Sigma = S = 0$ we find

$$\partial_\Sigma(A_c^{1/2})|_{\Sigma=S=0} = \frac{2}{3}\partial_\Sigma\vartheta_c\nabla q(\vartheta_c)q^{-1/3}(\vartheta_c)\frac{(t + O(|y|^2) + O(a|y|))}{4N\sqrt{a}} - \frac{1}{2N}(1 + O(a^{3/2}) + O(\lambda^{-2})),$$

which, together with (3.78) and (3.73), allows to obtain the explicit form of the derivative of A_c with respect to Σ (similar computations hold for S). $\qquad\square$

Remark 3.29. When $|N| \geq \lambda^2$, stationary phase in ρ turns out to be of particular interest: for a given t such that $t/\sqrt{a} \geq \lambda^2$, the cardinal of the set $\mathcal{N}_1(t,x,y)$ is large. Obtaining a bound of the integral defining $V_{N,a}$ better than the one given by integrating only with respect to Σ, S turns out to be crucial in order to obtain the desired dispersive estimates.

Let us consider the case $|N| > C\lambda^2$ for some constant $C > 1$. Let

$$\phi_{N,a}(t,x,y,\Sigma,S,\rho) := \Psi_{N,a,a}(t,x,y,\Sigma,S,A_c,\rho)$$

denote the critical value of $\Psi_{N,a,a}$. We compute derivatives of $\phi_{N,a}$ with respect to ρ. Since ϑ_c is independent of ρ, the phase $\phi_{N,a}$ is stationary in ρ when

$$0 = \partial_\rho\phi_{N,a} = (\partial_\rho\Psi_{N,a,a} + \partial_\rho A_c\partial_A\Psi_{N,a,a})|_{A_c} = \partial_\rho\Psi_{N,a,a}|_{A_c},$$

with A_c provided in (3.76), where the coefficients of t, Σ, S (in the first three terms of the sum) are independent of ρ (notice that in the equation (3.78) verified by A_c, the only term containing ρ is B_L'

which amounts for the terms with factor $O(\lambda^{-2})$ in (3.76)). Since $\Psi_{N,a,a} = \Psi_{0,a,a} - \frac{4}{3}a^{3/2}\rho N A^{3/2} + NhB_L(\rho\lambda A^{3/2})$, and since $\Psi_{0,a,a} - \frac{4}{3}a^{3/2}\rho N A^{3/2}$ is linear in ρ, the second derivative is

$$\partial_\rho^2\phi_{N,a} = \left(\partial_\rho^2\Psi_{N,a,a} + \partial_\rho A_c(\partial_\rho\partial_A\Psi_{N,a,a} + \partial_\rho A_c\partial_A^2\Psi_{N,a,a})\right)|_{A_c}.$$

For $|N| > \lambda^2 \gg \lambda$, the only part that matters here is the contribution from $\frac{N}{\lambda}B_L(\rho\lambda A^{3/2})$. Taking into account that the support of the symbol of $V_{N,a}$ in A is a fixed, compact set of $(0,\infty)$, the contribution of $\partial_\rho^2\Psi_{N,a,a}$ will be hN/λ. The main contribution of $\partial_\rho\partial_A\Psi_{N,a,a}|_{A_c}$ is also of size hN/λ, but it comes with a factor $\partial_\rho A_c$, which, from (3.76), can be at most $O(1/N)$, since the first three terms in the sum defining A_c do not depend on ρ. Eventually we obtain,

$$(3.80) \qquad\qquad \frac{1}{h}|\partial_\rho^2\phi_{N,a}| \sim \frac{|N|}{\lambda},$$

hence the stationary phase in ρ will produce a factor $(\frac{|N|}{\lambda})^{-1/2}$ which is (of interest only in the regime $|N| \geq C\lambda$ and) particularly useful for $\frac{|N|}{\lambda} > \lambda$. Let $\sigma_{V,h,a}(\Sigma, S, \rho)$ be obtained from

$$\psi(A)p_h(x,y,\theta,a|\theta|^{2/3}A,\sqrt{a}|\theta|^{1/3}\Sigma)\tilde{q}_h(\theta,a|\theta|^{2/3}A,\sqrt{a}|\theta|^{1/3}S)$$

after applying the stationary phase in both ϑ and A. We note that $\sigma_{V,h,a}$ is independent of N, of order zero and has compact support in Σ, S. We are left with estimating the oscillatory integral $\int e^{\frac{i}{h}\phi_{N,a}}\sigma_{V,h,a}d\Sigma dSd\rho$. Following the approach in [12] (see also [9]), we deal separately with $|N| > \lambda^{1/3}$ and $|N| \leq \lambda^{1/3}$, where we recall that $\lambda = \lambda_a = a^{3/2}/h$.

3.2.1. *Large number of reflections:* $|N| > \lambda^{1/3}$.

Lemma 3.30. *There exists C (independent of N) such that, if $|N| \geq \lambda^{1/3}$,*

$$(3.81) \qquad\qquad \left|\int e^{\frac{i}{h}\phi_{N,a}}\sigma_{V,h,a}(\Sigma,S,\rho)d\Sigma dSd\rho\right| \leq \frac{C}{\lambda^{2/3}}.$$

Remark 3.31. The proof will be split in two parts: first, we consider $\lambda^{1/3} \leq |N| \leq \lambda^2$ where we integrate with respect to S, Σ and then bound the remaining integral in ρ owning to its compact support; then for $|N| > \lambda^2$ we start with stationary phase in ρ, which provides additional decay; then we integrate with respect to S, Σ following closely the approach for $\lambda^{1/3} \leq |N| \leq \lambda^2$.

Proof. We compute derivatives of $\phi_{N,a}$ with respect to Σ, S. Start with $\lambda^{1/3} \leq |N| \leq \lambda^2$ and follow [12, Proposition 4] (see also [9, Lemma 2.24], where only the case $N < \lambda^2$ is considered). Using $\tilde{\Phi}_{0,a,a}$ as provided in (3.29) for $N = 0$ and γ replaced by a, we compute

$$(3.82) \quad \partial_\Sigma\phi_{N,a} = \partial_\Sigma\tilde{\Phi}_{0,a,a}|_{\vartheta_c,A_c,\rho=|\theta|} = a^{3/2}\rho\Big[\Sigma^2 + \frac{x}{a}q^{1/3}(\vartheta_c) - A_c$$
$$+ \frac{x}{a}\frac{\tau_q(aA,\vartheta_c)}{\sqrt{a}}\partial_\Sigma A_\Gamma\Big(x,y,\frac{\sqrt{a}\Sigma q^{1/3}(\vartheta_c)}{\tau_q(aA,\vartheta_c)},\frac{\vartheta_c}{\tau_q(aA,\vartheta_c)}\Big)\Big]$$
$$= a^{3/2}\rho\Big[\Sigma^2 + \frac{x}{a}q^{1/3}(\vartheta_c) - A_c + \frac{x}{a}q^{1/3}(\vartheta_c)\partial_\Xi A_\Gamma(x,y,\Xi,\frac{\vartheta_c}{\tau_q(aA,\vartheta_c)})|_{\Xi=\frac{\sqrt{a}\Sigma q^{1/3}(\vartheta_c)}{\tau_q(aA,\vartheta_c)}}\Big].$$

In the same way we have

$$\partial_S \phi_{N,a} = \partial_S \tilde{\Phi}_{0,a,a}|_{\vartheta_c, A_c, \rho=|\theta|} = -a^{3/2}\rho\Big[S^2 + q^{1/3}(\vartheta_c) - A_c$$
$$+ \frac{\tau_q(aA, \vartheta_c)}{\sqrt{a}} A_\Gamma\Big(a, 0, \frac{\sqrt{a}Sq^{1/3}(\vartheta_c)}{\tau_q(aA, \vartheta_c)}, \frac{\vartheta_c}{\tau_q(aA, \vartheta_c)}\Big)\Big]$$
$$= -a^{3/2}\rho\Big[S^2 + q^{1/3}(\vartheta_c) - A_c + q^{1/3}(\vartheta_c)\partial_\Xi A_\Gamma(a, 0, \Xi, \frac{\vartheta_c}{\tau_q(aA, \vartheta_c)})|_{\Xi=\frac{\sqrt{a}Sq^{1/3}(\vartheta_c)}{\tau_q(aA, \vartheta_c)}}\Big].$$

The critical points are such that

$$\Sigma^2 + \frac{x}{a}q^{1/3}(\vartheta_c)\Big(1 + \partial_\Xi A_\Gamma(x, y, \Xi, \frac{\vartheta_c}{\tau_q(aA, \vartheta_c)})|_{\Xi=\frac{\sqrt{a}\Sigma q^{1/3}(\vartheta_c)}{\tau_q(aA, \vartheta_c)}}\Big) = A_c,$$
$$S^2 + q^{1/3}(\vartheta_c)\Big(1 + \partial_\Xi A_\Gamma(a, 0, \Xi, \frac{\vartheta_c}{\tau_q(aA, \vartheta_c)})|_{\Xi=\frac{\sqrt{a}Sq^{1/3}(\vartheta_c)}{\tau_q(aA, \vartheta_c)}}\Big) = A_c,$$

where $\partial_\Xi A_\Gamma(x, y, \Xi, \Theta) = \ell(y, \vartheta_c) + 2\Xi\mu(y, \vartheta_c) + \mathcal{H}_{j\geq 2}$, $\ell(0, \vartheta_c) = 0$, and where homogeneous terms of order j come with small factors $a^{j/2}$. We will prove that, although the determinant of the matrix of second derivatives may vanish, we can still use degenerate stationary phase with critical point of order at most 2 to conclude. We compute the second derivative with respect to Σ using (3.82):

$$(3.83) \qquad \partial_\Sigma^2 \phi_{N,a} = a^{3/2}\rho\Big[2\Sigma - \partial_\Sigma A_c + \frac{x}{a}\partial_\Sigma\Big(q^{1/3}(\vartheta_c)\partial_\Xi A_\Gamma(x, y, \Xi, \frac{\vartheta_c}{\tau_q(aA_c, \vartheta_c)})|_{\Xi=\frac{\sqrt{a}\Sigma q^{1/3}(\vartheta_c)}{\tau_q(aA_c, \vartheta_c)}}\Big)\Big],$$

where homogeneous terms of order j in $\partial_\Xi A_\Gamma$ come with small factors $a^{j/2}$ and

$$(3.84) \quad \partial_\Sigma\Big(q^{1/3}(\vartheta_c)\partial_\Xi A_\Gamma(x, y, \Xi, \frac{\vartheta_c}{\tau_q(aA_c, \vartheta_c)})|_{\Xi=\frac{\sqrt{a}Sq^{1/3}(\vartheta_c)}{\tau_q(aA_c, \vartheta_c)}}\Big) = \partial_\Sigma\Big(q^{1/3}(\vartheta_c)(\ell(y, \vartheta_c)$$
$$+ 2\mu(y, \vartheta_c)\frac{\sqrt{a}\Sigma q^{1/3}(\vartheta_c)}{\tau_q(aA_c, \vartheta_c)} + \mathcal{H}_{j\geq 2})\Big) = \partial_\Sigma \vartheta_c \nabla_\vartheta\Big(\ell(y, \vartheta)q^{1/3}(\vartheta) + 2\sqrt{a}\Sigma\mu(y, \vartheta)q^{2/3}(\vartheta) + O(a)\Big)|_{\vartheta=\vartheta_c}$$
$$+ 2\sqrt{a}\mu(y, \vartheta_c)q^{2/3}(\vartheta_c) + O(a).$$

Using that $\sqrt{a} \lesssim \frac{1}{N}$, $x \lesssim 2a$, we obtain an estimation of the second derivatives of $\phi_{N,a}$

$$(3.85) \qquad\qquad \partial_\Sigma^2 \phi_{N,a} = a^{3/2}\rho(2\Sigma + O(a^{3/2}/|y|) + O(1/N)),$$
$$\partial_S^2 \phi_{N,a} = a^{3/2}\rho(2S + O(a^{3/2}/|y|) + O(1/N)),$$
$$\partial_{\Sigma,S}^2 \phi_{N,a} = O(a^{3/2}/N).$$

We rescale variables $(\Sigma, S) = (\lambda^{-1/3}\mathsf{x}, \lambda^{-1/3}\mathsf{y})$, so that we are left with proving

$$(3.86) \qquad\qquad \left|\int e^{\frac{i}{\hbar}\phi_{N,a}(\lambda^{-1/3}\mathsf{x}, \lambda^{-1/3}\mathsf{y}, \rho)}\sigma_{V,h,a}(\lambda^{-1/3}\mathsf{x}, \lambda^{-1/3}\mathsf{y}, \rho)\,d\mathsf{x}d\mathsf{y}\right| \leq C.$$

From the compact support of $\sigma_{V,h,a}$ we obviously have, for any multi-index ν,

$$|\partial_{(\mathsf{x},\mathsf{y})}^\nu \sigma_{V,h,a}(\lambda^{-1/3}\mathsf{x}, \lambda^{-1/3}\mathsf{y}, \rho)| \leq C_\nu(1 + |\mathsf{x}| + |\mathsf{y}|)^{-|\nu|}.$$

Let $\mathbb{A}_0$ be the main term of $A_c^{1/2}$ defined (3.77); we define $(\mathfrak{A}, \mathfrak{B})$ as follows

$$-\mathfrak{A}\lambda^{-2/3} = \frac{x}{a}q^{1/3}(\vartheta_c)(1 + \ell(y, \vartheta_c)) - \mathbb{A}_0^2, \qquad -\mathfrak{B}\lambda^{-2/3} = q^{1/3}(\vartheta_c) - \mathbb{A}_0^2$$

and we also write $(\mathfrak{A}, \mathfrak{B}) = r(\sin\mathfrak{S}, \cos\mathfrak{S})$, where $r^2 = \mathfrak{A}^2 + \mathfrak{B}^2$. In the new variables, since $\frac{\partial\Sigma}{\partial\mathbf{x}} = \frac{\partial S}{\partial\mathbf{y}} = \lambda^{-1/3}$ and using $\partial_\Xi A_\Gamma$ as provided in (3.84), we find

$$
\frac{1}{h}\partial_\mathbf{x}\phi_{N,a} = \frac{a^{3/2}}{h}\lambda^{-1/3}\rho\Big[\lambda^{-2/3}(\mathbf{x}^2 - \mathfrak{A}) + 2\frac{\lambda^{-1/3}\mathbf{x}}{2N}\Big(\mathbb{A}_0(1 - xE_1) + \frac{x}{a}(N\sqrt{a})\mu(y,\vartheta_c)q^{2/3}(\vartheta_c)\Big)
$$
$$
- 2\frac{\lambda^{-1/3}\mathbf{y}}{2N}\Big(\mathbb{A}_0(1 - aE_2)\Big) + \frac{\lambda^{-1/3}}{N}O(\frac{\mathbf{y}}{N},\frac{\mathbf{x}}{N}) + O(a)\Big]
$$
$$
= \rho\Big((\mathbf{x}^2 - \mathfrak{A}) + \frac{\lambda^{1/3}}{N}\Big(\mathbf{x}(\mathbb{A}_0(1 - xE_1) + O(N\sqrt{a})) - \mathbf{y}\mathbb{A}_0(1 - aE_2) + O(\frac{\mathbf{y}}{N},\frac{\mathbf{x}}{N})\Big) + O(\lambda^{2/3}a)\Big)
$$
$$
= \rho(\mathbf{x}^2 - \mathfrak{A} + O(\mathbf{x},\mathbf{y}) + O(1)),
$$

where the term $O(a)$ in the second comes from the terms homogeneous of order $j \geq 2$ in the term $\partial_\Xi A_\Gamma(x,y,\Xi,\frac{\vartheta_c}{\tau_q(aA_c,\vartheta_c)})\big|_{\Xi = \frac{\sqrt{a}Sq^{1/3}(\vartheta_c)}{\tau_q(aA_c,\vartheta_c)}}$ (see (3.84)). We have written $O(\lambda^{2/3}a) = O((\lambda^{1/3}/N \times N\sqrt{a})^2) = O(1)$ since $\lambda^{1/3}/N < 1$ and $(Na^{1/2}) < 1$. Similarly, derivatives with respect to $\mathbf{x}$ and $\mathbf{y}$ are,

$$
\frac{1}{h}\partial_\mathbf{x}\phi_{N,a} = \rho(\mathbf{x}^2 - \mathfrak{A} + O(\mathbf{x},\mathbf{y}) + O(1)), \qquad \frac{1}{h}\partial_\mathbf{y}\phi_{N,a} = \rho(\mathbf{y}^2 - \mathfrak{B} + O(\mathbf{x},\mathbf{y}) + O(1)).
$$

From (3.85), we obtain, in the new coordinates

$$
(3.87) \qquad \frac{1}{h}\partial_\mathbf{x}^2\phi_{N,a} = \lambda^{1/3}\rho(2\lambda^{-1/3}\mathbf{x} + O(N^{-1})) = \rho(2\mathbf{x} + O(1)),
$$
$$
\frac{1}{h}\partial_\mathbf{y}^2\phi_{N,a} = \lambda^{1/3}\rho(2\lambda^{-1/3}\mathbf{y} + O(N^{-1})) = \rho(2\mathbf{y} + O(1)),
$$

where we used $|N| > \lambda^{1/3}$ in the last step together with $|y| \geq c_0 t$ which yields $a^{3/2}/|y| \lesssim a^{3/2}/t \sim a/N$, as $t/\sqrt{a} \sim 4N$ when the phase is stationary in A. Now, going back to (3.86), the integral is bounded for $0 \leq r \leq r_0$, for some $r_0 > 0$, by integration by parts for large $(\mathbf{x},\mathbf{y})$ (recall that the support of the integrand is now of radius $\lambda^{1/3}$). For $r_0 < r \lesssim \lambda^{2/3}$, set $(\mathbf{x},\mathbf{y}) = r^{1/2}(\mathbf{x}',\mathbf{y}')$, $\phi_{N,a} = r^{3/2}\tilde\phi_{N,a}$, $\tilde\sigma_{V,h,a}(\mathbf{x}',\mathbf{y}',.) = \sigma_{V,h,a}(r^{1/2}\lambda^{-1/3}\mathbf{x}', r^{1/2}\lambda^{-1/3}\mathbf{y}',\rho)$, and as $r^{1/2}\lambda^{-1/3}$ is bounded, we retain the decay $|\partial_{(\mathbf{x}',\mathbf{y}')}^\nu\tilde\sigma_{V,h,a}(\mathbf{x}',\mathbf{y}',.)| \leq C_\nu(1 + |\mathbf{x}'| + |\mathbf{y}'|)^{-|\nu|}$. It remains to prove

$$
r\Big|\int e^{i\frac{r^{3/2}}{h}\tilde\phi_{N,a}}\tilde\sigma_{V,h,a}\,d\mathbf{x}'d\mathbf{y}'\Big| \leq C.
$$

To begin with, notice that

$$
\partial_{\mathbf{x}'}(\frac{r^{3/2}}{h}\tilde\phi_{N,a}) = r^{3/2}\rho\Big(\mathbf{x}'^2 - \sin\mathfrak{S} + r^{-1/2}O(\mathbf{x}',\mathbf{y}') + r^{-1}O(1)\Big),
$$
$$
\partial_{\mathbf{y}'}(\frac{r^{3/2}}{h}\tilde\phi_{N,a}) = r^{3/2}\rho\Big(\mathbf{y}'^2 - \cos\mathfrak{S} + r^{-1/2}O(\mathbf{x}',\mathbf{y}') + r^{-1}O(1)\Big).
$$

If $|\sin\mathfrak{S}| \geq 1/100$, then $\frac{r^{3/2}}{h}\tilde\phi_{N,a}$ has two critical points in $\mathbf{x}'$, namely $\mathbf{x}'_\pm = \pm|\sin\mathfrak{S}|^{1/2} + O(r^{-1/2})$, and these critical points are non degenerate as $|\partial_{\mathbf{x}'}^2(\frac{1}{h}\tilde\phi_{N,a})| \geq \rho/10$. By stationary phase, we get

$$
\int e^{i\frac{r^{3/2}}{h}\tilde\phi_{N,a}}\tilde\sigma_{V,h,a}\,d\mathbf{x}'d\mathbf{y}' = (r^{3/2})^{-1/2}\Big(\int e^{i\frac{r^{3/2}}{h}\tilde\phi_{N,h,a,+}}\tilde\sigma_{V,h,a,+}\,d\mathbf{y}' + \int e^{i\frac{r^{3/2}}{h}\tilde\phi_{N,a,-}}\tilde\sigma_{V,h,a,-}\,d\mathbf{y}'\Big),
$$

for some symbols $\tilde{\sigma}_{V,h,a,\pm}$ of order 0. Using (3.87), one may check that $|\partial^3_{\mathsf{y}',\mathsf{y}',\mathsf{y}'}(\frac{1}{h}\tilde{\phi}_{N,a})| \geq c > 0$, and by degenerate stationary phase (or Van der Corput lemma) we get

$$| \int e^{i\frac{r^{3/2}}{h}\tilde{\phi}_{N,a,\pm}}\tilde{\sigma}_{V,h,a,\pm}\, d\mathsf{y}'| \leq C(r^{3/2})^{-1/3} = Cr^{-1/2}\,,$$

and for $r \geq 1$, we get the desired decay, and even an extra $r^{-1/4}$ on the right hand side. If $|\cos\mathfrak{S}| \geq 1/100$, we proceed in the same way, exchanging x' and y'.

