



Littlewood-Paley Theorem, Nikolskii Inequality, Besov Spaces, Fourier and Spectral Multipliers on Graded Lie Groups

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Abstract

In this paper we investigate Besov spaces on graded Lie groups. We prove a Nikolskii type inequality (or the Reverse Hölder inequality) on graded Lie groups and as consequence we obtain embeddings of Besov spaces. We prove a version of the Littlewood-Paley theorem on graded Lie groups. The results are applied to obtain embedding properties of Besov spaces and multiplier theorems for both spectral and Fourier multipliers in Besov spaces on graded Lie groups. In particular, we give a number of sufficient conditions for the boundedness of Fourier multipliers in Besov spaces.

Keywords Nikolskii inequality · Besov spaces · Littlewood-Paley theorem · Fourier multipliers · Spectral multipliers · Graded Lie groups · Nilpotent Lie groups

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1 Introduction

In this paper we are interested in advancing the notions and results of harmonic analysis in the setting of graded Lie groups, building up on the fundamental book [12] of Folland and Stein, as well as on more recent developments over the decades, in particular summarised in the recent book [9] by Véronique Fischer and the second author. Indeed, as it was pointed out by Folland and Stein, the setting of homogeneous groups is ideal for the distillation of those results of harmonic analysis that depend only on the group and dilation structures of the underlying space, while the setting of graded Lie groups allows one to also use the techniques coming from the theory of partial differential operators. The difference between the classes of nilpotent, homogeneous and graded Lie groups is rather small, with the majority

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of nilpotent Lie groups allowing for a compatible graded structure, see [9, Chapter 3] for a detailed explanation. In particular, this setting includes the class of stratified groups [11] when the Rockland operator can be chosen to be the sub-Laplacian. We also mention that general Rockland operators on graded Lie groups naturally appear when one is dealing with questions concerning general partial differential operators on manifolds, as their liftings following the celebrated lifting procedure of Rothschild and Stein [24].

Summarising the research of this paper, here we obtain the following results:

- establish the Nikolskii (or the reverse Hölder) inequality in the setting of graded Lie groups in terms of its homogeneous dimension. We believe such a result to be new already on stratified groups, and even on the Heisenberg group;
- prove the Littlewood-Paley theorem on graded Lie groups for the dyadic decomposition associated to positive Rockland operators;
- investigate homogeneous and inhomogeneous Besov spaces in terms of Rockland operators and prove their embedding properties. We show that the Besov spaces in this context are also the interpolation spaces between Sobolev spaces, and prove that they are independent of a particular choice of the Rockland operator used to define them. We also prove their embedding properties with the usual (locally defined) Besov spaces on \mathbb{R}^n ;
- apply these results to establish multiplier theorems for spectral and Fourier multipliers in Besov spaces on graded Lie groups. More precisely, we give negative results on the boundedness of invariant operators in Besov spaces. For Fourier multipliers, we show that the boundedness between L^p -spaces implies the boundedness on Besov spaces and give several applications of this result to Fourier multipliers using Hörmander-Mihlin type and other theorems in this setting.

Nikolskii-type inequalities, following the usual terminology, are, roughly speaking, inequalities between different metrics of the same function (usually trigonometric polynomials). Nikolskii [19] in 1951 proved the inequalities for $1 \leq p \leq q \leq \infty$:

$$\|T_{L_1, L_2, \dots, L_n}\|_{L^q[0, 2\pi]} \leq 2^n [(2\pi)^n L_1 L_2 \cdots L_n]^{\frac{1}{p} - \frac{1}{q}} \|T_{L_1, L_2, \dots, L_n}\|_{L^p[0, 2\pi]}, \quad (1.1)$$

for trigonometric polynomials of the form

$$T_{L_1, L_2, \dots, L_n} = \sum_{k=1}^n \sum_{j_k=-L_k}^{L_k} c_{j_1, j_2, \dots, j_k} e^{i(j_1 x_1 + \dots + j_k x_k)}, \quad (1.2)$$

as well as for entire functions of exponential type. Sometimes such inequality is also called the reverse Hölder inequality in the literature.

On \mathbb{R}^n , the Nikolskii inequality takes the form

$$\|f\|_{L^q(\mathbb{R}^n)} \leq C[\mu[\text{c.h.}[\text{supp}(\widehat{f})]]]^{\frac{1}{p} - \frac{1}{q}} \|f\|_{L^p(\mathbb{R}^n)}, \quad (1.3)$$

for every function $f \in L^p(\mathbb{R}^n)$ with Fourier transform \widehat{f} of compact support. Here, c.h.(E) denotes the convex hull of the set E . Recently, the Nikolskii inequality has been considered in the setting of Lie groups G . In [23], Pesenson has obtained the Nikolskii inequality for symmetric spaces G/K of non-compact type. For the formulation of the Nikolskii inequality on arbitrary compact manifolds we refer the reader to Pesenson [22]. On the other hand, for compact homogeneous manifolds G/K , in [20] the following Nikolskii inequality was obtained:

$$\|T_L\|_{L^q(G/K)} \leq N(\rho L)^{\frac{1}{p} - \frac{1}{q}} \|T_L\|_{L^p(G/K)}, \quad (1.4)$$

for $0 < p < q \leq \infty$; here, if $0 < p \leq 2$, $\rho := 1$, and for $2 < p \leq \infty$, $\rho := [\frac{p}{2}] + 1$, $N(L) \simeq L^{\dim G/K}$ is the Weyl eigenvalue counting function for the elliptic pseudo-differential operator $(I - \mathcal{L}_{G/K})^{\frac{1}{2}}$, where $\mathcal{L}_{G/K}$ is the Laplacian on G/K .

In this paper we prove a Nikolskii type inequality in the framework of graded Lie groups G . We believe this to be new also on stratified groups, even on the Heisenberg group.

This inequality is important in mathematical analysis because it is a fundamental tool in the proof of several embeddings properties of important function spaces such as Besov spaces. The Besov spaces form scales $B_{p,q}^r(G)$ carrying three indices $r \in \mathbb{R}$, $0 < p, q \leq \infty$, and they can be obtained by interpolation of suitable Sobolev spaces. As it was discussed in [10], Sobolev spaces can be defined on \mathbb{R}^n , and on compact and non-compact Lie groups in various equivalent ways. In a recent work of the second author with V. Fischer, Sobolev spaces were introduced on arbitrary graded Lie groups by using positive Rockland operators (see [10]). It is important to mention that Sobolev spaces on stratified Lie groups were introduced by Folland in [11] by using sub-Laplacians, and it was proved (see also [12]) that these spaces are different from their Euclidean counterpart defined by the Fourier transform or by using the local properties of the Laplace operators. The Folland's Sobolev spaces coincide with those introduced in [10] on graded Lie groups in the setting of stratified groups. We also refer to [2] for a number of useful inequalities on graded Lie groups.

In this paper we use positive Rockland operators in order to introduce Besov spaces on graded Lie groups, and later on, we prove that our Besov spaces can be obtained by interpolation of the Sobolev spaces introduced in [10]. For special cases of parameters p, q and r , Besov spaces were also considered by Bahouri, Gérard and Xu in [3]. Apart of the trivial embeddings that can be obtained on the q parameters for Besov spaces $B_{p,q}^r(G)$, the Nikolskii inequality will be a useful tool in order to establish embeddings that involve the parameters r and p .

As a substitute of the Plancherel theorem on $L^2(G)$, in $L^p(G)$ spaces, we prove a version of the Littlewood-Paley theorem and we will use both, our Nikolskii inequality and our Littlewood-Paley theorem in order to get boundedness of Fourier multipliers and spectral multipliers on Besov spaces. For the case of Fourier multipliers we will use the version of the Hörmander-Mihlin theorem in the nilpotent setting [8].

We note that in the case of the sub-Laplacian, a wealth of results is available, to mention only a few, see e.g. Folland [11] and Saka [25] for Sobolev spaces and Besov spaces on stratified groups, respectively; Furioli, Melzi and Veneruso [13] and Alexopoulos [1] for the Littlewood-Paley theorem and Besov spaces, and for spectral multiplier theorems for the sub-Laplacian on Lie groups of polynomial growth, respectively. There are also many results on functions of sub-Laplacians in the fundamental monograph by Varopoulos, Saloff-Coste and Coulhon [26].

The novelty of this paper is that we are working with Rockland operators; these are linear invariant homogeneous hypoelliptic partial differential operators, in view of the Helffer and Nourrigat's resolution of the Rockland conjecture in [16]. Such operators always exist on graded Lie groups and, in fact, the existence of such operators on nilpotent Lie groups does characterise the class of graded Lie groups, see [9, Section 4.1] for further details and references. As the literature concerning the analysis based on sub-Laplacians is immense, we do not review it here, but refer to the introduction in [9] for a more extensive presentation of the subject. Some results of this paper were announced in [5].

This paper is organised as follows. In Section 2 we present some preliminaries on the Fourier analysis of graded Lie groups and its homogeneous structure, and we present positive Rockland operators and elements of their functional calculus. For this we follow [9]. In

Section 3 we prove our version of the Nikolskii inequality for functions defined on graded Lie groups. In Section 4 we prove our version of the Littlewood-Paley theorem. In Section 5 we define Besov spaces and we prove some embedding properties for these spaces. In Section 6 we prove that Besov spaces can be obtained by interpolation of Sobolev spaces in the nilpotent setting and in Section 6.1 we prove further interpolation properties. In Section 7 we show embedding properties between localisation of these Besov spaces and the usual (Euclidean) Besov spaces.

Finally, in Section 8 we study the boundedness of Fourier multipliers and spectral multipliers in Besov spaces. In the case of Fourier multipliers, we prove that $L^p(G)$ -multipliers on graded nilpotent Lie groups generate multipliers in Besov spaces $B_{p,q}^r(G)$. As a consequence of this fact, we end Section 8 with several examples on multipliers.

2 Preliminaries

In this section, we recall some preliminaries on graded and homogeneous Lie groups G . The unitary dual of these groups will be denoted by \widehat{G} . We also present the notion of Rockland operators and Sobolev spaces on G and on the unitary dual \widehat{G} by following [8], to which we refer for further details on constructions presented in this section.

2.1 Homogeneous and Graded Lie Groups

Let G be a graded Lie group. This means that G is a connected and simply connected Lie group whose Lie algebra \mathfrak{g} may be decomposed as the sum of subspaces $\mathfrak{g} = \mathfrak{g}_1 \oplus \mathfrak{g}_2 \oplus \cdots \oplus \mathfrak{g}_s$ such that $[\mathfrak{g}_i, \mathfrak{g}_j] \subset \mathfrak{g}_{i+j}$, and $\mathfrak{g}_{i+j} = \{0\}$ if $i + j > s$. This implies that the group G is nilpotent because the sequence

$$\mathfrak{g}_{(1)} := \mathfrak{g}, \quad \mathfrak{g}_{(n)} := [\mathfrak{g}, \mathfrak{g}_{(n-1)}]$$

defined inductively terminates at $\{0\}$ in a finite number of steps. Examples of such groups are the Heisenberg group \mathbb{H}^n and more generally any stratified groups where the Lie algebra \mathfrak{g} is generated by \mathfrak{g}_1 . The exponential mapping from \mathfrak{g} to G is a diffeomorphism, then, we can identify G with \mathbb{R}^n or $\mathfrak{g}_1 \times \mathfrak{g}_2 \times \cdots \times \mathfrak{g}_s$ as manifolds. Consequently we denote by $\mathcal{S}(G)$ the Schwartz space of functions on G , by considering the identification $G \cong \mathbb{R}^n$. Here, n is the topological dimension of G , $n = n_1 + \cdots + n_s$, where $n_k = \dim \mathfrak{g}_k$. A family of dilations D_r , $r > 0$, on a Lie algebra \mathfrak{g} is a family of linear mappings from \mathfrak{g} to itself satisfying the following two conditions:

- For every $r > 0$, D_r is a map of the form

$$D_r = \text{Exp}(\ln(r)A)$$

for some diagonalisable linear operator A on \mathfrak{g} .

- $\forall X, Y \in \mathfrak{g}$, and $r > 0$, $[D_r X, D_r Y] = D_r[X, Y]$.

We call the eigenvalues of A , $\nu_1, \nu_2, \dots, \nu_n$, the dilations weights or weights of G . A homogeneous Lie group is a connected simply connected Lie group whose Lie algebra \mathfrak{g} is equipped with a family of dilations D_r . In such case, and with the notation above, the homogeneous dimension of G is given by

$$Q = \text{Tr}(A) = \sum_{l=1}^s l \cdot \dim \mathfrak{g}_l.$$

We can transport dilations D_r of the Lie algebra \mathfrak{g} to the group by considering the family of maps

$$\exp_G \circ D_r \circ \exp_G^{-1}, \quad r > 0,$$

where $\exp_G : \mathfrak{g} \rightarrow G$ is the usual exponential function associated to the Lie group G . We denote this family of dilations also by D_r and we refer to them as dilations on the group. If we write $rx = D_r(x)$, $x \in G$, $r > 0$, then a relation on the homogeneous structure of G and the Haar measure dx on G is given by

$$\int_G (f \circ D_r)(x) dx = r^{-Q} \int_G f(x) dx.$$

2.2 The Unitary Dual and the Plancherel Theorem

We will always equip a graded Lie group with the Haar measure dx . For simplicity, we will write $L^p(G)$ for $L^p(G, dx)$. We denote by \widehat{G} the unitary dual of G , that is the set of equivalence classes of unitary, irreducible, strongly continuous representations of G acting in separable Hilbert spaces. The unitary dual can be equipped with the Plancherel measure $d\mu$. So, the Fourier transform of every function $\varphi \in \mathcal{S}(G)$ at $\pi \in \widehat{G}$ is defined by

$$(\mathcal{F}_G \varphi)(\pi) \equiv \widehat{\varphi}(\pi) = \int_G \varphi(x) \pi(x)^* dx,$$

and the corresponding Fourier inversion formula is given by

$$\varphi(x) = \int_{\widehat{G}} \text{Tr}(\pi(x) \widehat{\varphi}(\pi)) d\mu(\pi).$$

In this case, we have the Plancherel identity

$$\|\varphi\|_{L^2(G)} = \left(\int_{\widehat{G}} \text{Tr}(\widehat{\varphi}(\pi) \widehat{\varphi}(\pi)^*) d\mu(\pi) \right)^{\frac{1}{2}} = \|\widehat{\varphi}\|_{L^2(\widehat{G})}.$$

We also denote $\|\widehat{\varphi}\|_{\text{HS}}^2 = \text{Tr}(\widehat{\varphi}(\pi) \widehat{\varphi}(\pi)^*)$ the Hilbert-Schmidt norm of operators. A Fourier multiplier is formally defined by

$$T_\sigma u(x) = \int_{\widehat{G}} \text{Tr}(\pi(x) \sigma(\pi) \widehat{u}(\pi)) d\mu(\pi), \quad u \in C_0^\infty(G), \quad (2.1)$$

where the symbol $\sigma(\pi)$ is defined on the unitary dual \widehat{G} of G . It is easy to see e.g. that for $\sigma \in L^\infty(\widehat{G})$, that is $\|\sigma\|_{L^\infty(\widehat{G})} := \sup_{\pi \in \widehat{G}} \|\sigma(\pi)\|_{\text{op}} < \infty$, $T_\sigma : L^2(G) \rightarrow L^2(G)$ is bounded. For a rather comprehensive treatment of this quantization we refer to [9] and to references therein.

