

ON THE K-THEORY OF \mathbb{Z} -CATEGORIES.

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ABSTRACT. We establish connections between the concepts of Noetherian, regular coherent, and regular n -coherent categories for \mathbb{Z} -linear categories with finitely many objects and the corresponding notions for unital rings. These connections enable us to obtain a negative K -theory vanishing result, a fundamental theorem, and a homotopy invariance result for the K -theory of \mathbb{Z} -linear categories.

1. INTRODUCTION

Let R be an associative ring with a unit. The fundamental theorem in K -theory also known as the Bass-Heller-Swan theorem, expresses the K -groups of $R[t, t^{-1}]$ in terms of the K -groups and Nil-groups of R

$$K_i(R[t, t^{-1}]) \simeq K_{i-1}(R) \oplus K_i(R) \oplus \text{Nil}_{i-1}(R) \oplus \text{Nil}_{i-1}(R).$$

The groups $\text{Nil}_i(R)$ for $i \in \mathbb{Z}$, and the K -groups $K_i(R)$ for $i < 0$, are known to vanish when R is right regular (i.e. right Noetherian and right regular coherent). Swan [18] proved that $\text{Nil}_i(R)$ also vanishes when R is right regular coherent and $i \geq 0$, using Quillen's resolution and devissage theorems as the main tools. In [10], we extended the study to n -coherent rings, where $n \geq 0$ (here 1-coherent ring is the same as coherent ring, and 0-coherent ring is the same as Noetherian ring). We derived a new expression for $\text{Nil}_i(R)$ for a n -regular and n -coherent ring R , but its vanishing status remains unknown. Our current focus is on exploring various methods for computing these groups.

The algebraic K -theory of a ring with a unit can be generalized to categories that have additional structure, and even to non-unital rings. In the context of K -theory, it is often more convenient to use additive categories instead of rings. With this motivation in mind, Bartels and Lück extended the notions of regularity and regular coherence to additive categories. In [3] they proved the following result:

Theorem 1.1. [3, Corollary 12.2] *Let \mathcal{C} be an additive category which is regular. Then $K_i(\mathcal{C}) = 0$ for all $i \leq -1$.*

The focus of this paper is to extend the notion of regular n -coherence and some vanishing results in K -theory from rings to \mathbb{Z} -linear categories. Let \mathcal{C} be a \mathbb{Z} -linear category. We define a right \mathcal{C} -module as a contravariant \mathbb{Z} -linear functor $F : \mathcal{C}^{op} \rightarrow \text{Ab}$. We denote the category of right \mathcal{C} -modules as $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$. Using the Yoneda lemma, we embed \mathcal{C} into $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ with the purpose of using homological constructions in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ which a priori make no sense in \mathcal{C} . The finiteness conditions for $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ are defined in [5]. As the category of R -modules, $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ is a Grothendieck category with a generating set of finitely generated projective objects. A right \mathcal{C} -module F is said to be of type \mathcal{FP}_n if and

only if there exists an exact sequence

$$P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow F \rightarrow 0$$

where P_i is a finitely generated and projective right \mathcal{C} -module for every $0 \leq i \leq n$. A right \mathcal{C} -module F is of type \mathcal{FP}_∞ if it is of type \mathcal{FP}_n for all $n \geq 0$.

We say that \mathcal{C} is right n -coherent if the category $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ is n -coherent in the sense of [5, Definition 4.6]. In other words, \mathcal{C} is right n -coherent if and only if the \mathcal{C} -modules of type \mathcal{FP}_n in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ coincide with those of type \mathcal{FP}_∞ . We say that \mathcal{C} is right *regular n -coherent* if \mathcal{C} is right n -coherent and every \mathcal{C} -module F of type \mathcal{FP}_n has finite projective dimension. In Proposition 2.7, we prove that this homological property of \mathcal{C} also holds for \mathcal{C}_\oplus .

Let $n \geq 1$ and $f : x \rightarrow y$ be a morphism in \mathcal{C} . Following [6], we say that f has a *pseudo n -kernel* if there exists a chain of morphisms:

$$x_n \xrightarrow{f_n} x_{n-1} \xrightarrow{f_{n-1}} x_{n-2} \rightarrow \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} x \xrightarrow{f} y$$

such that the following sequence of abelian groups is exact:

$$\text{hom}_{\mathcal{C}}(-, x_n) \xrightarrow{f_n^*} \cdots \rightarrow \text{hom}_{\mathcal{C}}(-, x_1) \xrightarrow{f_1^*} \text{hom}_{\mathcal{C}}(-, x) \xrightarrow{f^*} \text{hom}_{\mathcal{C}}(-, y).$$

In Proposition 2.9, we establish necessary and sufficient intrinsic conditions on \mathcal{C} for it to be right regular n -coherent. Specifically, we demonstrate that an additive category \mathcal{C} is right regular n -coherent if and only if the following conditions hold:

- i) Every morphism in \mathcal{C} with a pseudo $(n-1)$ -kernel has a pseudo n -kernel.
- ii) For every morphism $f : x \rightarrow y$ in \mathcal{C} with a pseudo ∞ -kernel, there exist $k \in \mathbb{N}$ and a morphism $\alpha : x_{k-1} \rightarrow x_{k-1}$ such that the following diagram commutes:

$$\begin{array}{ccccccc} x_k & \xrightarrow{f_k} & x_{k-1} & \xrightarrow{f_{k-1}} & x_{k-2} & \xrightarrow{f_{k-2}} & \cdots & \cdots & \xrightarrow{f_2} & x_1 & \xrightarrow{f_1} & x & \xrightarrow{f} & y \\ & \searrow 0 & \downarrow \alpha & \nearrow f_{k-1} & & & & & & & & & & \\ & & x_{k-1} & & & & & & & & & & & \end{array}$$

The algebraic K-theory of a \mathbb{Z} -linear category \mathcal{C} is defined using the non-connective spectrum $\mathbf{K}^\infty(\mathcal{C}_\oplus)$, which was introduced in [16]. Furthermore, \mathcal{C} is associated with a ring defined as

$$\mathcal{A}(\mathcal{C}) = \bigoplus_{a, b \in \text{ob}\mathcal{C}} \text{hom}_{\mathcal{C}}(a, b),$$

where $\text{ob}\mathcal{C}$ denotes the objects of \mathcal{C} . The multiplication and addition in $\mathcal{A}(\mathcal{C})$ are defined naturally, resulting in a ring with local units. It is important to note that $\mathcal{A}(\mathcal{C})$ is unital only when $\text{ob}\mathcal{C}$ is finite. Furthermore, it is worth mentioning that there exists a weak equivalence between the spectrum of the algebraic K-theory of \mathcal{C} and the spectrum of the algebraic K-theory of $\mathcal{A}(\mathcal{C})$, as shown in [7, Sec. 4.2]. Therefore, the K-theory groups of \mathcal{C} and $\mathcal{A}(\mathcal{C})$ coincide.

In Section 3, we compare the notions of Noetherianity, regular coherence, and regular n -coherence of $\mathcal{A}(\mathcal{C})$ with the corresponding notions for \mathcal{C} when \mathcal{C} is a \mathbb{Z} -linear category with finitely many objects. This comparison allows us to establish a relationship between the properties of \mathcal{C} and the properties of $\mathcal{A}(\mathcal{C})$. It is important to note that although some of the results we have used do not require the condition \mathcal{C} having finitely many objects, this condition is necessary to guarantee that the ring

$\mathcal{A}(\mathcal{C})$ has a unit. A \mathbb{Z} -linear category \mathcal{C} is right regular n -coherent if and only if the additive category \mathcal{C}_\oplus associated with \mathcal{C} has this property, as stated in Proposition 2.7. The reason for working with \mathbb{Z} -linear categories instead of additive categories is that the ring $\mathcal{A}(\mathcal{C}_\oplus)$ does not have a unit due to the fact that \mathcal{C}_\oplus has infinitely many objects. By considering \mathbb{Z} -linear categories, we are able to address this issue and ensure the existence of a unit for the corresponding ring.

We see in Proposition 3.4 that the category $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ is equivalent to $\text{Mod-}\mathcal{A}(\mathcal{C})$, where $\text{Mod-}\mathcal{A}(\mathcal{C})$ denotes the category of unital right modules over $\mathcal{A}(\mathcal{C})$. Furthermore, in Proposition 3.8, we prove that \mathcal{C} is a \mathbb{Z} -linear category with finitely many objects then a \mathcal{C} is right Noetherian (n -coherent or regular n -coherent) if and only if $\mathcal{A}(\mathcal{C})$ is (strong n -coherent or n -regular and strong n -coherent). We use Proposition 3.8 and results for rings in order to obtain information about the K-theory of a \mathbb{Z} -linear category. These results are not completely original but the way to obtain them is.