Let us now deal with $|N| \geq \lambda^2$. Stationary phase applies in ρ and from (3.80),

$$(3.88) \qquad\qquad \frac{1}{h}|\partial^2_\rho\phi_{N,a}| \sim \frac{|N|}{\lambda}\,,$$

hence the stationary phase in ρ yields a factor $(\frac{|N|}{\lambda})^{-1/2}$. In the following we apply exactly the same method as in the previous step; the only difference is that now we have an additional function ρ_c which depends on the variables Σ, S and whose derivatives with respect to these variables are small only for $|N| > \lambda^2$ (which explains why we did not perform stationary phase with respect to ρ earlier, for any $|N| > \lambda^{1/3}$). Indeed, from $\partial_\rho\phi_{N,a} = \partial_\rho\Psi_{N,a,a}|_{A_c} = 0$ we get

$$(3.89) \qquad \Psi_{0,a,a} - \frac{4}{3}a^{3/2}NA_c^{3/2} = Nh\lambda A_c^{3/2}B_L'(\rho\lambda A_c^{3/2}) = -\frac{NhA_c^{3/2}}{\rho^2\lambda A_c^3}(b_1 + O(1/\lambda)), \quad b_1 \neq 0\,,$$

where the term in the left hand side doesn't depend on ρ. Since $\rho = |\theta| \sim 1$ and since the support of the symbol in A_c is a given compact set of $(0,\infty)$, we have

$$\rho_c^2 = -\frac{Nh}{\lambda A_c^{3/2}}(b_1 + O(1/\lambda))\Big(\Psi_{0,a,a}(t,x,y,\Sigma,S,1) - \frac{4}{3}a^{3/2}NA_c^{3/2}\Big)^{-1}\,.$$

Taking the derivatives with respect to S, Υ of (3.89), with A_c bounded, provides a factor λ^2/N which is small in the regime we consider here, $|N| > C\lambda^2$ for some $C > 1$ sufficiently large (indeed, the terms containing Σ, S in (3.89) come with a factor $a^{3/2} = h\lambda$). Since $\partial_\Sigma\rho_c$ and $\partial_S\rho_c$ are now sufficiently small, we can follow the same earlier steps to estimate the integral in the remaining variables Σ, S. $\qquad\qquad\square$

3.2.2. *Moderately large $0 < |N| \leq \lambda^{1/3}$.* In this case the contribution from the integral in ρ is uniformly bounded due to its compact support. According to Remark 3.25, we bring the factor e^{iNB_L} into the symbol and work with the phase $\Psi_{0,a,a} - \frac{4}{3}a^{3/2}\rho NA^{3/2}$, which is linear in ρ. Therefore, the critical point A_c satisfies an equation similar to (3.78), but without the factor $1 - \frac{3}{4}B_L'(\rho\lambda A^{3/2})$, which leads to an explicit expression of the form (3.76) where $f_0 = 0$. We start in the same way as in the proof of Lemma 3.30, replacing $\lambda^{1/3}$ by $|N|$: we rescale variables with $(\Sigma, S) = (\frac{\mathsf{x}}{|N|}, \frac{\mathsf{y}}{|N|})$.

Lemma 3.32. *For $|N| \geq 1$, set $\Lambda = \lambda|N|^{-3}$ and assume $\Lambda \geq 1$, then we have*

$$(3.90) \qquad\qquad \left|\int e^{\frac{i}{h}\phi_{N,a}}\sigma_{V,h,a}(\frac{\mathsf{x}}{|N|}, \frac{\mathsf{y}}{|N|}, \rho)\, d\mathsf{x}d\mathsf{y}\right| \leq C\Lambda^{-3/4}\,.$$

Proof. Here we need all the first three terms in the formula (3.76). In our new variables, $\mathbb{A}_0$, as defined in (3.77), does not change and $\mathbb{A}_1 = \frac{1}{2N^2}(\mathbb{x}(1 - xE_1) - \mathbb{y}(1 - aE_2))$, hence, using (3.76)

$$\frac{1}{h}\partial_{\mathbb{x}}\phi_{N,a} = \Lambda r\Big(\mathbb{x}^2 + N^2(\frac{x}{a}q^{1/3}(\vartheta_c)(1 + \ell(y, \vartheta_c)) - \mathbb{A}_0^2)\Big) + \mathbb{A}_0\Big(\mathbb{x}(1 - xE_1) - \mathbb{y}(1 - aE_2)\Big)$$
$$+ 2N\sqrt{a}\mathbb{x}\frac{x}{a}\mu(y, \vartheta_c)q^{2/3}(\vartheta_c) - \frac{1}{4N^2}(\mathbb{x} - \mathbb{y})^2 + O(\frac{\mathbb{y}^2}{N^2}, \frac{\mathbb{x}^2}{N^2}) + N^2 O(a)\Big),$$

$$\frac{1}{h}\partial_{\mathbb{y}}\phi_{N,a} = -\Lambda r\Big(\mathbb{y}^2 + N^2(q^{1/3}(\vartheta_c) - \mathbb{A}_0^2) + \mathbb{A}_0\Big(\mathbb{x}(1 - xE_1) - \mathbb{y}(1 - aE_2)\Big)$$
$$- 2N\sqrt{a}\mathbb{y}\mu(0, \vartheta_c)q^{2/3}(\vartheta_c) - \frac{1}{4N^2}(\mathbb{x} - \mathbb{y})^2 + O(\frac{\mathbb{y}^2}{N^2}, \frac{\mathbb{x}^2}{N^2}) + N^2 O(a)\Big),$$

where $N^2 O(a)$ in the last two formulas are homogeneous terms of order $j \geq 2$ in the expression of $\partial_\Xi A_\Gamma$. Next, we compute the second derivatives : notice that (3.73) yields $|\nabla_{\Sigma,S}\vartheta_c| \sim O(a^{3/2}/|y|) \sim O(a/N)$ since $|y| \geq c_0 t$ and $t/\sqrt{a} \sim 4N$. Using (3.83) and (3.77) we find

$$\frac{1}{h}\partial_{\mathbb{x}}^2\phi_{N,a} = \Lambda\rho\Big(2\mathbb{x} + 2N\sqrt{a}\frac{x}{a}\mu(y, \vartheta_c)q^{2/3}(\vartheta_c) + \mathbb{A}_0(1 - xE_1) - \frac{1}{2N^2}(\mathbb{x} - \mathbb{y}) + O(\frac{\mathbb{y}^2}{N^2}, \frac{\mathbb{x}}{N^2})\Big),$$

$$\frac{1}{h}\partial_{\mathbb{y}}^2\phi_{N,a} = -\Lambda\rho\Big(2\mathbb{y} + 2N\sqrt{a}\mu(0, \vartheta_c)q^{2/3}(\vartheta_c)(1 + O(\sqrt{a})) + \mathbb{A}_0(1 - aE_2) - \frac{1}{2N^2}(\mathbb{x} - \mathbb{y})$$
$$+ O(\frac{\mathbb{y}}{N^2}, \frac{\mathbb{x}^2}{N^2})\Big)$$

$$\frac{1}{h}\partial_{\mathbb{x}}\partial_{\mathbb{y}}\phi_{N,a} = \Lambda\rho\Big(\mathbb{A}_0 - \frac{(\mathbb{x} - \mathbb{y})}{2N^2} + O(\frac{\mathbb{y}}{N^2}, \frac{\mathbb{x}}{N^2})\Big).$$

In fact, we infer that we can write

(3.91)
$$\frac{1}{h}\partial_{\mathbb{x}}\phi_{N,a} = \Lambda\rho\Big(\mathbb{x}^2 - \mathfrak{A} + (b\mathbb{x} - d\mathbb{y}) - \frac{1}{4N^2}(\mathbb{x} - \mathbb{y})^2 + O(\frac{\mathbb{x}^2}{N^2}, \frac{\mathbb{y}^2}{N^2})\Big),$$
$$\frac{1}{h}\partial_{\mathbb{y}}\phi_{N,a} = -\Lambda\rho\Big(\mathbb{y}^2 - \mathfrak{B} + (d\mathbb{x} - c\mathbb{y}) - \frac{1}{4N^2}(\mathbb{x} - \mathbb{y})^2 + O(\frac{\mathbb{x}^2}{N^2}, \frac{\mathbb{y}^2}{N^2})\Big),$$

where $(\mathfrak{A}, \mathfrak{B})$ and b, c, d are defined as follows

$$-\frac{\mathfrak{A}}{N^2} = \frac{x}{a}q^{1/3}(\vartheta_c)(1 + \ell(y, \vartheta_c)) - \mathbb{A}_0^2, \quad -\frac{\mathfrak{B}}{N^2} = q^{1/3}(\vartheta_c) - \mathbb{A}_0^2,$$
$$b = \mathbb{A}_0(1 + O(x)) + N\sqrt{a}\frac{x}{a}\mu(y, \vartheta_c)q^{2/3}(\vartheta_c)(1 + O(\sqrt{a})),$$
$$c = \mathbb{A}_0(1 + O(a)) - N\sqrt{a}\mu(0, \vartheta_c)q^{2/3}(\vartheta_c)(1 + O(\sqrt{a}))$$
$$d = \mathbb{A}_0(1 + O(x; a)).$$

This follows easily from writing the Taylor development for the first order derivatives of $\phi_{N,a}$ with respect to $\mathbb{x}, \mathbb{y}$ and use that $\partial_{\mathbb{x}}\partial_{\mathbb{y}}\phi_{N,a} = \partial_{\mathbb{y}}\partial_{\mathbb{x}}\phi_{N,a}$. We also have

$$\frac{1}{h}\partial_{\mathbb{x}}^2\phi_{N,a} = \Lambda\rho\Big(2\mathbb{x} + b - \frac{1}{2N^2}(\mathbb{x} - \mathbb{y}) + O(\frac{\mathbb{y}^2}{N^2}, \frac{\mathbb{x}}{N^2})\Big),$$
$$\frac{1}{h}\partial_{\mathbb{y}}^2\phi_{N,a} = -\Lambda\rho\Big(2\mathbb{y} - c + \frac{1}{2N^2}(\mathbb{x} - \mathbb{y}) + O(\frac{\mathbb{y}}{N^2}, \frac{\mathbb{x}^2}{N^2})\Big),$$
$$\frac{1}{h}\partial_{\mathbb{x}}\partial_{\mathbb{y}}\phi_{N,a} = -\Lambda\rho\Big(d - \frac{(\mathbb{x} - \mathbb{y})}{2N^2} + O(\frac{\mathbb{y}}{N^2}, \frac{\mathbb{x}}{N^2})\Big).$$

Let $\Lambda\rho M(\mathtt{x},\mathtt{y})$ denote the matrix of second order derivatives (i.e. the Hessian of $\frac{1}{h}\phi_{N,a}$); we compute its determinant

$$(3.92) \qquad \det M(\mathtt{x},\mathtt{y}) = -\Lambda^2\rho^2\Big(4\mathtt{x}\mathtt{y} + 2(b\mathtt{y} - c\mathtt{x}) + d^2 - bc + \frac{(\mathtt{x}-\mathtt{y})^2}{N^2} + O(\frac{\mathtt{x}}{N^2}, \frac{\mathtt{y}}{N^2})\Big).$$

Set $(\mathfrak{A}, \mathfrak{B}) = r(\sin\mathfrak{S}, \cos\mathfrak{S})$. We will again deal separately with $r \geq r_0$, for some large $r_0 > 1$, and then $r < r_0$. We start with large $r \geq r_0 > 1$ and we prove

$$\left|\int e^{\frac{i}{h}\phi_{N,a}}\sigma_{V,h,a}(\frac{\mathtt{x}}{|N|}, \frac{\mathtt{y}}{|N|})\, d\mathtt{x}d\mathtt{y}\right| \leq C\Lambda^{-5/6},$$

which has better decay than required. First, observe that from the hypothesis on r and (3.91), we may integrate by parts in $(\mathtt{x},\mathtt{y})$ in a region $|(\mathtt{x},\mathtt{y})| \leq cr^{1/2}$ with c small enough: for any $k \geq 1$,

$$\left|\int_{|(\mathtt{x},\mathtt{y})\leq cr^{1/2}} e^{\frac{i}{h}\phi_{N,a}}\sigma_{V,h,a}(\frac{\mathtt{x}}{|N|}, \frac{\mathtt{y}}{|N|})\, d\mathtt{x}d\mathtt{y}\right| \leq C_k(r\Lambda)^{-k}.$$

Thus we are left with the region $|(\mathtt{x},\mathtt{y})| \geq cr^{1/2}$. We rescale again $(\mathtt{x},\mathtt{y}) = r^{1/2}(\mathtt{x}',\mathtt{y}')$ and, in the new variables, we prove the following

$$(3.93) \qquad \left|r\int_{|(\mathtt{x}',\mathtt{y}')\geq c} e^{\frac{i}{h}r^{3/2}\tilde\phi_{N,a}}\tilde\sigma_{V,h,a}(\mathtt{x}', \mathtt{y}', .)\, d\mathtt{x}'d\mathtt{y}'\right| \leq C\Lambda^{-5/6},$$

where we set, like in the previous section, $\phi_{N,a}(\mathtt{x},\mathtt{y},.) = r^{3/2}\tilde\phi_{N,a}(\mathtt{x}',\mathtt{y}',.)$ and $\tilde\sigma_{V,h,a}(\mathtt{x}',\mathtt{y}') = \sigma_{V,h,a}(r^{1/2}\mathtt{x}/N, r^{1/2}\mathtt{y}/N, .)$. Taking $b = r^{1/2}b'$, $d = r^{1/2}d'$ and $c = r^{1/2}c'$, we have

$$\partial_{\mathtt{x}'}\Big(\frac{r^{3/2}}{h}\tilde\phi_{N,a}\Big) = \Lambda\rho r^{3/2}\Big(\mathtt{x}'^2 - \sin\mathfrak{S} + 2(b'\mathtt{x}' - d'\mathtt{y}') - \frac{(\mathtt{x}'-\mathtt{y}')^2}{4N^2} + O(1/N^2)\Big)$$

$$\partial_{\mathtt{y}'}\Big(\frac{r^{3/2}}{h}\tilde\phi_{N,a}\Big) = -\Lambda\rho r^{3/2}\Big(\mathtt{y}'^2 - \cos\mathfrak{S} + 2(d'\mathtt{x}' - c'\mathtt{y}') - \frac{(\mathtt{x}'-\mathtt{y}')^2}{4N^2} + O(1/N^2)\Big).$$

From these expressions of $\nabla_{(\mathtt{x}',\mathtt{y}')}(\frac{r^{3/2}}{h}\tilde\phi_{N,a})$, if $(\mathtt{x}',\mathtt{y}')$ is large, we get decay by integrations by parts, as $|N| \geq 2$. Therefore, we can assume that $c \leq |(\mathtt{x}',\mathtt{y}')| \leq C$ where C is a large, fixed constant (we avoid a neighbourhood of $(0,0)$ since we have assumed $|(\mathtt{x},\mathtt{y})| > cr^{1/2}$). If $|N| \geq 2$, a is small and r_0 is large, the set $\{\det M(\mathtt{x},\mathtt{y}) = 0\}$ is a smooth curve which does not intersect the origin. Away from this set, we may use stationary phase, which will provide $r(r^{3/2}\Lambda)^{-1}$ decay on the left of (3.93). In the region close to $\{\det M(r^{1/2}\mathtt{x}', r^{1/2}\mathtt{y}') = 0\}$, we can apply [9, Lemma 2.21 (a)], i.e. degenerate stationary phase along a curve, to obtain

$$\left|\int_{c\leq|(\mathtt{x}',\mathtt{y}')\leq C} e^{\frac{i}{h}r^{3/2}\tilde\phi_{N,a}}\sigma_{V,h,a}(\mathtt{x}', \mathtt{y}', .)\, d\mathtt{x}'d\mathtt{y}'\right| \leq C(r^{3/2}\Lambda)^{-5/6},$$

and therefore we get (3.93) as the extra factor r on the lefthand side is canceled by the $r^{-5/4}$ on the righthand side (recall $r \geq r_0 > 1$).

We can now focus on $r = |(\mathfrak{A}, \mathfrak{B})| \leq r_0$. Notice that we may further restrict support again, this time to $|(\mathtt{x},\mathtt{y})| \leq 2r^{1/2} \leq 2r_0^{1/2}$ as we get $\Lambda^{-\infty}$ decay by integration by parts if $|(\mathtt{x},\mathtt{y})|$ is larger than this value, as we did before. We now aim at proving

$$\left|\int_{|(\mathtt{x},\mathtt{y})|\leq 2r^{1/2}} e^{\frac{i}{h}\phi_{N,a}}\sigma_{V,h,a}(\frac{\mathtt{x}}{|N|}, \frac{\mathtt{y}}{|N|})\, d\mathtt{x}d\mathtt{y}\right| \leq C\Lambda^{-3/4},$$

for which we will apply [9, Lemma 2.21 (b)]. The determinant of the Hessian $M(\mathtt{x},\mathtt{y})$ of $\frac{1}{h}\phi_{N,a}$ is given in (3.92) and the set $\{\det M(\mathtt{x},\mathtt{y}) = 0\}$ is a smooth curve in $\{|(\mathtt{x},\mathtt{y})| \leq 2r^{1/2}\}$, at

least for small a: it will be close to $(2\mathtt{x} + b)(2\mathtt{y} - c) + d^2 = -(\mathtt{x} - \mathtt{y})^2$ when $|N| = 1$ and $(2\mathtt{x} + b)(2\mathtt{y} - c) + d^2 = -(\mathtt{x} - \mathtt{y})^2/N^2$ for $|N| \geq 2$ (and using $b \sim c \sim d \sim \mathbb{A}_0(1 + O(a))$, we see that it is close to the hyperbola $4\mathtt{xy} + 2\mathbb{A}_0(\mathtt{x} - \mathtt{y}) = -(\mathtt{x} - \mathtt{y})^2/N^2$ for $|N| > 1$ and to the parabola $2\mathbb{A}_0(\mathtt{x} - \mathtt{y}) = -(\mathtt{x} + \mathtt{y})^2$ when $|N| = 1$).

Let $(\mathtt{x}_0, \mathtt{y}_0)$ such that $|(\mathtt{x}_0, \mathtt{y}_0)| \leq 2r^{1/2}$: if $\det M(\mathtt{x}_0, \mathtt{y}_0) \neq 0$, then the usual stationary phase applies, unless we are in the special condition where [9, Lemma 2.21] takes over and we recall it now: let $H(\xi)$ be a smooth function defined in a neighborhood of $(0,0)$ in $\mathbb{R}^2$, such that $H(0) = 0$ and $\nabla H(0) = 0$. We assume that the Hessian H'' satisfies $\mathrm{rank}(H''(0)) = 1$ and $\nabla \det(H'')(0) \neq 0$. Then $\det(H'')(q, p) = 0$ defines a smooth curve $\mathcal{C}$ near $0 \in \mathbb{R}^2$ with $0 \in \mathcal{C}$. Let $s \to \xi(s)$ be a smooth parametrization of $\mathcal{C}$, with $\xi(0) = 0$, and define the curve $\Xi(s) := H'(\xi(s))$ in $\mathbb{R}^2$.

Lemma 3.33. *([9, Lemma 2.21]) Let $K = \{\xi \in \mathbb{R}^2, |\xi| \leq R_0\}$, and $a(\xi, \Lambda)$ a classical symbol of order 0 in $\Lambda \geq 1$ with $a(\xi, \Lambda) = 0$ for $\xi \notin K$. Set for $(P, Q) \in \mathbb{R}^2$ close to 0*

$$I(\cdot, \Lambda) := \int e^{i\Lambda(<\xi, \cdot> - H(\xi))} a(\xi, \Lambda) d\xi \,.$$

Then for $R_0 > 0$ small enough, the following holds true:
  - *(a) If $\Xi'(0) \neq 0$, there exists C such that for all (P, Q) close to 0, $|I(\cdot, \Lambda)| \leq C\Lambda^{-5/6}$.*
  - *(b) If $\Xi'(0) = 0$ and $\Xi''(0) \neq 0$ there exists C such that for all $\cdot$ close to 0, $|I(\cdot, \Lambda)| \leq C\Lambda^{-3/4}$. Moreover, if a is elliptic at $\xi = 0$, there exists C' such that $|I((0,0), \Lambda)| \geq C'\Lambda^{-3/4}$.*

Let $(\mathtt{x}_0, \mathtt{y}_0)$ be such $|(\mathtt{x}_0, \mathtt{y}_0)| \leq 2r_0^{1/2}$ and $\det M(\mathtt{x}_0, \mathtt{y}_0) = 0$. For $(\mathtt{x}, \mathtt{y})$ near $(\mathtt{x}_0, \mathtt{y}_0)$, $|(\mathtt{x}, \mathtt{y})| \leq 2r_0^{1/2}$ let $\xi = (\mathtt{x} - \mathtt{x}_0, \mathtt{y} - \mathtt{y}_0)$ and let $R_0 = 4r_0^{1/2}$, then $|\xi| \leq R_0$. We set

$$H(\xi) = \Phi_N(\mathtt{x}, \mathtt{y}) - \Phi_N(\mathtt{x}_0, \mathtt{y}_0) - \nabla\phi_{N,a}(\mathtt{x}_0, \mathtt{y}_0) \cdot \xi, \quad (\mathtt{x}, \mathtt{y}) = (\mathtt{x}_0, \mathtt{y}_0) + \xi.$$

We see that $H(0) = 0$, $H'(0) = 0$ and $H''(0) = \nabla^2\phi_{N,a}(\mathtt{x}_0, \mathtt{y}_0) = M(\mathtt{x}_0, \mathtt{y}_0)$. The matrix M has two eigenvalues, 0 with normalized eigenvector $v(\mathtt{x}_0, \mathtt{y}_0)$ and $\lambda_1 = \mathrm{tr}(M(\mathtt{x}_0, \mathtt{y}_0))$ with normalized eigenvector $u(\mathtt{x}_0, \mathtt{y}_0)$. Let Q be the matrix formed with column vectors u and v, then ${}^tQM(\mathtt{x}_0, \mathtt{y}_0)Q = \mathrm{diag}(\lambda_1/2, 0)$, and $\lambda_1 \neq 0$ as the rank of $M(\mathtt{x}_0, \mathtt{y}_0) \geq 1$ (thanks to $|N| \geq 2$ and (t, x, y) close to 0). Using (3.92), a simple computation yields

$$|\nabla \det H''(0)|^2 = 16\Big(\mathtt{x}_0 + b/2 - \frac{(\mathtt{x}_0 - \mathtt{y}_0)}{2N^2}\Big)^2 + 16\Big(\mathtt{y}_0 - c/2 + \frac{(\mathtt{x}_0 - \mathtt{y}_0)}{2N^2}\Big)^2 + O\big(\frac{\mathtt{x}_0}{N^2}, \frac{\mathtt{y}_0}{N^2}\big).$$

We need to prove that there exists a positive constant $C_0 > 0$ such that $|\nabla \det H''(0)|^2 \geq C_0 > 0$. Since $b = \mathbb{A}_0(1 + O(a)) + O(a^{1/2}N)$, $c = \mathbb{A}_0(1 + O(a)) + O(a^{1/2}N)$ it follows that for $(\mathtt{x}_0, \mathtt{y}_0) \in \{M(\mathtt{x}, \mathtt{y}) = 0\}$ we can write $|\nabla \det H''(0)|^2 = 8\mathbb{A}_0^2 + 16(\mathtt{x}_0 - \mathtt{y}_0)^2(1 - \frac{1}{2N^2}) + O(a^{1/2}N)$, provided $a^{1/2}N$ is small enough and therefore $|\nabla \det H''(0)| \geq C_0 > 0$ for some positive constant C_0. This is enough to apply Lemma 3.33 (case [b]) and get the desired bounds. When $N = 1$, one may inspect the previous proofs to check that, knowing from support conditions that $(\mathtt{x}, \mathtt{y})$ is bounded, for large $(\mathfrak{A}, \mathfrak{B})$ integrations by parts provide decay. On the other hand, in the range $(\mathfrak{A}, \mathfrak{B})$ bounded, one may proceed as before, using Lemma 3.33. $\qquad \square$

We moreover remark that the last statement in Lemma 3.33, together with ellipticity of the symbol $\sigma_{V,h,a}$, provides, at fixed $\theta \in \mathbb{S}^{d-2}$, a sequence (t_N, x_N, y_N) where the bound (3.90) saturates, which is exactly at the swallowtail singularity in space-time. This is a key point in proving Theorem 1.3: for now, we have $V_N(t_N, x_N, y_N) \sim \|V_N\|_\infty$, and at (t_N, x_N, y_N), we have $\mathfrak{A} \sim \mathfrak{B} \sim 0$, i.e. $\mathbb{A}_0 \sim q^{1/6}(\theta)$, implying $t_N \sim 4Nq^{-1/2}(y/|y|)\sqrt{a}$, $x_N \sim a$ and $y_N/|y_N| \sim \vartheta_c \sim \theta$. Moreover, at (t_M, x_N, y_N) with $M \neq N$ and $|M - N| \leq M_0$, $\mathfrak{B} \sim CN$ and $|V_N(t_M, x_N, y_N)|$ decays much faster

($\Lambda^{-5/6}$ rather than $\Lambda^{-3/4}$). Therefore when summing over M at fixed N, only V_N contributes to saturating the bound and Theorem 1.3 holds.