2.3 Homogeneous Linear Operators and Rockland Operators

Let us denote by $\mathcal{D}(G)$ the family of compactly supported smooth functions $C_0^\infty(G)$ endowed with its standard Fréchet structure and let us denote by $\mathcal{D}'(G)$ its topological dual space. A linear operator $T : \mathcal{D}(G) \rightarrow \mathcal{D}'(G)$ is homogeneous of degree $\nu \in \mathbb{C}$ if for every $r > 0$

$$T(f \circ D_r) = r^\nu (Tf) \circ D_r \quad (2.2)$$

holds for every $f \in \mathcal{D}(G)$.

If for every representation $\pi \in \widehat{G}$, $\pi : G \rightarrow U(H_\pi)$, we denote by H_π^∞ the set of smooth vectors (also called Gårding vectors), that is, the space of vectors $v \in H_\pi$ such that the function $x \mapsto \pi(x)v$, $x \in \widehat{G}$, is smooth, a Rockland operator is a left-invariant partial differential operator

$$\mathcal{R} = \sum_{|\alpha| \leq m} a_\alpha X^\alpha : C^\infty(G) \rightarrow C^\infty(G)$$

which is homogeneous of positive degree $\nu = \nu_{\mathcal{R}}$ and such that, for every unitary irreducible non-trivial representation $\pi \in \widehat{G}$, its symbol $\pi(\mathcal{R})$ defined via the Fourier inversion formula by

$$\mathcal{R}f(x) = \int_{\widehat{G}} \text{Tr}[\pi(x)\pi(\mathcal{R})\widehat{f}(\pi)]d\pi, \quad x \in G, \quad (2.3)$$

is injective on H_π^∞ .

It can be shown that a Lie group G is graded if and only if there exists a differential Rockland operator on G , see e.g. [9, Page 172]. If the Rockland operator is formally self-adjoint, then \mathcal{R} and $\pi(\mathcal{R})$ admit self-adjoint extensions on $L^2(G)$ and \mathcal{H}_π , respectively. Now if we preserve the same notation for their self-adjoint extensions and we denote by E and E_π their spectral measures, by functional calculus we have

$$\mathcal{R} = \int_{-\infty}^{\infty} \lambda dE(\lambda), \quad \text{and} \quad \pi(\mathcal{R}) = \int_{-\infty}^{\infty} \lambda dE_\pi(\lambda).$$

We now recall a lemma on dilations on the unitary dual \widehat{G} , which will be useful in our analysis of spectral multipliers. For the proof, see Lemma 4.3 of [8].

Lemma 2.1 *For every $\pi \in \widehat{G}$ let us define $D_r(\pi) = \pi^{(r)}$ by $D_r(\pi)(x) = \pi(rx)$ for every $r > 0$ and $x \in G$. Then, if $f \in L^\infty(\mathbb{R})$ then $f(\pi^{(r)}(\mathcal{R})) = f(r^\nu \pi(\mathcal{R}))$.*

We refer to [9, Chapter 4] and references therein for an exposition of further properties of Rockland operators and their history, and to ter Elst and Robinson [6] for their spectral properties.

2.4 Sobolev Spaces and the Hörmander-Mihlin Theorem

In order to define Sobolev spaces, we choose a positive left-invariant Rockland operator \mathcal{R} of homogeneous degree $\nu > 0$. With notations above one defines Sobolev spaces as follows (c.f [9]).

Definition 2.2 Let $r \in \mathbb{R}$, the homogeneous Sobolev space $\dot{H}^{r,p}(G)$ consists of those $f \in \mathcal{D}'(G)$ satisfying

$$\|f\|_{\dot{H}^{r,p}(G)} := \|\mathcal{R}^{\frac{r}{\nu}} f\|_{L^p(G)} < \infty. \quad (2.4)$$

Analogously, the inhomogeneous Sobolev space $H^{r,p}(G)$ consists of those distributions $f \in \mathcal{D}'(G)$ satisfying

$$\|f\|_{H^{r,p}(G)} := \|(I + \mathcal{R})^{\frac{r}{\nu}} f\|_{L^p(G)} < \infty. \quad (2.5)$$

We record that a homogeneous quasi-norm on a graded Lie group G , is a continuous non-negative function $|\cdot| : G \rightarrow [0, \infty)$ such that: (i) $\forall x \in G$, $|x| = |x^{-1}|$, (ii) $|\cdot|$ is 1-homogeneous, i.e. $\forall x \in G$, $\forall r > 0$, $|rx| = r|x|$, and (iii) $|x| = 0$ if and only if $x = e_G$

is the identity element of G . In terms of the coordinate functions $x \mapsto x_i$ on G , a concrete example of a homogeneous quasi-norm is the following one

$$|x| := \left(\sum_{j=1}^n |x_j|^{\frac{p}{v_j}} \right)^{\frac{1}{p}}, \quad p := 2v_1 \cdots v_n. \quad (2.6)$$

Any two different homogeneous quasi-norms on a graded Lie group are equivalents, see e.g. Proposition 3.1.35 in [9, Page 110].

By using a quasi-norm $|\cdot|$ on G we can introduce for every $r \geq 0$, the inhomogeneous Sobolev space of order r on \widehat{G} , $H^r(\widehat{G})$ which is defined by

$$H^r(\widehat{G}) = \mathcal{F}_G(L^2(G, (1 + |\cdot|)^{2r} dx)),$$

where \mathcal{F}_G is the Fourier transform on the group G . In a similar way, for $r \geq 0$ the homogeneous Sobolev space $\dot{H}^r(\widehat{G})$ is defined by

$$\dot{H}^r(\widehat{G}) = \mathcal{F}_G(L^2(G, |\cdot|^{2r} dx)).$$

As usual if $r = 0$ we denote $L^2(\widehat{G}) = \dot{H}^0(\widehat{G}) = H^0(\widehat{G})$. Characterisations of Sobolev spaces on G and on the unitary dual \widehat{G} in terms of homogeneous norms on G can be found in [8] and [9], respectively.

Finally we present the Hörmander-Mihlin theorem for graded nilpotent Lie groups. This theorem will be useful in our proof of the Littlewood-Paley theorem. The formulation of such result requires a local notion of Sobolev space on the dual space \widehat{G} . We introduce this as follows. Let $s \geq 0$, we say that the field $\sigma = \{\sigma(\pi) : \pi \in \widehat{G}\}$ is locally uniformly in right- $H^s(\widehat{G})$ (resp. left- $H^s(\widehat{G})$) if there exists a positive Rockland operator \mathcal{R} and a function $\eta \in \mathcal{D}(G)$ satisfying

$$\|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}} := \sup_{r>0} \|\{\sigma(\pi^{(r)})\eta(\pi(\mathcal{R}))\}\|_{H^s(\widehat{G})} < \infty, \quad (2.7)$$

respectively,

$$\|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}} := \sup_{r>0} \|\{\eta(\pi(\mathcal{R}))\sigma(\pi^{(r)})\}\|_{H^s(\widehat{G})} < \infty. \quad (2.8)$$

It is important to mention that if ϕ is another function in $\mathcal{D}(0, \infty)$ then (see [8])

$$\|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}} \asymp \|\sigma\|_{H^s, l.u., R, \phi, \mathcal{R}}, \quad \text{and} \quad \|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}} \asymp \|\sigma\|_{H^s, l.u., L, \phi, \mathcal{R}}. \quad (2.9)$$

The following lemma shows how Sobolev spaces on the unitary dual interact with the family of dilations.

Lemma 2.3 *Let $\sigma \in L^2(\widehat{G})$. If $r > 0$ and $s \geq 0$ then*

$$\|\sigma \circ D_r\|_{\dot{H}^s(\widehat{G})} = r^{s-\frac{Q}{2}} \|\sigma\|_{\dot{H}^s(\widehat{G})}. \quad (2.10)$$

This implies that $\sigma \in \dot{H}^s(\widehat{G})$ if and only if for every $r > 0$, $\sigma \circ D_r \in \dot{H}^s(\widehat{G})$. Also, if \mathcal{R}, \mathcal{S} are positive Rockland operators and $\eta, \zeta \in \mathcal{D}(0, \infty)$, $\eta, \zeta \neq 0$, then there exists $C > 0$ such that

$$\|\sigma\|_{H^s, l.u., L, \zeta, \mathcal{S}} \leq C \|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}} \quad (2.11)$$

and

$$\|\sigma\|_{H^s, l.u., R, \zeta, \mathcal{S}} \leq C \|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}}. \quad (2.12)$$

Proof By Lemma 2.1 or Lemma 4.3 of [8] we have

$$\begin{aligned}\|\sigma \circ D_r\|_{\dot{H}^s(\widehat{G})} &= \| |\cdot|^s \mathcal{F}_G^{-1}(\sigma \circ D_r) \|_{L^2(G)} = \| |\cdot|^s r^{-Q} \mathcal{F}_G^{-1}(\sigma)(r^{-1} \cdot) \|_{L^2(G)} \\ &= r^{-\frac{Q}{2}} \| |r \cdot|^s \mathcal{F}_G^{-1}(\sigma) \|_{L^2(G)} \\ &= r^{s-\frac{Q}{2}} \|\sigma\|_{\dot{H}^s(\widehat{G})}.\end{aligned}$$

With the equality above, it is clear that $\sigma \in \dot{H}^s(\widehat{G})$ if and only if for every $r > 0$, $\sigma \circ D_r \in \dot{H}^s(\widehat{G})$. The second part of the Lemma has been shown in Proposition 4.6 of [8].

Now, we state the Hörmander-Mihlin theorem on the graded nilpotent Lie group G (c.f. Theorem 4.11 of [8]):

Theorem 2.4 *Let G be a graded Lie group. Let $\sigma \in L^2(\widehat{G})$. If*

$$\|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}}, \|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}} < \infty, \quad (2.13)$$

with $s > \frac{Q}{2}$, then the corresponding multiplier T_σ extends to a bounded operator on $L^p(G)$ for all $1 < p < \infty$. Moreover

$$\|T_\sigma\|_{\mathcal{L}(L^p(G))} \leq C \max\{\|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}}, \|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}}\}. \quad (2.14)$$

The following remarks will be useful in our formulation of the Littlewood-Paley theorem.

Remark 2.5 (On the proof of the Hörmander-Mihlin Theorem) The proof of the Hörmander-Mihlin theorem (c.f. Theorem 4.11 of [8]) on graded Lie groups uses a suitable Littlewood-Paley decompositions of the symbol. Indeed, for σ satisfying (2.13), the L^p -boundedness of T_σ is proved in Theorem 4.11 of [8], by decomposing

$$T_\sigma = \sum_{j \geq 0} T_j, \quad T_j := T_\sigma \psi_j(\mathcal{R}), \quad (2.15)$$

and using that the right-convolution kernels of the family T_j , $k_j := \mathcal{F}^{-1}(\sigma T_j)$, summed on j , provide the distributional kernel of T , $k = \sum_j k_j$, which agrees with a locally integrable function on $G \setminus \{0\}$, such that, for every $c > 0$,

$$\mathcal{J}_\ell := \sup_{z \in G} \int_{|x| > 4c|z|} |2^{-\ell Q} \kappa_\ell(2^{-\ell} \cdot z^{-1}x) - 2^{-\ell Q} \kappa_\ell(2^{-\ell} \cdot x)| dx, \quad (2.16)$$

satisfies (see [8], p. 26), $\mathcal{J}_\ell \lesssim 2^{-\ell \varepsilon_0} \max\{\|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}}, \|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}}\}$, for some $\varepsilon_0 > 0$, depending only of $c > 0$. The proof of the Hörmander-Mihlin theorem developed by V. Fischer and the second author consists of proving that this kernel estimates are sufficiently good in order that

$$\|T_j\|_{\mathcal{B}(L^p(G))} \leq \mathcal{J}_j \max\{\|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}}, \|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}}\}$$

and consequently

$$\|T\|_{\mathcal{B}(L^p(G))} \lesssim \sum_j 2^{-j \varepsilon_0} \max\{\|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}}, \|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}}\}.$$

In particular, if $T = I$, is the identity operator on $L^p(G)$, $\sigma(\pi) = I_{H_\pi}$, is the identity operator on H_π , and the right convolution kernel κ_ℓ associated with $\psi_\ell(\mathcal{R})$, satisfies the estimate

$$\mathcal{J}_\ell \lesssim 2^{-\ell \varepsilon_0}. \quad (2.17)$$

The inequality (2.17) will be useful in our proof in the Littlewood-Paley theorem.

In the next sections, we present our main results. We start with a formulation of the Nikolskii inequality on graded Lie groups.