In Section 4, we prove that if $\mathcal{D} = \mathcal{C}$, $\mathcal{D} = \mathcal{C}_\oplus$ or $\mathcal{D} = \text{colim}_{f \in F} \mathcal{C}_f$ with \mathcal{C} and \mathcal{C}_f regular \mathbb{Z} -linear categories with finitely many objects, then $K_i(\mathcal{D}) = 0 \ \forall i < 0$. We also prove that if $\mathcal{D} = \mathcal{C}$, $\mathcal{D} = \mathcal{C}_\oplus$ or $\mathcal{D} = \text{colim}_{f \in F} \mathcal{C}_f$ with \mathcal{C} and \mathcal{C}_f regular coherent \mathbb{Z} -linear categories with finitely many objects, then $K_{-1}(\mathcal{D}) = 0$,

$$K_i(\mathcal{D}) \simeq K_i(\mathcal{D}[t]) \quad \text{and} \quad K_{i+1}(\mathcal{D}[t, t^{-1}]) \simeq K_{i+1}(\mathcal{D}) \oplus K_i(\mathcal{D}) \quad \forall i \geq 0.$$

In Proposition 4.10 we obtain a generalization of [10, Thm 3.2].

2. MODULES OVER \mathbb{Z} -LINEAR CATEGORIES

A \mathbb{Z} -linear category is a category \mathcal{C} such that for every two objects $a, b \in \mathcal{C}$, the set of morphisms $\text{hom}_{\mathcal{C}}(a, b)$ is an abelian group, and for any other object $c \in \mathcal{C}$, the composition

$$\text{hom}_{\mathcal{C}}(b, c) \times \text{hom}_{\mathcal{C}}(a, b) \rightarrow \text{hom}_{\mathcal{C}}(a, c)$$

is a bilinear map. Throughout this paper, we assume that \mathbb{Z} -linear categories \mathcal{C} are small, i.e. the collection of objects is a set. A \mathbb{Z} -linear category is *additive* if it has an initial object and finite products. We consider the free additive category \mathcal{C}_\oplus as follow. The objects of \mathcal{C}_\oplus are finite tuples of objects in \mathcal{C} . A morphism from $\mathbf{a} = (a_1, \dots, a_k)$ to $\mathbf{c} = (c_1, \dots, c_m)$ for $a_i, c_j \in \mathcal{C}$ is given by $m \times k$ matrix of morphisms in \mathcal{C} (the composition is given by the usual row-by-column multiplication of matrices),

- $\text{ob}\mathcal{C}_\oplus = \{(c_1, \dots, c_k) : c_i \in \mathcal{C}, k \in \mathbb{N}\}$
- $\text{hom}_{\mathcal{C}_\oplus}(\mathbf{a}, \mathbf{c}) = \prod_{i=1}^k \prod_{j=1}^m \text{hom}_{\mathcal{C}}(a_i, c_j)$.

There is an obvious embedding $\mathcal{C} \rightarrow \mathcal{C}_\oplus$ which maps objects and morphisms to their associated 1-tuple. If \mathcal{C} is a \mathbb{Z} -linear category then \mathcal{C}_\oplus is a small additive category.

The *idempotent completion* $\text{Idem}(\mathcal{C}_\oplus)$ of \mathcal{C}_\oplus is defined to be the following small additive category.

- $\text{ob}(\text{Idem}(\mathcal{C}_\oplus)) = \{(\mathbf{c}, p) : \mathbf{c} \in \text{ob}\mathcal{C}_\oplus, p : \mathbf{c} \rightarrow \mathbf{c} \text{ such that } p^2 = p\}$
- $\text{hom}_{\text{Idem}(\mathcal{C}_\oplus)}((\mathbf{c}_1, p_1), (\mathbf{c}_2, p_2)) = \{w : \mathbf{c}_1 \rightarrow \mathbf{c}_2 \text{ such that } w = p_2 w p_1\}$.

By construction $\mathcal{C} \simeq \mathcal{C}_\oplus$ if \mathcal{C} is additive and $\mathcal{C}_\oplus \simeq \text{Idem}(\mathcal{C}_\oplus)$ if idempotents split in \mathcal{C}_\oplus . Recall the additive category \mathcal{C}_\oplus is equivalent to $\text{Idem}(\mathcal{C}_\oplus)$ if and only if every idempotent has a kernel.

Example 2.1. Given a ring R , consider $\mathcal{C} = \underline{R}$ the category which has one object \star and $\text{hom}_{\mathcal{C}}(\star, \star) = R$. The multiplication on R gives the composition on \underline{R} . The

category \mathcal{C}_\oplus is the category whose objects are natural numbers $m > 0$ and the morphisms are the matrices with coefficients in R , $\text{hom}_{\mathcal{C}_\oplus}(m, n) = M_{n \times m}(R)$.

Example 2.2. Let R be an associative ring with unity. If \mathcal{C} is the category of finitely generated free R -modules, then $\text{Idem}(\mathcal{C})$ is equivalent to the category of finitely generated projective R -modules.

2.1. Pseudo n -kernels and pseudo n -cokernels. Given a \mathbb{Z} -linear category \mathcal{C} we recall that a *pseudo kernel* of a morphism $f : x \rightarrow y$ in \mathcal{C} is a morphism $g : k \rightarrow x$ with $f \circ g = 0$, such that for any morphism $h : c \rightarrow x$ with $f \circ h = 0$, there exists $t : c \rightarrow k$ with $g \circ t = h$. Equivalently, a morphism $g : k \rightarrow x$ in \mathcal{C} is said to be a pseudo kernel of f if, for any $c \in \text{ob}\mathcal{C}$, the following sequence of abelian groups is exact

$$\text{hom}_{\mathcal{C}}(c, k) \rightarrow \text{hom}_{\mathcal{C}}(c, x) \rightarrow \text{hom}_{\mathcal{C}}(c, y).$$

Pseudo-kernels have been introduced by Freyd [11] as weak kernels. *Pseudo-cokernels* are pseudo kernels in \mathcal{C}^{op} . By [15, Corollary 1.1] the categories \mathcal{C} , \mathcal{C}_\oplus and $\text{Idem}(\mathcal{C}_\oplus)$ all have pseudo kernels or they don't. Let us remark that any triangulated or abelian category has pseudo-kernels and pseudo-cokernels.

Let $n \geq 1$ and $f : x \rightarrow y$ be a morphism in \mathcal{C} . Following [6], we say that f has a *pseudo n -kernel* if there exists a chain of morphisms

$$x_n \xrightarrow{f_n} x_{n-1} \xrightarrow{f_{n-1}} x_{n-2} \rightarrow \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} x \xrightarrow{f} y$$

such that the following sequence of abelian groups is exact

$$\text{hom}_{\mathcal{C}}(-, x_n) \xrightarrow{f_{n*}} \cdots \rightarrow \text{hom}_{\mathcal{C}}(-, x_1) \xrightarrow{f_{1*}} \text{hom}_{\mathcal{C}}(-, x) \xrightarrow{f_*} \text{hom}_{\mathcal{C}}(-, y).$$

We denote the pseudo n -kernel by $(f_n, f_{n-1}, \dots, f_1)$. The case $n = 1$ gives us the classic pseudo-kernels. For convenience, we let $x_0 := x$. Furthermore, any morphism f in \mathcal{C} will be assumed to be a pseudo 0-kernel of itself. We say that f has a *pseudo ∞ -kernel* if there exists a chain of morphisms

$$\cdots \rightarrow x_{n+1} \xrightarrow{f_{n+1}} x_n \xrightarrow{f_n} x_{n-1} \rightarrow \cdots \xrightarrow{f_2} x_1 \xrightarrow{f_1} x \xrightarrow{f} y$$

such that the following sequence of abelian groups is exact

$$\cdots \rightarrow \text{hom}_{\mathcal{C}}(-, x_{n+1}) \xrightarrow{f_{n+1*}} \text{hom}_{\mathcal{C}}(-, x_n) \xrightarrow{f_{n*}} \cdots \xrightarrow{f_{1*}} \text{hom}_{\mathcal{C}}(-, x) \xrightarrow{f_*} \text{hom}_{\mathcal{C}}(-, y).$$

Pseudo n -cokernels are defined as pseudo n -kernels in \mathcal{C}^{op} .

2.2. Categories of \mathbb{Z} -linear functors. The category of abelian groups will be denoted by Ab . For any \mathbb{Z} -linear category \mathcal{C} , we define a *left \mathcal{C} -module* as a \mathbb{Z} -linear functor $F : \mathcal{C} \rightarrow \text{Ab}$. We consider natural transformations as morphisms of \mathcal{C} -modules. Define a *right \mathcal{C} -module* as a \mathbb{Z} -linear functor $F : \mathcal{C}^{op} \rightarrow \text{Ab}$. Recall that a \mathbb{Z} -linear functor $F : \mathcal{C}^{op} \rightarrow \text{Ab}$ satisfies that $F(f + g) = F(f) + F(g)$ where $f, g \in \text{hom}_{\mathcal{C}^{op}}(x, y)$. In these categories limits and colimits of functors are defined objectwise. Denote by $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ the category of right \mathcal{C} -modules. This category is cocomplete and abelian. If c is an object of \mathcal{C} then there is the corresponding representable functor $\text{hom}_{\mathcal{C}}(-, c) : \mathcal{C}^{op} \rightarrow \text{Ab}$.