3.2.3. *The sum over $2 \leq N \lesssim \sqrt{a}$: completing the proof of Proposition 3.23.* We now prove (3.71). For $N \notin \mathcal{N}_1(t,x,y)$, Proposition 3.13 provides an $O_{C^\infty}(h^\infty)$ contribution. Recall that if $(\theta = (\rho, \vartheta), A, \Sigma, S)$ is a critical point in the phase integral defining $V_{N,a}$, then $t/(4\sqrt{a}|N|) \sim A^{1/2}$ and on the support of the symbol $\psi(A)$ we have $A \sim 1$. Set $\sharp N = |\mathcal{N}_1(t,x,y)|$. If $\sharp N \leq 2C$, the sum reduces to a finite sum, discarding all $N \notin \mathcal{N}_1(x,y,t)$. For those $N \in \mathcal{N}_1(t,x,y)$, if $|N| \leq \lambda^{1/3}$, then, collecting (3.72) (stationary phase in ϑ), (3.75) (stationary phase in A) and (3.90) (degenerate stationary phase in (Σ, S)),

$$|V_{N,a}(t,x,y)| \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{a^2}{h} \frac{1}{|N|^{1/2}\lambda^{1/2}} |N|^{1/4} \lambda^{-3/4}$$

from which, with $t \sim 4a^{1/2}N$ we get the desired result:

$$|V_{N,a}(t,x,y)| \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{a^{1/8}h^{1/4}}{|N|^{1/4}} \leq \frac{C_0}{h^d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{a^{1/4}h^{1/4}}{t^{1/4}}.$$

If $|N| \geq \lambda^{1/3}$, we collect the same bounds but with (3.90) replaced by (3.81):

$$|V_{N,a}(t,x,y)| \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} \frac{a^2}{h} \frac{1}{|N|^{1/2}\lambda^{1/2}} \frac{1}{\lambda^{2/3}}$$

and one easily checks that $a^{1/2}N^{-1/2}\lambda^{-1/6} \leq a^{1/8}h^{1/4}N^{-1/4}$ is equivalent to $|N| \geq \lambda^{1/3}$ (using $1/N \leq \lambda^{-1/3}$ we get $h^{1/3}$ instead of $a^{1/8}h^{1/4}/|N|^{1/4}$.) Moreover, if only a finite number of $V_{N,a}(t,x,y)$ contributes, we have $a > h^{4/7}$. We proceed with large $\sharp N (\geq 2C)$: as $\sharp N \leq C + C\frac{t}{\sqrt{a}\lambda^2}$, we have $t \geq c\sqrt{a}\lambda^2$. We also have $|N| \sim t/(4\sqrt{a})$ and from $t \lesssim 1$, we have $a \lesssim h^{4/7}$. For those N,

$$\left| \sum_{N \in \mathcal{N}_1(t,x,y), |N| \sim t/\sqrt{a}} V_{N,a} \right| \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} a^{1/2}\lambda^{1/3} \frac{1}{|N|} \sharp N \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{\frac{d-2}{2}} h^{1/3}\lambda^{-4/3},$$

and one checks that $h^{1/3}\lambda^{-4/3} \leq h^{1/3}$ for $a \gtrsim h^{2/3}$. This completes the proof of Proposition 3.23.

3.3. **The transverse part** $4a \leq \gamma \ll 1$. We go back to $\Psi_{N,a,\gamma}(t,x,y,\Sigma,S,A,\rho)$, which is the critical value of $\tilde{\Phi}_{N,a,\gamma}$ after the stationary phase in ϑ: when $4a \leq \gamma$ we start by applying stationary phase in S, as in the proof of Proposition 3.13. The phase has two distinct saddle points $S_\pm$,

$$S_\pm^2 + \frac{a}{\gamma}q^{1/3}(\vartheta_c)\left(1 + \partial_\Xi\left(A_\Gamma\left(a, 0, \Xi, \frac{\vartheta_c}{\tau_q(\gamma A, \vartheta_c)}\right)\right)\Big|_{\Xi = \frac{\sqrt{\gamma}S_\pm q^{1/3}(\vartheta_c)}{\tau_q(\gamma A, \vartheta_c)}}\right) = A.$$

Using (3.73), ϑ_c (function of Σ, S, A) does not affect the second derivative in S significantly and we get $\partial_S^2 \Psi_{N,a,\gamma}|_{S_\pm} \sim 2$. Moreover, the critical points $S_\pm$ depend on Σ only through ϑ_c and we have $S_\pm = S_0 \pm \sqrt{A - O(a/\gamma)}$, where S_0 is the unique solution to $\partial_S^2 \Psi_{N,a,\gamma}(t,x,y,\Sigma,S,A,\rho) = 0$ and it satisfies $S_0 = O(a/\sqrt{\gamma})$ (indeed, from Corollary 6.10, the unique solution s_0 to $\partial_s^2 \tilde{\Phi}_{N,a,\gamma}$ satisfies $s_0 = O(a)$; making the change of variables $s = \sqrt{\gamma}\rho^{1/3}S$ gives $S_0 = O(a/\sqrt{\gamma})$). We are left with an integral with respect to A, Σ, ρ and we pick a factor $\lambda_\gamma^{1/2}$.

When $1 \leq |N| \lesssim \lambda_\gamma^2$, we ignore ρ: we first perform stationary phase with respect to A, and then prove that the remaining phase has critical points that may be degenerate of order at most two in Σ. For each $S_\pm$, the critical point with respect to A, denoted $A_\pm$, is such that, with $\tau = \tau_q(\gamma A, \vartheta_c)$,

$$(3.94) \quad 2NA^{1/2}\left(1 - \frac{3}{4}B_L'(\rho\lambda_\gamma A^{3/2})\right) = \frac{q^{2/3}(\vartheta_c)\left(t - B_2(y, \vartheta_c) + \sum_{k\geq 2}\partial_\tau\left(\frac{(1-\tau)^k}{\tau^{k-1}}\right)B_{2k}(y, \vartheta_c)\right)}{2\sqrt{\gamma}\sqrt{1 + \gamma Aq^{2/3}(\vartheta_c)}}$$

$$- \Sigma + S_\pm + \frac{\gamma q^{2/3}(\vartheta_c)}{\sqrt{1 + \gamma Aq^{2/3}(\vartheta_c)}}\left[\frac{x}{\gamma^{3/2}}\partial_\tau\left(\tau A_\Gamma(x, y, \sqrt{\gamma}\Sigma q^{1/3}(\vartheta_c)/\tau, \vartheta_c/\tau)\right)\right.$$

$$\left. - \frac{a}{\gamma^{3/2}}\partial_\tau\left(\tau A_\Gamma(a, 0, \sqrt{\gamma}S_\pm q^{1/3}(\vartheta_c)/\tau, \vartheta_c/\tau)\right)\right].$$

The second derivative with respect to A behaves like $N/\sqrt{A}$ and with $A \sim 1$ on the support of the symbol ψ, stationary phase provides a factor $\lambda_\gamma^{-1/2}|N|^{-1/2}$.

When $|N| > \lambda_\gamma^2$ we apply stationary phase in A just like we did, and then stationary phase in ρ. We then prove that the remaining integral can be degenerate of order at most two. To apply (additional) stationary phase in ρ we act exactly like in the case $\gamma \sim a$ and obtain, as in (3.88)

$$\frac{1}{h}|\partial_\rho^2\Psi_{N,a,\gamma}(t, x, y, \Sigma, S_+, A_+, \rho)| \sim \frac{1}{h}|\partial_\rho^2\Psi_{N,a,\gamma}(t, x, y, \Sigma, S_+, A_+, \rho)| \sim \frac{|N|}{\lambda_\gamma}.$$

Let $\phi_{N,a,\gamma,+}(t, x, y, \Sigma, \rho) = \Psi_{N,a,\gamma}(t, x, y, \Sigma, S_+, A_+, \rho)$ and, respectively, $\phi_{N,a,\gamma,-}(t, x, y, \Sigma, \rho) = \Psi_{N,a,\gamma}(t, x, y, \Sigma, S_-, A_-, \rho)$, denote the critical values of $\Psi_{N,a,\gamma}$ after the the stationary phase in S and then in A (notice that A_+ (and A_-, respectively) is the critical point of the phase $\Psi_{N,a,\gamma}(t, x, y, S_+, A, \rho)$ (and $\Psi_{N,a,\gamma}(t, x, y, S_-, A, \rho)$, respectively), so the critical points S and A are paired with same sign, either $+$ or $-$). Let $\sigma_{V,h,\gamma,\pm}(\Sigma, \rho)$ be the symbol obtained from

$$\psi(A)p_h(x, y, \theta, \gamma|\theta|^{2/3}A, \sqrt{\gamma}|\theta|^{1/3}\Sigma)\tilde{q}_h(\theta, \gamma|\theta|^{2/3}A, \sqrt{\gamma}|\theta|^{1/3}S)$$

after applying stationary phase in ϑ, A and S, evaluated at $(S, A)_\pm$; $\sigma_{V,h,\gamma}$ is independent of N, of order zero and has compact support in Σ. We are left with estimating $\int e^{\frac{i}{h}\phi_{N,a,\gamma,\pm}}\sigma_{V,h,\gamma,\pm}d\Sigma d\rho$. Following [12], we state different estimates for $|N| \geq 2$, $N = 0$ and $N = \pm 1$.

Lemma 3.34. *For $|N| \geq 2$ and a given point (t, x, y) with $t \sim 4N\sqrt{\gamma}$, the phase $\phi_{N,a,\gamma,\pm}(t, x, y, \Sigma, \rho)$ has at most one degenerate critical point, which is of order two.*

Proof. The first derivative of the phase $\phi_{N,a,\gamma,\pm}$ is

$$(3.95) \quad \partial_\Sigma\phi_{N,a,\gamma,\pm} = \partial_\Sigma\Psi_{N,a,\gamma}(t, x, y, \Sigma, S, A, \rho)|_{A_\pm, S_\pm}$$

$$= \Sigma^2 + \frac{x}{\gamma}q^{1/3}(\vartheta_c)\left(1 + \partial_\Xi\left(A_\Gamma(x, y, \Xi, \frac{\vartheta_c}{\tau_q(\gamma A, \vartheta_c)})\right)\right)\Big|_{\Xi = \frac{\sqrt{\gamma}Sq^{1/3}(\vartheta_c)}{\tau_q(\gamma A, \vartheta_c)}} - A_c$$

Using (3.94) and then (3.95), we estimate derivatives w.r.t. Σ as follows

$$\partial_\Sigma(A_\pm^{1/2}) = -\frac{1}{2N \mp 1}(1 + O(\sqrt{\gamma}) + O(\lambda^{-2}) + O(\partial_\Sigma\omega_c)) = -\frac{1}{2N \mp 1}(1 + O(\sqrt{\gamma})),$$

$$\partial_\Sigma^2\phi_{N,a,\gamma,\pm} = 2\Sigma(1 + O(\sqrt{\gamma})) + \frac{A_\pm^{1/2}}{2N \mp 1}(1 + O(\sqrt{\gamma})), \quad \partial_\Sigma^3\phi_{N,a,\gamma,\pm} = 2 + O(1/N) + O(\sqrt{\gamma}).$$

For a given t, x, y, N, the equation $\partial_\Sigma^2\phi_{N,a,\gamma,\pm} = 0$ has at most one solution which is a saddle point when $\partial_\Sigma\phi_{N,a,\gamma,\pm} = 0$. Since the third order derivative stays close to 2, degenerate stationary phase (or Van der Corput Lemma) in Σ provides a factor $\lambda_\gamma^{-1/3}$. $\qquad\square$

Lemma 3.35. *Let $N = 0$, $|t| > \gamma$, then for $|\Sigma| \lesssim 1$ we have $\partial_\Sigma^2 \phi_{0,a,\gamma,\pm}(t, x, y, \Sigma, \rho) \sim t/\sqrt{\gamma}$.*

Proof. Let $N = 0$, $|t| > \gamma$, then $A_\pm$ solve (3.94) where the term in the left hand side vanishes. The main term containing A comes from $S_\pm = S_0 \pm \sqrt{A - O(a/\gamma)}$ and $A_\pm$ satisfy

$$(3.96) \qquad S_0 \pm \sqrt{A_\pm - O(a/\gamma)} = \Sigma(1 + O(\sqrt{\gamma})) - q^{2/3}(\vartheta_c)\frac{(t + O(|y|))}{2\sqrt{\gamma}}(1 + O(\gamma)).$$

Taking the derivative of (3.96) with respect to Σ, using that S_0 depends on Σ only through ϑ_c (which satisfies (3.73)), that $A_\pm$ stays close to 1 and that $t/\sqrt{\gamma}$ is bounded yields

$$\partial_\Sigma A_\pm = \pm 2\sqrt{A_\pm - O(a/\gamma)} + O(\sqrt{\gamma}) + O(\gamma^{3/2}/|y|) \times \frac{(t + O(|y|))}{2\sqrt{\gamma}}$$

$$= 2\Big(\Sigma(1 + O(\sqrt{\gamma})) - q^{2/3}(\vartheta_c)\frac{(t + O(|y|))}{2\sqrt{\gamma}}(1 + O(\gamma))\Big) + O(\sqrt{\gamma}),$$

and therefore, using that the saddle point Σ is bounded since $\Sigma^2 \leq A_\pm$, we find

$$\partial_\Sigma^2 \phi_{0,a,\gamma,\pm}|_{\partial_\Sigma \phi_{0,a,\gamma,\pm}=0} = 2\Sigma(1 + O(\sqrt{\gamma})) - \partial_\Sigma A_\pm = q^{2/3}(\vartheta_c)\frac{(t + O(|y|))}{\sqrt{\gamma}}(1 + O(\gamma)) + O(\sqrt{\gamma}).$$

When $|t| > \gamma$ we obtain the result. When $|t| \lesssim \gamma$, the wave has no time to reach the boundary. $\qquad\square$

Lemma 3.36. *For $N = \pm 1$: each phase function $\phi_{\pm 1,a,\gamma,\pm}(t, x, y, \Sigma, \rho)$ has at most one degenerate critical point Σ_c of order exactly two; for $t \neq 0$, the equation $\partial_\Sigma^2 \phi_{\pm 1,a,\gamma,\mp} = 0$ has an unique solution $\Sigma \sim \mp 1/\gamma$, while for $|\Sigma| \lesssim 1$, we have $\partial_\Sigma^2 \phi_{\pm 1,a,\gamma,\mp} \sim (t/\sqrt{\gamma})(1 + O(\gamma))$.*

Proof. For $\phi_{\pm 1,a,\gamma,\pm}$, the proof of Lemma 3.34 applies, for $\phi_{\pm 1,a,\gamma,\mp}$ the proof for $N = 0$ does. $\qquad\square$

Lemma 3.37. *For $|N| \geq \lambda_\gamma^2$, after stationary phase in ρ, the critical value $\phi_{N,a,\gamma,\pm}(t, x, y, \Sigma, \rho_c)$ has at most one degenerate critical point of order exactly two in Σ.*

Proof. The proof is essentially the same as the one of Lemma 3.34 since the contribution from the derivatives of the critical point with respect to ρ do not affect significantly the third order derivative of the phase with respect to Σ . $\qquad\square$

3.3.1. *Estimates for the sum over N and end of the proof of Proposition 3.22.* We now proceed with estimating the sum over N, i.e. proving (3.70). Again, for $N \notin \mathcal{N}_1(t, x, y)$, Proposition 3.13 provides an $O_{C^\infty}(h^\infty)$ contribution. With $\sharp N = |\mathcal{N}_1(t, x, y)|$, if $\sharp N \leq 2C$, the sum reduces to a finite sum. For those $N \in \mathcal{N}_1(t, x, y)$, we have $|N| \leq \lambda^2$, then, collecting (3.72) (stationary phase in ϑ), factors from stationary phase in S, A and from degenerate stationary phase in Σ,

$$|V_{N,\gamma}(t, x, y)| \lesssim \frac{1}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\frac{\gamma^2}{h}\frac{1}{\lambda_\gamma^{1/2}}\frac{1}{|N|^{1/2}\lambda_\gamma^{1/2}}\frac{1}{\lambda_\gamma^{1/3}} \sim \frac{1}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\frac{h^{1/3}}{\sqrt{N}}.$$

If $|N| \geq \lambda_\gamma^2$, we collect the same bounds but with an additional stationary phase in ρ,

$$|V_{N,\gamma}(t, x, y)| \lesssim \frac{1}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\frac{\gamma^2}{h}\frac{1}{\lambda_\gamma^{1/2}}\frac{\lambda_\gamma^{1/2}}{\sqrt{N}}\frac{1}{|N|^{1/2}\lambda_\gamma^{1/2}}\frac{1}{\lambda_\gamma^{1/3}} \sim \frac{1}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\frac{h^{1/3}\lambda_\gamma^{1/2}}{N}.$$

As $\sharp N \leq C + C\frac{T}{\lambda^2}$ and $T \sim N$, we complete the proof of Proposition 3.22 with

$$\left|\sum_{N \in \mathcal{N}_1(t,x,y)} V_{N,\gamma}\right| \lesssim \frac{1}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\frac{h^{1/3}\lambda_\gamma^{1/2}}{N}\sharp N \lesssim \frac{1}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\frac{h^{1/3}}{\lambda_\gamma^{3/2}}.$$

Remark 3.38. In this transverse regime, we used $\gamma \geq h^{2/3-\varepsilon}$ rather than $a \geq h^{2/3-\varepsilon}$. As such, all estimates hold with any $a > 0$. This will be of importance in the next section.

4. Dispersion for small $a \lesssim h^{2/3-\epsilon}$

We now obtain dispersion for $\mathcal{P}_{h,a}$ for small $a \in (0, h^{2/3-\epsilon})$ with a small $0 < \epsilon < 1/12$ using Propositions 2.18 and 2.19. Write $\mathcal{P}_{h,a}(t,x,y) := \mathcal{P}_{h,a}^1 + \mathcal{P}_{h,a}^2$, where $\mathcal{P}_{h,a}^j$ is given by (2.39) with $g_{h,a}$ replaced by $g_{h,a,j}$, $j \in \{1,2\}$. Dispersion for $\mathcal{P}_{h,a}^2$ easily follows using exactly the arguments from the transverse case (for $a > h^{\frac{2}{3}-\epsilon}$): ω is large and stationary phase arguments apply. We are left with $\mathcal{P}_{h,a}^1$.

Proposition 4.1. *For $h < t \leq T_0 \leq 1$*

$$(4.1) \qquad \|\mathcal{P}_{h,a}^1(t,.)\|_{L^\infty_{x>0,y}} \lesssim h^{-d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\left(\frac{h}{t}\right)^{1/3}.$$

Proof. We need the following lemma, relying on Corollary 2.31:

Lemma 4.2. *There exists symbols $\sigma(\eta,\omega_k)$ and $r_j(a,\eta,\omega_k)$, of order 0, σ, r_0 elliptic such that*

$$(4.2) \quad K_{\omega_k}(g_{h,a,1})(t,x,y) = \int e^{it\tau_q(\omega_k,\eta)} G(x,y,\eta,\omega_k) q^{\frac{1}{6}}(\eta)\varkappa(h\eta)\varkappa(h\tau_q(\omega_k,\eta))r(\eta,\omega_k)\sigma(\eta,\omega_k)$$

$$\chi^\flat\left(\frac{\omega_k}{\omega_{K_\epsilon}}\right)q^{\frac{1}{6}}(\eta)\Big(r_0 Ai(-\zeta(a,0,\eta,\omega_k)) + iq^{-\frac{1}{6}}(\eta)r_1 Ai'(-\zeta(a,0,\eta,\omega_k))\Big)d\eta,$$

where ζ is the phase introduced in Theorem 2.1.