3 Nikolskii Inequality on Graded Lie Groups

Let G be a graded Lie group with a family of dilations D_t , $t > 0$. Let \mathcal{R} be a positive Rockland operator of homogeneous degree $\nu > 0$, and for every $L > 0$, let us consider the linear operator $\psi_L(\mathcal{R}^{\frac{1}{\nu}})$, defined by the functional calculus, where $\psi_L(t) := \psi(L^{-1}t)$ and $\psi \in \mathcal{D}(0, \infty) = C_0^\infty(0, \infty)$ is a function with compact support in $[\frac{1}{2}, 2]$. In terms of the spectral resolution $(E(\lambda))_{\lambda \geq 0}$ associated with $\mathcal{R}^{\frac{1}{\nu}}$, we have

$$\mathbf{t}_L f \equiv \psi_L(\mathcal{R}^{\frac{1}{\nu}})f := \int_0^\infty \psi(L^{-1}\lambda) dE(\lambda) f, \quad (3.1)$$

for every $f \in \mathcal{S}(G)$. Then \mathbf{t}_L is a spectral multiplier and

$$\mathcal{F}_G(\mathbf{t}_L f)(\pi) = \left(\int_0^\infty \psi_L(\lambda) dE_\pi(\lambda) \right) \widehat{f}(\pi),$$

where $(E_\pi(\lambda))_{\lambda \geq 0}$ is the spectral resolution of $\pi(\mathcal{R})$. We define

$$E_\pi(L) = \int_0^\infty \phi(L^{-1}\lambda) dE_\pi(\lambda),$$

where $\phi \in \mathcal{D}(0, \infty)$ is a function satisfying $\phi = 1$ on $[\frac{1}{2}, 1]$. In terms of the Fourier inversion formula we have

$$\mathbf{t}_L f(x) = \int_{\widehat{G}} \text{Tr}[\pi(x) E_\pi(L) \psi_L(\pi) \widehat{f}(\pi)] d\pi. \quad (3.2)$$

With the notations above we present our version of the Nikolskii inequality in the following theorem.

Theorem 3.1 *Let G be a graded Lie group of homogeneous dimension Q , and let us consider the operator \mathbf{t}_L as in (3.2). If $1 \leq p \leq q \leq \infty$ then*

$$\|\mathbf{t}_L f\|_{L^q} \leq \|\mathcal{F}_G^{-1}[E_\pi(1)]\|_{L^r} L^{Q(\frac{1}{p} - \frac{1}{q})} \|\mathbf{t}_L f\|_{L^p}, \quad (3.3)$$

where $r = (1 + (1/q - 1/p))^{-1}$. Since $\mathcal{F}_G^{-1}[E_\pi(1)] \in \mathcal{S}(G)$, its L^r -norm is finite.

Remark 3.2 As the reviewer of this paper pointed out, an important difference between (3.3) and (1.3) is that the description in (3.3) is based on the “one dimensional” spectral calculus while the formulation of (1.3) is based on the Fourier transform on \mathbb{R}^n . One reason for this is that our analysis is formulated in terms of the spectral calculus of Rockland operators.

Proof Let us define for every $L > 0$, the function

$$g_L := L^{-Q}(\mathbf{t}_L f) \circ D_{L^{-1}},$$

i.e., $g_L(x) = L^{-Q}(\mathbf{t}_L f)(L^{-1}x)$, $x \in G$. For every $\pi \in \widehat{G}$ we have

$$\begin{aligned}\widehat{g}_L(\pi) &= \int_G L^{-Q}(\mathbf{t}_L f)(L^{-1}x) \pi(x)^* dx \\ &= \int_G (\mathbf{t}_L f)(y) \pi(Ly)^* dy \\ &= \widehat{\mathbf{t}_L f}(\pi(L \cdot)).\end{aligned}$$

In view of Lemma 2.1, (or [8, Lemma 4.3]) we have that

$$\widehat{g}_L(\pi) = \widehat{\mathbf{t}_L f}(\pi(L \cdot)) = \psi(L^{-1}\pi(L^{\frac{1}{p}})(\mathcal{R})) = \psi(L^{-1}L\pi(\mathcal{R})) = \psi(\pi(\mathcal{R})) = E_\pi(1)\psi(\pi(\mathcal{R})),$$

and consequently, we have shown that

$$\widehat{g}_L(\pi) = E_\pi(1)\widehat{g}_L(\pi),$$

in view of the properties of the functional calculus. Hence

$$g_L(x) = g_L * \mathcal{F}_G^{-1}[E_\pi(1)](x), \quad x \in G.$$

By applying Young inequality we have

$$\|g_L\|_{L^q} \leq \|\mathcal{F}_G^{-1}[E_\pi(1)]\|_{L^r} \|g_L\|_{L^p}, \quad (3.4)$$

provided that $\frac{1}{p} + \frac{1}{r} = \frac{1}{q} + 1$. We observe that the condition $1 \leq p \leq q \leq \infty$ implies that $0 \leq \frac{1}{r} = 1 + \frac{1}{q} - \frac{1}{p} \leq 1$ and consequently $1 \leq r \leq \infty$. Observe that for every $a > 0$, we have

$$\begin{aligned}\|g_L\|_{L^a(G)} &= \left(\int_G |g_L(x)|^a dx \right)^{\frac{1}{a}} = \left(\int_G L^{-Qa} |\mathbf{t}_L f(L^{-1}x)|^a dx \right)^{\frac{1}{a}} \\ &= \left(\int_G L^{[Q-Qa]} |\mathbf{t}_L f(y)|^a dy \right)^{\frac{1}{a}} \\ &= L^{Q(\frac{1}{a}-1)} \|\mathbf{t}_L f\|_{L^a}.\end{aligned}$$

So, by the inequality (3.4), we have

$$L^{Q(\frac{1}{q}-1)} \|\mathbf{t}_L f\|_{L^q} \leq \|\mathcal{F}_G^{-1}[E_\pi(1)]\|_{L^r} L^{Q(\frac{1}{p}-1)} \|\mathbf{t}_L f\|_{L^p}. \quad (3.5)$$

Thus, we obtain

$$\|\mathbf{t}_L f\|_{L^q} \leq \|\mathcal{F}_G^{-1}[E_\pi(1)]\|_{L^r} L^{Q(\frac{1}{p}-\frac{1}{q})} \|\mathbf{t}_L f\|_{L^p}. \quad (3.6)$$

This completes the proof.

4 A Vector Valued-Inequality for Littlewood-Paley Decompositions and the Littlewood-Paley Theorem on Graded Lie Groups

The Littlewood-Paley theory provides a partial substitute in L^p spaces for the results derived from the Plancherel theorem. The main notion in the Littlewood-Paley theory is the concept of a dyadic decomposition. Here, the sequence $\{\psi_l\}_{l \in \mathbb{N}_0}$ is a dyadic decomposition, defined as follows: we choose a function $\psi_0 \in C_0^\infty(\mathbb{R})$, $\psi_0(\lambda) = 1$, if $|\lambda| \leq 1$, and $\psi_0(\lambda) = 0$, for

$|\lambda| \geq 2$. For every $j \geq 1$, let us define $\psi_j(\lambda) = \psi_0(2^{-j}\lambda) - \psi_0(2^{-j+1}\lambda)$. For $\psi(\lambda) := \psi_0(\lambda) - \psi_0(2\lambda)$, $\psi_j(\lambda) = \psi(2^{-j}\lambda)$. In particular, we have

$$\sum_{l \in \mathbb{N}_0} \psi_l(\lambda) = 1, \quad \text{for every } \lambda > 0. \quad (4.1)$$

For versions of the Littlewood-Paley theorem for the sub-Laplacian on the Heisenberg group we can refer to Bahouri, Gérard and Xu [3], and for sub-Laplacians on groups of polynomial growth see Furioli, Melzi and Veneruso [13]. Here we prove it for general Rockland operators on graded groups. Now we present the Littlewood-Paley theorem in the form of the following result.

Theorem 4.1 *Let $1 < p < \infty$ and let G be a graded Lie group. If \mathcal{R} is a positive Rockland operator then there exist constants $0 < c_p, C_p < \infty$ depending only on p and ψ_0 such that*

$$c_p \|f\|_{L^p(G)} \leq \left\| \left(\sum_{l=0}^{\infty} |\psi_l((1 + \mathcal{R})^{\frac{1}{p}}) f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \leq C_p \|f\|_{L^p}, \quad (4.2)$$

holds for every $f \in L^p(G)$. Moreover, for $p = 1$, there exists a constant $C > 0$ independent of $f \in L^1(G)$ and $t > 0$, such that

$$\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1 + \mathcal{R})^{\frac{1}{p}}) f(x)|^2 \right)^{\frac{1}{2}} > t \right\} \right| \leq \frac{C}{t} \|f\|_{L^1(G)}. \quad (4.3)$$

For the proof of Theorem 4.1, we will assume for a moment the following theorem.

Theorem 4.2 *Let $1 < p, r < \infty$ and let G be a graded Lie group. If \mathcal{R} is a positive Rockland operator then there exist constants $C_p > 0$ depending only on p and ψ_0 , such that*

$$\left\| \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1 + \mathcal{R})^{\frac{1}{p}}) f_{\ell}|^r \right)^{\frac{1}{r}} \right\|_{L^p(G)} \leq C_p \left\| \left(\sum_{\ell=0}^{\infty} |f_{\ell}(x)|^r \right)^{\frac{1}{r}} \right\|_{L^p(G)} =: C_p \|\{f_{\ell}\}\|_{L^p(G, \ell^r(\mathbb{N}_0))}. \quad (4.4)$$

Moreover, for $p = 1$, there exists a constant $C > 0$ independent of $\{f_{\ell}\} \in L^1(G, \ell^r(\mathbb{N}_0))$ and $t > 0$, such that

$$\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1 + \mathcal{R})^{\frac{1}{p}}) f_{\ell}(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \leq \frac{C}{t} \|\{f_{\ell}\}\|_{L^1(G, \ell^r(\mathbb{N}_0))}. \quad (4.5)$$

Proof of Theorem 4.1 First we will prove that for every positive function $f \in L^p(G) \cap L^1(G)$, the estimate

$$\left\| \left(\sum_{l=0}^{\infty} |\psi_l((1 + \mathcal{R})^{\frac{1}{p}}) f|^2 \right)^{\frac{1}{2}} \right\|_{L^p(G)} \leq C_p \|f\|_{L^p(G)}, \quad (4.6)$$

holds true for every $1 < p < \infty$, and the inequality in the right hand side of (4.6) can be extended to general $f \in L^p(G)$ by the density argument. We will employ an argument of interpolation. First, let us prove (4.3). Indeed, it is equivalent to the fact that the vector-valued operator

$$W(f) = \{\psi_{\ell}((1 + \mathcal{R})^{1/p}) f\}_{\ell=0}^{\infty}, \quad (4.7)$$

admits a bounded extension from $L^1(G)$ into $L^{1,\infty}(G, \ell^2(\mathbb{N}_0^n))$. In view of the almost orthogonality of the functions $x \mapsto f_\ell(x) := \psi_\ell((1 + \mathcal{R})^{1/\nu})f(x)$, on $L^2(G)$, which is a consequence of the following property on the supports of the functions ψ_ℓ ,

$$\text{supp}(\psi_\ell) \cap \text{supp}(\psi_{\ell'}) = \emptyset, \quad |\ell - \ell'| \geq 2, \quad (4.8)$$

we have

$$\begin{aligned} \int_G \sum_{\ell=0}^{\infty} |\psi_\ell((1 + \mathcal{R})^{1/\nu})f(x)|^2 dx &\asymp \sum_{\ell=0}^{\infty} \int_G |\psi_\ell((1 + \mathcal{R})^{1/\nu})f(x)|^2 dx \\ &\asymp \int_G \left| \sum_{\ell=0}^{\infty} \psi_\ell((1 + \mathcal{R})^{1/\nu})f(x) \right|^2 dx \\ &= \int_G |f(x)|^2 dx, \end{aligned}$$

which implies that W admits a bounded extension from $L^2(G)$ into $L^2(G, \ell^2(\mathbb{N}_0^n))$. So, if we prove (4.3), interpolating with the $L^2(G) - L^2(G, \ell^2(\mathbb{N}_0^n))$ -boundedness of W , we obtain that W extends to a bounded operator from $L^p(G)$ into $L^p(G, \ell^2(\mathbb{N}_0^n))$, for all $1 < p \leq 2$. We will then extend the boundedness of W for all $2 \leq p < \infty$, by using the duality argument. So, our proof consists of the following steps.

- **Step 0.** Assume that f is a non-negative function in $L^p(G)$.
- **Step 1.** Prove the weak $(1, 1)$ -inequality (4.3).
- **Step 2.** Interpolation between (4.3) and the boundedness of W from $L^2(G)$ into $L^2(G, \ell^2(\mathbb{N}_0^n))$, in order to prove (4.6) for all $1 < p \leq 2$.
- **Step 3.** Apply the duality argument for extending (4.6) for all $2 \leq p < \infty$.
- **Step 4.** Proof of the left hand side of (4.2).
- **Step 5.** Extend (4.2) and (4.3) to general real-valued functions in $L^p(G)$.
- **Step 6.** Extend (4.2) and (4.3) to general complex-valued functions in $L^p(G)$.

Step 1. Apply the Calderón-Zygmund decomposition Lemma to the non-negative function $f \in L^p(G) \cap L^1(G) \subset L^1(G)$, under the identification $G \simeq \mathbb{R}^n$, (see, e.g. Hebisch [15]) in order to obtain a disjoint collection $\{I_j\}_{j=0}^{\infty}$ of disjoint open sets such that

- $f(x) \leq t$, for a.e. $x \in G \setminus \bigcup_{j \geq 0} I_j$,
- $\sum_{j \geq 0} |I_j| \leq \frac{C}{t} \|f\|_{L^1(G)}$, and
- $t|I_j| \leq \int_{I_j} f(x) dx \leq 2|I_j|t$, for all j .

Moreover, for every j , let us define R_j by

$$R_j := \sup\{R > 0 : B(z_j, R) \subset I_j, \text{ for some } z_j \in I_j\}, \quad (4.9)$$

where $B(z_j, R) = \{x \in I_j : |z_j^{-1}x| < R\}$. Then, we can assume that every I_j is diffeomorphic to an open cube on \mathbb{R}^n , that it is bounded, and that $I_j \subset B(z_j, 2R_j)$, where $z_j \in I_j$ (see Hebisch [15]).

Remark 4.3 Before of continuing with the proof note that by assuming $f(e_G) > t$, (this just re-defining $f \in L^p(G) \cap L^1(G)$ at the identity element) we should have that

$$e_G \in \bigcup_j I_j, \quad (4.10)$$

because $f(x) \leq t$, for a.e. $x \in G \setminus \bigcup_{j \geq 0} I_j$.