Lemma 2.3. (*Yoneda Lemma*) *Let \mathcal{C} be any \mathbb{Z} -linear category. Take $c \in \mathcal{C}$ and F a right \mathcal{C} -module. Then there is a natural identification*

$$\text{hom}_{\text{Fun}(\mathcal{C}^{op}, \text{Ab})}(\text{hom}_{\mathcal{C}}(-, c), F(-)) \cong F(c).$$

By Yoneda Lemma, the family $\{\text{hom}_{\mathcal{C}}(-, c)\}_{c \in \mathcal{C}}$ is a generating set of finitely generated projective modules in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$. A right \mathcal{C} -module M is *free* if it is isomorphic to $\bigoplus_{i \in I} \text{hom}_{\mathcal{C}}(-, a_i)$. It is free and finitely generated if I is finite.

Let R be a ring and \underline{R} be the \mathbb{Z} -linear category defined in Example 2.1. Note that

$$\begin{aligned} R\text{-Mod} &\cong \text{Fun}(\underline{R}, \text{Ab}) \\ \text{Mod-}R &\cong \text{Fun}(\underline{R}^{op}, \text{Ab}). \end{aligned}$$

2.3. Finitely n -presented objects and n -coherent categories. Let $n \geq 1$ be a positive integer. According to [5, Definition 2.1] a right \mathcal{C} -module F is said to be *finitely n -presented* or *of type \mathcal{FP}_n* if the functors $\text{Ext}_{\text{Fun}(\mathcal{C}^{op}, \text{Ab})}^i(F, -)$ preserves direct limits for all $0 \leq i \leq n - 1$. Denote by \mathcal{FP}_0 to the set of finitely generated objects. Then, a right \mathcal{C} -module M is of type \mathcal{FP}_0 if there exists a collection of objects $\{c_j : j \in J\}$ in \mathcal{C} for some finite set J and an epimorphism $\bigoplus_{j \in J} \text{hom}_{\mathcal{C}}(-, c_j) \rightarrow M$. Furthermore, a right \mathcal{C} -module F is said to be of type \mathcal{FP}_{∞} if it is of type \mathcal{FP}_n for all $n \geq 0$.

Recall that a *Grothendieck category* is a cocomplete abelian category, with a generating set and with exact direct limits. A Grothendieck category is *locally finitely generated (presented)* if it has a set of finitely generated (presented) generators. In other words, each object is a direct union (limit) of finitely generated (presented) objects. A Grothendieck category is *locally type \mathcal{FP}_n* [5, Definition 2.3], if it has a generating set consisting of objects of type \mathcal{FP}_n .

According to [13, Example 3.2] any finitely generated projective right \mathcal{C} -module is of type \mathcal{FP}_n for all $n \geq 0$. Then, the functor category $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ is a locally type \mathcal{FP}_{∞} Grothendieck category. Therefore, by the [5, Corollary 2.14], a right \mathcal{C} -module F is of type \mathcal{FP}_n if and only if there exists an exact sequence

$$P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow F \rightarrow 0$$

where P_i is finitely generated and projective right \mathcal{C} -module for every $0 \leq i \leq n$.

Recall from [5, Definition 4.1] that a right \mathcal{C} -module F is *n -coherent* if satisfies the following conditions:

- (1) F is of type \mathcal{FP}_n .
- (2) If S is a subobject of F that is of type \mathcal{FP}_{n-1} then S is also of type \mathcal{FP}_n .

Definition 2.4. Let \mathcal{C} be a \mathbb{Z} -linear category and $n \geq 0$. We say that \mathcal{C} is right (left) *n -coherent* if every right (left) \mathcal{C} -module F of type \mathcal{FP}_n is n -coherent.

Note that the \mathbb{Z} -linear category \mathcal{C} is right n -coherent if the category $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ is n -coherent as a Grothendieck category in the sense of [5, Definition 4.6]. Thus by [5, Theorem 4.7], \mathcal{C} is right n -coherent if and only if the \mathcal{C} -modules of type \mathcal{FP}_n coincide with the \mathcal{C} -modules of type \mathcal{FP}_{∞} .

In particular, an additive category \mathcal{C} is Noetherian as defined in [3, Definition 5.2] if and only if it is right 0-coherent.¹ Moreover, if $1 \leq n \leq \infty$ and \mathcal{C} is any small additive category, then the following conditions are equivalent, as shown in [6, Proposition 5.4]:

- (1) \mathcal{C} is right n -coherent.

¹In [3], the word right is omitted, but we have chosen to include it in our notation.

- (2) If a morphism in \mathcal{C} has a pseudo $(n-1)$ -kernel, then it has a pseudo n -kernel.

Definition 2.5. Let \mathcal{C} be a \mathbb{Z} -linear category and $n \geq 0$. We say that \mathcal{C} is *right regular n -coherent* if it satisfies the following conditions:

- (1) \mathcal{C} is right n -coherent.
- (2) Every right \mathcal{C} -module F of type \mathcal{FP}_n has a projective dimension.

Let \mathcal{C} be a small additive category. Then, according to [3, Definition 5.2] \mathcal{C} is regular coherent if and only if it is right regular 1-coherent.

Example 2.6. Let \mathcal{C} be a small additive category and $n \geq 1$.

- I. **Additive category with kernels.** By a result due to Auslander [2, Theorem 2.2.b] a small additive category \mathcal{C} with kernels is 1-coherent and every \mathcal{C} -module of type \mathcal{FP}_1 has projective dimension at most 2. Then \mathcal{C} is right regular 1-coherent.
- II. **Von Neumann regular categories.** We recall that \mathcal{C} is called von Neumann regular if for any morphism $f : a \rightarrow b$ in \mathcal{C} there exists a morphism $g : b \rightarrow a$ such that $fgf = f$. By [4, Corollary 8.1.3] \mathcal{C} is right regular 1-coherent.
- III. **Locally finitely presented categories.** An object $c \in \mathcal{C}$ is finitely presented if the functor $\text{hom}_{\mathcal{C}}(c, -)$ preserves direct limits. The category \mathcal{C} is locally finitely presented if every directed system of objects and morphisms has a direct limit, the class of finitely presented objects of \mathcal{C} is skeletally small and every object of \mathcal{C} is the direct limit of finitely presented objects. Then by [12, Lemma 2.2], every locally finitely presented category is left 1-coherent.
- IV. **n -hereditary categories.** Suppose the following two conditions hold in \mathcal{C} :
 - (a) Every morphism in \mathcal{C} with a pseudo $(n-1)$ -kernel has a pseudo n -kernel.
 - (b) For every morphism $f : x \rightarrow y$ in \mathcal{C} with pseudo n -kernel (f_n, \dots, f_1) , there exists an endomorphism $\alpha : x_{n-1} \rightarrow x_{n-1}$ making the following diagram commute:

$$\begin{array}{ccccccc}
 x_k & \xrightarrow{f_k} & x_{k-1} & \xrightarrow{f_{k-1}} & x_{k-2} & \xrightarrow{f_{k-2}} & \cdots & \cdots & \xrightarrow{f_2} & x_1 & \xrightarrow{f_1} & x & \xrightarrow{f} & y \\
 & \searrow 0 & \downarrow \alpha & \nearrow f_{k-1} & & & & & & & & & & \\
 & & x_{k-1} & & & & & & & & & & &
 \end{array}$$

By [6, Theorem 5.5], \mathcal{C} is right n -coherent and every \mathcal{C} -module of type \mathcal{FP}_n has projective dimension less than or equal 1. Therefore, \mathcal{C} is right regular n -coherent.

Due to [15, Lemma 1.1, 1.2] we have the following equivalences of categories

$$\text{Fun}(\mathcal{C}^{op}, \text{Ab}) \simeq \text{Fun}(\mathcal{C}_{\oplus}^{op}, \text{Ab}) \simeq \text{Fun}(\text{Idem}(\mathcal{C}_{\oplus}^{op}), \text{Ab})$$

In other words \mathcal{C} , \mathcal{C}_{\oplus} and $\text{Idem}(\mathcal{C}_{\oplus})$ are Morita equivalents. In particular, we obtain the following result.

Proposition 2.7. *Let \mathcal{C} be a \mathbb{Z} -linear category. The following are equivalent:*

- (1) \mathcal{C} is right regular n -coherent.

- (2) \mathcal{C}_\oplus is right regular n -coherent.
- (3) $\text{Idem}(\mathcal{C}_\oplus)$ is right regular n -coherent.

Let R be a ring with unity. A finitely n -presented right R -module M is n -coherent if every finitely $(n-1)$ -presented submodule $N \subseteq M$ is finitely n -presented. The ring R is right n -coherent if R is n -coherent as a right R -module (i.e. if each finitely $(n-1)$ -presented right ideal of R is finitely n -presented). We say that R is strong right n -coherent if each finitely n -presented right R -module is finitely $(n+1)$ -presented. A strong n -coherence ring is equivalent to a n -coherence ring for $n = 1$, but it is an open question for $n \geq 2$. A coherent ring is a 1-coherent ring (strong 1-coherent ring) and it is regular if and only if every finitely presented module has finite projective dimension. Motivated by this we introduce in [10, Definition 2.9] the definition of n -regular ring. Let $n \geq 1$, a ring R is called right n -regular if each finitely n -presented right R -module has finite projective dimension.