Proof. Replacing $\hat{g}_{h,a,1}$ given by Corollary 2.31 in formula (2.24) yields

$$K_{\omega_k}(g_{h,a,1}) = \int e^{it\tau_q(\omega_k,\eta)} G(x,y,\eta,\omega_k)\chi^\#(\omega_k)q^{\frac{1}{6}}(\eta)\varkappa(h\eta)\varkappa(h\tau_q(\omega_k,\eta))r(\eta,\omega_k)\frac{\sqrt{L'(\omega_k)}}{\sqrt{2\pi}}I_a(\eta,\omega_k)d\eta,$$

with $I_a(\eta,\omega_k)$ defined in (2.81). To compute I_a use (2.63) and $\overline{e(x,y,\eta,\omega_k)} = e^{-i|\eta|B_0}\overline{\tilde{e}(x,y,\eta,\omega_k)}$

$$(4.3) \qquad I_a = \int e^{-i(\tilde{y}\cdot(\eta-\tilde{\eta})+|\eta|B_0(\tilde{y},\frac{\eta}{|\eta|}))}\varkappa(h\tilde{\eta})\chi^\flat\left(\frac{\omega_k}{\omega_{K_\epsilon}}\right)e_k(a,\tilde{\eta})\left[\int_{\mathbb{R}_+}\overline{\tilde{e}(\tilde{x},\tilde{y},\eta,\omega_k)}e_k(\tilde{x},\tilde{\eta})d\tilde{x}\right]d\tilde{\eta}d\tilde{y}.$$

As in the proof of Lemma 2.28, the bracket term in (4.3) behaves like a symbol. For small $a \lesssim h^{2/3-\epsilon}$ and $k \leq K_\epsilon$, $e_k(a,\tilde{\eta})$ can also be included in the symbol, and with $\eta = \theta/h$ and $\tilde{\eta} = \tilde{\theta}/h$, stationary phase applies in $\tilde{y}, \tilde{\theta}$ for the phase $-(\tilde{y}\cdot\theta + |\theta|B_0(\tilde{y},\vartheta)) + \tilde{y}\cdot\tilde{\theta}$, with large parameter $\frac{1}{h}$ and symbol

$$\frac{1}{h^{d-1}}\chi^\flat\left(\frac{\omega_k}{\omega_{K_\epsilon}}\right)\varkappa(\tilde{\theta})\chi_0(h^{2/3}\omega_k q^{2/3}(\tilde{\theta}))e_k(a,\tilde{\theta}/h)\left(\int_0^\infty \overline{\tilde{e}(\tilde{x},\tilde{y},\theta/h,\omega_k)}e_k(\tilde{x},\tilde{\theta}/h)d\tilde{x}\right).$$

Stationary points are such that $\tilde{y} = 0$, $\tilde{\theta} = \theta + |\theta|\partial_{\tilde{y}}B_0(\tilde{y},\vartheta) = \theta$ (as $\partial_y B_0(0,\vartheta) = 0$). All derivatives with respect to $\tilde{y}$ land on $\overline{\tilde{e}(\tilde{x},\tilde{y},\theta/h,\omega_k)}$ and we can use Lemma 2.28. Derivatives with respect to $\tilde{\theta}$ land either on cut-offs, $e_k(\tilde{x},\tilde{\theta}/h)$ or $e_k(a,\tilde{\theta}/h)$: using (2.76) for e_k,

$$\left|\partial_{\tilde{y}}^{\beta_1}\partial_{\tilde{\eta}}^{\beta_2}\left(\int_0^\infty \overline{\tilde{e}(\tilde{x},\tilde{y},\eta,\omega_k)}e_k(\tilde{x},\tilde{\eta})d\tilde{x}\right)\right| \leq \|\partial_{\tilde{y}}^{\beta_1}\tilde{e}(.,\omega_k)\|_{L^2_{x\geq 0}}\|\partial_{\tilde{\eta}}^{\beta_2}e_k\|_{L^2_{x\geq 0}} \lesssim (\omega_k|\eta|^{1/3})^{|\beta_1|+|\beta_2|}.$$

As $|\eta| \sim 1/h$ and $k \lesssim K_\epsilon < h^{-1/4+\epsilon}$ we find $\omega_k|\eta|^{1/3} \leq h^{-1/2+2\epsilon/3}$. As one derivative on $e_k(a,\tilde{\theta}/h)$ yields at most $\frac{a}{h}\sqrt{h^{2/3}\omega_k}$ and we have $\frac{a}{h}\sqrt{h^{2/3}\omega_k} \leq h^{-1/12-2\epsilon/3} \ll h^{-1/2}$ for $\epsilon < 5/8$, stationary phase applies in $\tilde{y}, \tilde{\theta}$. After stationary phase, $e_k(a,\tilde{\theta}/h)$ transforms into a linear combination of e_k and its derivative : indeed, repeated derivatives in $\tilde{\theta}$ lead to $Ai(aq^{1/3}(\tilde{\theta}/h) - \omega_k)$ and

$Ai'(aq^{1/3}(\tilde{\theta}/h)-\omega_k)$, and we denote r_0, r_1 the corresponding asymptotic expansions; r_0, r_1 are functions of $(a, \theta/h, \omega_k)$ and r_0 is elliptic with main contribution equal to 1. Taking the difference between phase functions of $e_k(a, \tilde{\theta}/h)$ and $e(a, 0, \theta/h, \omega_k)$ yields $a\tau_q(h^{2/3}\omega_k, \theta)A_\Gamma(a, 0, sq^{1/3}(\theta)/\tau_q, \theta/\tau_q)$, which behaves (at worst) like $a(s^2 - h^{2/3}\omega_k q^{2/3}(\theta) + O(s^3, s(h^{2/3}\omega_k))$ (where $s^2 \lesssim h^{2/3}\omega_k$); as $ah^{2/3}\omega_k \leq h^{7/6-\epsilon/3}$, $e^{\frac{i}{h}a\tau_q(h^{2/3}\omega_k, \theta)A_\Gamma(a, 0, sq^{1/3}(\theta)/\tau_q, \theta/\tau_q)}$ does not oscillate and can be brought into the symbol. The $L^2_{x>0}$ product in (4.3) becomes a new symbol $\sigma(\theta/h, \omega_k)$, with main contribution σ_0

$$\sigma_0 = \int_0^\infty \overline{\tilde{e}(\tilde{x}, 0, \theta/h, \omega_k)}e_k(\tilde{x}, \theta/h)d\tilde{x} = h^{-2/3}\int_0^\infty e^{\frac{i}{h}(-\tilde{s}^3/3-\tilde{s}(\tilde{x}q^{1/3}(\theta)-h^{2/3}\omega_k)+s^3/3+s(\tilde{x}q^{1/3}(\theta)-h^{2/3}\omega_k))}$$

$$\times \frac{2\pi}{L'(\omega_k)}\frac{q^{1/3}(\theta)}{h^{2/3}}\tilde{p}_h(\tilde{x}, \theta, h^{2/3}\omega_k, \tilde{s}, h/t)e^{-\frac{i}{h}\tilde{x}\tau_q(h^{2/3}\omega_k, \theta)A_\Gamma(\tilde{x}, 0, \sigma q^{1/3}(\theta)/\tau_q, \theta/\tau_q)}d\tilde{s}dsd\tilde{x},$$

for a symbol $\tilde{p}_h$ obtained from p_h in (2.17). For $\tilde{x} \gtrsim h^{2/3}\omega_k$, repeated integrations by parts in s yield an $O(h^\infty)$ contribution. For $\tilde{x} \lesssim h^{2/3}\omega_k$, apply stationary phase in $s, \tilde{x}$: as $\tilde{x}q^{1/3}(\theta) = h^{2/3}\omega_k - s^2$ and $sq^{1/3}(\theta) = \tilde{s}q^{1/3}(\theta) + \tau_q(A_\Gamma + \tilde{x}\partial_{\tilde{x}}A_\Gamma)$, we are left with an integral in $\tilde{s}$ with phase $e^{\frac{i}{h}(h^{2/3}\omega_k-\tilde{s}^2)q^{1/3}(-\theta)\tau_q(h^{2/3}\omega_k, \theta)A_\Gamma((h^{2/3}\omega_k-\tilde{s}^2)q^{-1/3}(\theta), 0, \tilde{s}q^{1/3}(\theta)/\tau_q, \theta/\tau_q)}$. One derivative with respect to θ on this phase yields at most $\frac{1}{h}(h^{2/3}\omega_k)^2 \lesssim h^{4\epsilon/3}$ for $k \leq K_\epsilon \leq h^{-1/4+\epsilon}$ and therefore σ_0 is of order 0. Rewriting $K_{\omega_k}(g_{h,a,1})$ after stationary phase, relabelling r_0 and r_1, we obtain (4.2). $\qquad\square$

Now we evaluate the L^∞ norm of $\mathcal{P}^1_{h,a}(t, .)$. If $d \geq 3$, we perform stationary phase in (4.2) w.r.t $\eta/|\eta| \in \mathbb{S}^{d-2}$: $\eta = \theta/h$, $\theta = |\theta|\vartheta$, $\theta \in \text{supp}(\varkappa)$, the phase of each $K_{\omega_k}(g_{h,a,1})(t, x, y)$ is

$$t\tau_q(h^{2/3}\omega_k, \theta) + |\theta|(y \cdot \vartheta + B_0(y, \vartheta)) + O(h^{2/3}\omega_k) = |\theta|(t + y \cdot \vartheta + B_0(y, \vartheta)) + O(h^{2/3}\omega_k),$$

where $O(h^{2/3}\omega_k)$ contains contributions from $B_\Gamma - B_0$, xA_Γ and aA_Γ. Critical points $\vartheta_\pm$ are given by (3.33) (with $\gamma = a$, $A = h^{2/3}\omega_k/(a|\theta|^{2/3})$, $S = s/(\sqrt{a}|\theta|^{1/3})$, $\Sigma = \sigma/(\sqrt{a}|\theta|^{1/3})$). As $\nabla^2_\vartheta(y \cdot \vartheta + B_0(y, \vartheta)) \sim |y|$ and $|y| \sim |t|$ (Lemma 3.2), stationary phase in ϑ yields $(h/|\theta||y|)^{d-2} \leq C(h/|t|)^{d-2}$ for ϑ near $\vartheta_\pm$. For ϑ outside a small neighborhood of $\vartheta_\pm$, we get $O(\frac{h}{t})^\infty$. Stationary phase yields new symbols as asymptotic expansions with small parameter $\frac{h}{t}$; the main contribution $K_{\omega_k}(g_{1,a,h})(t, x, y)$ remains similar to (4.2), with a front factor $(h/t)^{(d-2)/2}$, $\eta = |\theta|\vartheta_\pm/h$ and integration over η replaced by integration over $|\theta|$.

Lemma 4.3. ([9, Lemma 3.5]) *There exists C_0 such that for $L \geq 1$,*

$$(4.4) \quad \sup_{b \in \mathbb{R}}\Big(\sum_{1 \leq k \leq L}\omega_k^{-1/2}Ai^2(b - \omega_k)\Big) \leq C_0 L^{1/3}, \quad \sup_{b \in \mathbb{R}_+}\Big(\sum_{1 \leq k \leq L}\omega_k^{-1/2}h^{2/3}Ai'^2(b - \omega_k)\Big) \leq C_0 h^{2/3}L.$$

Applying Cauchy-Schwarz, dispersion for small t reduces to estimates like (4.4), as

$$(4.5) \quad \|\sum_{k=1}^L \frac{2\pi}{L'(\omega_k)}K_{\omega_k}(g_{h,a,1})(t, x, y)\|_{L^\infty} \lesssim h^{-d}\Big(\frac{h}{t}\Big)^{\frac{d-2}{2}}h^{1/3}\Big(L^{1/3} + h^{1/3}L^{2/3} + h^{2/3}L\Big),$$

where $h^{-(d-1)}(h/t)^{\frac{d-2}{2}}$ comes from stationary phase in ϑ, while $h^{-2/3} = h^{-1}h^{1/3}$ arises from $q^{1/3}(\theta/h)$ and L-related terms come from (4.4). Recall $h^{-2\epsilon} \leq K_\epsilon \leq h^{-1/4+\epsilon}$, hence $h^\epsilon \gg h^{2\epsilon} \geq \frac{1}{K_\epsilon}$. Let $L = h^{-\epsilon}$: for $t \leq h^\epsilon$ we bound $L \leq \frac{1}{t}$ in (4.5) and get (4.1) for the sum up to $h^{-\epsilon}$.

When $h^\epsilon \leq t$, we apply stationary phase in each oscillatory integral in the sum over k for $k \leq h^{-\epsilon}(\ll K_\epsilon)$. Using Corollary 6.10 in the Appendix, we reformulate the phase ψ of Theorem 2.1 as

$$\psi(x, y, \theta, \omega_k) = y \cdot \theta + \tau_q(h^{2/3}\omega_k, \theta)B_\Gamma(y, \theta/\tau_q) + \Upsilon(x, y, \theta, h^{2/3}\omega_k).$$

Lemma 4.4. *The stationary phase theorem with respect to $|\theta|$ applies in (4.2) with large parameter $t\omega_k h^{-1/3}$, with phase function $\phi_{h,k}(t, x, y, \theta) = t\tau_q(h^{2/3}\omega_k, \theta) + \psi(x, y, \theta, \omega_k)$ and with symbol*

(4.6)
$$\sigma_{h,k} = \left(p_0 Ai(-\zeta(x, y, \theta/h, \omega_k)) + ip_1 Ai'(-\zeta(x, y, \theta/h, \omega_k))\right)q^{\frac{1}{3}}(\theta)\varkappa(\theta)\varkappa(\tau_q(h^{\frac{2}{3}}\omega_k, \theta))\chi^\flat\left(\frac{\omega_k}{\omega_{K_\epsilon}}\right)$$
$$\times \sigma(\theta/h, \omega_k)r(\theta/h, \omega_k)\left(r_0 Ai(-\zeta(a, 0, \theta/h, \omega_k)) + iq^{-1/6}(\theta/h)r_1 Ai(-\zeta(a, 0, \theta/h, \omega_k))\right),$$

where p_0, p_1 were defined in (2.6) and $\theta = |\theta|\vartheta$ with $\vartheta \in \{\vartheta_\pm\}$.

Proof. Let $r = |\theta|$ and recall $\vartheta \in \{\vartheta_\pm\}$ (we already performed stationary phase in ϑ). We have

$$\phi_{h,k} = t\tau_q(h^{2/3}\omega_k, \theta) + \psi(x, y, \theta, \omega_k) = r\left(t + y \cdot \vartheta + B_0(y, \vartheta)\right)$$
$$+ \frac{r^{1/3}}{2}h^{2/3}\omega_k q^{2/3}(\vartheta)\left(t + B_0(y, \vartheta) - B_2(y, \vartheta) + O(|y|(h^{2/3}\omega_k))\right) + (rt)O((h^{2/3}\omega_k)^2)$$
$$- x\left(O(x) + O(h^{2/3}\omega_k)\right) - k_1\zeta\left(\zeta + O(xh^{2/3}\omega_k) + O((h^{2/3}\omega_k)^2)\right),$$

where, in the second line, $O(|y|(h^{2/3}\omega_k))$ comes from $B_\Gamma - B_0 - B_2$, and $(rt)O((h^{2/3}\omega_k)^2)$ comes from $t\tau_q(h^{2/3}\omega_k, r\vartheta)$ and where the last line represents the contribution from Υ as given in (6.31).

Here the critical points $\vartheta_\pm$ are solutions to Equation (3.33), but with γA replaced with $h^{2/3}\omega_k/r^{2/3}$ in the whole equation. Then, $y \cdot \vartheta_\pm + B_0(y, \vartheta_\pm) = \pm|y + \nabla B_0(y, \vartheta_\pm)| + O(a^2)$, $a^2 \ll h$. Moreover, $\partial_r \vartheta_\pm = O(h^{2/3}\omega_k)$ (deriving (3.33); originally in (3.33) we made a suitable change of variables and got rid of $r = |\theta|$, which is no longer possible as we now work with fixed ω_k). A critical point with respect to r for $\phi_{h,k}$ is such that

(4.7) $$0 = t \pm |y + \nabla B_0(y, \vartheta_\pm)| + O(a^2) + rO(|y|^2)O(h^{2/3}\omega_k)$$
$$+ \frac{2}{3}r^{-2/3}h^{2/3}\omega_k q^{2/3}(\vartheta_\pm)\left(t + B_0(y, \vartheta_\pm) - B_2(y, \vartheta_\pm) + O(|y|(h^{2/3}\omega_k))\right) + tO((h^{2/3}\omega_k)^2)$$
$$- \partial_r k_1\zeta^2(x, y, \theta, h^{2/3}\omega_k) + O(x^2, x(h^{2/3}\omega_k)),$$

where the last line is $\partial_r\Upsilon$, whose main contribution is $-\partial_r k_1\zeta^2(x, y, \theta, \omega_k)$: as $h^{2/3}\zeta(x, y, \theta/h, \omega_k) = \zeta(x, y, \theta, h^{2/3}\omega_k) = h^{2/3}\omega_k - xq^{1/3}(\theta)e_0(x, y, \vartheta, h^{2/3}\omega_k/|\theta|)$, $\partial_r\zeta$ yields a factor x, which puts everything else in $O(x^2, x(h^{2/3}\omega_k))$. Moreover, set $t(> h) > 0$ as usual, only ϑ_- is to be kept (at ϑ_+ the phase will be non-stationary). As $B_0(y, \vartheta_\pm) - B_2(y, \vartheta_\pm) = O(|y|^2) = O(t^2)$ and $x \lesssim h^{2/3-\epsilon}$, the main contribution of $\partial_r^2\phi_{h,k}$ comes from first terms on the second and last lines of (4.7)

$$-\frac{4}{9}r^{-5/3}h^{2/3}\omega_k q^{2/3}(\vartheta_\pm)t\left(1 + O(|y|^2/t) + O(|y|/t) \times (h^{2/3}\omega_k)) + O((h^{2/3}\omega_k)^2)\right)$$
$$- \partial^2 k_1\zeta^2(x, y, \theta, h^{2/3}\omega_k) + O(x^2, x(h^{2/3}\omega_k)).$$

Observe that $h^{2/3}\omega_k t \gg \zeta^2(x, y, \theta, h^{2/3}\omega_k)$ for all $t > h^\epsilon$: indeed, $h^\epsilon \gg h^{2/3}\omega_k$, which holds true for all $\omega_k \leq \omega_{K_\epsilon} \leq h^{-1/6+2\epsilon/3}$. At the stationary point satisfying (4.7), $|y|$ behaves like t, and therefore $\partial_r^2\phi_{h,k}$ behaves like $th^{2/3}\omega_k$ there. We are left to prove that stationary phase indeed applies with large parameter $(th^{2/3}\omega_k)/h \sim t\omega_k h^{-1/3}$, for $h^\epsilon \lesssim t$ and symbol $\sigma_{h,k}$ from (4.6). Computing $\partial_r^2\sigma_{h,k}$ yields at most $\omega_k^{11/4}$ which occurs when $\zeta(x, y, \theta/h, \omega_k)$ is large, $\zeta(a, 0, \theta/h, \omega_k)$ is bounded and when both derivatives fall on of the first Airy: hence, it suffices to check $\frac{h^{1/3}}{t\omega_k}\omega_k^{11/4} \ll 1$ for $t \geq h^\epsilon$, uniformly in $k \lesssim h^{-\epsilon}$: this does hold as long as $\epsilon < 2/13$ and we already set $\epsilon < 1/12$. $\square$

We then sum up to $k \leq h^{-\epsilon}$, for $t \geq h^\epsilon$ and $\epsilon < 1/12$, with $L'(\omega_k) \sim 2\omega_k^{\frac{1}{2}}$:

$$\|\mathcal{P}_{h,a}^1(t,x,y)\|_{L^\infty} \lesssim \frac{h^{1/3}}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}} \sum_{k \lesssim h^{-\epsilon}} \frac{1}{L'(\omega_k)}\left(\frac{h^{1/3}}{t\omega_k}\right)^{\frac{1}{2}} \sim \frac{1}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\left(\frac{h}{t}\right)^{\frac{1}{2}} h^{-\frac{\epsilon}{3}} \lesssim \frac{1}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\left(\frac{h}{t}\right)^{\frac{1}{3}}.$$

It remains to deal with $h^{-\epsilon} \leq k \leq K_\epsilon$ and $t \in (h, T_0]$. Taking $L = K_\epsilon$ in (4.5), we obtain (4.1) exactly as before for all $t \leq K_\epsilon^{-1}$ (up to $k \leq K_\epsilon$). Hence we are left with $t \geq K_\epsilon^{-1}$ and $h^{-\epsilon} \leq k \lesssim K_\epsilon$. It suffices to consider $0 \leq x, a \lesssim h^{\frac{2}{3}-\epsilon}$ (as for $x \geq h^{\frac{2}{3}-\epsilon}$, the arguments for the transverse case $a > h^{\frac{2}{3}-\epsilon}$ apply). For small values of both x, a, $|\zeta(x,y,\eta,\omega_k)| \geq \omega_k/2$ and $|\zeta(a,0,\eta,\omega_k)| \geq \omega_k/2$ ($k \geq h^{-\epsilon}$ is large so $\omega_k \geq h^{-2\epsilon/3} \gg h^{-\epsilon} \gtrsim a/h^{2/3}$). We can write $Ai(-\zeta) = \sum_{\pm} A_{\pm}(\zeta)$ which give rise to factors $\zeta(x,y,\eta,\omega_k)^{-1/4}\zeta(a,0,\eta,\omega_k)^{-1/4}$. If we decide to ignore the integral with respect to $r \in \text{supp}\varkappa$, we can immediately bound the sum of integrals with

$$\|\mathcal{P}_{h,a}^1(t,x,y)\|_{L^\infty} \lesssim \frac{h^{1/3}}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}} \sum_{h^{-\epsilon} \leq k \leq K_\epsilon} \frac{1}{\omega_k^{1/2}}\frac{1}{\omega_k^{1/2}} \lesssim h^{-d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}(hK_\epsilon)^{1/3},$$

and as $K_\epsilon \leq h^{-1/4+\epsilon}$, $(hK_\epsilon)^{1/3} \leq h^{1/4+\epsilon/3}$ which is not optimal (a similar crude bound was obtained in [9] in that regime.) To obtain sharper bounds when $t \geq K_\epsilon^{-1}$, $h^{-\epsilon} \leq k \leq K_\epsilon$, decompose $Ai(-z) = A_+(z) + A_-(z)$, with $A_{\pm}(z) = \Psi(e^{\mp i\pi/3}z)e^{\mp\frac{2}{3}iz^{3/2}}$ and $4\pi^{3/2}|\Psi(z)| \sim |z|^{-1/4}$ for large z. Then there are four different phases in $K_{\omega_k}(g_{h,a,1})$, where $\pm_1$ and $\pm_2$ mean idependent signs,

$$\frac{1}{h}\left(t\tau_q(h^{2/3}\omega_k,\theta) + \psi(x,y,\theta,\omega_k) \pm_1 \frac{2}{3}\zeta(x,y,\theta,h^{2/3}\omega_k)^{3/2} \pm_2 \frac{2}{3}\zeta(a,0,\theta,h^{2/3}\omega_k)^{3/2}\right)$$

and a symbol which behaves like $h^{-(d-1)-2/3}q^{1/3}(\theta)\varkappa(\theta)|\zeta(x,y,\theta/h,\omega_k)|^{-1}|\zeta(a,0,\theta/h,\omega_k)|^{-1/4}$, where $h^{-2/3}\zeta(x,y,\theta,h^{2/3}\omega_k) = \zeta(x,y,\theta/h,\omega_k) = \omega_k - xq^{1/3}(\theta)/h^{2/3}e_0(x,y,\theta/h,\omega_k) \geq \omega_k/2$ and also $\zeta(a,0,\theta/h,\omega_k) = \omega_k - aq^{1/3}(\theta/h)e_0(a,0,\theta/h,\omega_k) \geq \omega_k/2$. We prove that stationary phase applies with the same "large parameter" $th^{2/3}\omega_k/h$ but a new symbol. As in (4.7), we find

$$t - |y + \nabla B_0(y,\vartheta_-)| + rO(|y|^2)O(h^{\frac{2}{3}}\omega_k) + \frac{2}{3}r^{-\frac{2}{3}}h^{\frac{2}{3}}\omega_k q^{\frac{2}{3}}(\vartheta_-)\left(t + B_0(y,\vartheta_-) - B_2(y,\vartheta_-) - x\mu(y,\vartheta_-)\right)$$

$$+ O((h^{\frac{2}{3}}\omega_k)^2) \pm_1 x\partial_r(q^{\frac{1}{3}}(r\vartheta_-)e_0)\sqrt{\zeta(x,y,r\vartheta_-,h^{\frac{2}{3}}\omega_k)} \pm_2 a\partial_r(q^{\frac{1}{3}}(r\vartheta_-)e_0)\sqrt{\zeta(a,0,r\vartheta_-,h^{\frac{2}{3}}\omega_k)} = 0.$$