Let us define, for every $x \in I_j$,

$$g(x) := \frac{1}{|I_j|} \int_{I_j} f(y) dy, \quad b(x) = f(x) - g(x), \quad (4.11)$$

and for $x \in G \setminus \cup_{j \geq 0} I_j$,

$$g(x) = f(x), \quad b(x) = 0. \quad (4.12)$$

Observe that for every $x \in I_j$,

$$|g(x)| = \left| \frac{1}{|I_j|} \int_{I_j} f(y) dy \right| \leq 2t.$$

In view of the Minkowski inequality, we deduce that

$$\begin{aligned} & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) f(x)|^2 \right)^{\frac{1}{2}} > t \right\} \right| \\ & \leq \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) g(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ & \quad + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|. \end{aligned}$$

By the Chebyshev inequality, we have

$$\begin{aligned} & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) f(x)|^2 \right)^{\frac{1}{2}} > t \right\} \right| \\ & \leq \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) g(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ & \quad + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ & = \left| \left\{ x \in G : \sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) g(x)|^2 > \frac{t^2}{2^2} \right\} \right| \\ & \quad + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ & \leq \frac{2^2}{t^2} \int_G \sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) g(x)|^2 dx + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu}) b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|. \end{aligned}$$

In view of the almost orthogonality of the functions $x \mapsto g_\ell(x) := \psi_\ell((1 + \mathcal{R})^{1/\nu})g(x)$, on $L^2(G)$, we have

$$\begin{aligned} \int_G \sum_{\ell=0}^{\infty} |\psi_\ell((1 + \mathcal{R})^{1/\nu})g(x)|^2 dx &= \sum_{\ell=0}^{\infty} \int_G |\psi_\ell((1 + \mathcal{R})^{1/\nu})g(x)|^2 dx \\ &\asymp \int_G \left| \sum_{\ell=0}^{\infty} \psi_\ell((1 + \mathcal{R})^{1/\nu})g(x) \right|^2 dx = \int_G |g(x)|^2 dx. \end{aligned}$$

The estimate

$$\begin{aligned} \|g\|_{L^2(G)}^2 &= \int_G |g(x)|^2 dx = \sum_j \int_{I_j} |g(x)|^2 dx + \int_{G \setminus \cup_j I_j} |g(x)|^2 dx \\ &= \sum_j \int_{I_j} |g(x)|^2 dx + \int_{G \setminus \cup_j I_j} |f(x)|^2 dx \\ &\leq \sum_j \int_{I_j} (2t)^2 dx + \int_{G \setminus \cup_j I_j} f(x)^2 dx \lesssim t^2 \sum_j |I_j| + \int_{G \setminus \cup_j I_j} f(x)f(x) dx \\ &\leq t^2 \times \frac{C}{t} \|f\|_{L^1(G)} + t \int_{G \setminus \cup_j I_j} f(x) dx \lesssim t \|f\|_{L^1(G)}, \end{aligned}$$

implies that,

$$\begin{aligned} &\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_\ell((1 + \mathcal{R})^{1/\nu})f(x)|^2 \right)^{\frac{1}{2}} > t \right\} \right| \leq \frac{4}{t} \|f\|_{L^1(G)} \\ &+ \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_\ell((1 + \mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|. \end{aligned}$$

Taking into account that $b \equiv 0$ on $G \setminus \cup_j I_j$, we have that

$$b = \sum_k b_k, \quad b_k(x) = b(x) \cdot 1_{I_k}(x). \quad (4.13)$$

Let us assume that I_j^* is a open set, such that $I_j \subset I_j^*$, and $|I_j^*| = K|I_j|$ for some $K > 0$, and $\text{dist}(\partial I_j^*, \partial I_j) \geq 4c \text{dist}(\partial I_j, e_G)$, where $c > 0$ and e_G is the identity element of G . So, by the Minkowski inequality we have,

$$\begin{aligned} &\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_\ell((1 + \mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ &= \left| \left\{ x \in \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_\ell((1 + \mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \end{aligned}$$

$$\begin{aligned}
 & + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\
 & \leq \left| \left\{ x \in G : x \in \cup_j I_j^* \right\} \right| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|.
 \end{aligned}$$

Since

$$\left| \left\{ x \in G : x \in \cup_j I_j^* \right\} \right| \leq \sum_j |I_j^*|,$$

we have

$$\begin{aligned}
 & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\
 & \leq \sum_j |I_j^*| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\
 & = K \sum_j |I_j| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\
 & \leq \frac{CK}{t} \|f\|_{L^1(G)} + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|.
 \end{aligned}$$

The Chebyshev inequality implies that

$$\begin{aligned}
 & \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\
 & \leq \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} dx \\
 & = \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} \left| \left(\psi_{\ell}((1+\mathcal{R})^{1/\nu}) \left(\sum_k b_k \right) \right) (x) \right|^2 \right)^{\frac{1}{2}} dx \\
 & = \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \left\| \{ (\sum_k \psi_{\ell}((1+\mathcal{R})^{1/\nu})b_k(x)) \}_{\ell=0}^{\infty} \right\|_{\ell^2(\mathbb{N}_0)} dx \\
 & \leq \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \sum_k \left\| \{ (\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_k(x)) \}_{\ell=0}^{\infty} \right\|_{\ell^2(\mathbb{N}_0)} dx \\
 & = \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |(\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_k)(x)|^2 \right)^{\frac{1}{2}} dx.
 \end{aligned}$$

If κ_ℓ is the right convolution kernel of $\psi_\ell((1 + \mathcal{R})^{1/\nu})$, from the inequality,

$$\left(\sum_{\ell=0}^{\infty} |(\psi_\ell((1 + \mathcal{R})^{1/\nu})b_k)(x)|^2 \right)^{\frac{1}{2}} \leq \sum_{\ell=0}^{\infty} |(\psi_\ell((1 + \mathcal{R})^{1/\nu})b_k)(x)| \quad (4.14)$$

we deduce

$$\begin{aligned} & \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_\ell((1 + \mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ & \leq \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} |(\psi_\ell((1 + \mathcal{R})^{1/\nu})b_k)(x)| dx \\ & = \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} |b_k * \kappa_\ell(x)| dx \\ & = \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} \left| \int_{I_k} b_k(z) \kappa_\ell(z^{-1}x) dz \right| dx. \end{aligned}$$

By using that $\int_{I_k} b_k(z) dz = 0$, we have

$$\begin{aligned} & \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} \left| \int_{I_k} b_k(z) \kappa_\ell(z^{-1}x) dz \right| dx \\ & = \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} \left| \int_{I_k} b_k(z) \kappa_\ell(z^{-1}x) dz - \kappa_\ell(x) \int_{I_k} b_k(z) dz \right| dx \\ & = \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} \left| \int_{I_k} (\kappa_\ell(z^{-1}x) - \kappa_\ell(x)) b_k(z) dz \right| dx \\ & \leq \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} \int_{I_k} |(\kappa_\ell(z^{-1}x) - \kappa_\ell(x)) b_k(z)| dz dx \\ & = \frac{2}{t} \sum_k \int_{I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| dx |b_k(z)| dz. \end{aligned}$$

If we assume for a moment that

$$M = \sup_k \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_\ell(z^{-1}x) - \kappa_\ell(x)| dx < \infty, \quad (4.15)$$

then we have

$$\begin{aligned} & \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \leq \frac{2M}{t} \sum_k \int_{I_k} |b_k(z)| dz \\ &= \frac{2M}{t} \|b\|_{L^1(G)} \\ &\leq \frac{6M}{t} \|f\|_{L^1(G)}. \end{aligned}$$

So, if we prove (4.15) we obtain the weak (1,1) inequality (4.3) and we finish the first step of the proof. The proof of (4.15) lies in the estimates of the Calderón-Zygmund kernel of every operator $\psi_{\ell}(\mathcal{R})$. Because $\text{dist}(\partial I_j^*, \partial I_j) \geq 4c \text{dist}(\partial I_j, e_G)$, for $x \in G \setminus \cup_j I_j^*$, for $z \in I_k$, $4c|z| = 4c \times \text{dist}(z, e_G) \lesssim \text{dist}(\partial I_k^*, \partial I_k) \leq |x|$. Indeed, fix $\varepsilon > 0$, and let us take $w \in \partial I_k$, and $w' \in \partial I_k^*$ such that $d(w, w') \leq \text{dist}(\partial I_k, \partial I_k^*) + \varepsilon$. Then, from the triangle inequality, we have

$$\begin{aligned} & d(z, e_G) \\ &\leq d(z, w) + d(w, w') + d(w', e_G) \leq \text{diam}(I_k) + \text{dist}(\partial I_k, \partial I_k^*) + \text{dist}(\partial I_k^*, e_G) + \varepsilon \\ &\lesssim \text{diam}(I_k) + \text{dist}(\partial I_k, \partial I_k^*) + \text{dist}(\partial I_k, e_G) + \varepsilon \\ &\lesssim \text{diam}(I_k) + \text{dist}(\partial I_k, \partial I_k^*) + \frac{1}{4c} \text{dist}(\partial I_k, \partial I_k^*) + \varepsilon \\ &\asymp \text{dist}(\partial I_k, \partial I_k^*) + \varepsilon, \end{aligned} \tag{4.16}$$

where in the last line we have assumed that $\text{diam}(I_k) \asymp \text{dist}(\partial I_k, \partial I_k^*)$, (with constants of proportionality independent in k) and that $\text{dist}(\partial I_k, \partial I_k^*)$ is proportional to R_k in view of the relation $|I_k^*| = K|I_k|$. Assuming (4.16), one has that for all $\varepsilon > 0$, $d(z, e_G) \lesssim \text{dist}(\partial I_k, \partial I_k^*) + \varepsilon$, which implies that

$$d(z, e_G) \lesssim \text{dist}(\partial I_k, \partial I_k^*). \tag{4.17}$$

To show that the proportionality constant in (4.17) is uniform in k , let us recall the definition of the radii R'_k in (4.9), that $B(z_k, R_k) \subset I_k \subset B(z_k, 2R_k)$, and that $B(z_k, R_k/C) \subset I_k^* \subset B(z_k, CR_k)$ for some $C > 2$ independent of k , where for any k , $z_k \in I_k$. From this remark observe that:

- The condition $B(z_k, R_k) \subset I_k \subset B(z_k, 2R_k)$, implies that $2R_k \leq \text{diam}(I_k) \leq 4R_k$.
- That $B(z_k, R_k) \subset I_k \subset I_k^* \subset B(z_k, CR_k)$, implies that

$$\text{dist}(\partial I_k, \partial I_k^*) \leq \text{dist}(\partial B(z_k, R_k), \partial B(z_k, CR_k)) = (C-1)R_k.$$

On the other hand, by observing that in every step above we can replace $I_k^* := B(z_k, CR_k)$, in view of the inclusion

$$I_k \subset B(z_k, 2R_k) \subset I_k^* := B(z_k, CR_k),$$

we have

$$(C-2)R_k = \text{dist}(\partial I_k^*, \partial B(z_k, 2R_k)) \leq \text{dist}(\partial I_k, \partial I_k^*).$$

Consequently,

$$\begin{aligned} \text{diam}(I_k) &\asymp R_k \asymp \text{dist}(\partial B(z_k, 2R_k), \partial B(z_k, CR_k)) \\ &\asymp \text{dist}(\partial I_k, \partial I_k^*). \end{aligned}$$

To show that $\text{dist}(\partial I_k^*, \partial I_k) \leq |x|$, observe that from Remark 4.10, $e_G \in \cup_j I_j$, and because of $x \in G \setminus \cup_j I_j$,

$$\text{dist}(\partial I_k^*, \partial I_k) \lesssim \text{diam}(\cup_j I_j) \lesssim d(x, e_G) = |x|.$$

So, we have guaranteed the existence of a positive constant, which we again denote by $c > 0$, such that,

$$\{x \in G : x \in G \setminus \cup_j I_j^*\} \subset \{x \in G : \text{for all } z \in I_k, \quad 4c|z| \leq |x|\}.$$

So, by a suitable variable change of variables and by using (2.17), we have

$$\begin{aligned} M_k &:= \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_{\ell}(z^{-1}x) - \kappa_{\ell}(x)| dx \\ &= \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |2^{-\ell} Q_{\kappa_{\ell}}(2^{-\ell} \cdot z^{-1}x) - 2^{-\ell} Q_{\kappa_{\ell}}(2^{-\ell} \cdot x)| dx \\ &\leq \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{|x| > 4c|z|} |2^{-\ell} Q_{\kappa_{\ell}}(2^{-\ell} \cdot z^{-1}x) - 2^{-\ell} Q_{\kappa_{\ell}}(2^{-\ell} \cdot x)| dx \\ &\leq \sum_{\ell=0}^{\infty} \sup_{z \in G} \int_{|x| > 4c|z|} |2^{-\ell} Q_{\kappa_{\ell}}(2^{-\ell} \cdot z^{-1}x) - 2^{-\ell} Q_{\kappa_{\ell}}(2^{-\ell} \cdot x)| dx \\ &= \sum_{\ell=0}^{\infty} \mathcal{J}_{\ell} \lesssim \sum_{\ell=0}^{\infty} 2^{-\ell \varepsilon_0} = O(1). \end{aligned}$$

Because

$$M_k := \sup_{z \in I_k} \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_{\ell}(z^{-1}x) - \kappa_{\ell}(x)| dx \lesssim \sum_{\ell=0}^{\infty} 2^{-\ell \varepsilon_0},$$

with the right hand side of the inequality being independent of k , we conclude that M in (4.15) is finite.

According to Step 2, the vector-valued interpolation between the (4.3) and the boundedness of W from $L^2(G)$ into $L^2(G, \ell^2(\mathbb{N}_0^n))$, allows us to conclude (4.6) for all $1 < p \leq 2$.