Corollary 2.8. *Let R be a ring with unity and $n \geq 1$. Then the following are equivalent.*

- (1) The ring R is strong right n -coherent or right n -regular and strong right n -coherent respectively;
- (2) The additive category \underline{R}_\oplus is right n -coherent or right regular n -coherent respectively;
- (3) The additive category $\text{Idem}(\underline{R}_\oplus)$ is right n -coherent or right regular n -coherent respectively.

Let \mathcal{C} be a small additive category. By [3, Lemma 6.8], \mathcal{C} is right Noetherian if and only if each object c has the following property. Consider any directed set I and collections of morphisms $\{f_i : a_i \rightarrow c\}_{i \in I}$ with c as target such that $f_i \subseteq f_j$ holds for $i \leq j$, then there exists $i_0 \in I$ with $f_i \subseteq f_{i_0}$ for all $i \in I$. Our aim is to find out intrinsic condition of \mathcal{C} which guarantees that $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ is regular n -coherent.

Proposition 2.9. *Let \mathcal{C} be a small additive category and $n \geq 1$. The following are equivalent*

- (1) \mathcal{C} is right regular n -coherent.
- (2) The following conditions hold in \mathcal{C} :
 - i) Every morphism in \mathcal{C} with a pseudo $(n-1)$ -kernel has a pseudo n -kernel.
 - ii) For every morphism $f : x \rightarrow y$ in \mathcal{C} with pseudo ∞ -kernel there exists $k \in \mathbb{N}$ and $\alpha : x_{k-1} \rightarrow x_{k-1}$ making the following diagram commute:

$$\begin{array}{ccccccc}
 x_k & \xrightarrow{f_k} & x_{k-1} & \xrightarrow{f_{k-1}} & x_{k-2} & \xrightarrow{f_{k-2}} & \cdots & \cdots & \xrightarrow{f_2} & x_1 & \xrightarrow{f_1} & x & \xrightarrow{f} & y \\
 & & \searrow 0 & \downarrow \alpha & \nearrow f_{k-1} & & & & & & & & & \\
 & & & x_{k-1} & & & & & & & & & &
 \end{array}$$

Proof. (1 \Rightarrow 2) Suppose that \mathcal{C} is right regular n -coherent. First, we note that (i) is clear by [6, Prop 5.4]. Now suppose that $f : x \rightarrow y$ is a morphism in \mathcal{C} with a pseudo ∞ -kernel (\cdots, f_3, f_2, f_1) . Here, we let $f_0 := f$. Thus $\text{coker}(f_*)$ is of type \mathcal{FP}_∞ in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ because there exists an exact sequence of the form

$$\cdots \xrightarrow{f_{2*}} \text{hom}_{\mathcal{C}}(-, x_1) \xrightarrow{f_{1*}} \text{hom}_{\mathcal{C}}(-, x) \xrightarrow{f_*} \text{hom}_{\mathcal{C}}(-, y) \rightarrow \text{coker}(f_*) \rightarrow 0.$$

There exists $k \in \mathbb{N}$ such that $\text{coker}(f_*)$ has projective dimension $\leq k$. It implies that $\ker(f_{k-2*}) = \text{im}(f_{k-1*})$ is projective, and therefore $\ker(f_{k-1*}) = \text{im}(f_{k*})$ is projective too. Consider

$$\begin{array}{ccccc} \cdots & \rightarrow & \text{hom}_{\mathcal{C}}(-, x_k) & \xrightarrow{f_{k*}} & \text{hom}_{\mathcal{C}}(-, x_{k-1}) & \rightarrow & \cdots \\ & & & \searrow \sigma & \uparrow \iota & & \\ & & & & \text{im}(f_{k*}) & & \end{array}$$

where $\iota : \text{im}(f_{k*}) \hookrightarrow \text{hom}_{\mathcal{C}}(-, x_{k-1})$ and $\sigma : \text{hom}_{\mathcal{C}}(-, x_k) \twoheadrightarrow \text{im}(f_{k*})$ are the canonical morphisms. There exists $\iota' : \text{im}(f_{k*}) \rightarrow \text{hom}_{\mathcal{C}}(-, x_k)$ and $\sigma' : \text{hom}_{\mathcal{C}}(-, x_{k-1}) \rightarrow \text{im}(f_{k*})$ such that $\sigma \circ \iota' = \text{id}_{\text{im}(f_{k*})}$ and $\sigma' \circ \iota = \text{id}_{\text{im}(f_{k*})}$. By Yoneda Lemma and using the same techniques [6, Theorem 5.5] there exists $h : x_{k-1} \rightarrow x_k$ in \mathcal{C} such that $h_* : \text{hom}_{\mathcal{C}}(-, x_{k-1}) \rightarrow \text{hom}_{\mathcal{C}}(-, x_k)$ satisfy $h_* \circ f_{k*} = \iota' \circ \sigma$. The morphism $\alpha := \text{id}_{x_{k-1}} - f_k \circ h$ satisfies the desired condition.

(2 \Rightarrow 1) Suppose that the affirmation (2) is satisfied for $n \geq 1$. Using the condition (2-i), we deduce that \mathcal{C} is right n -coherent [6, Prop 5.4] and thus $\mathcal{F}\mathcal{P}_n = \mathcal{F}\mathcal{P}_{\infty}$. Now, for each $F : \mathcal{C}^{op} \rightarrow \text{Ab}$ of type $\mathcal{F}\mathcal{P}_n$ we get an exact sequence of the form

$$\cdots \rightarrow \text{hom}_{\mathcal{C}}(-, x_n) \rightarrow \cdots \rightarrow \text{hom}_{\mathcal{C}}(-, x_1) \xrightarrow{f_{1*}} \text{hom}_{\mathcal{C}}(-, x) \xrightarrow{f_*} \text{hom}_{\mathcal{C}}(-, y) \rightarrow F \rightarrow 0$$

where $f : x \rightarrow y$ is a morphism in \mathcal{C} . It implies that f has a pseudo ∞ -kernel, and therefore, there is $k \in \mathbb{N}$ and an endomorphism $\alpha : x_{k-1} \rightarrow x_{k-1}$ making the following diagram commute:

$$\begin{array}{ccccc} x_k & \xrightarrow{f_k} & x_{k-1} & \xrightarrow{f_{k-1}} & x_{k-2} \\ & \searrow 0 & \downarrow \alpha & \nearrow f_{k-1} & \\ & & x_{k-1} & & \end{array}$$

Next, we show that $\text{im}(f_{k-1*}) = \ker(f_{k-2*})$ is a projective functor. Consider

$$\begin{array}{ccccc} \text{hom}_{\mathcal{C}}(-, x_k) & \xrightarrow{f_{k*}} & \text{hom}_{\mathcal{C}}(-, x_{k-1}) & \xrightarrow{f_{k-1*}} & \text{hom}_{\mathcal{C}}(-, x_{k-2}) \\ & \searrow 0 & \downarrow \alpha_* & \nearrow \iota & \parallel \\ & & \text{im}(f_{k-1*}) & & \\ & & \downarrow \sigma & & \\ & & \text{hom}_{\mathcal{C}}(-, x_{k-1}) & \xrightarrow{f_{k-1*}} & \text{hom}_{\mathcal{C}}(-, x_{k-2}) \end{array}$$

where $\sigma : \text{hom}_{\mathcal{C}}(-, x_{k-1}) \rightarrow \text{im}(f_{k-1*})$ and $\iota : \text{im}(f_{k-1*}) \rightarrow \text{hom}_{\mathcal{C}}(-, x_{k-2})$ are the canonical natural transformations. Note that $\text{im}(f_{k-1*}) = \text{coker}(f_{k*})$, then there exists unique natural transformation $t : \text{im}(f_{k-1*}) \rightarrow \text{hom}_{\mathcal{C}}(-, x_{k-1})$ such that $t \circ \sigma = \alpha_*$. Moreover, applying the same techniques [6, Theorem 5.5] we have

$$\iota \circ \text{id}_{\text{im}(f_{k-1*})} \circ \sigma = \iota \circ \sigma = f_{k-1*} = (f_{k-1} \circ \alpha)_* = f_{k-1*} \circ \alpha_* = f_{k-1*} \circ t \circ \sigma = \iota \circ \sigma \circ t \circ \sigma$$

which implies that

$$\text{id}_{\text{im}(f_{k-1*})} = \sigma \circ t.$$

Then σ is a split epimorphism, and therefore, $\text{im}(f_{k-1*})$ is projective. \square

According to [3], a small additive category \mathcal{C} is considered to be right regular if it satisfies two conditions: it is both right Noetherian and right regular 1-coherent. It's important to note that this usage of regular should not be confused with the concept of von Neumann regular.