The second derivative is

$$-\frac{4}{9}r^{-5/3}h^{2/3}\omega_k q^{2/3}(\vartheta_-)(t + O(|y|^2)) + O((h^{2/3}\omega_k)^2)) \mp_1 xO(\sqrt{h^{2/3}\omega_k})(1 + O((h^{2/3}\omega_k)^2)$$

$$\mp_2 aO(\sqrt{h^{2/3}\omega_k})(1 + O((h^{2/3}\omega_k)^2)),$$

and for $t \geq K_\epsilon^{-1}$, $h^{-\epsilon} \leq k \leq K_\epsilon$, the main contribution comes from the first term : indeed, $t\sqrt{h^{2/3}\omega_k} \geq K_\epsilon^{-1}h^{1/3(1-\epsilon)} \geq h^{7/12-4\epsilon/3} \gg h^{2/3-\epsilon}$, hence $t(h^{2/3}\omega_k) \gg \max\{x,a\}O(\sqrt{h^{2/3}\omega_k})$ for all $x, a \lesssim h^{2/3-\epsilon}$. We then check that $th^{2/3}\omega_k/h$ is a large parameter for $t > K_\epsilon^{-1}$ (recall that in Lemma 4.4 we had $t \geq h^\epsilon \gg K_\epsilon^{-1}$): indeed, $t\omega_k \geq K_\epsilon^{-1}h^{-2\epsilon/3} \geq h^{1/4-5\epsilon/3} \gg h^{1/3}$. We find

$$\|\mathcal{P}_{h,a}^1(t,x,y)\|_{L^\infty} \leq \frac{h^{1/3}}{h^d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}} \sum_{1 \leq k \lesssim K_\epsilon} \frac{1}{L'(\omega_k)\omega_k^{1/4+1/4}}\left(\frac{h^{1/3}}{t\omega_k}\right)^{1/2} \sim h^{-d}\left(\frac{h}{t}\right)^{\frac{d-2}{2}}\left(\frac{h}{t}\right)^{1/2}\log h^{-1},$$

where we use the symbol of both factors $A_{\pm}$ to get $|\zeta|^{-1/4} < 2/\omega_k^{1/4}$. We proved (4.1) and therefore Theorem 3.1. □

5. Strichartz estimates

Standard duality and interpolation arguments would lead to Theorem 1.5 with $\gamma(d) = 1/4$ if one only uses dispersion formula (1.3). We aim at improving on this straightforward application of our dispersion estimate, combining three ingredients, where the first two were already proven earlier: in our estimates from Propositions 3.21, 3.22, 3.23 and 4.1, we either have better decay or additional (small) factors of a or γ; we also have the usual dispersion on time intervals with lengh less than $\sqrt{\gamma}$, in the sum of reflections regime (corresponding to dispersion on the V_0 term in the expansion over N); and finally, we claim that the additional cut-off that we introduced to localize $h^{2/3}\omega \sim \gamma$ is not only a useful technical device but also essentially commutes with the flow itself. Assuming these facts, one can then prove Strichartz estimates at fixed γ: on intervals of length one, with the $\gamma(d) = 1/4$ loss but with a constant that depends on a positive power of γ; on intervals of size $\sqrt{\gamma}$, one has the usual Strichartz estimates, with $\gamma(d) = 0$, and by iteration, one has a corresponding Strichartz estimate on time length one but with a constant that depends on a negative power of γ; by interpolation, one recovers a Strichartz estimate, at fixed γ, that improves upon $\gamma(d) = 1/4$, and then summation over γ yields the theorem, with strict inequality on $\gamma(d)$ because of that last summation over γ. Such a strategy was implemented in [11] and is facilitated there because the γ localization commutes with the wave flow. Moreover, a refined analysis of the dispersion estimates around the exceptional times that force $\gamma(d) = 1/4$ allows for a better Strichartz estimate to interpolate with, thus yielding a better result on the 2D model. We expect such result to generalize to the 2D general convex case, while higher dimensions will require additional arguments related to the space-time localization of swallowtails (which can no longer be a set of discrete, exceptional, times). As such, we believe that Theorem 1.5 is of interest as it illustrates, in a relatively simple way, that the rate of dispersion does tell the whole story as far as Strichartz estimates are concerned.

We now turn to the details: we relabel $\mathcal{P}_h(t, x, y, a, b)$ our parametrix but with source $(a, b) \in \mathbb{R}^d$, and similarly $\mathcal{P}_{h,\gamma}(t, x, y, a, b)$ where an additional cut-off in α was inserted. Inhomogeneous Strichartz estimates follow from estimating the inhomogeneous operator P_h (and/or $P_{h,\gamma}$ with obvious notations)

$$P_h f(t, x, y) = \int_s \mathcal{P}_h(t - s, x, y, a, b) f(s, a, b) \, da \, db \, ds \,.$$

We remark that, with such notations, the homogeneous operator approximating the half-wave flow with data $f_0(a, b)$ at time $t = 0$ is

$$(W_h f_0)(t, x, y) = P_h(\delta_{s=0} f_0(a, b))(t, x, y) \,,$$

and we define similarly $W_{h,\gamma}$. Now, suppose we restrict ourselves to $P_h^\flat = \sum_{\gamma < h^{1/3}} P_{h,\gamma}$: for $\mathcal{P}_{h,\gamma}$ with $\gamma < h^{1/3}$, we have $a^{1/4} h^{1/4} \lesssim h^{1/3}$ and, collecting all bounds from Propositions 3.21, 3.22, 3.23 and 4.1 in our regime, we do have

$$(5.1) \qquad |\mathcal{P}_h^\flat(t, x, y, a, b)| \lesssim \frac{1}{h^d} \left(\frac{h}{t}\right)^{(d-2)/2} \min\left(1, \left(\frac{h}{t}\right)^{1/3}\right) \,.$$

Commuting the wave flow and the $\alpha \sim \gamma$ localization in the parametrix is an issue, to be adressed in Proposition 5.1 below; we already observe that $\mathcal{P}_{h,\gamma}$ is an approximate solution to the wave equation, and as such, satisfies energy estimates. We then obtain inhomogeneous Strichartz estimates but with $\gamma(d) = 1/6$ rather than $1/4$, after interpolation between (5.1) and energy estimates, followed by the argument from [14] for the endpoint (only required for $d \geq 4$) and a dependence

on h dictated by scaling, which we encode with $\lesssim_h$:

$$\|P_{h,\gamma}f\|_{L_t^2 L_x^{2\frac{3d-4}{3d-7}}} \lesssim_h \|f\|_{L_t^2 L_x^{2\frac{3d-4}{3d+3}}} , \quad \text{or, for } d=3, \quad \|P_{h,\gamma}f\|_{L_t^{\frac{1}{2}5} L_x^\infty} \lesssim_h \|f\|_{L_t^{\frac{12}{7}} L_x^1} .$$

For now, we focus on a given $\gamma > h^{1/3}$. First, we remark that $P_{h,\gamma}$ satisfies inhomogeneous Strichartz estimates with $\gamma(d) = 1/4$, but with a constant related to γ (from the $a^{1/4}$, $\gamma^{1/4}$ factors in (3.69), (3.70), (3.71))

$$(5.2) \qquad \|P_{h,\gamma}f\|_{L_t^2 L_x^{2\frac{2d-3}{2d-7}}} \lesssim_h \gamma^{\frac{1}{2d-3}}\|f\|_{L_t^2 L_x^{2\frac{2d-3}{2d+4}}} , \quad \text{or, for } d=3, \quad \|P_{h,\gamma}f\|_{L_t^{\frac{8}{3}} L_x^\infty} \lesssim_h \gamma^{\frac{1}{4}}\|f\|_{L_t^{\frac{8}{5}} L_x^1} .$$

This follows from performing the usual duality and interpolation argument directly on $P_{h,\gamma}$ rather than P_h, using Proposition 5.1 to handle commutation of the localization with the flow. At the same time, $W_{h,\gamma}$ satisfies the usual short time dispersion (either from [2] or from (3.68) and close inspection of the $N = 1$ cases, especially Lemma 3.36), hence homogeneous Strichartz estimates on a time interval of size $\gamma^{1/2}$ (this may be seen as a direct consequence of short time Strichartz estimates proven in a more general context in [2].) With the help of Proposition 5.1, one may sum such L_t^2 estimates over $\gamma^{-1/2}$ intervals of length $\gamma^{-1/2}$, obtaining

$$\|W_{h,\gamma}f_0\|_{L_t^2 L_x^{2\frac{d-1}{d-3}}} \lesssim_h \gamma^{-\frac{1}{4}}\|f_0\|_2$$

and then revert to inhomogenous estimates,

$$(5.3) \qquad\qquad\qquad \|P_{h,\gamma}f\|_{L_t^2 L_x^{2\frac{d-1}{d-3}}} \lesssim_h \gamma^{-\frac{1}{2}}\|f\|_{L_t^2 L_x^{2\frac{d-1}{d+1}}} .$$

(abusing the enpoint for $d = 3$, for which exponents are to be shifted to avoid the forbidden endpoint). One then interpolates between (5.2) and (5.3) to retain $\gamma^{\varepsilon(q)}$, with $\varepsilon(q) > 0$ dictated by interpolation (to sum over γ later),

$$\|P_{h,\gamma}f\|_{L_t^2 L_x^q} \lesssim_h \gamma^{\varepsilon(q)}\|f\|_{L_t^2 L_x^{q'}} , \quad \text{or, for } d=3, \quad \|P_{h,\gamma}f\|_{L_t^{\frac{12}{5-12\varepsilon}} L_x^{\frac{1}{\varepsilon}}} \lesssim_h \gamma^\varepsilon\|f\|_{L_t^{\frac{12}{7+12\varepsilon}} L_x^{\frac{1}{1-\varepsilon}}} .$$

with $1/2 - 1/q > 2d/((d-1)(2d-1))$ for $d \geq 4$: this corresponds to $\gamma(d) > 1/4 - 1/4d$ for $d \geq 3$. Taking advantage of γ^ε, we sum over dyadic γ's to get estimates for $P_h^\sharp = \sum_{\gamma > h^{1/3}} P_{h,\gamma}$, and the resulting estimate is always worse than (5.2) for $P_h^\flat$, that is to say, $q > 2(3d-4)/(3d-7)$. As such, $P_h = P_h^\sharp + P_h^\flat$ satisfies the same set of estimates as $P_h^\sharp$, that is, Strichartz estimates with $\gamma(d) > 1/4 - 1/4d = 1/6 + (1/4)(1/3 - 1/d)$. In particular, for $d = 3$, we obtain (except for the endpoint) the set of Strichartz estimates that would result from dispersion with a loss of only $1/6$, which is the loss resulting from cusp-like singularities in the wave front. This completes the proof of Theorem 1.5, up to proving Proposition 5.1.

It remains to handle the issue of localization with respect to α in our parametrix. For this, we check that we do have a suitable form of the group property for the operator $W_{h,\gamma}(t)$: assume that $\tilde{W}_{h,\gamma}$ is the same operator (with kernel denoted by $\tilde{\mathcal{P}}_{h,\gamma}$) but with a cut-off in α, denoted $\tilde{\chi}_1$, which is the identity on the support of the cut-off χ_1 in $W_{h,\gamma}$.

Proposition 5.1. *We have* $\tilde{W}_{h,\gamma}(t_1) \circ W_{h,\gamma}(t_2) = W_{h,\gamma}(t_1+t_2)$, *modulo* $O(h^\infty)$. *In terms of kernels,*

$$\mathcal{P}_{h,\gamma}(t_1 + t_2, x, y; a, b) = \int_{\tilde{x},\tilde{y}} \tilde{\mathcal{P}}_{h,\gamma}(t_1, x, y; \tilde{x}, \tilde{y}) \mathcal{P}_{h,\gamma}(t_2, \tilde{x}, \tilde{y}; a, b) \, d\tilde{x}d\tilde{y} + O(h^\infty) .$$

Similarly, for γ_1 and γ_2 such that the corresponding cut-offs have disjoint support, $W_{h,\gamma_1}(t_1) \circ W_{h,\gamma_2}(t_2) = O(h^\infty)$ and

$$(5.4) \qquad \int_{\tilde{x},\tilde{y}} \mathcal{P}_{h,\gamma_1}(t_1, x, y; \tilde{x}, \tilde{y}) \mathcal{P}_{h,\gamma_2}(t_2, \tilde{x}, \tilde{y}; a, b) \, d\tilde{x}d\tilde{y} = O(h^\infty)\,.$$

Proof. Recall that $\mathcal{P}_{h,\gamma}$ is given by (3.5) with $K_{\omega,\gamma}$ defined in (3.4). Then we rewrite

$$\int_{\tilde{x},\tilde{y}} \tilde{\mathcal{P}}_{h,\gamma}(t_1, x, y; \tilde{x}, \tilde{y}) \mathcal{P}_{h,\gamma_1}(t_2, \tilde{x}, \tilde{y}; a, b) \, d\tilde{x}d\tilde{y} = \sum_{N_1, N_2} \int e^{-iN_1 L(\omega)} \chi^\flat(h^{2/3}\omega/\epsilon_0) J_{N_2,\gamma_1,\gamma_2}(t_1, t_2, x, y, \omega) d\omega,$$

where we set

$$J_{N_2,\gamma_1,\gamma_2}(t_1, t_2, x, y, \omega) = \int_{\tilde{x},\tilde{y}} \tilde{K}_{\omega,\gamma_1}(g_{h,(\tilde{x},\tilde{y})})(t_1, x, y) F_{N_2,t_2,\gamma_2}(\tilde{x}, \tilde{y}) d\tilde{x}d\tilde{y}\,,$$

$$F_{N_2,t_2,\gamma_2}(\tilde{x}, \tilde{y}) = e^{-iN_2 L(\tilde{\omega})} \chi^\flat(h^{2/3}\tilde{\omega}/\epsilon_0) K_{\tilde{\omega},\gamma_2}(g_{h,(a,b)})(t_2, \tilde{x}, \tilde{y})\,,$$

and $\tilde{K}_{\omega,\gamma_1}(g_{h,(\tilde{x},\tilde{y})})$ has the same form as (3.4) but with a cut-off $\tilde{\chi}_1(\frac{\omega}{\gamma_1|\eta|^{2/3}})$ instead of $\chi_1(\frac{\omega}{\gamma_1|\eta|^{2/3}})$, supported near 1 and equal to 1 on the support of χ_1.

From (3.4), $F_{N_2,t_2,\gamma_2}(\tilde{x}, \tilde{y})$ expands as

$$\int G(\tilde{x}, \tilde{y}, \tilde{\eta}, \tilde{\omega}) \chi^\#(\tilde{\omega}) q^{1/6}(\tilde{\eta}) \varkappa(h\tilde{\eta}) \varkappa(h\tau_q(\tilde{\omega}, \tilde{\eta})) \chi_1(\frac{\tilde{\omega}}{\gamma_2|\tilde{\eta}|^{2/3}}) e^{-iN_2 L(\tilde{\omega}) + it_2 \tau_q(\tilde{\omega}, \tilde{\eta})} \hat{g}_{h,(a,b)}(\tilde{\eta}, \frac{\tilde{\omega}}{h^{1/3}}) d\tilde{\omega}d\tilde{\eta},$$

hence $F_{N_2,t_2,\gamma_2}(\tilde{x}, \tilde{y}) = J(f_{N_2,t_2,\gamma_2})$, where we have set

$$(5.5) \qquad \hat{f}_{N_2,t_2,\gamma_2}(\tilde{\eta}, \frac{\tilde{\omega}}{h^{1/3}}) = \chi_1(\frac{\tilde{\omega}}{\gamma_2|\tilde{\eta}|^{2/3}}) e^{-iN_2 L(\tilde{\omega}) + it_2 \tau_q(\tilde{\omega}, \tilde{\eta})} \hat{g}_{h,(a,b)}(\tilde{\eta}, \frac{\tilde{\omega}}{h^{1/3}}).$$

We turn to $\tilde{K}_{\omega,\gamma_1}(g_{h,(\tilde{x},\tilde{y})})(t_1, x, y)$ which expands as

$$(5.6) \qquad \int G(x, y, \eta, \omega) \chi^\#(\omega) q^{1/6}(\eta) \varkappa(h\eta) \varkappa(h\tau_q(\omega, \eta)) \tilde{\chi}_1(\frac{\omega}{\gamma_1|\eta|^{2/3}}) e^{it_1 \tau_q(\omega, \eta)} \hat{g}_{h,(\tilde{x},\tilde{y})}(\eta, \frac{\omega}{h^{1/3}}) d\eta.$$

Lemma 5.2. *We have*

$$\int_{\tilde{x},\tilde{y}} g_{h,(\tilde{x},\tilde{y})}(y', \rho) F_{N_2,t_2,\gamma_2}(\tilde{x}, \tilde{y}) d\tilde{x}d\tilde{y} = J^{-1}(F_{N_2,t_2,\gamma_2})(y', \rho).$$

Proof. Recall from (2.33) that, in the variables $\theta = h\eta$, $\alpha = h^{2/3}\omega$, we may write

$$g_{h,(\tilde{x},\tilde{y})}(y', \rho) = \frac{1}{h^{2d+1}} \int e^{\frac{i}{h}(y' \cdot \theta' - \Phi(\overline{x}, \overline{y}, \theta', \alpha, s) + \rho\alpha + (\overline{x} - \tilde{x})\sigma + (\overline{y} - \tilde{y}) \cdot \theta)}$$

$$\times q_h(\overline{x}, \overline{y}, \theta', \alpha, s) q^{-1/6}(\theta') \tilde{\chi}_0(\sigma) \varkappa(\theta) d\overline{x}d\overline{y}d\theta' d\sigma ds d\alpha d\theta,$$

where $\tilde{\chi}_0$ is supported near 0 and equals 1 on the support of χ_0 (of Lemma 2.11). We thus obtain

$$\int_{\tilde{x},\tilde{y}} g_{h,(\tilde{x},\tilde{y})}(y', \rho) F_{N_2,t_2,\gamma_2}(\tilde{x}, \tilde{y}) d\tilde{x}d\tilde{y} = \frac{1}{h^{2d+1}} \int e^{\frac{i}{h}(y' \cdot \theta' - \Phi(\overline{x}, \overline{y}, \theta', \alpha, s) + \rho\alpha + \overline{x}\sigma + \overline{y} \cdot \theta)} q_h(\overline{x}, \overline{y}, \theta', \alpha, s) q^{-1/6}(\theta')$$

$$\times e^{-\frac{i}{h}(\tilde{x}\sigma + \tilde{y} \cdot \theta)} \tilde{\chi}_0(\sigma) \varkappa(\theta) \hat{F}_{N_2,t_2,\gamma_2}(\sigma/h, \theta/h).$$

As $\hat{F}_{N_2,t_2,\gamma_2} = \tilde{\chi}_0(\sigma) \hat{F}_{N_2,t_2,\gamma_2}(\sigma/h, \theta/h)$ (which contains $\chi_0(\sigma)$), the last integral equals $J^{-1}(F_{N_2,t_2,\gamma_2})$. $\square$

Using the lemma and $F_{N_2,t_2,\gamma_2}(\tilde{x},\tilde{y}) = J(f_{N_2,t_2,\gamma_2})$ yields

$$(5.7) \quad \int_{\tilde{x},\tilde{y}} g_{h,(\tilde{x},\tilde{y})}(y',\rho) F_{N_2,t_2,\gamma_2}(\tilde{x},\tilde{y}) d\tilde{x}d\tilde{y} = J^{-1} \circ J(f_{N_2,t_2,\gamma_2})(y',\rho) = f_{N_2,t_2,\gamma_2}(y',\rho) + O(h^\infty),$$

hence taking the Fourier transform of $g_{h,(\tilde{x},\tilde{y})}(y',\rho)$ (as needed in (5.6)) gives $\hat{f}_{N_2,t_2,\gamma_2}(\theta/h,\alpha/h)$ modulo $O(h^\infty)$ terms. Hence, $J_{N_2,\gamma_1,\gamma_2}(t_1,t_2,x,y,\omega)$ becomes

$$J_{N_2,\gamma_1,\gamma_2}(t_1,t_2,x,y,\omega) = \int G(x,y,\eta,\omega)\chi^\#(\omega)q^{1/6}(\eta)\varkappa(h\eta)\varkappa(h\tau_q(\omega,\eta))\tilde{\chi}_1(\frac{\omega}{\gamma_1|\eta|^{2/3}})e^{it_1\tau_q(\omega,\eta)}\hat{f}_{N_2,t_2,\gamma_2}(\eta,\omega/h^{1/3}),$$

with $\hat{f}_{N_2,t_2,\gamma_2}$ given in (5.5). We find, modulo $O(h^\infty)$ terms from (5.7) (which stay $O(h^\infty)$ as the sums over N_1, N_2 below are finite),

$$(5.8)$$
$$\int_{\tilde{x},\tilde{y}} \tilde{\mathcal{P}}_{h,\gamma_1}(t_1,x,y;\tilde{x},\tilde{y})\mathcal{P}_{h,\gamma_2}(t_2,\tilde{x},\tilde{y};a,b)\,d\tilde{x}d\tilde{y} = \sum_{N,\tilde{N}}\int e^{it_1\tau_q(\omega,\eta)-iN_1L(\omega)}\chi^\flat(h^{2/3}\omega/\epsilon_0)G(x,y,\eta,\omega)$$

$$\times \chi^\#(\omega)q^{1/6}(\eta)\varkappa(h\eta)\varkappa(h\tau_q(\omega,\eta))\tilde{\chi}_1(\frac{\omega}{\gamma_1|\eta|^{2/3}})\hat{f}_{N_2,t_2,\gamma_2}(\eta,\omega/h^{1/3})d\omega$$

$$= \sum_{N_1,N_2}\int e^{i(t_1+t_2)\tau_q(\omega,\eta)-i(N_1+N_2)L(\omega)}G(x,y,\eta,\omega)\chi^\#(\omega)q^{1/6}(\eta)\varkappa(h\eta)\varkappa(h\tau_q(\omega,\eta))$$

$$\tilde{\chi}_1(\frac{\omega}{\gamma_1|\eta|^{2/3}})\chi_1(\frac{\omega}{\gamma_2|\eta|^{2/3}})\hat{g}_{h,(a,b)}(\eta,\frac{\omega}{h^{1/3}}).$$

Recall that $\tilde{\chi}_1 = 1$ on the support of χ_1. Therefore, if $\gamma_1,\gamma_2 \in 1/2^{\mathbb{N}}$, with $\gamma_1 \neq \gamma_2$, then $\tilde{\chi}_1(\frac{\omega}{\gamma_1|\eta|^{2/3}})\chi_1(\frac{\omega}{\gamma_2|\eta|^{2/3}}) = 0$ and in this case we obtain (5.4).