Step 3. Let us assume that $2 \leq p < \infty$, and let $f \in L^p(G, \ell^2)$. So, for a.e. $x \in G$, $f(x)$ is a sequence in ℓ^2 . If p' is the conjugate exponent of p , $1 < p' \leq 2$, by using that $\psi_j((1 + \mathcal{R})^{\frac{1}{p}})$ is self-adjoint, we have

$$\begin{aligned} &\|Wf\|_{L^p(G, \ell^2)} \\ &= \sup_{\|h\|_{L^{p'}(G, \ell^2)}=1} \int_G (Wf(x), h(x))_{\ell^2} dx = \sup_{\|h\|_{L^{p'}(G, \ell^2)}=1} \int_G \sum_{j=0}^{\infty} \psi_j((1 + \mathcal{R})^{\frac{1}{p}}) f(x) h_j(x) dx \\ &= \sup_{\|h\|_{L^{p'}(G, \ell^2)}=1} \int_G \sum_{j=0}^{\infty} f(x) \psi_j((1 + \mathcal{R})^{1/p})^* h_j(x) dx \\ &= \sup_{\|h\|_{L^{p'}(G, \ell^2)}=1} \int_G (f(x), Wh(x))_{\ell^2} dx, \quad Wh(x) := \{\psi_{\ell}((1 + \mathcal{R})^{1/p}) h_{\ell}\}_{\ell=0}^{\infty}, \\ &\leq \|f\|_{L^p(G, \ell^2)} \sup_{\|h\|_{L^{p'}(G, \ell^2)}=1} \|Wh\|_{L^{p'}(G, \ell^2)}. \end{aligned}$$

By using Theorem 4.2 with $r = 2$, for $\|h\|_{L^{p'}(G, \ell^2)} = 1$, we have that

$$\begin{aligned} \|Wh\|_{L^{p'}(G, \ell^2)} &= \left\| \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{\frac{1}{\nu}})h_{\ell}|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(G)} \leq C_{p'} \left\| \left(\sum_{\ell=0}^{\infty} |h_{\ell}(x)|^2 \right)^{\frac{1}{2}} \right\|_{L^{p'}(G)} \\ &= C_{p'}, \end{aligned}$$

where the constant $C_{p'} > 0$, came from Theorem 4.2. Consequently, we have proved (4.6) for all $2 \leq p < \infty$, in view of the boundedness of W from $L^p(G)$ into $L^p(G, \ell^2)$ for all $2 \leq p < \infty$.

Step 4. The proof of the left hand side of (4.2) for non-negative f is as follows. Now, let us denote by $(E(\lambda))_{\lambda \geq 0}$ the spectral resolution associated to $(1+\mathcal{R})^{1/\nu}$, and for every $\pi \in \widehat{G}$ denote by $(E_{\pi}(\lambda))_{\lambda \geq 0}$ the spectral resolution of $\pi((1+\mathcal{R})^{1/\nu})$. We observe that by duality

$$\begin{aligned} \|f\|_{L^p(G)} &\lesssim \sup\{|\int_G f(x)g(x)dx| : g \in \mathcal{D}(G), g \geq 0, \|g\|_{L^{p'}} = 1\} \\ &= \sup\{|\int_G \sum_{l \in \mathbb{N}_0} [\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})f](x)g(x)dx| : g \in \mathcal{D}(G), g \geq 0, \|g\|_{L^{p'}} = 1\} \\ &= \sup\{|\int_G \sum_{l \in \mathbb{N}} [E^{(l)}\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})f](x)g(x)dx \\ &\quad + \int_G [E^{(0)}\psi_0((1+\mathcal{R})^{\frac{1}{\nu}})f](x)g(x)dx| : g \in \mathcal{D}(G), g \geq 0, \|g\|_{L^{p'}} = 1\} \\ &= \sup\{|\int_G \sum_{l \in \mathbb{N}} [\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})f](x)E^{(l)}g(x)dx \\ &\quad + \int_G [\psi_0((1+\mathcal{R})^{\frac{1}{\nu}})f](x)E^{(0)}g(x)dx| : g \in \mathcal{D}(G), g \geq 0, \|g\|_{L^{p'}} = 1\}, \end{aligned}$$

where $E^{(l)} := \psi_{l-1}((1+\mathcal{R})^{\frac{1}{\nu}}) + \psi_l((1+\mathcal{R})^{\frac{1}{\nu}}) + \psi_{l+1}((1+\mathcal{R})^{\frac{1}{\nu}})$, for $l \geq 1$, and $E^{(0)} := \psi_0((1+\mathcal{R})^{1/\nu}) + \psi_1((1+\mathcal{R})^{1/\nu}) + \psi_2((1+\mathcal{R})^{1/\nu})$. Consequently, we have

$$\begin{aligned} &\|f\|_{L^p(G)} \\ &\leq \sup_{g \in \mathcal{D}(G), g \geq 0, \|g\|_{L^{p'}} = 1} \left\{ \int_G \left| \sum_{l \in \mathbb{N}_0} |[\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})f](x)|^2 \right|^{\frac{1}{2}} \left| \sum_{l \in \mathbb{N}_0} |E^{(l)}g(x)|^2 \right|^{\frac{1}{2}} dx \right\} \\ &\leq \sup_{g \in \mathcal{D}(G), g \geq 0, \|g\|_{L^{p'}} = 1} \left\{ \left\| \left[\sum_{l \in \mathbb{N}_0} |[\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})f](x)|^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)} \left\| \left[\sum_{l \in \mathbb{N}_0} |E^{(l)}g(x)|^2 \right]^{\frac{1}{2}} \right\|_{L^{p'}(G)} \right\}. \end{aligned}$$

Because

$$\left\| \left[\sum_{l \in \mathbb{N}_0} |E^{(l)}g(x)|^2 \right]^{\frac{1}{2}} \right\|_{L^{p'}(G)} \asymp \left\| \left[\sum_{l \in \mathbb{N}_0} |\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})g(x)|^2 \right]^{\frac{1}{2}} \right\|_{L^{p'}(G)} \lesssim \|g\|_{L^{p'}} = 1, \quad (4.18)$$

we obtain

$$\|f\|_{L^p(G)} \lesssim \left\| \left[\sum_{l \in \mathbb{N}_0} |[\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})f](x)|^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)}$$

for all $1 < p < \infty$.

Step 5. Let us assume that $f \in L^p(G)$ is real-valued. Decompose $f = f^+ - f^-$, as the difference of two non-negative functions, where $f^+, f^- \in L^p(G)$, and $|f| = f^+ + f^-$. Because, $f^+, f^- \leq |f|$, the Minkowski inequality implies

$$\begin{aligned} & \left\| \left[\sum_{l \in \mathbb{N}_0} |[\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})f](x)|^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)} \\ & \leq \left\| \left[\sum_{l \in \mathbb{N}_0} |[\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})f^+](x)|^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)} + \left\| \left[\sum_{l \in \mathbb{N}_0} |[\psi_l((1+\mathcal{R})^{\frac{1}{\nu}})f^-](x)|^2 \right]^{\frac{1}{2}} \right\|_{L^p(G)} \\ & \leq C_p (\|f^+\|_{L^p(G)} + \|f^-\|_{L^p(G)}) \leq 2C_p \|f\|_{L^p(G)}. \end{aligned}$$

So, we have proved the right hand side of (4.2). For the proof of the left hand side, we only need to repeat the proof made in Step 4. The proof for the weak (1,1) inequality is similar. Indeed,

$$\begin{aligned} & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})f(x)|^2 \right)^{\frac{1}{2}} > t \right\} \right| \\ & \leq \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})f^+(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ & \quad + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})f^-(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\ & \leq \frac{2C}{t} \|f^+\|_{L^1(G)} + \frac{2C}{t} \|f^-\|_{L^1(G)} \\ & \leq \frac{4C}{t} \|f\|_{L^1(G)}. \end{aligned}$$

A similar analysis can be used for the proof of Step 6. So, the proof of the Littlewood-Paley Theorem is complete.

We end this section with the proof of the vector-valued inequalities presented in Theorem 4.2.

Proof of Theorem 4.2 Define the vector-valued operator

$$W : L^2(G, \ell^2(\mathbb{N}_0))_f \rightarrow L^2(G, \ell^2(\mathbb{N}_0)), \quad (4.19)$$

by

$$W(\{f_{\ell}\}_{\ell=0}^{\infty}) = \{\psi_{\ell}((1+\mathcal{R})^{1/\nu})f_{\ell}\}_{\ell=0}^{\infty}, \quad (4.20)$$

where $L^2(G, \ell^2(\mathbb{N}_0))_f$ is the set of sequences $\{f_{\ell}\}_{\ell=0}^{\infty}$, with compact support in the ℓ -variables. We claim that $W : L^2(G, \ell^2(\mathbb{N}_0)) \rightarrow L^2(G, \ell^2(\mathbb{N}_0))$ extends to a bounded

operator. Indeed, let us observe that

$$\begin{aligned} \|W(\{f_\ell\}_{\ell=0}^\infty)\|_{L^2(\ell^2)}^2 &= \int_G \sum_\ell |\psi_\ell((1+\mathcal{R})^{1/\nu})f_\ell(x)|^2 dx \leq C^2 \sum_\ell \|f_\ell\|_{L^2(G)}^2 \\ &= \|\{f_\ell\}_{\ell=0}^\infty\|_{L^2(\ell^2)}^2, \end{aligned}$$

where

$$\begin{aligned} C &= \sup_{\ell \geq 0} \|\psi_\ell((1+\mathcal{R})^{1/\nu})\|_{\mathcal{B}(L^2(G))} = \sup_{\ell \geq 0} \|\psi_\ell(1+(\pi(\mathcal{R}))^{1/\nu})\|_{\mathcal{B}(L^2(\widehat{G}))} \\ &\leq \sup_{\ell \geq 0, \lambda \geq 0} |\psi(2^{-\ell}\lambda)| = O(1). \end{aligned}$$

Now, we will assume for a moment that for every $\ell \in \mathbb{N}_0$, the family of operators $\{\psi_\ell((1+\mathcal{R})^{1/\nu})\}$ is uniformly bounded from $L^1(G)$ into $L^{1,\infty}(G)$, this is, every operator $\psi_\ell((1+\mathcal{R})^{1/\nu})$ is of weak $(1, 1)$ type with the $(L^1(G), L^{1,\infty}(G))$ -operator norm bounded with a constant independent of ℓ ,

$$\Omega := \sup_\ell \|\psi_\ell((1+\mathcal{R})^{1/\nu})\|_{\mathcal{B}(L^1(G), L^{1,\infty}(G))} < \infty. \quad (4.21)$$

This assumption allows us to show that

$$W : L^1(G, \ell^1(\mathbb{N}_0)) \rightarrow L^{1,\infty}(G, \ell^1(\mathbb{N}_0)), \quad (4.22)$$

extends to a bounded operator. Indeed, if we define $e_{\ell'}(\ell) = \delta_{\ell, \ell'}$, observe that

$$\begin{aligned} \|W(\{f_\ell\}_{\ell=0}^\infty)\|_{L^{1,\infty}(G, \ell^1(\mathbb{N}_0))} &= \|\{\psi_\ell((1+\mathcal{R})^{1/\nu})f_\ell\}_{\ell=0}^\infty\|_{L^{1,\infty}(G, \ell^1(\mathbb{N}_0))} \\ &= \left\| \sum_{\ell'=0}^\infty \{e_{\ell'}(\ell)\psi_\ell((1+\mathcal{R})^{1/\nu})f_\ell\}_{\ell=0}^\infty \right\|_{L^{1,\infty}(G, \ell^1(\mathbb{N}_0))} \\ &\lesssim \sum_{\ell'=0}^\infty \|\{e_{\ell'}(\ell)\psi_\ell((1+\mathcal{R})^{1/\nu})f_\ell\}_{\ell=0}^\infty\|_{L^{1,\infty}(G, \ell^1(\mathbb{N}_0))}. \end{aligned}$$

The fact that $\|\{e_{\ell'}(\ell)\psi_\ell((1+\mathcal{R})^{1/\nu})f_\ell\}_{\ell=0}^\infty\|_{L^{1,\infty}(G, \ell^1(\mathbb{N}_0))} = \sup_{t>0} t \cdot |\{x \in G : |\psi_{\ell'}((1+\mathcal{R})^{1/\nu})f_{\ell'}(x)| > t\}|$, implies that

$$\begin{aligned} \|W(\{f_\ell\}_{\ell=0}^\infty)\|_{L^{1,\infty}(G, \ell^1(\mathbb{N}_0))} &= \sum_{\ell'=0}^\infty \sup_{t>0} t \cdot |\{x \in G : |\psi_{\ell'}((1+\mathcal{R})^{1/\nu})f_{\ell'}(x)| > t\}| \\ &= \sum_{\ell'=0}^\infty \|\psi_{\ell'}((1+\mathcal{R})^{1/\nu})f_{\ell'}\|_{L^{1,\infty}(G)} \\ &\leq \Omega \sum_{\ell'=0}^\infty \|f_{\ell'}\|_{L^1(G)} = \int_G \sum_{\ell'=0}^\infty |\psi_{\ell'}((1+\mathcal{R})^{1/\nu})f_{\ell'}(x)| dx \\ &= \|\{f_\ell\}_{\ell=0}^\infty\|_{L^1(G, \ell^1(\mathbb{N}_0))}. \end{aligned}$$

Now, if $1 < p < 2$, there exists $\theta \in (0, 1)$ such that $\frac{1}{p} = \frac{\theta}{1} + \frac{1-\theta}{2}$. In view of (4.19) and (4.22), by the Lions-Peetre vector-valued interpolation theorem we have

$$W : L^p(G, \ell^p(\mathbb{N}_0)) \rightarrow L^p(G, \ell^p(\mathbb{N}_0)), \quad (4.23)$$

extends to a bounded operator for all $1 < p \leq 2$. Here, we have used that in this vector-valued context,

$$L^p(G, \ell^p(\mathbb{N}_0)) = (L^{1,\infty}(G, \ell^1(\mathbb{N}_0)), L^2(G, \ell^2(\mathbb{N}_0)))_{\theta,p}, \quad (4.24)$$

with the usual notation of real interpolation (see Section 6 below). Because W is a symmetric operator on $L^2(G)$, the duality argument allows us to prove that (4.23) extends to a bounded operator for all $2 \leq p < \infty$. So, the boundedness of W for all $1 < p < \infty$ is proved once that we have proved the estimate (4.21). For this, we observe that in view of the weak (1,1) estimate in the Hörmander-Mihlin theorem (see Corollary 4.12 of [8]), we have

$$\begin{aligned} \Omega &:= \sup_{\ell} \|\psi_{\ell}((1 + \mathcal{R})^{1/\nu})\|_{\mathcal{B}(L^1(G), L^{1,\infty}(G))} \lesssim \sup_{\ell; \alpha \leq [Q/2]+1, t>0} t^{\alpha} |\psi_{\ell}^{(\alpha)}(t)| \\ &= \sup_{\ell; \alpha \leq [Q/2]+1, t>0} t^{\alpha} 2^{-\ell\alpha} |\psi^{(\alpha)}(2^{-\ell}t)| \\ &\lesssim 1, \end{aligned}$$

where in the last line we have used that the compactly supported function ψ , satisfies estimate of the type

$$|\psi^{(\alpha)}(\lambda)| \leq C_{\alpha} \lambda^{-\alpha}, \quad \lambda \neq 0, \quad (4.25)$$

and so

$$t^{\alpha} 2^{-\ell\alpha} |\psi^{(\alpha)}(2^{-\ell}t)| \leq C_{\alpha} t^{\alpha} 2^{-\ell\alpha} (2^{-\ell}t)^{-\alpha} = C_{\alpha}.$$

Now, we claim that

$$W : L^p(G, \ell^r(\mathbb{N}_0)) \rightarrow L^p(G, \ell^r(\mathbb{N}_0)), \quad 1 < r < \infty. \quad (4.26)$$

To do so, we will prove that

$$W : L^1(G, \ell^r(\mathbb{N}_0)) \rightarrow L^{1,\infty}(G, \ell^r(\mathbb{N}_0)), \quad 1 < r < \infty. \quad (4.27)$$

extends to a bounded operator together with a interpolation argument coming from (4.19). For the proof of (4.27), we need to show that there exists a constant $C > 0$ independent of $\{f_{\ell}\} \in L^1(G, \ell^r(\mathbb{N}_0))$ and $t > 0$, such that

$$\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1 + \mathcal{R})^{1/\nu}) f_{\ell}(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \leq \frac{C}{t} \|\{f_{\ell}\}\|_{L^1(G, \ell^r(\mathbb{N}_0))}. \quad (4.28)$$

So, fix $\{f_{\ell}\} \in L^1(G, \ell^r(\mathbb{N}_0))$ and $t > 0$, and let $h(x) := \left(\sum_{\ell=0}^{\infty} |f_{\ell}(x)|^r \right)^{\frac{1}{r}}$, apply the Calderón-Zygmund decomposition Lemma to $h \in L^1(G)$, under the identification $G \simeq \mathbb{R}^n$, (see e.g. Hebisch [15]) in order to obtain a disjoint collection $\{I_j\}_{j=0}^{\infty}$ of disjoint open sets such that

- $h(x) \leq t$, for a.e. $x \in G \setminus \cup_{j \geq 0} I_j$,
- $\sum_{j \geq 0} |I_j| \leq \frac{C}{t} \|h\|_{L^1(G)}$, and
- $t \leq \frac{1}{|I_j|} \int_{I_j} h(x) dx \leq 2t$, for all j .