Corollary 2.10. *Let \mathcal{C} be a small additive category. The following are equivalent*

- (1) \mathcal{C} is right regular.
- (2) The following conditions hold in \mathcal{C} :
 - i) Every object c in \mathcal{C} has the following property. Consider any directed set I and collections of morphisms $\{f_i : a_i \rightarrow c\}_{i \in I}$ with c as target such that $f_i \subseteq f_j$ holds for $i \leq j$. Then there exists $i_0 \in I$ with $f_i \subseteq f_{i_0}$ for all $i \in I$.
 - ii) For every morphism $f : x \rightarrow y$ in \mathcal{C} with pseudo ∞ -kernel there exists $k \in \mathbb{N}$ and $\alpha : x_{k-1} \rightarrow x_{k-1}$ making the following diagram commute:

$$\begin{array}{ccccccccccc}
 x_k & \xrightarrow{f_k} & x_{k-1} & \xrightarrow{f_{k-1}} & x_{k-2} & \xrightarrow{f_{k-2}} & \cdots & \cdots & \xrightarrow{f_2} & x_1 & \xrightarrow{f_1} & x & \xrightarrow{f} & y \\
 & \searrow 0 & \downarrow \alpha & \nearrow f_{k-1} & & & & & & & & & & \\
 & & x_{k-1} & & & & & & & & & & &
 \end{array}$$

In [3] another type of regularity is introduced due to bad behavior of regularity with respect to infinity products. Let R be a ring with unity. Specifically, R is right l -uniformly regular coherent, if every finitely presented right R -module M admits a l -dimensional finite projective resolution, i.e. there exists an exact sequence

$$0 \rightarrow P_l \rightarrow P_{l-1} \rightarrow \cdots \rightarrow P_0 \rightarrow M \rightarrow 0$$

where each P_i is finitely generated and projective right R -module. This concept is extended to additive categories in [3, Section 6]. Let \mathcal{C} be a \mathbb{Z} -linear category and $l \geq 1$. We say that \mathcal{C} is right l -uniformly regular coherent, if every right \mathcal{C} -module F of type \mathcal{FP}_1 admits a l -dimensional finite projective resolution, i.e. there exists an exact sequence

$$0 \rightarrow P_l \rightarrow P_{l-1} \rightarrow \cdots \rightarrow P_0 \rightarrow F \rightarrow 0$$

where each P_i is finitely generated and projective right \mathcal{C} -module.

The equivalence $\text{Fun}(\mathcal{C}^{op}, \text{Ab}) \simeq \text{Fun}(\mathcal{C}_{\oplus}^{op}, \text{Ab})$ implies that \mathcal{C} is right l -uniformly regular coherent if and only if \mathcal{C}_{\oplus} is right l -uniformly regular coherent. Note that, if \mathcal{C} is right 1-coherent and every right \mathcal{C} -module F of type \mathcal{FP}_1 has a projective dimension $\leq l$ then \mathcal{C} is right l -uniformly regular coherent.

Corollary 2.11. *Let $l \geq 1$ and let \mathcal{C} be a small additive category. Suppose that \mathcal{C} is right 1-coherent. Then, the following are equivalent:*

- (1) \mathcal{C} is right l -uniformly regular coherent.
- (2) For every morphism $f : x \rightarrow y$ in \mathcal{C} there exists $l \in \mathbb{N}$, an pseudo l -kernel $(f_l, f_{l-1}, \dots, f_1)$ of f and $\alpha : x_{l-1} \rightarrow x_{l-1}$ making the following diagram commute:

$$\begin{array}{ccccccc}
x_l & \xrightarrow{f_l} & x_{l-1} & \xrightarrow{f_{l-1}} & x_{l-2} & \xrightarrow{f_{l-2}} & \cdots & \cdots & \xrightarrow{f_2} & x_1 & \xrightarrow{f_1} & x & \xrightarrow{f} & y \\
& & \searrow 0 & \downarrow \alpha & \nearrow f_{l-1} & & & & & & & & & \\
& & & x_{l-1} & & & & & & & & & &
\end{array}$$

3. THE RING $\mathcal{A}(\mathcal{C})$ AND THE \mathbb{Z} -LINEAR CATEGORY \mathcal{C}

In this section we study the relation between some properties of a \mathbb{Z} -linear category \mathcal{C} with the properties of a ring $\mathcal{A}(\mathcal{C})$ associated with it. We prove the categories $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ and $\text{Mod-}\mathcal{A}(\mathcal{C})$ are equivalent.

3.1. The ring $\mathcal{A}(\mathcal{C})$. Let \mathcal{C} be a \mathbb{Z} -linear category. Recall from [7]

$$(3.1) \quad \mathcal{A}(\mathcal{C}) = \bigoplus_{a,b \in \text{ob}\mathcal{C}} \text{hom}_{\mathcal{C}}(a, b).$$

If $f \in \mathcal{A}(\mathcal{C})$ write $f_{a,b}$ for the component in $\text{hom}_{\mathcal{C}}(b, a)$. The following multiplication law

$$(3.2) \quad (fg)_{a,b} = \sum_{c \in \text{ob}\mathcal{C}} f_{a,c} g_{c,b}$$

makes $\mathcal{A}(\mathcal{C})$ into an associative ring, which is unital if and only if $\text{ob}\mathcal{C}$ is finite. Whatever the cardinal of $\text{ob}\mathcal{C}$ is, $\mathcal{A}(\mathcal{C})$ is always a ring with *local units*, i.e. a filtering colimit of unital rings.

3.2. The $\mathbb{Z}\mathcal{C}$ -modules. Recall that M is a unital right $\mathcal{A}(\mathcal{C})$ -module if $M \cdot \mathcal{A}(\mathcal{C}) = M$. Consider $\text{Mod-}\mathcal{A}(\mathcal{C})$ the category of unital right $\mathcal{A}(\mathcal{C})$ -modules. Let us define functors

$$\mathcal{S}(-) : \text{Fun}(\mathcal{C}^{op}, \text{Ab}) \rightarrow \text{Mod-}\mathcal{A}(\mathcal{C}) \quad (-)_{\mathcal{C}} : \text{Mod-}\mathcal{A}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Ab})$$

Let $M \in \text{Fun}(\mathcal{C}^{op}, \text{Ab})$

$$\mathcal{S}(M) = \bigoplus_{a \in \text{ob}\mathcal{C}} M(a)$$

Let $N \in \text{Mod-}\mathcal{A}(\mathcal{C})$

$$N_{\mathcal{C}} : \mathcal{C}^{op} \rightarrow \text{Ab} \quad a \mapsto N \cdot \text{id}_a.$$

Lemma 3.3. *If N is a unital right $\mathcal{A}(\mathcal{C})$ -module then*

$$\bigoplus_{a \in \text{ob}\mathcal{C}} N \cdot \text{id}_a = N.$$

Proof. For every $a \in \text{ob}\mathcal{C}$ we have $N \cdot \text{id}_a \subseteq N$ then $\bigoplus_{a \in \text{ob}\mathcal{C}} N \cdot \text{id}_a \subseteq N$. Let $n \in N$, because N is unital $N = N \cdot \mathcal{A}(\mathcal{C})$ then $n = \sum_{i=1}^{i=m} n_i \cdot f_i$ with $n_i \in N$ and $f_i \in \text{hom}_{\mathcal{C}}(a_i, b_i)$. Let $I = \{a \in \text{ob}\mathcal{C} : a = a_i, \text{ for some } i = 1, \dots, m\}$ then

$$n = \sum_{i=1}^{i=m} n_i \cdot f_i = \left(\sum_{i=1}^{i=m} n_i \cdot f_i \right) \cdot \left(\sum_{a \in I} \text{id}_a \right) = n \cdot \sum_{a \in I} \text{id}_a$$

We conclude $N \subseteq \bigoplus_{a \in \text{ob}\mathcal{C}} N \cdot \text{id}_a$. \square

Proposition 3.4. *Let \mathcal{C} be a \mathbb{Z} -linear category then*

$$\mathcal{S}(-) : \text{Fun}(\mathcal{C}^{op}, \text{Ab}) \rightarrow \text{Mod-}\mathcal{A}(\mathcal{C}) \quad (-)_{\mathcal{C}} : \text{Mod-}\mathcal{A}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Ab})$$

are an equivalence of categories.

Proof. Let $N \in \text{Mod-}\mathcal{A}(\mathcal{C})$ and $M \in \text{Fun}(\mathcal{C}^{op}, \text{Ab})$ then

$$\begin{aligned} S(N_{\mathcal{S}}) &= \bigoplus_{a \in \text{ob}\mathcal{C}} N_{\mathcal{C}}(a) = \bigoplus_{a \in \text{ob}\mathcal{C}} N \cdot \text{id}_a = N \\ (S(M))_{\mathcal{C}}(c) &= S(M) \cdot \text{id}_c = \bigoplus_{a \in \text{ob}\mathcal{C}} M(a) \cdot \text{id}_c = M(c) \quad \forall c \in \text{ob}\mathcal{C}. \end{aligned}$$

□

The abelian structure of $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ comes from the abelian structure in Ab . A sequence $M \xrightarrow{f} N \xrightarrow{g} R$ is exact in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ if for each object $c \in \mathcal{C}$ the sequence $M(c) \xrightarrow{f(c)} N(c) \xrightarrow{g(c)} R(c)$ is exact in Ab .