It remains to deal with $\gamma_1 = \gamma_2 =: \gamma$: in this case, as $\tilde{\chi}_1\chi_1 = \chi_1$, the last sum in (5.8) equals

$$(5.9) \quad \sum_{N_1,N_2} < e^{-i(N_1+N_2)L(\omega)}, \chi^\flat(h^{2/3}\omega/\epsilon_0)K_{\omega,\gamma}(g_{h,(a,b),\gamma})(t_1+t_2,x,y) >_\omega +O(h^\infty),$$

and we need to show that the double sum (over N_1, N_2) yields indeed $\mathcal{P}_{h,\gamma}(t_1+t_2,x,y;a,b)$. This is the goal of the next lemma.

Lemma 5.3. *The sum in* (5.9) *can be written as*

$$(5.10) \quad \sum_N < e^{-iNL(\omega)}, \chi^\flat(h^{2/3}\omega/\epsilon_0)K_{\omega,\gamma}(g_{h,(a,b),\gamma})(t_1+t_2,x,y) >_\omega .$$

Proof. We use arguments similar to those in Lemma 2.16, i.e. we check when the phase function of $\int e^{-i(N_1+N_2)L(\omega)}K_{\omega,\gamma}(g_{h,(a,b),\gamma})(t_1+t_2,x,y)d\omega$ may be stationary with respect to $\alpha = h^{2/3}\omega$.

Recall first that, independently of the size of a (or $\tilde{x}$), the phase function of $\hat{g}_{h,(a,b)}$ (or of $\hat{g}_{h,(\tilde{x},\tilde{y})}$) equals that of $G(a,b,\eta,\omega)$ (or of $G(\tilde{x},\tilde{y},\tilde{\eta},\tilde{\omega})$) : this follows from Propositions 2.12 and and 2.35 in case $a > h^{2/3-\epsilon}$ and from Lemma 2.35 when $a \lesssim h^{2/3-\epsilon}$ and $\omega > h^{-\epsilon}$ (for some $\epsilon > 0$). (This property also holds when $a \lesssim h^{2/3-\epsilon}$ and ω is bounded). These constructions of $g_{h,(a,b)}(y',\rho)$ (resp. $g_{h,(\tilde{x},\tilde{y})}(y',\rho)$) in both cases are very similar and these functions are compactly supported for ρ near 0. As in Lemma 2.16, the phase function of $< e^{-iNL(\omega)}, \chi^\flat(h^{2/3}\omega/\epsilon_0)K_{\omega,\gamma}(g_{h,(a,b),\gamma})(t_1+t_2,x,y) >_\omega$ (with $\eta = \theta/h$, $\omega = \alpha/h^{2/3}$) is given by (2.45) and its critical points with respect to α,σ satisfy the equations (2.46) and (2.52). Therefore, taking $\alpha = \gamma A$, $\sigma = \sqrt{\gamma}\Sigma$, $T_1 = t_1/(2\sqrt{\gamma})$, $X = x/\gamma$ we must have

$$T_1 - \Sigma = \rho + 2N_1\sqrt{A}, \text{ where } \Sigma^2 + Xq^{1/3}(\theta)(1+\ell+O(\gamma)) = A \text{ and } \rho \text{ is small.}$$

If there are two non-trivial contributions for the same $t_1 > 0$, corresponding to N_1 and $N_1 - 1$, then there exists $A_1 \sim 1$ and Σ_1 satisfying $\Sigma_1^2 + Xq^{1/3}(\theta)(1 + \ell + O(\gamma)) = A_1$ such that

$$T_1 - \Sigma_1 = \rho + 2(N_1 - 1)\sqrt{A_1} = \rho + 2N_1\sqrt{A_1} - 2\sqrt{A_1}.$$

The only solutions satisfying the last equations verify $\Sigma/\sqrt{A} \sim -1$ and $\Sigma_1/\sqrt{A_1} \sim 1$. On the other hand, there are no solutions if, instead of $N_1 - 1$ and N_1 we consider $N_1 - 2$, N_1. Therefore, each time that we obtain non-trivial contributions from two consecutive values of N_1 in the sum (5.9), the corresponding critical points in Σ satisfy $\Sigma^2/A \sim 1$. In the same way, if, for fixed t_2, N_2 and $N_2 + 1$ provide nontrivial contributions in $f_{N_2,t_2,\gamma}(y', \rho)$ and $f_{N_2+1,t_2,\gamma}(y', \rho)$, then the (normalized) Airy variable $\tilde{\Sigma}$ in $G(a, b, \tilde{\eta}, \tilde{\omega})$ has to satisfy $\tilde{\Sigma}^2/\tilde{A} \sim 1$.

We introduce χ, supported in $[-3/4, 3/4]$ and equal to 1 on $[-1/2, 1/2]$ and $\chi_\pm(z) := (1-\chi)(z)$ if $\pm z > 0$, and equal to 0 otherwise: then $\chi + \sum_\pm \chi_\pm = 1$ everywhere. If σ, s denote the Airy variables of $G(x, y, \eta, \omega)$ and $G(a, b, \eta, \omega)$, we split its symbol in two parts, $p_h = p_h\chi(\sigma^2/\alpha) + p_h(1 - \chi(\sigma^2/\alpha))$ and write $K_{\omega,\gamma}(f) = K^0_{\omega,\gamma}(f) + K^\pm_{\omega,\gamma}(f)$. We then do the same with $g_{h,(a,b),\gamma}$, that we write as a sum $g_{h,(a,b),\gamma} = g^0_{h,(a,b),\gamma} + g^\pm_{h,(a,b),\gamma}$, where in the integral form of $\hat{g}_{h,(a,b),\gamma}$ we added the cut-offs $\chi(s^2/\alpha)$ and $(1 - \chi)(s^2/\alpha)$.

Using the same arguments as above, we notice that, with $N_t = [t/4\sqrt{\gamma}]$, the only non-trivial contribution in last line of (5.8) comes from pairs $(N_t, N_{\tilde{t}}), (N_t \mp 1, N_{\tilde{t}}), (N_t, N_{\tilde{t}} \pm 1), (N_t \mp 1, N_{\tilde{t}} \pm 1)$ which corresponds to the following products of cutoffs (with respect to $(\sigma/\alpha), (s/\alpha))$: (χ, χ), $(\chi_\pm, \chi), (\chi, \chi_\pm)$ and $(\chi_\pm, \chi_\pm)$. Summing up all these contributions allows to obtain (5.10) (as we recover $(\chi + \sum_\pm \chi_\pm)(\cdot)(\chi + \sum_\pm \chi_\pm)(\cdot))$; the sum of the contributions coming from $|N - N_t| \geq 2$ or $|\tilde{N} - \tilde{N}_{\tilde{t}}| \geq 2$ equals $O(h^\infty)$ by repeated integrations by parts and using that these sums stay finite. This achieves the proof of Proposition 5.1. $\qquad\square$

From the last Lemma we conclude the proof of Proposition 5.1. $\qquad\square$

6. APPENDIX

We provide details on the proof of Lemma 2.28 and on obtaining $\boldsymbol{\chi}_M$ (from Proposition 2.4).

6.1. Proof of Lemma 2.28. We start with twisted modes $\tilde{e}$ (see Definition 2.24):

Lemma 6.1. *Let k such that $\omega_k h^{2/3} \leq \epsilon_0$ with $\epsilon_0 < 1/100$. There exists constants $l_0 > 0$ and $0 < c_0 < C_0$ independent of k, h, a such that, for all y such that $|y| \leq l_0$,*

$$(6.1) \qquad\qquad c_0 \leq \|\tilde{e}(\cdot, y, \eta, \omega_k)\|_{L^2} \leq C_0.$$

Proof. By definition, $\tilde{e}(x, y, \eta, \omega_k) = \frac{q(\eta)^{\frac{1}{6}}}{\sqrt{L'(\omega_k)}}e^{-i(y\cdot\eta + |\eta|B_0(y, \eta/|\eta|))}G(x, y, \eta, \omega_k)$ and, using (2.22), $L'(\omega_k) = 2\pi\|Ai(\cdot - \omega_k)\|_{L^2}^2 \sim \omega_k^{1/2}$, for all $k \geq 1$. From (2.6), with e_0, p_0 and p_1 from Theorem 2.1,

$$(6.2) \quad \tilde{e}(x, y, \eta, \omega_k) = \frac{q(\eta)^{1/6}}{\sqrt{L'(\omega_k)}}e^{i(\psi(x,y,\eta,\omega_k) - y\cdot\eta - |\eta|B_0(y,\eta/|\eta|))}\left(p_0 Ai((-\zeta) + ip_1 q^{-1/6}(\eta)Ai'(-\zeta)\right),$$

where $\zeta(x, y, \eta, \omega_k) = -\omega_k + x|\eta|^{2/3}e_0(x, y, \eta/|\eta|, \omega_k/|\eta|^{2/3})$. Using Cauchy-Schwarz,

$$(6.3) \quad \int_0^\infty |\tilde{e}(x, y, \eta, \omega_k)|^2 dx \leq \frac{q^{1/3}(\eta)}{L'(\omega_k)}\left[\int_0^\infty \left|p_0 Ai(-\zeta)\right|^2 dx + 2\left(\int_0^\infty \left|p_0 Ai(-\zeta)\right|^2 dx\right)^{1/2}\right.$$

$$\left. \times \left(\int_0^\infty \left|p_1 q^{-1/6}(\eta)Ai'(-\zeta)\right|^2 dx\right)^{1/2} + \int_0^\infty \left|p_1 q^{-1/6}(\eta)Ai'(-\zeta)\right|^2 dx\right].$$

For values $x|\eta|^{2/3}e_0 > \omega_k$ both $Ai(-\zeta)$ and $Ai'(-\zeta)$ are exponentially decreasing. Setting $X := xe_0(x, y, \eta/|\eta|, \omega_k/|\eta|^{2/3})$ yields $x = x(X, y, \eta/|\eta|, \omega_k/|\eta|^{2/3})$ and

$$\frac{q^{1/3}(\eta)}{L'(\omega_k)} \int_0^\infty \left| p_0 Ai(-\zeta) \right|^2 dx = \frac{q^{1/3}(\eta)}{L'(\omega_k)} \int_0^\infty \left| \tilde{p}_0 Ai(X|\eta|^{2/3} - \omega_k) \right|^2 \frac{dx}{dX} dX,$$

where $\tilde{p}_0(X, y, \eta, \omega_k) := p_0(x(X, y, \eta/|\eta|, \omega_k/|\eta|^{2/3})$. Here $\omega_k/|\eta|^{2/3} \ll 1$ from $\eta \sim 1/h$ and $h^{2/3}\omega_k \ll 1$. Moreover $\frac{dx}{dX} = \frac{1}{e_0(0,\cdot)} + O(X)$, $e(0, \cdot)$ is close to 1 and $X \sim x \leq \omega_k|\eta|^{2/3} \ll 1$ (as for large X, $Ai(\cdot)$ is exponentially decreasing.) With $C := \sup |\tilde{p}_0|^2|\frac{dx}{dX}| < \infty$ and rescaling variables $\tilde{X} = X|\eta|^{2/3}$,

$$(6.4) \qquad \frac{q^{1/3}(\eta)}{L'(\omega_k)} \int_0^\infty \left| p_0 Ai(-\zeta) \right|^2 dx \leq C \frac{q^{1/3}(\eta)/|\eta|^{2/3}}{L'(\omega_k)} \int_0^{\omega_k} Ai(\tilde{X} - \omega_k)^2 dX = Cq^{1/3}(\eta/|\eta|).$$

For the integral with Ai' we proceed similarly: p_1 is bounded, $\|Ai'(\tilde{X} - \omega_k)\|^2_{L^2_{\tilde{X}}} \lesssim \omega_k^{3/2}$ and $\tilde{X} \lesssim \omega_k$ as for large values we have exponential decay. We compute, with $X = xe_0(x, \cdot)$, then $\tilde{X} = X|\eta|^{2/3}$,

$$(6.5) \quad \frac{q^{1/3}(\eta)}{L'(\omega_k)} \int_0^\infty \left| p_1 q^{-1/6}(\eta) Ai'(-\zeta) \right|^2 dx \leq \frac{C'}{L'(\omega_k)} \int_0^\infty |X Ai'(X|\eta|^{2/3} - \omega_k)|^2 dX$$

$$\leq \frac{C'|\eta|^{-2/3}}{L'(\omega_k)} \int_0^{\omega_k} \tilde{X}^2 |\eta|^{-4/3} |Ai'(\tilde{X} - \omega_k)|^2 d\tilde{X} \leq C'|\eta|^{-2} \omega_k^{3+1/2}/L'(\omega_k) \sim C'(\omega_k h^{2/3})^3 \ll C',$$

where we used $|Ai'(z)^2| \leq (1 + |z|)^{1/2}$ and that $p_1|_{x=0} = 0$. Collecting all bounds yields the upper bound in (6.1), which will be enough for proving Lemma 2.28 below. We now prove the lower bound, which was used for the proof of Proposition 2.25. From (6.3), using (6.4), (6.5) and $\omega_k h^{2/3} \leq \epsilon_0$, we have

$$\int_0^\infty |\tilde{e}(x, y, \eta, \omega_k)|^2 dx \geq \frac{q^{1/3}(\eta)}{L'(\omega_k)} \left[\int_0^\infty \left| p_0 Ai(-\zeta) \right|^2 dx - 2 \left(\int_0^\infty \left| p_0 Ai(-\zeta) \right|^2 dx \right)^{1/2} \right.$$

$$\times \left. \left(\int_0^\infty \left| x p_1 q^{-1/6}(\eta) Ai'(-\zeta) \right|^2 dx \right)^{1/2} - \int_0^\infty \left| x p_1 q^{-1/6}(\eta) Ai'(-\zeta) \right|^2 dx \right]$$

$$\geq \frac{q^{1/3}(\eta)}{L'(\omega_k)} \int_0^\infty \left| p_0 Ai(-\zeta) \right|^2 dx - O(\omega_k h^{2/3})^{3/2}.$$

As $\omega_k h^{2/3} \leq \epsilon_0$, we are left to prove that $\frac{q^{1/3}(\eta)}{L'(\omega_k)} \int_0^\infty \left| p_0 Ai(-\zeta) \right|^2 dx$ can be bounded from below by a constant independently of k, h, a. From ellipticity of $p_0 \sim 1$, there exists $\varepsilon_1 > 0$ such that $p_0(x, y, \eta, \omega_k) \geq 1/2$ for all (x, y) such that $|(x, y)| \leq \varepsilon_1$. On the other hand, for values $x|\eta|^{2/3}e_0(x, y, \eta/|\eta|, \omega_k/|\eta|^{2/3}) > \omega_k$ with $|\eta| \sim 1/h$, $Ai(-\zeta)$ is exponentially decreasing: thus, the bulk of the L^2 norm of $p_0 Ai(-\zeta)$ is located for $x \lesssim \omega_k h^{2/3} \leq \epsilon_0 \ll 1$ and $\int_{4\epsilon_0}^\infty |p_0 Ai(-\zeta)|^2 dx = O(h^\infty)$. Taking ϵ_0 smaller if necessary such that $\epsilon_0 < \varepsilon_1/4$, we have, for all $|y| \leq \varepsilon_1$

$$(6.6) \qquad \frac{q^{1/3}(\eta)}{L'(\omega_k)} \int_0^\infty \left| p_0 Ai(-\zeta) \right|^2 dx \geq \frac{q^{1/3}(\eta)}{4L'(\omega_k)} \int_0^{4\epsilon_0} |Ai(-\zeta)|^2 dx + O(h^\infty).$$

Ellipticity of e_0 near $(x, y) = (0, 0)$ provides $c > 0$ and $\varepsilon_2 > 0$ such that $e_0(x, y, \cdot) \geq c$ for all $|(x, y)| \leq \varepsilon_2$. Taken ε_0 smaller if necessary (so that $\varepsilon_0 < \varepsilon_2/4$), we can assume that $e_0(x, y, \cdot) \geq c$ for all $x \leq 2\varepsilon_0$ and all $|y| \leq \varepsilon_2$. Let $X = xe_0(x, y, \eta/|\eta|, \omega_k/|\eta|^{2/3})$ for $x \leq \varepsilon_0$ and $|y| \leq l_0 :=$

$\min\{\varepsilon_1, \varepsilon_2\}$, then $|\frac{dx}{dX}| \geq \frac{1}{c}$ for all $0 \leq x \leq 4\varepsilon_0$ and $|y| \leq l_0$ and

$$\frac{q^{1/3}(\eta)}{4L'(\omega_k)} \int_0^{4\epsilon_0} |Ai(-\zeta)|^2 dx \geq \frac{q^{1/3}(\eta/|\eta|)}{4cL'(\omega_k)} \int_0^{4\epsilon_0|\eta|^{2/3}} Ai^2(\tilde{X} - \omega_k) d\tilde{X}.$$

As $\omega_k \leq 4\varepsilon_0|\eta|^{2/3}$ for $h|\eta| \in [1/2, 2]$ and $\omega_k h^{2/3} \leq \varepsilon_0$, we find

$$\int_0^{4\epsilon_0|\eta|^{2/3}} Ai^2(\tilde{X} - \omega_k) d\tilde{X} = \int_0^\infty Ai^2(\tilde{X} - \omega_k) d\tilde{X} - \int_{4\epsilon_0|\eta|^{2/3}}^\infty Ai^2(\tilde{X} - \omega_k) d\tilde{X} = L'(\omega_k) + O(h^\infty)$$

and therefore the left hand side term in (6.6) is bounded from below by $\inf_{\Theta \in \mathbb{S}^{d-2}} q^{1/3}(\Theta)/(4c)$ for all y with $|y| \leq l_0$. As q is positive definite, this completes the proof of Lemma 6.1. $\qquad\square$

Proof. (of Lemma 2.28) Using $B_{2j}(y, \cdot) = O(y)$ for all $j \geq 2$ and Corollary 6.10, the phase $\psi(x, y, \eta, \omega_k) - y \cdot \eta - |\eta| B_0(y, \eta/|\eta|)$ of $\tilde{e}(x, y, \eta, \omega_k)$ reads (see (6.2))

$$(\tau_q(\omega_k, \eta) - |\eta|)(B_0 + B_2) + \tau_q(\omega_k, \eta)(O(x)\mathcal{H}_{j\geq 2} + O(\zeta/|\eta|^{2/3})\mathcal{H}_{j\geq 2} + O(y)\mathcal{H}_{j\geq 3}).$$

Using Lemma 6.1, $\|\tilde{e}(., \omega_k)\|_{L^2(x\geq 0)} \lesssim 1$. From $\tau_q(\omega, \eta) - |\eta| = \frac{\omega q^{2/3}(\eta)}{|\eta| + \tau_q(\omega, \eta)} \sim \omega|\eta|^{1/3}$ and $x \lesssim \omega_k/|\eta|^{2/3}$, taking derivatives with respect to y or η of the phase of (6.2) provides, at each step, $\omega_k|\eta|^{1/3} \sim \omega_k/h^{1/3}$. On the other hand, taking the derivatives (with respect to y or η) in the last factor of the right hand side of (6.2) and using that $Ai''(-\zeta) = \zeta Ai(-\zeta)$ provides

$$\partial_y^{\beta_1} \partial_\eta^{\beta_2} \left(p_0 Ai(-\zeta) + ip_1 q^{-1/6}(\eta) Ai'(-\zeta)\right) = (\omega|\eta|^{1/3})^{\beta_1 + \beta_2} \left(p_0^{(\beta_1, \beta_2)} Ai(-\zeta) + ip_1^{(\beta_1, \beta_2)} q^{-1/6}(\eta) Ai'(-\zeta)\right),$$

where $p_0^{(\beta_1, \beta_2)}$ and $p_1^{(\beta_1, \beta_2)}$ are asymptotic expansions with main contributions homogeneous of degree 0 and small parameter $(\omega_k|\eta|^{1/3})^{-1}$. Then, (2.76) follows from bounds like in (6.4) and (6.5). $\qquad\square$

6.2. **The generating function φ_Γ of χ_M.** We aim at proving (2.10). Set $\Theta = \varrho\vartheta$ with $\varrho = |\Theta|$ near 1 and $\vartheta = \Theta/|\Theta|$. Functions A_Γ, B_Γ are to be defined near the glancing set $\mathcal{GL} = \{x = 0, \Xi = 0, \varrho - 1 = 0\}$ and for (y, ϑ) near $\{0\} \times \mathbb{S}^{d-1}$. We work with formal Taylor expansions F near $\mathcal{GL}$ such that $F = \sum_{a,b,c} f_{a,b,c}(y, \vartheta) X_M{}^a (\varrho - 1)^b \xi^c$. We attribute a degree to each factor $x, \varrho - 1, \Xi$: a monomial of the form $x^a(\varrho - 1)^b\Xi^c$ is homogeneous of degree k if and only if $c + 2(a + b) = k$. For such a formal serie $F(x, y, \Xi, \varrho, \vartheta)$, defined near $\mathcal{GL}$, we write $F = \sum_{k\geq 0} F_k$, where F_k is homogeneous of degree k; we also write $F \in \mathcal{H}_{\geq j}$ if and only if $F = \sum_{k\geq j} F_k$. Therefore, $F_0 = f_0(y, \vartheta)$, $F_1 = \Xi f_1(y, \vartheta)$, $F_2 = X_M f_2^0(y, \vartheta) + (\varrho - 1)f_2^1(y, \vartheta) + \Xi^2 f_2^2(y, \vartheta)$, and so on. Replacing X_M, ξ, η by their formulas (2.11) (as functions of (x, y, Ξ, Θ)) and using that from (2.9)

$$(6.7) \qquad \xi^2 + R(x, y, \eta) = 1 \quad \text{if and only if} \quad \Xi^2 + |\Theta|^2 + X_M q(\Theta) = 1,$$

(where we notice that there is no Y_M in the second equation), we get

$$(6.8) \qquad B_\Gamma = \sum_{j\geq 0} (\varrho - 1)^j B_{2j}(y, \vartheta), \quad A_\Gamma = \sum_{k\geq 1} A_k.$$

Using the third equation from (2.11) and $\xi|_{\mathcal{GL}} = 0$, we have $A_0 = 0$. We also have $\xi(x, y, \Xi, \Theta) \in \mathcal{H}_{\geq 1}$ and $X_M(x, y, \Xi, \Theta) \in \mathcal{H}_{\geq 2}$. Moreover, from the proof of Melrose's theorem [15], if formal series of the form (6.8) satisfy (2.11) and (6.7), then there exist C^∞ functions A_Γ, B_Γ with the same Taylor development near $\mathcal{GL}$ satisfying (2.11) and (6.7).