Now, we will define a suitable decomposition of f_{ℓ} , for every $\ell \geq 0$. Recall that every I_j is diffeomorphic to an open cube on \mathbb{R}^n , that it is bounded, and that $I_j \subset B(z_j, 2R_j)$, where $z_j \in I_j$ (see Hebisch [15]). Let us define, for every ℓ , and $x \in I_j$,

$$g_{\ell}(x) := \frac{1}{|I_j|} \int_{I_j} f_{\ell}(y) dy, \quad b_{\ell}(x) = f_{\ell}(x) - g_{\ell}(x). \quad (4.29)$$

and for $x \in G \setminus \cup_{j \geq 0} I_j$,

$$g_\ell(x) = f_\ell(x), \quad b_\ell(x) = 0. \quad (4.30)$$

So, for a.e. $x \in G$, $f_\ell(x) = g_\ell(x) + b_\ell(x)$. Note that for every $1 < r < \infty$, $\|\{g_\ell\}\|_{L^r(\ell^r)}^r \leq t^{r-1} \|\{f_\ell\}\|_{L^1(\ell^r)}$, indeed for $x \in I_j$, Minkowsky integral inequality gives,

$$\begin{aligned} \left(\sum_{\ell=0}^{\infty} |g_\ell(x)|^r \right)^{\frac{1}{r}} &\leq \left(\sum_{\ell=0}^{\infty} \left| \frac{1}{|I_j|} \int_{I_j} f_\ell(y) dy \right|^r \right)^{\frac{1}{r}} \leq \frac{1}{|I_j|} \int_{I_j} \left(\sum_{\ell=0}^{\infty} |f_\ell(y)|^r \right)^{\frac{1}{r}} dy \\ &\leq 2t. \end{aligned}$$

Consequently, we have

$$\begin{aligned} \|\{g_\ell\}\|_{L^r(\ell^r)}^r &= \int_G \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx = \sum_j \int_{I_j} \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx + \int_{G \setminus \cup_j I_j} \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx \\ &= \sum_j \int_{I_j} \sum_{\ell=0}^{\infty} |g_\ell(x)|^r dx + \int_{G \setminus \cup_j I_j} \sum_{\ell=0}^{\infty} |f_\ell(x)|^r dx \\ &\leq \sum_j \int_{I_j} (2t)^r dx + \int_{G \setminus \cup_j I_j} h(x)^r dx \\ &\lesssim t^r \sum_j |I_j| + \int_{G \setminus \cup_j I_j} h(x)^{r-1} h(x) dx \\ &\leq t^r \times \frac{C}{t} \|h\|_{L^1(G)} + t^{r-1} \int_{G \setminus \cup_j I_j} h(x) dx \lesssim t^{r-1} \|h\|_{L^1(G)} \\ &= t^{r-1} \|\{f_\ell\}\|_{L^1(\ell^r)}. \end{aligned}$$

Now, by using the Minkowski and the Chebyshev inequality, we obtain

$$\begin{aligned} &\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_\ell((1+\mathcal{R})^{1/\nu}) f_\ell(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \\ &\leq \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_\ell((1+\mathcal{R})^{1/\nu}) g_\ell(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\ &\quad + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_\ell((1+\mathcal{R})^{1/\nu}) b_\ell(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\ &\leq \frac{2^r}{t^r} \int_G \sum_{\ell=0}^{\infty} |\psi_\ell((1+\mathcal{R})^{1/\nu}) g_\ell(x)|^r dx \\ &\quad + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_\ell((1+\mathcal{R})^{1/\nu}) b_\ell(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|. \end{aligned}$$

In view of (4.23), $W : L^r(G, \ell^r(\mathbb{N}_0)) \rightarrow L^r(G, \ell^r(\mathbb{N}_0))$, extends to a bounded operator and

$$\int_G \sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})g_{\ell}(x)|^r dx = \|W\{g_{\ell}\}\|_{L^r(\ell^r)}^r \lesssim \|\{g_{\ell}\}\|_{L^r(\ell^r)}^r \leq t^{r-1} \|\{f_{\ell}\}\|_{L^1(\ell^r)}. \quad (4.31)$$

Consequently,

$$\begin{aligned} & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})f_{\ell}(x)|^r \right)^{\frac{1}{r}} > t \right\} \right| \\ & \leq \frac{1}{t} \|\{f_{\ell}\}\|_{L^1(\ell^r)} + \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|. \end{aligned}$$

Now, we only need to prove that

$$\left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \lesssim \frac{1}{t} \|\{f_{\ell}\}\|_{L^1(\ell^r)}. \quad (4.32)$$

Taking into account that $b_{\ell} \equiv 0$ on $G \setminus \cup_j I_j$, we have that

$$b_{\ell} = \sum_k b_{\ell,k}, \quad b_{\ell,k}(x) = b_{\ell}(x) \cdot 1_{I_k}(x). \quad (4.33)$$

Let us assume that I_j^* is a open set, such that $|I_j^*| = K|I_j|$ for some $K > 0$, and $\text{dist}(\partial I_j^*, \partial I_j) \geq 4c \text{dist}(\partial I_j, e_G)$, where c is defined in (2.16) and e_G is the identity element of G . So, by the Minkowski inequality we have,

$$\begin{aligned} & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\ & = \left| \left\{ x \in \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\ & \quad + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\ & \leq \left| \left\{ x \in G : x \in \cup_j I_j^* \right\} \right| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right|. \end{aligned}$$

Since

$$\left| \left\{ x \in G : x \in \cup_j I_j^* \right\} \right| \leq \sum_j |I_j^*|,$$

we have

$$\begin{aligned}
 & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\
 & \leq \sum_j |I_j^*| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\
 & = K \sum_j |I_j| + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right| \\
 & \leq \frac{CK}{t} \|f\|_{L^1(G, \ell^r)} + \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^2 \right)^{\frac{1}{2}} > \frac{t}{2} \right\} \right|.
 \end{aligned}$$

Observe that the Chebyshev inequality implies

$$\begin{aligned}
 & \left| \left\{ x \in G \setminus \cup_j I_j^* : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \\
 & \leq \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell}(x)|^r \right)^{\frac{1}{r}} dx \\
 & = \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} \left| \left(\psi_{\ell}((1+\mathcal{R})^{1/\nu}) \left(\sum_k b_{\ell,k} \right) \right) (x) \right|^r \right)^{\frac{1}{r}} dx \\
 & = \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \| \{ (\psi_{\ell}((1+\mathcal{R})^{1/\nu}) \left(\sum_k b_{\ell,k} \right) (x))_{\ell=0}^{\infty} \|_{\ell^r(\mathbb{N}_0)} dx \\
 & = \frac{2}{t} \int_{G \setminus \cup_j I_j^*} \| \sum_k (\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell,k})(x) \|_{\ell^r(\mathbb{N}_0)}^{\infty} dx \\
 & \leq \frac{2}{t} \sum_k \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |(\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell,k})(x)|^r \right)^{\frac{1}{r}} dx.
 \end{aligned}$$

Now, if κ_{ℓ} is the right convolution Calderón-Zygmund kernel of $\psi_{\ell}((1+\mathcal{R})^{1/\nu})$, (see Remark 2.5), and by using that $\int_{I_k} b_{k,\ell}(y)dy = 0$, we have that

$$\left(\sum_{\ell=0}^{\infty} |(\psi_{\ell}((1+\mathcal{R})^{1/\nu})b_{\ell,k})(x)|^r \right)^{\frac{1}{r}} = \left(\sum_{\ell=0}^{\infty} |b_{\ell,k} * \kappa_{\ell}(x)|^r \right)^{\frac{1}{r}}$$

$$\begin{aligned}
&= \left(\sum_{\ell=0}^{\infty} \left| \int_{I_k} \kappa_{\ell}(y^{-1}x) b_{\ell,k}(y) dy - \kappa_{\ell}(x) \int_{I_k} b_{\ell,k}(y) dy \right|^r \right)^{\frac{1}{r}} \\
&= \left(\sum_{\ell=0}^{\infty} \left| \int_{I_k} (\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)) b_{\ell,k}(y) dy \right|^r \right)^{\frac{1}{r}}.
\end{aligned}$$

Now, we will proceed as follows. By using that $|b_{\ell,k}(y)|^r \leq \sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r$, we have, by an application of the Minkowsky integral inequality,

$$\begin{aligned}
&\left(\sum_{\ell=0}^{\infty} |(\psi_{\ell}((1+\mathcal{R})^{1/\nu}) b_{\ell,k})(x)|^r \right)^{\frac{1}{r}} = \left(\sum_{\ell=0}^{\infty} \left| \int_{I_k} (\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)) b_{\ell,k}(y) dy \right|^r \right)^{\frac{1}{r}} \\
&\leq \int_{I_k} \left(\sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)|^r |b_{\ell,k}(y)|^r \right)^{\frac{1}{r}} dy \\
&\leq \int_{I_k} \left(\sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} \left(\sum_{\ell=0}^{\infty} |\kappa_{\ell}(xy^{-1}) - \kappa_{\ell}(x)|^r \right)^{\frac{1}{r}} dy.
\end{aligned}$$

Consequently, we deduce,

$$\begin{aligned}
&\frac{2}{t} \sum_k \int_{G \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |(\psi_{\ell}((1+\mathcal{R})^{1/\nu}) b_{\ell,k})(x)|^r \right)^{\frac{1}{r}} dx \\
&\leq \frac{2}{t} \sum_k \int_{G \cup_j I_j^*} \int_{I_k} \left(\sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} \left(\sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)|^r \right)^{\frac{1}{r}} dy dx \\
&= \frac{2}{t} \sum_k \int_{I_k} \int_{G \cup_j I_j^*} \left(\sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} \left(\sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)|^r \right)^{\frac{1}{r}} dx dy \\
&= \frac{2}{t} \sum_k \int_{I_k} \left(\sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} \int_{G \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)|^r \right)^{\frac{1}{r}} dx dy.
\end{aligned}$$

Because $\text{dist}(\partial I_j^*, \partial I_j) \geq 4c \text{dist}(\partial I_j, e_G)$, for $x \in G \setminus \cup_j I_j^*$, for $y \in I_k$, the analysis in (4.16), shows that $4c|y| = 4c \times \text{dist}(y, e_G) \lesssim \text{dist}(\partial I_k^*, \partial I_k) \leq |x|$. So,

$$\{x \in G : x \in G \setminus \cup_j I_j^*\} \subset \{x \in G : \text{for all } z \in I_k, \quad 4c|z| \leq |x|\}.$$

Now, from Remark 2.5, the identity (2.16), and the estimate (2.17), we deduce

$$\begin{aligned}
 & \int_{G \setminus \cup_j I_j^*} \left(\sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)|^r \right)^{\frac{1}{r}} dx \leq \int_{G \setminus \cup_j I_j^*} \sum_{\ell=0}^{\infty} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)| dx \\
 & \leq \sum_{\ell=0}^{\infty} \int_{G \setminus \cup_j I_j^*} |\kappa_{\ell}(y^{-1}x) - \kappa_{\ell}(x)| dx \\
 & \leq \sum_{\ell=0}^{\infty} \int_{|x| > 4c|y|} |2^{-\ell Q} \kappa_{\ell}(2^{-\ell} \cdot y^{-1}x) - 2^{-\ell Q} \kappa_{\ell}(2^{-\ell} \cdot x)| dx \\
 & \lesssim \sum_{\ell=0}^{\infty} 2^{-\ell \varepsilon_0} = O(1).
 \end{aligned}$$

Thus, we have proved that

$$\begin{aligned}
 & \left| \left\{ x \in G : \left(\sum_{\ell=0}^{\infty} |\psi_{\ell}((1 + \mathcal{R})^{1/\nu}) b_{\ell}(x)|^r \right)^{\frac{1}{r}} > \frac{t}{2} \right\} \right| \lesssim \frac{2}{t} \sum_k \int_{I_k} \left(\sum_{\ell'=0}^{\infty} |b_{\ell',k}(y)|^r \right)^{\frac{1}{r}} dy \\
 & = \frac{2}{t} \int_{\cup_k I_k} \left(\sum_{\ell'=0}^{\infty} |b_{\ell'}(y)|^r \right)^{\frac{1}{r}} dy \\
 & \lesssim \frac{1}{t} \|\{f_{\ell}\}\|_{L^1(\ell^r)}.
 \end{aligned}$$

This, the proof of the weak (1,1) inequality is complete and we have that

$$W : L^1(G, \ell^r(\mathbb{N}_0)) \rightarrow L^{1,\infty}(G, \ell^r(\mathbb{N}_0)), \quad 1 < r < \infty, \quad (4.34)$$

extends to a bounded operator. As an application of the vector-valued Lions-Peetre interpolation theorem between (4.19) and (4.34) we obtain that W in (4.26) extends to a bounded operator and together with the duality argument we complete the proof.