Proposition 3.5. *Let \mathcal{C} be a \mathbb{Z} -linear category then*

$$\mathcal{S}(-) : \text{Fun}(\mathcal{C}^{op}, \text{Ab}) \rightarrow \text{Mod-}\mathcal{A}(\mathcal{C}) \quad (-)_{\mathcal{C}} : \text{Mod-}\mathcal{A}(\mathcal{C}) \rightarrow \text{Fun}(\mathcal{C}^{op}, \text{Ab})$$

are exact functors.

Proof. Let $M \xrightarrow{f} N \xrightarrow{g} R$ be an exact sequence in $\text{Mod-}\mathcal{A}(\mathcal{C})$. Let us prove $M_{\mathcal{C}} \xrightarrow{f_{\mathcal{C}}} N_{\mathcal{C}} \xrightarrow{g_{\mathcal{C}}} R_{\mathcal{C}}$ is exact in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ showing $M_{\mathcal{C}}(a) \xrightarrow{f_{\mathcal{C}}(a)} N_{\mathcal{C}}(a) \xrightarrow{g_{\mathcal{C}}(a)} R_{\mathcal{C}}(a)$ is exact for every object a in \mathcal{C} . By functoriality $\text{im}(f_{\mathcal{C}}(a)) \subseteq \ker(g_{\mathcal{C}}(a))$. Let $n \cdot \text{id}_a \in \ker(g_{\mathcal{C}}(a))$ then

$$g_{\mathcal{C}}(a)(n \cdot \text{id}_a) = g(n) \cdot \text{id}_a = g(n \cdot \text{id}_a) = 0$$

then $n \cdot \text{id}_a \in \ker(g) = \text{im}(f)$. There exists $m \in M$ such that $f(m) = n \cdot \text{id}_a$ then

$$f_{\mathcal{C}}(a)(m \cdot \text{id}_a) = f(m \cdot \text{id}_a) = f(m) \cdot \text{id}_a = (n \cdot \text{id}_a) \cdot \text{id}_a = n \cdot \text{id}_a$$

then $n \cdot \text{id}_a \in \text{im}(f_{\mathcal{C}}(a))$. We conclude $(-)_{\mathcal{C}}$ is exact.

We proceed to show that \mathcal{S} is exact. Let $M \xrightarrow{f} N \xrightarrow{g} R$ be an exact sequence in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$. Consider

$$S(M) = \bigoplus_{a \in \text{ob}\mathcal{C}} M(a) \xrightarrow{S(f)} S(N) = \bigoplus_{a \in \text{ob}\mathcal{C}} N(a) \xrightarrow{S(g)} S(R) = \bigoplus_{a \in \text{ob}\mathcal{C}} R(a)$$

Similarly as above, let $\sum_{a \in \mathcal{C}} x_a \in \ker S(g)$ then

$$\begin{aligned} S(g)(\sum_{a \in \mathcal{C}} x_a) = \sum_{a \in \mathcal{C}} g(a)(x_a) = 0 &\Rightarrow g(a)(x_a) = 0 \quad \forall x_a \in N(a) \\ &\Rightarrow x_a \in \ker g(a) = \text{im } f(a) \quad \forall x_a \in N(a) \\ &\Rightarrow \exists y_a \in M(a) \text{ such that } f(a)(y_a) = x_a \end{aligned}$$

□

Corollary 3.6. *Let \mathcal{C} be a \mathbb{Z} -linear category.*

- (1) *If $p : M \rightarrow N$ is an epimorphism in $\text{Mod-}\mathcal{A}(\mathcal{C})$ then $p_{\mathcal{C}} : M_{\mathcal{C}} \rightarrow N_{\mathcal{C}}$ is an epimorphism in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$.*
- (2) *If $\pi : M \rightarrow N$ is an epimorphism in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$ then $\mathcal{S}(\pi) : \mathcal{S}(M) \rightarrow \mathcal{S}(N)$ is an epimorphism in $\text{Mod-}\mathcal{A}(\mathcal{C})$.*
- (3) *$(M \oplus N)_{\mathcal{C}} = M_{\mathcal{C}} \oplus N_{\mathcal{C}}$ in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$.*
- (4) *$\mathcal{S}(M \oplus N) = \mathcal{S}(M) \oplus \mathcal{S}(N)$ in $\text{Mod-}\mathcal{A}(\mathcal{C})$.*

Let A be a ring with local units. From [19] we recall that an A -module is *quasi-free* if it is isomorphic to a direct sum of modules of the form $e \cdot A$ with $e^2 = e$, $e \in A$. Quasi-free modules over a ring with local units play the same role as free modules over a ring with unity. Also recall that M is a finitely generated module if and only if it is an image of a finitely generated quasi-free module. A finitely generated module M is projective if and only if it is a direct summand of a finitely generated quasi-free modules. In this paper we work with $A = \mathcal{A}(\mathcal{C})$ and we say that M is a quasi-free right $\mathcal{A}(\mathcal{C})$ -module if it is isomorphic to a finite sum of modules $\text{id}_a \cdot \mathcal{A}(\mathcal{C})$.

Lemma 3.7. *Let \mathcal{C} be a \mathbb{Z} -linear category.*

- (1) *If F is a free finitely generated right \mathcal{C} -module, then $S(F)$ is a quasi-free finitely generated right $\mathcal{A}(\mathcal{C})$ -module.*
- (2) *If P is a projective finitely generated right \mathcal{C} -module, then $S(P)$ is projective and finitely generated right $\mathcal{A}(\mathcal{C})$ -module.*
- (3) *If M is a quasi-free finitely generated right $\mathcal{A}(\mathcal{C})$ -module then $M_{\mathcal{C}}$ is a finitely generated free right \mathcal{C} module.*
- (4) *If P is a projective finitely generated right $\mathcal{A}(\mathcal{C})$ -module then $P_{\mathcal{C}}$ is projective and finitely generated right \mathcal{C} -module.*

Proof. (1) Let I be a finite subset of objects in \mathcal{C} such that $F = \bigoplus_{b \in I} \text{hom}_{\mathcal{C}}(-, b)$. Then we have

$$S(F) = \bigoplus_{b \in I} S(\text{hom}_{\mathcal{C}}(-, b)) = \bigoplus_{b \in I} \text{id}_b \cdot \mathcal{A}(\mathcal{C})$$

This shows that $S(F)$ is a quasi-free finitely generated right $\mathcal{A}(\mathcal{C})$ -module.

- (2) Suppose P is a finitely generated projective right \mathcal{C} -module. Then there exists a module Q such that $P \oplus Q = F$, where F is a free module. Moreover, we have $S(P) \oplus S(Q) = S(F)$, where $S(F)$ is quasi-free and finitely generated. Therefore, $S(P)$ is also projective.
- (3) Suppose M is a quasi-free finitely generated right $\mathcal{A}(\mathcal{C})$ -module. Then there exists a finite set I such that $M = \bigoplus_{b \in I} \text{id}_b \cdot \mathcal{A}(\mathcal{C})$. Note that for any object a in \mathcal{C} ,

$$M_{\mathcal{C}}(a) = M \cdot \text{id}_a = \left(\bigoplus_{b \in I} \text{id}_b \cdot \mathcal{A}(\mathcal{C}) \right) \cdot \text{id}_a = \bigoplus_{b \in I} \text{hom}_{\mathcal{C}}(a, b)$$

Therefore, we have

$$M_{\mathcal{C}} = \bigoplus_{b \in I} \text{hom}_{\mathcal{C}}(-, b)$$

which is a free finitely generated module in $\text{Fun}(\mathcal{C}^{op}, \text{Ab})$.

- (4) Suppose P is a projective finitely generated $\mathcal{A}(\mathcal{C})$ -module. Then there exists a module Q such that $P \oplus Q = F$, where F is a quasi-free finitely generated $\mathcal{A}(\mathcal{C})$ -module. We have

$$P_{\mathcal{C}} \oplus Q_{\mathcal{C}} = F_{\mathcal{C}}.$$

Therefore, $P_{\mathcal{C}}$ is also projective and finitely generated. □

Proposition 3.8. *Let \mathcal{C} be a \mathbb{Z} -linear category with finitely many objects and $n \geq 1$.*

- (1) *The category \mathcal{C} is right Noetherian if and only if $\mathcal{A}(\mathcal{C})$ is a right Noetherian ring.*

- (2) The category \mathcal{C} is right n -coherent if and only if $\mathcal{A}(\mathcal{C})$ is a strong right n -coherent ring.
- (3) The category \mathcal{C} is regular n -coherent if and only if $\mathcal{A}(\mathcal{C})$ is a right n -regular and strong right n -coherent ring.