We first consider in (6.7) homogeneous terms of order ≤ 5 in the expansion of Γ: using $X_M(x, y, \Xi, \Theta) \in \mathcal{H}_{\geq 2}$, we are to write the explicit form of A_Γ up to $\mathcal{H}_{j\leq 3}$ and B_Γ up to $\mathcal{H}_{j\leq 5}$. Let

$A_1 = \Xi\ell(Y, \vartheta)$, $A_2 = \alpha(y, \vartheta)x + \beta(y, \vartheta)(\varrho - 1) + \mu(y, \vartheta)\Xi^2$, $A_3 = \alpha_1(y, \vartheta)x\Xi + \beta_1(y, \vartheta)(\varrho - 1)\Xi + \mu_1(y, \vartheta)\Xi^3$ and $A_\Gamma = A_1 + A_2 + A_3 + \mathcal{H}_{j\geq 4}$.

Remark 6.2. Understanding this process will allow us to proceed with the expansion at any order. Observe that A_{2j} and A_{2j+1} always have the same number of terms: the only way to obtain homogeneous terms of order $2j+1$ is to add a factor Ξ to homogeneous terms of order $2j$. Moreover, each A_{2j} and A_{2j+1} will have $2j + 1$ terms. This is of importance to understand why all A_k may be obtained from the system of equations that will follow below.

Using these expansions for A_Γ and B_Γ, and omitting variables for the functions ℓ, μ, α, β, μ, α_1, β_1 and μ_1, (2.11) yields

$$(6.9) \qquad X_M = x(1 + \ell + 2\mu\Xi + \alpha_1 x + \beta_1(\varrho - 1) + 3\mu_1\Xi^2 + \mathcal{H}_{j\geq 3})$$

$$(6.10) \qquad \xi = (1 + \ell)\Xi + (2\alpha x + \beta(\varrho - 1) + \mu\Xi^2) + [2\alpha_1 x\Xi + \beta_1(\varrho - 1)\Xi + \mu_1\Xi^3] + \mathcal{H}_{j\geq 4}$$

$$(6.11) \qquad \eta = \vartheta + \nabla_y B_0 + (\varrho - 1)(\vartheta + \nabla_y B_2) + x\Xi\nabla_y\ell + x^2\nabla_y\alpha + x(\varrho - 1)\nabla_y\beta$$
$$+ x\Xi^2\nabla_y\mu + (\varrho - 1)^2\nabla_y B_4 + \mathcal{H}_{j\geq 5}.$$

Using $x \in \mathcal{H}_{j\geq 2}$ we rewrite $x = x_2 + x_3 + x_4 + \mathcal{H}_{j\geq 5}$ where $x_j \in \mathcal{H}_j$; in the same way, $\eta = \eta_0 + \eta_1 + \eta_2 + \eta_3 + \eta_4 + \mathcal{H}_{j\geq 5}$, with $\eta_j \in \mathcal{H}_j$. From (6.11) we obtain

$$(6.12) \qquad \eta_0 = \vartheta + \nabla_y B_0, \quad \eta_1 = 0, \quad \eta_2 = (\varrho - 1)(\vartheta + \nabla_y B_2).$$

Then, from $x \in \mathcal{H}_{j\geq 2}$, η_3 is homogeneous of order 3 and $\eta_3 = x_2\Xi\nabla_y\ell$, while

$$\eta_4 = x_3\Xi\nabla_y\ell + x_2^2\nabla_y\alpha + x_2(\varrho - 1)\nabla_y\beta + x_2\Xi^2\nabla_y\mu + (\varrho - 1)^2\nabla_y B_4.$$

Similarly, $\xi = \xi_1 + \xi_2 + \xi_3 + \mathcal{H}_{j\geq 4}$, $\xi_j \in \mathcal{H}_j$, depending on x_2, x_3, x_4 (notice that $\xi_0 = 0$). For ξ, an expansion up to $\mathcal{H}_{j\geq 3}$ is sufficient as we work with ξ^2. From (6.10),

$$\xi_1 = (1 + \ell)\Xi, \quad \xi_2 = 2\alpha x_2 + \beta(\varrho - 1) + \mu\Xi^2, \quad \xi_3 = 2\alpha x_3 + 2\alpha_1 x_2\Xi + \beta_1(\varrho - 1)\Xi + \mu_1\Xi^3.$$

Lemma 6.3. *Let* $\mathcal{L} := \{(x, y, \vartheta, \varrho, \Xi), X_M q(\Theta) = 1 - \Xi^2 - |\Theta|^2\}$, *where* $X_M = X_M(x, y, \vartheta, \varrho, \Xi)$ *is given by* (6.9), $\Theta = \varrho\vartheta$. *If* $(x, y, \vartheta, \varrho, \Xi) \in \mathcal{L}$ *then* $x = x_2 + x_3 + x_4 + \mathcal{H}_{j\geq 5}$, *with*

$$(6.13) \qquad x_2 = -\frac{(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)}, \quad x_3 = \frac{2\mu\Xi(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)^2},$$

$$(6.14) \qquad x_4 = \frac{1}{q(\vartheta)(1 + \ell)}\Big[\Xi^4\Big(\frac{3\mu_1}{(1 + \ell)} - \frac{4\mu^2}{(1 + \ell)^2} - \frac{\alpha_1}{q(\vartheta)(1 + \ell)^2}\Big)$$
$$+ (\varrho - 1)^2\Big(\frac{2}{(1 + \ell)}\big(\frac{-2\alpha_1}{q(\vartheta)(1 + \ell)} + 3 + \beta_1\big)\Big)$$
$$+ \Xi^2(\varrho - 1)\Big(2 + 2\big(\frac{3\mu_1}{(1 + \ell)} - \frac{4\mu^2}{(1 + \ell)^2} - \frac{\alpha_1}{q(\vartheta)(1 + \ell)^2}\big)$$
$$+ \frac{1}{(1 + \ell)}\big(\beta_1 - \frac{2\alpha_1}{q(\vartheta)(1 + \ell)}\big)\Big)\Big].$$

Proof. Using $X_M q(\Theta) = 1 - \Xi^2 - |\Theta|^2$, $q(\Theta) = \varrho^2 q(\vartheta)$ and (6.9) yields

$$(6.15) \qquad x(1 + \ell + 2\mu\Xi + \alpha_1 x + \beta_1(\varrho - 1) + 3\mu_1\Xi^2 + \mathcal{H}_{j\geq 3})\varrho^2 q(\vartheta) = 1 - \varrho^2 - \Xi^2.$$

which immediately provides x in terms of (y, ϑ), Ξ and $\varrho - 1$ up to $\mathcal{H}_{j\geq 3}$ as follows

$$x|_\mathcal{L} = \frac{(1 - \varrho^2 - \Xi^2)}{\varrho^2 q(\vartheta)(1 + \ell)}\Big(1 - \frac{2\Xi\mu}{1 + \ell} + \mathcal{H}_{j\geq 2}\Big);$$

hence, using $1 - \varrho^2 = -2(\varrho - 1) - (\varrho - 1)^2$, we get (6.13). In order to obtain homogeneous terms of order 4 we write $x|_{\mathcal{L}} = x_2 + x_3 + x_4 + \mathcal{H}_{j \geq 5}$ and replace x_2, x_3, x_4 in (6.15); using moreover $\frac{1}{\varrho^2} = 1 - 2(\varrho - 1) + 3(\varrho - 1)^2 + O((\varrho - 1)^3)$, we find

$$(x_2 + x_3 + x_4 + \mathcal{H}_{j \geq 5})(1 + \ell + 2\mu\Xi + \alpha_1 x_2 + \beta_1(\varrho - 1) + 3\mu_1\Xi^2 + \mathcal{H}_{j \geq 3})$$
$$= -\frac{1}{q(\vartheta)}(\Xi^2 + 2(\varrho - 1) + (\varrho - 1)^2) \times (1 - 2(\varrho - 1) + 3(\varrho - 1)^2 + \mathcal{H}_{j \geq 6})$$
$$= -\frac{1}{q(\vartheta)}\Big((\Xi^2 + 2(\varrho - 1)) - (\varrho - 1)(2\Xi^2 + 3(\varrho - 1)) + \mathcal{H}_{j \geq 6}\Big).$$

Identifying homogeneous terms of order $2, 3, 4$, we obtain (again) x_2, x_3 as well as x_4:

$$x_4(1 + \ell) + 2\mu\Xi x_3 + (\alpha_1 x_2 + \beta_1(\varrho - 1) + 3\mu_1\Xi^2)x_2 = \frac{1}{q(\vartheta)}(\varrho - 1)(2\Xi^2 + 3(\varrho - 1)),$$

which yields (6.14) by substitution. This completes the proof of Lemma 6.3. $\qquad\square$

We now replace (6.13) and (6.14) in (6.10) and (6.11) and then x, ξ, η in $\xi^2 + R(x, y, \eta) = 1$ to obtain a system of equations with unknown $B_0, B_2, B_4, \ell, \alpha, \beta, \mu, \alpha_1, \beta_1, \mu_1$ as follows. First,

$$\xi^2 = (\xi_1 + \xi_2 + \xi_3 + \mathcal{H}_{j \geq 4})^2 = (1 + \ell)^2\Xi^2 + 2(1 + \ell)\Xi(2\alpha x_2 + \beta(\varrho - 1) + \mu\Xi^2)$$
$$+ 2(1 + \ell)\Xi(2\alpha x_3 + 2\alpha_1 x_2\Xi + \beta_1(\varrho - 1)\Xi + \mu_1\Xi^3) + (2\alpha x_2 + \beta(\varrho - 1) + \mu\Xi^2)^2 + \mathcal{H}_{j \geq 5}.$$

Write $\xi^2 = (\xi^2)_2 + (\xi^2)_3 + (\xi^2)_4 + \mathcal{H}_{j \geq 5}$, where $(\xi^2)_j \in \mathcal{H}_j$; replacing x_2 and x_3 by (6.13), we find

$$(6.16)\; (\xi^2)_2 = (1 + \ell)^2\Xi^2 \in \mathcal{H}_2,$$

$$(6.17)\; (\xi^2)_3 = 2(1 + \ell)\Xi\Big(-2\alpha\frac{(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)} + \beta(\varrho - 1) + \mu\Xi^2\Big) \in \mathcal{H}_3,$$

$$(6.18)\; (\xi^2)_4 = \Big(-2\alpha\frac{(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)} + \beta(\varrho - 1) + \mu\Xi^2\Big)^2 + 2(1 + \ell)\Xi$$
$$\times \Big(2\alpha\frac{2\mu\Xi(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)^2} - 2\alpha_1\Xi\frac{(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)} + \beta_1(\varrho - 1)\Xi + \mu_1\Xi^3\Big) \in \mathcal{H}_4$$
$$= (\Xi^2 + 2(\varrho - 1))^2\Big(\mu - \frac{2\alpha}{q(\vartheta)(1 + \ell)}\Big)^2 + 2(1 + \ell)\mu_1\Xi^4 + 2(1 + \ell)\beta_1\Xi^2(\varrho - 1)$$
$$+ \frac{4\Xi^2}{q(\vartheta)}\Big(\frac{2\alpha\mu}{(1 + \ell)} - \alpha_1\Big)(\Xi^2 + 2(\varrho - 1)).$$

We do the same for $\eta = \eta_0 + \eta_2 + \eta_3 + \eta_4 + \mathcal{H}_{j \geq 5}$, for which it remains to replace x_2 and x_3 obtained in (6.13) in the expression of η_3 and η_4 that we have already obtained from (6.11). We get
(6.19)

$$\eta_3 = -\Xi\nabla_y\ell\frac{(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)}, \quad \eta_4 = (\varrho - 1)^2\nabla_y B_4 - \frac{(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)}\Big(\Xi^2\nabla_y\mu + (\varrho - 1)\nabla_y\beta\Big)$$
$$+ \nabla_y\ell\frac{2\mu\Xi^2(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)^2} + \nabla_y\alpha\frac{(\Xi^2 + 2(\varrho - 1))^2}{q^2(\vartheta)(1 + \ell)^2}.$$

We expand $R(x, y, \eta) R_0(y, \eta) + x R_1(y, \eta) + \frac{1}{2} x^2 R_2(y, \eta) + O(x^3)$ (η_0, η_2 were obtained in (6.12)),

$$(6.20) \quad R(x, y, \eta) = R_0(y, \eta_0) + (\eta_2 + \eta_3 + \eta_4) \nabla_\eta R_0(y, \eta_0) + \frac{1}{2} \eta_2^2 \nabla_{\eta,\eta}^2 R_0(y, \eta_0) + \mathcal{H}_{j \geq 5}$$

$$+ (x_2 + x_3 + x_4)(R_1(y, \eta_0) + \eta_2 \nabla_\eta R_1(y, \eta_0)) + \frac{1}{2} x_2^2 R_2(y, \eta_0) + \mathcal{H}_{j \geq 5} .$$

We set $R(x, y, \eta) = (R)_0 + (R)_1 + (R)_2 + (R)_3 + (R)_4 + \mathcal{H}_{j \geq 5}$, with $(R)_j$ the homogeneous term of order j in $R(x, y, \eta)$; from (6.20) and (6.12) we get $(R)_0 = R_0(y, \vartheta + \nabla_y B_0)$, $(R)_1 = 0$, and

$$(6.21) \qquad (R)_2 = \eta_2 \nabla_\eta R_0(y, \eta_0) + x_2 R_1(y, \eta_0), \quad \eta_0 = \vartheta + \nabla_y B_0(y, \vartheta),$$

$$(6.22) \qquad (R)_3 = \eta_3 \nabla_\eta R_0(y, \eta_0) + x_3 R_1(y, \eta_0),$$

$$(6.23) \qquad (R)_4 = \eta_4 \nabla_\eta R_0(y, \eta_0) + \frac{1}{2} \eta_2^2 \nabla_{\eta,\eta}^2 R_0(y, \eta_0) + x_2 \eta_2 \nabla_\eta R_1(y, \eta_0) + \frac{1}{2} x_2^2 R_2(y, \eta_0) .$$

Recall from (2.1) that we had set $R_0(y, \eta) = R(0, y, \eta) = |\eta|^2 + O(y)$ and $R_1(y, \eta) = \frac{\partial R}{\partial x}(0, y, \eta)$, $q(\eta) := R_1(0, \eta)$. From (6.7) it follows that for $(x, y, \vartheta, \varrho, \Xi) \in \mathcal{L}$ we must have

$$(6.24) \qquad\qquad\qquad \xi^2 + R(x, y, \eta) = 1 .$$

On $\mathcal{L}$ we have obtained $x|_{\mathcal{L}}$ as a sum of homogeneous terms of the form (6.13), (6.14) which in turn has allowed to do the same for ξ^2 and $R(x, y, \eta)$; it remains to get homogeneous terms of order j for $j \in \{0, 2, 3, 4\}$ in (6.24) to obtain a system of equations whose unknown are the coefficients of A_Γ and B_Γ (notice there are no terms of order $j = 1$). First, we have $R_0(y, \vartheta + \nabla_y B_0) = 1$, (homogeneous terms of order 0). The next lemma easily follows from solving transport equations:

Lemma 6.4. *For $\eta \in \mathbb{R}^{d-1} \setminus 0$, there exists an unique function $\phi(y, \eta)$, homogeneous of degree 1 in η, solving the eikonal equation $R_0(y, \nabla_y \phi) = |\eta|^2$, with $\phi_{|y \cdot \eta = 0} = 0$.*

We then obtain B_0: from $R_0(y, \eta) = |\eta|^2 + O(y)$, we have $\phi(y, \eta) = y \cdot \eta(1 + O(y))$, and

$$(6.25) \qquad\qquad\qquad B_0(y, \vartheta) = \phi(y, \vartheta) - y \cdot \vartheta .$$

As a consequence we have $B_0(0, \vartheta) = 0$ and $\nabla_y B_0(0, \vartheta) = 0$. Back to (6.24), consider homogeneous terms of order 2 such that $(\xi^2)_2 + (R)_2 = 0$; using (6.16) and (6.21), this translates into

$$(6.26) \qquad (1+\ell)^2 \Xi^2 + (\varrho - 1)(\vartheta + \nabla_y B_2) \cdot \nabla_\eta R_0(y, \eta_0) - \frac{(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1+\ell)} R_1(y, \eta_0) = 0 .$$

We first match coefficients of Ξ^2 in (6.26): $(1+\ell)^2 = \frac{R_1(y,\eta_0)}{q(\vartheta)(1+\ell)}$ which yields $1+\ell = \left(\frac{R_1(y, \vartheta + \nabla_y B_0)}{q(\vartheta)} \right)^{1/3}$. As $R_1(0, \vartheta) = q(\vartheta)$ and $\nabla_y B_0(0, \vartheta) = 0$, we obtain $\ell(0, \vartheta) = 0$. We now match coefficients of $\varrho - 1$,

$$(6.27) \qquad\qquad (\vartheta + \nabla_y B_2) \cdot \nabla_\eta R_0(y, \eta_0) = \frac{2 R_1(y, \eta_0)}{q(\vartheta)(1+\ell)} ,$$

which is a linear transport equation for $B_2(y, \vartheta)$ and we can take $B_{2|y \cdot \vartheta = 0} = 0$: at $y = 0$, the transport field is $2\vartheta \cdot \nabla_y$, as $\nabla_\eta R_0(y, \eta_0) = 2\eta_0 + O(y)$, $\eta_0 = \vartheta + \nabla_y B_0(y, \vartheta)$ and $\nabla_y B_0(0, \vartheta) = 0$. The first three equations involving B_0, B_2 and ℓ can be solved explicitly using only homogeneous contributions up to order 2. We consider now homogeneous terms of order 3 in (6.24), i.e. $(\xi^2)_3 +$

$(R)_3 = 0$. This yields, using (6.17) together with (6.22), (6.19) and (6.13)

$$2(1 + \ell)\Xi\left(- 2\alpha\frac{(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)} + \beta(\varrho - 1) + \mu\Xi^2\right)$$
$$- \Xi\frac{(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)}\nabla_y\ell \cdot \nabla_\eta R_0(y, \eta_0) + \frac{2\mu\Xi(\Xi^2 + 2(\varrho - 1))}{q(\vartheta)(1 + \ell)^2}R_1(y, \eta_0) = 0\,,$$

in which there are only Ξ^3 and $\Xi(\varrho - 1)$ terms. Exactly like we did for (6.26), we match coefficients for these terms separately. Unknown functions are α, β and μ (we already chose ℓ and B_0, B_2.) Using that $q(\vartheta)(1 + \ell)^3 = R_1(y, \eta_0)$, we get

(6.28)
$$\begin{cases} 4\mu(1 + \ell) = \frac{4\alpha}{q(\vartheta)} + \frac{\nabla_y\ell}{q(\vartheta)(1+\ell)} \cdot \nabla_\eta R_0(y, \eta_0), \text{ terms in } \Xi^3; \\ (2\mu + \beta)(1 + \ell) = \frac{4\alpha}{q(\vartheta)} + \frac{\nabla_y\ell}{q(\vartheta)(1+\ell)} \cdot \nabla_\eta R_0(y, \eta_0)\,, \text{ terms in } \Xi(\varrho - 1). \end{cases}$$

The last system of two equations and three unknown functions implies $\beta = 2\mu$, and provides a relation between α and μ (given by the first equation in (6.28)). Moreover, at $y = 0$, we have $q(\vartheta) = R_1(0, \vartheta)$ and $3\nabla_y\ell(0, \vartheta) = \frac{\nabla_y R_1(0, \vartheta)}{q(\vartheta)}$. We summarize what we obtained so far:

Proposition 6.5. *The phase function $\Gamma(x, \cdot) = B_\Gamma + xA_\Gamma$ is such that, near the glancing set $\mathcal{GL}$,*

(6.29)
$$\begin{cases} A_\Gamma(x, y, \Xi, \Theta) = \Xi\ell(y, \vartheta) + \alpha(y, \vartheta)x + \mu(y, \vartheta)(\Xi^2 + |\Theta|^2 - 1) + \mathcal{H}_{j\geq 3}\,, \\ B_\Gamma(y, \Theta) = B_0(y, \vartheta) + (\varrho - 1)B_2(y, \vartheta) + \mathcal{H}_{j\geq 3}\,, \end{cases}$$

where $\vartheta = \Theta/|\Theta|$, $\varrho = |\Theta|$ and the functions B_0 B_2 were defined in (6.25), (6.27). Morover,

$$\beta(y, \vartheta) = 2\mu\,, \quad and \quad \mu(y, \vartheta) = \frac{\alpha}{q(\vartheta)(1 + \ell)} - \frac{\nabla_\eta R_0(y, \vartheta + \nabla_y B_0)}{4q(\vartheta)} \cdot \nabla_y\left(\frac{1}{1 + \ell}\right).$$

We also have $\ell(0, \vartheta) = 0$, $B_0(0, \vartheta) = 0$, $\nabla_y B_0(0, \vartheta) = 0$, $B_2(0, \vartheta) = 0$ and $\nabla_y B_2(0, \vartheta) = 0$, B_0 (resp. B_2) is homogeneous of order 1 (resp. of order 0) in the second variable.

Remark 6.6. The restriction of $\boldsymbol{\chi}_M$ to $\mathcal{GL}_M$ is given by $x = \xi = 0$ and $Y_M = y + \nabla_\Theta B_\Gamma(y, \Theta)|_{\varrho=1}$, $\eta = \Theta + \nabla_y B_0(y, \Theta)(y, \Theta)|_{\varrho=1}$. It preserves the canonical foliation: $\boldsymbol{\chi}_M\left(\{Y_M = Y_0 + 2s\vartheta_0, \Theta = \vartheta_0, \vartheta_0^2 = 1\}\right)$ is an integral curve of H_{R_0} on $R_0 = 1$.