5 Homogeneous and Inhomogeneous Besov Spaces

Let \mathcal{R} be a (left-invariant) positive Rockland operator on a graded Lie group G . In order to define the family of Besov spaces on G , let us assume that \mathcal{R} is homogeneous of degree $\nu > 0$ and let us fix a dyadic decomposition of its spectrum: we choose a function $\psi \in C_0^{\infty}(\mathbb{R})$ supported in $[1/4, 2]$, $\psi = 1$ on $[1/2, 1]$. Denote by ψ_I the function $\psi_I(t) = \psi(2^{-I}t)$, $t \in \mathbb{R}$. For some smooth compactly supported function ψ_0 we have

$$\sum_{I \in \mathbb{N}_0} \psi_I(\lambda) = 1, \quad \text{for every } \lambda > 0. \quad (5.1)$$

With notations above we define (left) Besov spaces associated to a (left-invariant) positive Rockland operator as follows.

Definition 5.1 Let $r \in \mathbb{R}$, $0 < p < \infty$ and $0 < q \leq \infty$. The homogeneous Besov space $\dot{B}_{p,q,\psi,\mathcal{R}}^r(G)$ associated to $(\mathcal{R}, (\psi_l)_l)$ consists of those $f \in \mathcal{D}'(G)$ satisfying

$$\|f\|_{\dot{B}_{p,q,\psi,\mathcal{R}}^r(G)} := \left(\sum_{l \in \mathbb{N}_0} 2^{l r q} \|\psi_l(\mathcal{R}^{\frac{1}{v}}) f\|_{L^p(G)}^q \right)^{\frac{1}{q}} < \infty, \quad (5.2)$$

for $0 < q < \infty$, and for $q = \infty$,

$$\|f\|_{\dot{B}_{p,\infty,\psi,\mathcal{R}}^r(G)} := \sup_{l \in \mathbb{N}_0} 2^{l r} \|\psi_l(\mathcal{R}^{\frac{1}{v}}) f\|_{L^p(G)} < \infty. \quad (5.3)$$

Analogously, the inhomogeneous Besov space $B_{p,q,\psi,\mathcal{R}}^r(G)$ is defined as the space of distributions $f \in \mathcal{D}'(G)$ satisfying

$$\|f\|_{B_{p,q,\psi,\mathcal{R}}^r(G)} := \left(\sum_{l \in \mathbb{N}_0} 2^{l r q} \|\psi_l((I + \mathcal{R})^{\frac{1}{v}}) f\|_{L^p(G)}^q \right)^{\frac{1}{q}} < \infty, \quad (5.4)$$

if $0 < q < \infty$ and, for $q = \infty$,

$$\|f\|_{B_{p,\infty,\psi,\mathcal{R}}^r(G)} := \sup_{l \in \mathbb{N}_0} 2^{l r} \|\psi_l((I + \mathcal{R})^{\frac{1}{v}}) f\|_{L^p(G)} < \infty. \quad (5.5)$$

Homogeneous and inhomogeneous Besov spaces do not depend on a particular choice of a positive Rockland operator \mathcal{R} and of the sequence of smooth functions ψ_l . We will prove this fact in the following section (see Theorem 6.1). Now, we prove the following embedding properties of Besov spaces. We use the simplified notation motivated by Theorem 6.1,

$$(\dot{B}_{p,q}^r(G), \|\cdot\|_{\dot{B}_{p,q}^r(G)}) = (\dot{B}_{p,q,\psi,\mathcal{R}}^r(G), \|\cdot\|_{\dot{B}_{p,q,\psi,\mathcal{R}}^r(G)})$$

and

$$(B_{p,q}^r(G), \|\cdot\|_{B_{p,q}^r(G)}) = (B_{p,q,\psi,\mathcal{R}}^r(G), \|\cdot\|_{B_{p,q,\psi,\mathcal{R}}^r(G)}).$$

For Sobolev spaces $H^{r,p}(G)$ and $\dot{H}^{r,p}(G)$ and their properties we refer to [9, Section 4].

We also note that similar results would hold if we chose right-invariant (instead of left-invariant) Rockland operator in the definition of Besov spaces, see Remark 5.4. Additionally, as the reviewer of this paper has observed, the following result follows automatically from the general theory of interpolation of linear operators from Theorem 6.1. We present a detailed proof of such a fact for completeness.

Theorem 5.2 Let G be a graded Lie group of homogeneous dimension Q and let $r \in \mathbb{R}$. Then

- (1) $\dot{B}_{p,q_1}^{r+\varepsilon}(G) \hookrightarrow \dot{B}_{p,q_1}^r(G) \hookrightarrow \dot{B}_{p,q_2}^r(G) \hookrightarrow \dot{B}_{p,\infty}^r(G)$, $\varepsilon > 0$, $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$.
- (2) $\dot{B}_{p,q_1}^{r+\varepsilon}(G) \hookrightarrow \dot{B}_{p,q_2}^r(G)$, $\varepsilon > 0$, $0 < p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.
- (3) $\dot{B}_{p_1,q}^{r_1}(G) \hookrightarrow \dot{B}_{p_2,q}^{r_2}(G)$, $1 \leq p_1 \leq p_2 \leq \infty$, $0 < q < \infty$, $r_1 \in \mathbb{R}$ and $r_2 = r_1 - Q(\frac{1}{p_1} - \frac{1}{p_2})$.
- (4) $\dot{H}^r(G) = \dot{B}_{2,2}^r(G)$ and $\dot{B}_{p,p}^r(G) \hookrightarrow \dot{H}^{r,p}(G) \hookrightarrow \dot{B}_{p,2}^r(G)$, $1 < p \leq 2$.
- (5) $\dot{B}_{p,1}^r(G) \hookrightarrow L^q(G)$, $1 \leq p \leq q \leq \infty$, $r = Q(\frac{1}{p} - \frac{1}{q})$.

Proof For the proof of (1) we observe that

$$\begin{aligned}\|f\|_{\dot{B}_{p,\infty}^r} &= \sup_{s \in \mathbb{N}_0} 2^{rs} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p} \leq \|\{2^{sr} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{l^{q_2}(\mathbb{N}_0)} \equiv \|f\|_{\dot{B}_{p,q_2}^r} \\ &\leq \|\{2^{sr} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{l^{q_1}(\mathbb{N}_0)} \equiv \|f\|_{\dot{B}_{p,q_1}^r} \\ &\leq \|\{2^{s(r+\varepsilon)} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{l^{q_1}(\mathbb{N}_0)} \\ &\equiv \|f\|_{\dot{B}_{p,q_1}^{r+\varepsilon}}.\end{aligned}$$

For the proof of (2) we use Hölder inequality as follows,

$$\begin{aligned}\|f\|_{\dot{B}_{p,q_2}^r} &= \|\{2^{sr} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{l^{q_2}(\mathbb{N}_0)} \\ &= \|\{2^{s(r+\varepsilon)-s\varepsilon} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p}\}_{s \in \mathbb{N}_0}\|_{l^{q_2}(\mathbb{N}_0)} \\ &\leq \|\{2^{s(r+\varepsilon)q_2} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p}^{q_2}\}_{s \in \mathbb{N}_0}\|_{l^{\frac{q_2}{q_1}}(\mathbb{N}_0)}^{\frac{1}{q_2}} \left[\sum_{s \in \mathbb{N}_0} 2^{-\frac{s\varepsilon q_2 q_1}{q_1 - q_2}} \right]^{\frac{1}{q_1} - \frac{1}{q_2}} \\ &\lesssim \|f\|_{\dot{B}_{p,q_1}^{r+\varepsilon}}.\end{aligned}$$

In order to prove (3) we use Nikolskii inequality from Theorem 3. So, by taking into account the estimate

$$\|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^{p_2}} \leq C 2^{sQ(\frac{1}{p_1} - \frac{1}{p_2})} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^{p_1}}, \quad (5.6)$$

we deduce

$$\left(\sum_{s \in \mathbb{N}_0} 2^{sr_2 q} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^{p_2}(G)}^q \right)^{\frac{1}{q}} \lesssim \left(\sum_{s \in \mathbb{N}_0} 2^{s[r_2 + Q(\frac{1}{p_1} - \frac{1}{p_2})]q} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^{p_1}(G)}^q \right)^{\frac{1}{q}}.$$

Now we will prove (4), that is $\dot{B}_{p,p}^r(G) \hookrightarrow \dot{H}^{r,p}(G) \hookrightarrow \dot{B}_{p,2}^r(G)$, for $1 < p \leq 2$. In fact, if $dE(\lambda)$ denotes the spectral resolution of $\mathcal{R}^{\frac{1}{v}}$, we have

$$\begin{aligned}\|f\|_{\dot{H}^{r,p}}^p &\equiv \|\mathcal{R}^{\frac{r}{v}}f\|_{L^p}^p = \left\| \int_0^\infty \lambda^r dE(\lambda) f \right\|_{L^p}^p \\ &= \left\| \sum_{s \in \mathbb{Z}} \int_{2^s}^{2^{s+1}} \lambda^r dE(\lambda) \psi_s(\mathcal{R}^{\frac{1}{v}})f \right\|_{L^p}^p \leq \sum_{s \in \mathbb{Z}} \left\| \int_{2^s}^{2^{s+1}} \lambda^r dE(\lambda) \psi_s(\mathcal{R}^{\frac{1}{v}})f \right\|_{L^p}^p \\ &= \sum_{s \in \mathbb{Z}} 2^{srp} \left\| \int_{2^s}^{2^{s+1}} 2^{-sr} \lambda^r dE(\lambda) \psi_s(\mathcal{R}^{\frac{1}{v}})f \right\|_{L^p}^p \\ &\asymp \sum_{s \in \mathbb{Z}} 2^{srp} \left\| \int_{2^s}^{2^{s+1}} dE(\lambda) \psi_s(\mathcal{R}^{\frac{1}{v}})f \right\|_{L^p}^p = \sum_{s \in \mathbb{N}_0} 2^{srp} \left\| \int_{2^s}^{2^{s+1}} \psi_s(\lambda) dE(\lambda) f \right\|_{L^p}^p \\ &= \sum_{s \in \mathbb{N}_0} 2^{srp} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p}^p = \|f\|_{\dot{B}_{p,p}^r}^p.\end{aligned}$$

For the other embedding we use the following version of the Minkowski integral inequality

$$\left(\sum_j \left(\int_X f_j(x) d\mu(x) \right)^\alpha \right)^{\frac{1}{\alpha}} \leq \int_X \left(\sum_j f_j^\alpha(x) \right)^{\frac{1}{\alpha}} d\mu(x), \quad f_j(x) \geq 0, \quad a.e. x \in X,$$

where (X, μ) is a σ -finite measure space, and $\alpha = \frac{2}{p}$. So, we get

$$\begin{aligned}
 \|f\|_{\dot{B}_{p,2}^r} &= \left(\sum_{s \in \mathbb{N}_0} 2^{2rs} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p}^2 \right)^{\frac{1}{2}} \\
 &= \left(\sum_{s \in \mathbb{N}_0} 2^{2rs} \left[\int_G |\psi_s(\mathcal{R}^{\frac{1}{v}})f(x)|^p dx \right]^{\frac{2}{p}} \right)^{\frac{p}{2} \cdot \frac{1}{p}} \\
 &\leq \left[\int_G \left[\sum_{s \in \mathbb{N}_0} 2^{2sr} |\psi_s(\mathcal{R}^{\frac{1}{v}})f(x)|^{\frac{2}{p}p} dx \right]^{\frac{p}{2}} \right]^{\frac{1}{p}} \\
 &= \left[\int_G \left[\sum_{s \in \mathbb{N}_0} 2^{2sr} |\psi_s(\mathcal{R}^{\frac{1}{v}})f(x)|^2 dx \right]^{\frac{p}{2}} \right]^{\frac{1}{p}} \\
 &= \| [\sum_{s \in \mathbb{N}_0} 2^{2sr} |\psi_s(\mathcal{R}^{\frac{1}{v}})f(x)|^2 dx]^{\frac{1}{2}} \|_{L^p} \asymp \| [\sum_{s \in \mathbb{N}_0} |\psi_s(\mathcal{R}^r)f(x)|^2 dx]^{\frac{1}{2}} \|_{L^p} \\
 &\asymp \|\mathcal{R}^r f\|_{L^p} = \|f\|_{\dot{H}^{r,p}},
 \end{aligned}$$

using Littlewood-Paley theorem (Theorem 4.1). We observe that in the embedding $\dot{B}_{p,p}^r(G) \hookrightarrow \dot{H}^{r,p}(G) \hookrightarrow \dot{B}_{p,2}^r(G)$, if $p = 2$ then $\dot{H}^{r,2}(G) = \dot{B}_{2,2}^r(G)$. Now, for the proof of (5) we use Nikolskii inequality,

$$\begin{aligned}
 \|f\|_{L^q} &= \left\| \int_{\widehat{G}} \text{Tr}[\pi(x) \widehat{f}(\pi)] d\pi \right\|_{L^q} \\
 &= \left\| \sum_{s \in \mathbb{N}_0} \int_{\widehat{G}} \text{Tr}[\pi(x) \psi_s[\pi(\mathcal{R})] \widehat{f}(\pi)] d\pi \right\|_{L^q} \\
 &\leq \sum_{s \in \mathbb{N}_0} \left\| \int_{\widehat{G}} \text{Tr}[\pi(x) \psi_s[\pi(\mathcal{R})] \widehat{f}(\pi)] d\pi \right\|_{L^q} \\
 &= \sum_{s \in \mathbb{N}_0} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^q} \leq \sum_{s \in \mathbb{N}_0} 2^{Q(\frac{1}{p} - \frac{1}{q})} \|\psi_s(\mathcal{R}^{\frac{1}{v}})f\|_{L^p} \\
 &= \|f\|_{\dot{B}_{p,1}^{Q(\frac{1}{p} - \frac{1}{q})}}.
 \end{aligned}$$

This completes the proof.

In the following theorem we present embeddings properties for inhomogeneous Besov spaces $B_{p,q}^r(G)$. The proof is similar to the homogeneous case, so we omit it.