Proof. (1) Let M be a finitely generated right $\mathcal{A}(\mathcal{C})$ -module and let N be a submodule. Consider the epimorphism

$$\mathcal{A}(\mathcal{C}) \oplus \dots \oplus \mathcal{A}(\mathcal{C}) \rightarrow M,$$

and let us apply Corollary 3.6 to obtain the following epimorphism:

$$\mathcal{A}(\mathcal{C})_{\mathcal{C}} \oplus \dots \oplus \mathcal{A}(\mathcal{C})_{\mathcal{C}} \rightarrow M_{\mathcal{C}}.$$

As $\mathcal{A}(\mathcal{C})_{\mathcal{C}} = \bigoplus_{b \in \text{ob} \mathcal{C}} \text{hom}_{\mathcal{C}}(-, b)$ we obtain that $M_{\mathcal{C}}$ is finitely generated.

$$\mathcal{A}(\mathcal{C})_{\mathcal{C}} \oplus \dots \oplus \mathcal{A}(\mathcal{C})_{\mathcal{C}} = \bigoplus_{j \in J} \bigoplus_{b_j \in \text{ob} \mathcal{C}} \text{hom}_{\mathcal{C}}(-, b_j)$$

Since \mathcal{C} is right Noetherian, we can conclude that $N_{\mathcal{C}}$ is also finitely generated. Moreover, there exists an epimorphism

$$\bigoplus_{i \in I} \text{hom}_{\mathcal{C}}(-, a_i) \rightarrow N_{\mathcal{C}}$$

then

$$\bigoplus_{i \in I} \mathcal{S}(\text{hom}_{\mathcal{C}}(-, a_i)) \rightarrow \mathcal{S}(N_{\mathcal{C}}) = N.$$

Consider the projection

$$p_i : \mathcal{A}(\mathcal{C}) \rightarrow \mathcal{S}(\text{hom}_{\mathcal{C}}(-, a_i)) = \bigoplus_{c \in \text{ob} \mathcal{C}} \text{hom}_{\mathcal{C}}(c, a_i)$$

Taking $n = \#I$ we obtain an epimorphism

$$\mathcal{A}(\mathcal{C})^n \rightarrow \bigoplus_{i \in I} \mathcal{S}(\text{hom}_{\mathcal{C}}(-, a_i)) \rightarrow N,$$

then N is finitely generated.

Conversely if $M \in \text{Fun}(\mathcal{C}^{op}, \text{Ab})$ is finitely generated let us show that every subobject is also finitely generated. Take N as a submodule of M . There is an epimorphism

$$\bigoplus_{i \in I} \text{hom}_{\mathcal{C}}(-, a_i) \rightarrow M$$

then we have an epimorphism

$$\bigoplus_{i \in I, c \in \text{ob} \mathcal{C}} \text{hom}_{\mathcal{C}}(c, a_i) = \bigoplus_{i \in I} \mathcal{S}(\text{hom}_{\mathcal{C}}(-, a_i)) \rightarrow \mathcal{S}(M).$$

We obtain that $\mathcal{S}(N)$ is a submodule of $\mathcal{S}(M)$ which is finitely generated, then $\mathcal{S}(N)$ is also finitely generated and $\mathcal{S}(N)_{\mathcal{C}} = N$ is finitely generated.

- (2) Let M be a finitely n -presented right $\mathcal{A}(\mathcal{C})$ -module. Consider $m_0, m_1, \dots, m_n \in \mathbb{N}$ such that

$$\mathcal{A}(\mathcal{C})^{m_n} \rightarrow \mathcal{A}(\mathcal{C})^{m_{n-1}} \rightarrow \dots \rightarrow \mathcal{A}(\mathcal{C})^{m_1} \rightarrow \mathcal{A}(\mathcal{C})^{m_0} \rightarrow M \rightarrow 0$$

is exact. By Proposition 3.5 the following is also an exact sequence

$$\mathcal{A}(\mathcal{C})_{\mathcal{C}}^{m_n} \rightarrow \mathcal{A}(\mathcal{C})_{\mathcal{C}}^{m_{n-1}} \rightarrow \dots \rightarrow \mathcal{A}(\mathcal{C})_{\mathcal{C}}^{m_1} \rightarrow \mathcal{A}(\mathcal{C})_{\mathcal{C}}^{m_0} \rightarrow M_{\mathcal{C}} \rightarrow 0$$

As $\mathcal{A}(\mathcal{C})_{\mathcal{C}} = \bigoplus_{b \in \text{ob } \mathcal{C}} \text{hom}_{\mathcal{C}}(-, b)$ we obtain that $M_{\mathcal{C}}$ is of type \mathcal{FP}_n . Because \mathcal{C} is right n -coherent there exists an exact sequence

$$\cdots \rightarrow P_{n+1} \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M_{\mathcal{C}} \rightarrow 0$$

where each P_i is both projective and finitely generated. Then,

$$\cdots \rightarrow S(P_{n+1}) \rightarrow S(P_n) \rightarrow \cdots \rightarrow S(P_1) \rightarrow S(P_0) \rightarrow M \rightarrow 0$$

is exact and by Lemma 3.7 $S(P_i)$ is projective and finitely generated. Therefore, $\mathcal{A}(\mathcal{C})$ is a strong right n -coherent ring.

Conversely, if $F \in \text{Fun}(\mathcal{C}^{op}, \text{Ab})$ is of type \mathcal{FP}_n then $S(F)$ is an $\mathcal{A}(\mathcal{C})$ -module of the type \mathcal{FP}_n . As $\mathcal{A}(\mathcal{C})$ is a strong right n -coherent ring there exists P_i projective finitely generated $\mathcal{A}(\mathcal{C})$ -modules such that

$$\cdots \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow S(F) \rightarrow 0$$

Then

$$\cdots \rightarrow (P_n)_{\mathcal{C}} \rightarrow \cdots \rightarrow (P_1)_{\mathcal{C}} \rightarrow (P_0)_{\mathcal{C}} \rightarrow F \rightarrow 0$$

where $(P_i)_{\mathcal{C}}$ are projective and finitely generated by Lemma 3.7(4).

- (3) Let M be a finitely n -presented right $\mathcal{A}(\mathcal{C})$ -module. By the previous item, we know that $M_{\mathcal{C}}$ is of type \mathcal{FP}_n .

Since \mathcal{C} is right n -regular, there exists an exact sequence

$$0 \rightarrow P_k \rightarrow P_{k-1} \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M_{\mathcal{C}} \rightarrow 0$$

where each P_i is a finitely generated projective module. Then, the sequence

$$0 \rightarrow S(P_k) \rightarrow S(P_{k-1}) \rightarrow \cdots \rightarrow S(P_1) \rightarrow S(P_0) \rightarrow M \rightarrow 0$$

is exact, and by Lemma 3.7, $S(P_i)$ is also finitely generated and projective. Therefore, $\mathcal{A}(\mathcal{C})$ is a right n -regular and strong right n -coherent ring. The conversely is similar. \square

Example 3.9. Let us consider some examples of \mathbb{Z} -linear categories with finitely many objects.

- (1) Let R be a ring and $G = \mathbb{Z}_n$. Consider $\tilde{R} = \frac{R[t]}{\langle t^n \rangle}$. The category $\mathcal{C}_{\tilde{R}}$ is the category with n objects and

$$\text{hom}_{\mathcal{C}_{\tilde{R}}}(p, q) = \tilde{R}_{q-p} = R$$

Note $\mathcal{A}(\mathcal{C}_{\tilde{R}}) = M_{n \times n}(R)$. If R is a Noetherian ring, then $\mathcal{A}(\mathcal{C}_{\tilde{R}})$ is also Noetherian. By Proposition 3.8, then $\mathcal{C}_{\tilde{R}}$ is Noetherian.

- (2) We recall from [8] that a ring R is said to be (n, d) -ring if every n -presented R -module has projective dimension at most d . Remark that if $n \leq n'$ and $d \leq d'$, then every (n, d) -ring is also a (n', d') -ring.

Let R, S be a finite direct sum of fields and \mathcal{C} be the \mathbb{Z} -linear category with two objects a and b such that $\text{hom}_{\mathcal{C}}(a, b) = \text{hom}_{\mathcal{C}}(b, a) = 0$, $\text{hom}_{\mathcal{C}}(a, a) = R$ and $\text{hom}_{\mathcal{C}}(b, b) = S$. Notice $\mathcal{A}(\mathcal{C}) = R \oplus S$, by [8, Theorem 1.3 (i)] $\mathcal{A}(\mathcal{C})$ is a $(0, 0)$ -ring and hence a Noetherian and regular coherent ring.

- (3) Let G be a finite commutative group. An associative ring R graded by G is

$$R = \bigoplus_{g \in G} R_g$$

such that the multiplication satisfies $R_g R_h \subseteq R_{g+h}$ for all $g, h \in G$. A (left) graded module over R is an R -module M together with a decomposition $M = \bigoplus_{g \in G} M_g$ such that $R_g M_h \subseteq M_{g+h}$. We denote by $R\text{-GrMod}$ the category of graded R -modules. The category \mathcal{C}_R is the \mathbb{Z} -linear category whose set of objects is $\{g : g \in G\}$ and whose morphism groups are given by $\text{hom}_{\mathcal{C}_R}(g, h) = R_{h-g}$. By [9, Lemma 2.2] there is an equivalence between $R\text{-GrMod}$ and the additive functor category $\text{Fun}(\mathcal{C}_R, \text{Ab})$.