To complete the proof of Proposition 2.4, we are to identify homogeneous terms of order at least 4 for A_Γ and B_Γ in (6.7). One gets a cascade of linear equations (similar to those obtained by identifying homogeneous terms of order $0, 1, 2, 3$ in order to chose α, β, μ) which may be solved by induction. We only do it for homogeneous terms of order 4 in (6.24) as an example: let $(\xi^2)_4 + (R)_4 = 0$ and match coefficients for Ξ^4 and $\Xi^2(\varrho - 1)$ using (6.18), (6.23), (6.19), (6.12), (6.13),

$$\frac{4\alpha_1}{q(\vartheta)} - 2(1 + \ell)\mu_1 = 9\mu^2 + \frac{R_2(y, \eta_0)}{2q^2(\vartheta)(1 + \ell)^2} + \frac{\nabla_\eta R_0(y, \eta_0)}{q(\vartheta)(1 + \ell)} \cdot \nabla_y\left(\frac{\nabla_\eta R_0(y, \eta_0)}{4q(\vartheta)} \cdot \nabla_y\left(\frac{1}{(1 + \ell)}\right)\right)$$

$$\frac{4\alpha_1}{q(\vartheta)} - (1 + \ell)\beta_1 = 10\mu^2 + \frac{R_2(y, \eta_0)}{q^2(\vartheta)(1 + \ell)^2} - \frac{\nabla_\eta R_1(y, \eta_0)}{2q(\vartheta)(1 + \ell)} \cdot (\vartheta + \nabla_y B_2)$$
$$+ \frac{2\nabla_\eta R_0(y, \eta_0)}{q(\vartheta)(1 + \ell)} \cdot \nabla_y\left(\frac{\nabla_\eta R_0(y, \eta_0)}{4q(\vartheta)} \cdot \nabla_y\left(\frac{1}{(1 + \ell)}\right)\right),$$

where both RHSs only contain μ^2 and known function such as ℓ, B_0, B_2, R_0, R_1, R_2.

Remark 6.7. If we require $\beta_1 = 2\mu_1$ (which implies $A_3 = \Xi(x\alpha_1 + (\Xi^2 + 2(\varrho - 1))\mu_1))$, then by difference between the last two equations μ^2 is uniquely determined (and therefore α using Proposition 6.5). Similarly, we could ask for

$$A_{2j} = x\alpha_j + (\Xi^2 + 2(\varrho - 1))\mu_j, \qquad A_{2j+1} = x\Xi\alpha_{j-1} + (\Xi^2 + 2(\varrho - 1))\Xi\mu_{j-1}, \quad j \geq 2,$$

but this would determine B_{2j} in an unique way. Indeed, for $j = 1$ requiring $\beta_1 = 2\mu_1$ provides a unique μ and then a unique α. Moreover, identifying the coefficients of $(\varrho - 1)^2$ in homogeneous terms of degree 4 in (6.24) does not involve α_1, β_1, μ_1, but only $\nabla_y B_4$ (with $\nabla_\eta R_0(y, \eta_0 \neq 0)$ and α, μ, ℓ (this is the first occurrence of B_4.) Indeed, η_4 does not contain α_1, β_1, μ_1, which appear only in $(\xi^2)_4$ (with Ξ^4 or $\Xi^2(\varrho - 1)$). Therefore for given α, μ, this equation (obtained by identifying coefficients of $(\varrho - 1)^2$) determines $\nabla_y B_4$ (and therefore $\nabla_y B_4(0, \vartheta) \neq 0$ unlike for B_0, B_2.)

Remark 6.8. That the formal expansion is not uniquely defined reflects that the group of canonical transformations which preserves the model $\{X_M = 0, \Xi^2 + |\Theta|^2 + X_M q(\Theta) = 1\}$ is not trivial.

6.3. **Equivalence of phase functions for** $G(x, y, \eta, \omega)$. Both phases $\psi + s^3/3 - s\zeta$ (from (2.6) in Theorem 2.1) and $y \cdot \eta + \sigma^3/3 + \sigma(xq^{1/3}(\eta) - \omega) + \tau_q \Gamma(x, y, \sigma q^{1/3}(\eta)/\tau_q, \eta/\tau_q)$ (from (2.12)) define the same Lagrangian. We now explain how they are related. From a classical result (see [3]) on the normal form of integrals whose phases have degenerate critical points of order 2, we have:

Lemma 6.9. *Let* $\phi(x, y, \theta, \alpha, \sigma) = \sigma^3/3 + \sigma(xq^{1/3}(\theta) - \alpha) + x\tau_q A_\Gamma(x, y, \sigma q^{1/3}(\theta)/\tau_q, \theta/\tau_q)$. *There exists a unique map* $\sigma \to s$ *and* $\Upsilon(x, y, \theta, \alpha) \in C^\infty$ *such that* $\phi(x, y, \theta, \alpha, \sigma) = s^3/3 - s\zeta(x, y, \theta, \alpha) + \Upsilon(x, y, \theta, \alpha)$ *and* $\frac{ds}{d\sigma} \notin \{0, \infty\}$. *Let* $w := (x, y, \theta, \alpha)$ *and denote* $\sigma_0(w)$ *the unique solution to* $\partial^2_{\sigma,\sigma}\phi(w, \sigma) = 0$; *then the two saddle points of* ϕ, *that we denote* $\sigma_\pm(w)$, *correspond to the critical points* $s_\pm(w) := \pm\sqrt{\zeta(w)}$ *and such that* $\sigma_\pm(w) = \sigma_0(w) \pm \sqrt{\zeta(w)}k(\pm\sqrt{\zeta(w)}, w)$, *with* $k(u, w) = 1 + \sum_{j \geq 1} k_j(w)u^j$, *where* k_j *are smooth functions of* w. *Moreover,*

$$(6.30) \quad \frac{3}{4}\zeta^{3/2}(w) = \phi(w, \sigma_-(w)) - \phi(w, \sigma_+(w)), \quad \Upsilon(w) := \frac{1}{2}\Big(\phi(w, \sigma_+(w)) + \phi(w, \sigma_-(w))\Big).$$

Corollary 6.10. *We may write* $\psi(x, y, \theta, \alpha) = y \cdot \theta + \tau_q(\alpha, \theta)B_\Gamma(y, \theta/\tau_q) + \Upsilon(x, y, \theta, \alpha)$, *with*

$$(6.31) \quad \Upsilon(x, y, \theta, \alpha) = -x\big(x\mu(y, \theta/|\theta|)q(\theta)/\tau_q(\alpha, \theta)(1 + \ell(y, \theta/|\theta|) + \mathcal{H}_{j \geq 4}\big) - k_1\zeta(\zeta + x\mathcal{H}_{j \geq 2} + \zeta\mathcal{H}_{j \geq 2}).$$

Proof. Compute σ_0: as $\partial^2_{\sigma,\sigma}\phi(w, \sigma) = 2\sigma + 2x\big(\mu(y, \theta/|\theta|)q^{2/3}(\theta)/\tau_q(\alpha, \theta) + \mathcal{H}_{j \geq 1}\big) = 0$, where $\mathcal{H}_{j=1}$ contains only multiples of σ, we get $\sigma_0(x, y, \theta, \alpha) = -x\big(\mu(y, \theta/|\theta|)q^{2/3}(\theta)/\tau_q(\alpha, \theta) + \mathcal{H}_{j \geq 2}\big)$, where all $\mathcal{H}_{j \geq 2}$ in the RHS come with weights x and α. We develop $\phi(w, \sigma)$ near $\sigma = 0$ and replace σ by $\sigma_\pm(w)$: as $\frac{1}{2}(\sigma_+ + \sigma_-) = \sigma_0 + k_1(w)\zeta(w)(1 + O(\zeta(w)))$, using (6.30) yields (6.31). Moreover, ψ contains B_Γ as it does not depend on σ. $\square$

7. INDEX OF NOTATIONS

Below is a commented list of the main notations, with reference to their very first occurence.

7.1. **General notations (used consistently throughout the paper).**
- $(\Omega, g) = d$ dimensional manifold, $d \geq 2$, Δ_g its Laplace Beltrami operator, section 1.
- $\gamma(d)$ encodes the loss in Strichartz estimates w.r.t. the case without boundary, Theorem 1.5.
- (x, y), boundary normal coordinates; t the time variable; locally, $\Omega = \{(x, y) : x > 0, y \in \mathbb{R}^{d-1}\}$, section 2.

- (ξ, η, τ), dual variables: $(x, y, t, \xi, \eta, \tau) \in T^*(\Omega \times \mathbb{R}_t)$. For (x, y) near $(0, y)$, the metric is $\xi^2 + R(x, y, \eta)$. In a neighborhood of $(0, 0) \in \partial\Omega$,

$$R_0(y, \partial_y) := R(0, y, \eta), \quad R_1(y, \eta) := \partial_x R(0, y, \partial),$$

  section 2 and (2.1).
- $\Delta_M = \partial_x^2 + \sum_j \partial_{y_j}^2 + x \sum_{j,k=1}^{d-1} R_1^{j,k}(0)\partial_{y_j}\partial_{y_k}$: model Laplace operator, (2.2); Multipliers

$$q(\eta) = \sum_{j,k=1}^{d-1} R_1^{j,k}(0)\eta_j\eta_k, \quad \tau_q(\omega, \eta) := \sqrt{|\eta|^2 + \omega q(\eta)^{2/3}}.$$

- $\{e_k(x, \eta)\}_{k \geq 0}$: in the spectral decomposition of $-\Delta_M$ (section 2.3.1), an explicit orthonormal base of eigenfunctions associated to eigenvalues $\lambda_k(\eta)$, where

$$\lambda_k(\eta) = |\eta|^2 + \omega_k q(\eta)^{2/3} = \tau_q^2(\omega_k, \eta).$$

- $\{-\omega_k\}_{k \geq 0}$: zeros of the Airy function in decreasing order. Everywhere in the paper $\omega > 1$ and serves as a substitute to the ξ variable: if Q_y is the differential operator with symbol q, $\alpha = h^{2/3}\omega$ quantizes the operator $x - Q_y^{-1}\partial_x^2$.
- s, σ: integration variables in Airy type oscillatory integrals, (2.12), (2.4).
- (a, b) coordinates of the source point, mostly set with $b = 0$, Theorem 1.1.
- $h \in (0, 1)$ (Theorem 1.1), $\gamma \in (0, 1)$ with $1/\gamma \in 2^{\mathbb{N}}$ (Section 3.1): small parameters.
- $\lambda = a^{3/2}/h$, $\lambda_\gamma = \gamma^{3/2}/h$: large parameters, Section 3.1.
- (X, Y, T), rescaled coordinates (using some combination of a, h or λ, λ_γ as rescaling parameters), Section 3.1.
- Σ, S, A: rescaled variables in Airy-type oscillatory integrals, Section 3.1.
- ω: (2.2), parameter and integration variable, successively rescaled to α ((2.16)) and then A (Section 3.1). Stationary phases in oscillatory integrals are performed with respect to α, σ, s or their rescaled versions A, Σ, S, less frequently η, with a large parameter being $1/h$, λ or λ_γ.
- θ: (Section 2) rescaled η, near $\mathbb{S}^{d-1}$, and $\rho = |\theta|$, $\vartheta = \theta/\rho$.

7.2. Localisations in phase space.
- We localize $\tau_q(\omega, \eta) \sim 1/h$ and $|\eta| \sim 1/h$. For small x, this corresponds to large frequencies $-\Delta \sim -\Delta_M \sim 1/h^2$ and "tangent" directions: the number of reflections on the boundary may be quite large.
- A further localization is to values $\omega/|\eta|^{2/3} \sim \gamma$. Informally, it relates to the angle of incidence at the boundary for a ray starting tangentially from $(\gamma, 0)$.
- Cut-offs : $\varkappa \geq 0$ is a cut-off function in $C_0^\infty(\mathbb{R}^m)$ with $m = 1$ or with $m = d - 1$, localizing around a small neighbourhood of 1 (for $d = 1$), or near $\mathbb{S}^{m-1}$ for $m = d - 1$; $\varkappa_1$ is a 1-d $\varkappa$. We also have $\chi^\flat \in C^\infty(\mathbb{R})$ such that $\chi^\flat = 1$ on $(-\infty, 1]$ and $\chi^\flat = 0$ on $[2, \infty)$ and $\chi^\sharp = 1 - \chi^\flat$. Also, $\chi_0 \in C_0^\infty(\mathbb{R})$ is supported is a small, fixed neighborhood of 0.

7.3. Operators, kernels and quasimodes.
- $G(x, y, \eta, \omega)$: a quasimode, (2.7) or (2.12); satisfies (2.7) $-\Delta G = \tau_q^2 G + O_{C^\infty}(\tau_q^{-\infty})$.
- $K_\omega(f)(t, x, y)$: operator related to wave flow (2.24), acting on smooth f.
- $J(f)(x, y)$: Fourier integral operator, (2.19).
- $\mathcal{P}_{h,a}(t, x, y)$, (2.37) and (2.38), our parametrix for the wave equation
- V_N: a wave in the expansion over N of $\mathcal{P}_{h,a}$, (2.42).
- $V_{N,\gamma}$: further localized with $\chi_1(\omega/(\gamma|\eta|^{2/3}))$, (3.6).

- $\mathcal{P}_{h,a,\gamma}$: the corresponding sum over N, (3.7).
- $\mathcal{E}_M(\cdot, \omega_k)$: galery modes for the model Laplacian Δ_M, (2.63).
- $e_k(x, \eta)$: eigenfunctions of $\mathcal{F}_y(\Delta_M)$, (2.60).
- $e(x, y, \eta, \omega)$: quasimodes for Δ_g, (2.68).
- $g_{h,a}$, $g_{h,a,j}$, $j \in \{1, 2\}$: functions to serve as arguments to J and K_ω to construct a suitable smoothed out Dirac data, (2.31), Propositions 2.18 and 2.19.
- $F_{\omega_k}(g)(x, y)$: operator acting on functions $g \in L^2(\mathbb{R}^{d-1})$, average (with density $\hat{g}(\eta)$) of quasimodes $e(x, y, \eta, \omega_k)$, (2.69).
- $\mathfrak{L}(f)(y)$: operator actiong on $f \in L^2(\mathbb{R}^{d-1})$ which allows to "get rid" of the term B_0 in the phase of $e(x, y, \eta, \omega)$, (2.70).
- $\tilde{F}_{\omega_k}(f)(x, y) = F_{\omega_k} \circ \mathfrak{L}(f)(x, y)$: its main property is that it can be inverted.

7.4. **Phase functions and canonical transformation.**

- $\zeta(x, y, \eta, \omega)$, $\psi(x, y, \eta, \omega)$: the phase functions of $G(x, y, \eta, \omega)$ from Theorem 2.1.
- Σ_0: (2.8), a neighborhood of a glancing point in the model case.
- $\boldsymbol{\chi}_M$, (2.9), the canonical transformation defined in a conic neighborhood of Σ_0 mapping the model case (variables (X_M, Y_M, Ξ, Θ)) to the general case (variables (x, y, ξ, η)).
- $\varphi_\Gamma(x, y, \Xi, \Theta) = x\Xi + y\Theta + \Gamma(x, y, \Xi, \Theta)$: Proposition 2.4, the generating function for $\boldsymbol{\chi}_M$, with $\Gamma(x, y, \Xi, \Theta) = B_\Gamma(y, \Theta) + xA_\Gamma(x, y, \Xi, \Theta)$ from (2.10).
- A_Γ, B_Γ: phase functions that are formal series (6.8) from Section 6.2, defined near $\mathcal{GL} = \{x = 0, \Xi = 0, \varrho - 1 = 0\}$ and for (y, ϑ) near $\{0\} \times \mathbb{S}^{d-1}$, where $\Theta = \varrho\vartheta$. Their explicit form is given in (6.29).
- $\mathcal{H}_{\geq j} = \{F \text{ such that } F = \sum_{k \geq j} F_k, \text{ with } F_k \text{ homogeneous of degree } k \geq 1\}$, where a monomial of the form $x^a(\varrho - 1)^b\Xi^c$ is homogeneous of degree k if $c + 2(a + b) = k$.
- $\ell(y, \vartheta)$ (which defines $A_1 = \Xi\ell$), $\alpha(y, \vartheta)$, $\beta(y, \vartheta)$, $\mu(y, \vartheta)$ (which define $A_2 = \alpha(y, \vartheta)x + \beta(y, \vartheta)(\varrho - 1) + \mu(y, \vartheta)\Xi^2$) so that $A_\Gamma = A_1 + A_2 + \mathcal{H}_{\geq 3}$: other functions related to Γ from Section 6.2.
- $B_\Gamma(y, \Theta) = B_0(y, \vartheta) + (\varrho - 1)B_2(y, \vartheta) + \mathcal{H}_{j \geq 3}$, whose properties are stated in Proposition 6.5.
- $\mathcal{L} = \{(x, y, \vartheta, \varrho, \Xi), p_{M,2}(X_M, Y_M, \Xi, \Theta) = 0\}$ defined in Lemma 6.3.
- WF_h, the semiclassical wavefront set (see [25]).
- $\Phi_{N,a,\gamma}$: the phase function of $V_{N,\gamma}$ defined in (3.9).
- $\Phi_{N,a,\gamma}^M$: the phase function in the model case (Ω, Δ_M). (In general, a notation with an additional M indicates that we consider the model situation.)
- $\tilde{\Phi}_{N,a,\gamma}$, a rescaled $\Phi_{N,a,\gamma}$ (see (3.28)) with $\sigma = \sqrt{\gamma}|\theta|^{1/3}\Sigma$, $\alpha = \gamma|\theta|^{2/3}A$, $s = \sqrt{\gamma}|\theta|^{1/3}S$.
- $\Psi_{N,a,\gamma}$: relabeled $\tilde{\Phi}_{N,a,\gamma}$ after rescaling $x = \gamma X$, $t = \sqrt{\gamma}T$, $y = \sqrt{\gamma}Y$.
- $\mathcal{N}(t, x, y)$: the set of N with significant contributions of the phase $\Phi_{N,a,\gamma}(t, x, y, \cdot)$, defined in (3.8).
- $\mathcal{C}_\gamma$: a cylinder defined by (3.15).
- $\mathcal{N}_d^1$: enlargement of $\mathcal{N}$ defined by $\mathcal{N}_d^1(t, x, y) = \cup_{\mathcal{C}_\gamma(t,x,y)}\mathcal{N}(t', x', y')$ in (3.14).
- $\tilde{\Gamma}_\gamma$, $\check{\Gamma}_\gamma$ (see (3.35)), $\mathcal{E}_\pm$, $\tilde{\mathcal{E}}$ (see Lemma 3.8), $\tilde{\mathcal{E}}$ (see Lemma 3.10): remainder terms in phase functions (they do not influence the behaviour of the corresponding phase functions.)

References

[1] K. G. Andersson and R. B. Melrose. The propagation of singularities along gliding rays. *Invent. Math.*, 41(3):197–232, 1977.

[2] Matthew D. Blair, Hart F. Smith, and Christopher D. Sogge. Strichartz estimates for the wave equation on manifolds with boundary. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(5):1817–1829, 2009.

[3] C. Chester, B. Friedman, and F. Ursell. An extension of the method of steepest descents. *Proc. Cambridge Philos. Soc.*, 53:599–611, 1957.

[4] Gregory Eskin. Parametrix and propagation of singularities for the interior mixed hyperbolic problem. *J. Analyse Math.*, 32:17–62, 1977.

[5] Lars Hörmander. *The analysis of linear partial differential operators. I.* Classics in Mathematics. Springer-Verlag, Berlin, 2003. Distribution theory and Fourier analysis, Reprint of the second (1990) edition [Springer, Berlin; MR1065993 (91m:35001a)].

[6] Oana Ivanovici. Counterexamples to Strichartz estimates for the wave equation in domains. *Math. Ann.*, 347(3):627–673, 2010.

[7] Oana Ivanovici. Counterexamples to the Strichartz inequalities for the wave equation in general domains with boundary. *J. Eur. Math. Soc. (JEMS)*, 14(5):1357–1388, 2012.

[8] Oana Ivanovici and Gilles Lebeau. Dispersion for the wave and the Schrödinger equations outside strictly convex obstacles and counterexamples. *C. R. Math. Acad. Sci. Paris*, 355(7):774–779, 2017.

[9] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. Dispersion for the wave equation inside strictly convex domains I: the Friedlander model case. *Ann. of Math. (2)*, 180(1):323–380, 2014.

[10] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. Estimations de strichartz pour les ondes dans le modèle de friedlander en dimension 3. In *Séminaire Laurent Schwartz – EDP et applications, 2013–2014*, Sémin. Équ. Dériv. Partielles, pages Exp. No. III, 12. École Polytech., Palaiseau, 2014.

[11] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. New counterexamples to Strichartz estimates for the wave equation on a 2D model convex domain. *J. Éc. polytech. Math.*, 8:1133–1157, 2021.

[12] Oana Ivanovici, Gilles Lebeau, and Fabrice Planchon. Strichartz estimates for the wave equation on a 2D model convex domain. *J. Differential Equations*, 300:830–880, 2021.

[13] V. Ja. Ivriĭ. Propagation of the singularities of a solution of the wave equation near the boundary. *Dokl. Akad. Nauk SSSR*, 239(4):772–774, 1978.

[14] Markus Keel and Terence Tao. Endpoint Strichartz estimates. *Amer. J. Math.*, 120(5):955–980, 1998.

[15] R. B. Melrose. Equivalence of glancing hypersurfaces. *Invent. Math.*, 37(3):165–191, 1976.

[16] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. I. *Comm. Pure Appl. Math.*, 31(5):593–617, 1978.

[17] R. B. Melrose and J. Sjöstrand. Singularities of boundary value problems. II. *Comm. Pure Appl. Math.*, 35(2):129–168, 1982.

[18] Richard B. Melrose and Michael E. Taylor. *Boundary Problems for Wave Equations With Grazing and Gliding Rays*. Available at http://www.unc.edu/math/Faculty/met/wavep.html.

[19] Richard B. Melrose and Michael E. Taylor. Near peak scattering and the corrected Kirchhoff approximation for a convex obstacle. *Adv. in Math.*, 55(3):242–315, 1985.

[20] Richard B. Melrose and Michael E. Taylor. The radiation pattern of a diffracted wave near the shadow boundary. *Comm. Partial Differential Equations*, 11(6):599–672, 1986.

[21] Hart F. Smith and Christopher D. Sogge. On the critical semilinear wave equation outside convex obstacles. *J. Amer. Math. Soc.*, 8(4):879–916, 1995.

[22] Hart F. Smith and Christopher D. Sogge. On the L^p norm of spectral clusters for compact manifolds with boundary. *Acta Math.*, 198(1):107–153, 2007.

[23] Daniel Tataru. Strichartz estimates for second order hyperbolic operators with nonsmooth coefficients. III. *J. Amer. Math. Soc.*, 15(2):419–442 (electronic), 2002.

[24] Maciej Zworski. High frequency scattering by a convex obstacle. *Duke Math. J.*, 61(2):545–634, 1990.

[25] Maciej Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.