4. K-THEORY OF \mathbb{Z} -LINEAR CATEGORIES

4.1. Vanishing negative K-theory. In this section, we have a result of vanishing negative K-theory of \mathbb{Z} -linear categories. Recall from [7, Section 4] the definition of the K-theory spectrum of a \mathbb{Z} -linear category \mathcal{C} , the K-theory spectrum of the ring $\mathcal{A}(\mathcal{C})$ and the map

$$(4.1) \quad \varphi : K(\mathcal{C}) \rightarrow K(\mathcal{A}(\mathcal{C}))$$

which is a natural equivalence in \mathcal{C} , see [7, Proposition 4.2.8].

Theorem 4.2. *Let \mathcal{C} be a \mathbb{Z} -linear category with finitely many objects.*

- (1) *If \mathcal{C} is right regular, then $K_i(\mathcal{C}) = 0$ for all $i < 0$.*
- (2) *If \mathcal{C} is right regular coherent, then $K_{-1}(\mathcal{C}) = 0$.*

Proof. Assume that \mathcal{C} is a right regular category. Then, by Proposition 3.8, $\mathcal{A}(\mathcal{C})$ is a right regular ring. By the fundamental theorem of K-theory, we have $K_i(\mathcal{A}(\mathcal{C})) = 0$ for all $i < 0$. It follows that

$$K_i(\mathcal{C}) \simeq K_i(\mathcal{A}(\mathcal{C})) = 0 \quad \forall i < 0.$$

Now, assume that \mathcal{C} is a right regular coherent category. Then $\mathcal{A}(\mathcal{C})$ is a right regular coherent ring, by Proposition 3.8. By [1, Theorem 3.30], we have $K_{-1}(\mathcal{A}(\mathcal{C})) = 0$, and

$$K_{-1}(\mathcal{C}) \simeq K_{-1}(\mathcal{A}(\mathcal{C})) = 0.$$

□

Corollary 4.3. *Let $\mathcal{D} = \mathcal{C}_{\oplus}$ with \mathcal{C} be a \mathbb{Z} -linear category with finitely many objects.*

- (1) *If \mathcal{C} is right regular, then $K_i(\mathcal{D}) = 0$ for all $i < 0$.*
- (2) *If \mathcal{C} is right regular coherent, then $K_{-1}(\mathcal{D}) = 0$.*

Definition 4.4. A \mathbb{Z} -linear category \mathcal{C} is right *AF-regular* if there is $\{\mathcal{C}_f\}_{f \in F}$ a direct system of right regular \mathbb{Z} -linear categories with finitely many objects such that

$$\mathcal{C} = \text{colim}_{f \in F} \mathcal{C}_f$$

Similarly, we say that \mathcal{C} is right *AF-Noetherian* (*AF-regular coherent*) if

$$\mathcal{C} = \text{colim}_{f \in F} \mathcal{C}_f$$

with \mathcal{C}_i being directed systems of right Noetherian (regular coherent) \mathbb{Z} -linear categories with finitely many objects.

Theorem 4.5. *Let \mathcal{C} be a \mathbb{Z} -linear category.*

- (1) *If \mathcal{C} is right AF-regular, then $K_i(\mathcal{C}) = 0 \forall i < 0$.*
- (2) *If \mathcal{C} is right AF-regular coherent, then $K_{-1}(\mathcal{C}) = 0$.*

Proof. Assume that $\mathcal{C} = \text{colim}_{f \in F} \mathcal{C}_f$. Using the continuity of K -theory, we have $K_i(\mathcal{C}) = \text{colim}_{f \in F} K_i(\mathcal{C}_f)$ for all $i < 0$. The rest of the proof follows from Theorem 4.2. \square

4.2. Fundamental Theorem and homotopy invariance. Let \mathcal{C} be a \mathbb{Z} -linear category. We consider the category $\mathcal{C}[t]$ with the same objects of \mathcal{C} and morphisms are

$$\text{hom}_{\mathcal{C}[t]}(a, b) = \left\{ \sum_{i=0}^n f_i t^i : n \in \mathbb{N} \quad f_i \in \text{hom}_{\mathcal{C}}(a, b) \right\}.$$

Let us also consider the category $\mathcal{C}[t, t^{-1}]$ with the same objects of \mathcal{C} and morphisms are

$$\text{hom}_{\mathcal{C}[t, t^{-1}]}(a, b) = \left\{ \sum_{i=-n}^n f_i t^i : n \in \mathbb{N} \quad f_i \in \text{hom}_{\mathcal{C}}(a, b) \right\}.$$

Remark 4.6. If \mathcal{C} is a \mathbb{Z} -linear category then $\mathcal{C}[t]$ and $\mathcal{C}[t, t^{-1}]$ are \mathbb{Z} -linear categories and

$$\mathcal{A}(\mathcal{C}[t]) \cong \mathcal{A}(\mathcal{C})[t] \quad \mathcal{A}(\mathcal{C}[t, t^{-1}]) \cong \mathcal{A}(\mathcal{C})[t, t^{-1}].$$

Theorem 4.7. *Let \mathcal{C} be a right regular coherent \mathbb{Z} -linear category with finitely many objects then*

$$K_i(\mathcal{C}) \cong K_i(\mathcal{C}[t]) \quad K_{i+1}(\mathcal{C}[t, t^{-1}]) \cong K_{i+1}(\mathcal{C}) \oplus K_i(\mathcal{C}) \quad i \geq 0.$$

Proof. Because (4.1) is a weak equivalence then $K_i(\mathcal{C}) \cong K_i(\mathcal{A}(\mathcal{C}))$. Observe that $\mathcal{A}(\mathcal{C})$ is a right regular coherent ring, by Proposition 3.8. Using [18, Cor 5.3] and [18, Thm 6.1] we obtain

$$K_i(\mathcal{A}(\mathcal{C})) \cong K_i(\mathcal{A}(\mathcal{C})[t])$$

for $i \geq 0$. By Remark 4.6 and using again that (4.1) is a weak equivalence we obtain

$$K_i(\mathcal{C}) \cong K_i(\mathcal{A}(\mathcal{C})) \cong K_i(\mathcal{A}(\mathcal{C})[t]) \cong K_i(\mathcal{A}(\mathcal{C}[t])) \cong K_i(\mathcal{C}[t]) \quad i \geq 0$$

Similarly

$$\begin{aligned} K_{i+1}(\mathcal{C}[t, t^{-1}]) &\cong K_{i+1}(\mathcal{A}(\mathcal{C}[t, t^{-1}])) \\ &\cong K_{i+1}(\mathcal{A}(\mathcal{C})[t, t^{-1}]) \\ &\cong K_{i+1}(\mathcal{A}(\mathcal{C})) \oplus K_i(\mathcal{A}(\mathcal{C})) \quad \text{by [18, Thm 6.1]} \\ &\cong K_{i+1}(\mathcal{C}) \oplus K_i(\mathcal{C}) \quad i \geq 0 \end{aligned}$$

\square

Corollary 4.8. *Let \mathcal{C} be a right AF-regular coherent \mathbb{Z} -linear category then*

$$K_i(\mathcal{C}) \cong K_i(\mathcal{C}[t]) \quad K_{i+1}(\mathcal{C}[t, t^{-1}]) \cong K_{i+1}(\mathcal{C}) \oplus K_i(\mathcal{C}) \quad i \geq 0$$

Denote by $\mathbb{Z}\text{-Cat}$ to the category of \mathbb{Z} -linear categories. Consider \mathcal{F} a full subcategory of $\mathbb{Z}\text{-Cat}$. A functor $F : \mathbb{Z}\text{-Cat} \rightarrow \mathcal{D}$ is \mathcal{F} -homotopy invariant if

$$F(\iota) : F(\mathcal{C}) \rightarrow F(\mathcal{C}[t])$$

is an isomorphism for every \mathcal{C} in \mathcal{F} .

Corollary 4.9. *Let \mathcal{F} the full subcategory of right AF-regular coherent \mathbb{Z} -linear categories. Then K_i is \mathcal{F} -homotopy invariant for $i \geq 0$.*

Using [10, Thm 3.2] and Proposition 3.8 we obtain the following result.

Proposition 4.10. *Let \mathcal{C} be a \mathbb{Z} -linear category with finitely many objects. Suppose that \mathcal{C} is right regular n -coherent. Then*

$$K_i(\mathcal{C}) \simeq K_i(\mathcal{FP}_n(\mathcal{A}(\mathcal{C}))) \quad i \geq 0.$$

ACKNOWLEDGMENTS

The first author was partially supported by ANII. Both authors were partially supported by PEDECIBA, CSIC and by the grant ANII FCE-3-2018-1-148588. We would like to thank Willie Cortiñas and Carlos E. Parra for their interesting comments and discussions. We also thank to the referee for their corrections and comments.

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