Theorem 5.3 *Let G be a graded Lie group of homogeneous dimension Q and let $r \in \mathbb{R}$. Then*

- (1) $B_{p,q_1}^{r+\varepsilon}(G) \hookrightarrow B_{p,q_1}^r(G) \hookrightarrow B_{p,q_2}^r(G) \hookrightarrow B_{p,\infty}^r(G)$, $\varepsilon > 0$, $0 < p \leq \infty$, $0 < q_1 \leq q_2 \leq \infty$.
- (2) $B_{p,q_1}^{r+\varepsilon}(G) \hookrightarrow B_{p,q_2}^r(G)$, $\varepsilon > 0$, $0 < p \leq \infty$, $1 \leq q_2 < q_1 < \infty$.
- (3) $B_{p_1,q}^{r_1}(G) \hookrightarrow B_{p_2,q}^{r_2}(G)$, $1 \leq p_1 \leq p_2 \leq \infty$, $0 < q < \infty$, $r_1 \in \mathbb{R}$ and $r_2 = r_1 - Q(\frac{1}{p_1} - \frac{1}{p_2})$.

- (4) $H^r(G) = B_{2,2}^r(G)$ and $B_{p,p}^r(G) \hookrightarrow H^{r,p}(G) \hookrightarrow B_{p,2}^r(G)$, $1 < p \leq 2$.
 (5) $B_{p,1}^r(G) \hookrightarrow L^q(G)$, $1 \leq p \leq q \leq \infty$, $r = Q(\frac{1}{p} - \frac{1}{q})$.

Remark 5.4 (Right Besov spaces) Throughout this section we have considered Besov spaces associated to (left-invariant) positive Rockland operators. A similar formulation of homogeneous and inhomogeneous (right) Besov spaces can be obtained if we choose (right-invariant) positive Rockland operators. It can be shown that these spaces satisfy (right) versions of Theorems 5.2 and 5.3. When properties that we want to consider hold for left and right Besov spaces, we omit the prefixes left and right, nevertheless, we consider in the proofs the case of (left) Besov spaces.

6 Independence of Rockland Operators and Interpolation Properties

In this section we prove the independence of the choice of Rockland operator and the dyadic partition ψ_l in the definition of Besov spaces. For this, we show that Besov spaces can be obtained as interpolation of Sobolev spaces. If X_0 and X_1 are Banach spaces, the main notion in real interpolation theory is the K -functional, defined by

$$K(f, t) = \inf\{\|f_0\|_{X_0} + t\|f_1\|_{X_1} : f = f_0 + f_1, f_0 \in X_0, f_1 \in X_1\}, \quad t \geq 0. \quad (6.1)$$

If $0 < \theta < 1$ and $1 \leq q < \infty$, the real interpolation space $X_{\theta,q} := (X_0, X_1)_{\theta,q}$ is defined by those vectors $f \in X_0 + X_1$ satisfying

$$\|f\|_{\theta,q} = \left(\int_0^\infty (t^{-\theta} K(f, t))^q \frac{dt}{t} \right)^{\frac{1}{q}} < \infty \text{ if } q < \infty, \quad (6.2)$$

and for $q = \infty$

$$\|f\|_{\theta,q} = \sup_{t>0} t^{-\theta} K(f, t) < \infty. \quad (6.3)$$

For our purposes, the following discrete form (see [17], p. 1136) will be useful

$$\|f\|_{\theta,q} \asymp \inf \left\{ \left(\sum_{k \in \mathbb{Z}} \max\{\|f_k\|_{X_0}, 2^k \|f_k\|_{X_1}\}^q \right)^{\frac{1}{q}} : f = \sum_{k \in \mathbb{Z}} 2^{k\theta} f_k \right\}. \quad (6.4)$$

with $1 \leq q < \infty$.

Theorem 6.1 *Let G be a graded Lie group, and let \mathcal{R} and \mathcal{R}' be two positive Rockland operators with homogeneity degrees $\nu > 0$ and $\nu' > 0$, respectively. If $(\psi_l)_l$ and $(\psi'_l)_l$ are sequences satisfying (5.1), $1 < p < \infty$ and $1 \leq q < \infty$, the spaces $\dot{B}_{p,q,\psi,\mathcal{R}}^r(G)$ and $\dot{B}_{p,q,\psi',\mathcal{R}'}^r(G)$ coincide and have equivalent norms, as well as the spaces $B_{p,q,\psi,\mathcal{R}}^r(G)$ and $B_{p,q,\psi',\mathcal{R}'}^r(G)$. We also have the following interpolation properties:*

$$B_{p,q}^r(G) = (H^{b,p}(G), H^{a,p}(G))_{\theta,q}, \quad a < r < b, \quad r = b(1 - \theta) + a\theta, \quad (6.5)$$

and

$$\dot{B}_{p,q}^r(G) = (\dot{H}^{b,p}(G), \dot{H}^{b,a}(G))_{\theta,q}, \quad a < r < b, \quad r = b(1 - \theta) + a\theta. \quad (6.6)$$

Proof It was proved in [9, Theorem 4.4.20], that the definition of (homogeneous and inhomogeneous) Sobolev spaces $(\dot{H}^{r,p}(G)$ and $H^{r,p}(G)$, respectively) does not depend on the

choice of Rockland operators. Hence the independence of the choice of Rockland operators and of the dyadic decomposition ψ_l in the case of Besov spaces would follow if we show that Besov spaces can be obtained by interpolation of Sobolev spaces. So, it suffices to prove (6.5) and (6.6). First we will show that for $r > 0$, $\dot{B}_{p,q}^r = (\dot{H}^{r_1,p}, \dot{H}^{r_0,p})_{\theta,q}$ where $0 < r_0 < r < r_1 = r_0 + \nu$, $r = r_1 + (r_0 - r_1)\theta$, and later we will deduce the general case from this fact. For $f \in \dot{H}^{r_1,p} + \dot{H}^{r_0,p}$ we write

$$f = \sum_{l \geq 0} \psi_l(\mathcal{R}^{\frac{1}{\nu}})f = \sum_{l \geq 0} 2^{l\theta} f'_l, \quad f'_l = 2^{-l\theta} \psi_l(\mathcal{R}^{\frac{1}{\nu}})f. \quad (6.7)$$

Hence

$$\|f\|_{\theta,q}^q \lesssim \sum_{l \geq 0} \max\{\|f'_l\|_{\dot{H}^{r_1,p}}, 2^l \|f'_l\|_{\dot{H}^{r_0,p}}\}^q.$$

Now, if $(E_\lambda)_{\lambda \geq 0}$ denotes the spectral resolution associated to $\mathcal{R}^{\frac{1}{\nu}}$, we have

$$\begin{aligned} \|f\|_{\theta,q}^q &\lesssim \sum_{l \geq 0} 2^{-\theta l q} \max\{\|\psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{\dot{H}^{r_1,p}}, 2^l \|\psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{\dot{H}^{r_0,p}}\}^q \\ &= \sum_{l \geq 0} 2^{-\theta l q} \max\left\{\left\|\int_{2^{l-1}}^{2^{l+1}} \psi_l(\lambda) dE_\lambda f\right\|_{\dot{H}^{r_1,p}}, 2^l \left\|\int_{2^{l-1}}^{2^{l+1}} \psi_l(\lambda) dE_\lambda f\right\|_{\dot{H}^{r_0,p}}\right\}^q \\ &= \sum_{l \geq 0} 2^{-\theta l q} \max\left\{\left\|\int_{2^{l-1}}^{2^{l+1}} \lambda^{r_1} \psi_l(\lambda) dE_\lambda f\right\|_{L^p}, 2^l \left\|\int_{2^{l-1}}^{2^{l+1}} \lambda^{r_0} \psi_l(\lambda) dE_\lambda f\right\|_{L^p}\right\}^q \\ &\asymp \sum_{l \geq 0} 2^{-\theta l q} \max\left\{\|2^{r_1 l} \int_{2^{l-1}}^{2^{l+1}} \psi_l(\lambda) dE_\lambda f\|_{L^p}, 2^l \|2^{r_0 l} \int_{2^{l-1}}^{2^{l+1}} \psi_l(\lambda) dE_\lambda f\|_{L^p}\right\}^q \\ &= \sum_{l \geq 0} 2^{-\theta l q} \max\{\|2^{r_1 l} \psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{L^p}, 2^l \|2^{r_0 l} \psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{L^p}\}^q. \end{aligned}$$

Since

$$\max\{\|2^{r_1 l} \psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{L^p}, 2^l \|2^{r_0 l} \psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{L^p}\} = \max\{2^{r_1 l}, 2^{r_0 l + l}\} \|\psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{L^p},$$

we obtain

$$\begin{aligned} \|f\|_{\theta,q}^q &\lesssim \sum_{l \geq 0} 2^{-\theta l q} \max\{2^{r_1 l q}, 2^{r_0 l q + l q}\} \|\psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{L^p}^q \\ &= \sum_{l \geq 0} \max\{2^{r_1 l q - \theta l q}, 2^{r_0 l q + l q - \theta l q}\} \|\psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{L^p}^q \\ &= \sum_{l \geq 0} \max\{2^{-q l \theta (r_0 - r_1 + 1)}, 2^{q l (1 - \theta)(r_0 - r_1 + 1)}\} 2^{r q l} \|\psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{L^p}^q. \end{aligned}$$

Taking into account that $r_0 - r_1 + \nu = 0$ we have

$$\|f\|_{\theta,q}^q \lesssim \sum_{l \geq 0} 2^{r l q} \|\psi_l(\mathcal{R}^{\frac{1}{\nu}})f\|_{L^p}^q \equiv \|f\|_{\dot{B}_{p,q}^r}^q. \quad (6.8)$$

Now, in order to proof the converse inequality we use the following estimate on the operator norm of ψ_l for l large enough: $\|\psi_l(\mathcal{R}^{\frac{1}{\nu}})\|_{\mathcal{L}(L^p)} = O(1)$, which can be obtained by interpolation between the trivial estimate for $p = 2$, (4.21), and the duality argument. We

observe that by the Liouville theorem (see Geller [14] or [9, Section 3.2.8]), $\lambda = 0$ is not an eigenvalue of \mathcal{R} . So, we have,

$$\begin{aligned} \|f\|_{\dot{B}_{p,q}^r}^q &= \sum_{l \geq 0} 2^{rlq} \|\psi_l(\mathcal{R}^{\frac{1}{v}})f\|_{L^p}^q \\ &= \sum_{l \geq 0} 2^{rlq} \|\psi_l(\mathcal{R}^{\frac{1}{v}})\mathcal{R}^{-r_1/v}\mathcal{R}^{r_1/v}f\|_{L^p}^q \\ &= \sum_{l \geq 0} 2^{rlq} \left\| \int_{2^{l-1}}^{2^{l+1}} \psi_l(\lambda) \lambda^{-r_1} dE_\lambda \mathcal{R}^{r_1/v} f \right\|_{L^p}^q \\ &\asymp \sum_{l \geq 0} 2^{lrq-lr_1q} \left\| \int_{2^{l-1}}^{2^{l+1}} \psi_l(\lambda) dE_\lambda \mathcal{R}^{r_1/v} f \right\|_{L^p}^q \\ &= \sum_{l \geq 0} 2^{lrq-lr_1q} \|\psi_l(\mathcal{R}^{\frac{1}{v}})\mathcal{R}^{r_1/v}f\|_{L^p}^q \\ &\lesssim \sum_{l \geq 0} 2^{lrq-lr_1q} \|\mathcal{R}^{r_1/v}f\|_{L^p}^q \lesssim C_s \|\mathcal{R}^{r_1/v}f\|_{L^p}^q. \end{aligned}$$

Hence

$$\|f\|_{\dot{B}_{p,q}^r} \lesssim C_s \|f\|_{\dot{H}^{r_1,p}}.$$

In a similar way, we can prove the estimate

$$\|f\|_{\dot{B}_{p,q}^r} \lesssim \|f\|_{\dot{H}^{r_0,p}}.$$

So, we have the embedding $\dot{H}^{r_i,p} \hookrightarrow \dot{B}_{p,q}^r$ for $i = 0, 1$. Hence $(\dot{H}^{r_1,p}, \dot{H}^{r_0,p})_{\theta,q} \hookrightarrow \dot{B}_{p,q}^r$.

So we conclude that $\|f\|_{\dot{B}_{p,q}^r} \lesssim \|f\|_{\theta,q}$. In the case where $r < 0$ we observe that $(I + \mathcal{R})^{\frac{|r|-r}{v}} : \dot{B}_{p,q}^{|r|} \rightarrow \dot{B}_{p,q}^r$ is an isomorphism and for $0 < r_0 < |r| < r_0 + v = r_1$ we obtain,

$$\begin{aligned} \dot{B}_{p,q}^r &= (I + \mathcal{R})^{\frac{|r|-r}{v}} (\dot{B}_{p,q}^{|r|}) = ((I + \mathcal{R}))^{\frac{|r|-r}{v}} \dot{H}^{r_0,p}, (I + \mathcal{R}))^{\frac{|r|-r}{v}} \dot{H}^{r_1,p} \theta_{\theta,q} \\ &= (\dot{H}^{r_0+r-|r|,p}, \dot{H}^{r_1+r-|r|,p})_{\theta,q}, \end{aligned}$$

with $|r| = r_1 + \theta(r_0 - r_1)$. The general case where $a < r < b$ and $r = b(1 - \theta) + a\theta$ now follows if we consider $r_0 = r - \frac{v}{2}$, $r_1 = r + \frac{v}{2}$ and by observing that

$$r = \frac{1}{2}r_1 + \frac{1}{2}r_2 = r_1 + \frac{1}{2}(r_2 - r_1).$$

So we get

$$(\dot{H}^{b,p}, \dot{H}^{a,p})_{\theta,q} = (\dot{H}^{r_1,p}, \dot{H}^{r_0,p})_{\frac{1}{2},q}. \quad (6.9)$$

Since $(\dot{H}^{r_1,p}, \dot{H}^{r_0,p})_{\frac{1}{2},q} = \dot{B}_{p,q}^r$ we conclude the proof of the homogeneous case. An analogous proof can be adapted to the inhomogeneous case.

6.1 Interpolation Inequalities in Besov Spaces

In this subsection we consider the problem of interpolation inequalities on Besov spaces on graded Lie groups. The following theorem generalises a version for Besov spaces in \mathbb{R}^n proved by Machihara and Ozawa [18]. In turn, this extended many other known families of inequalities, we refer to [18] for the review of the literature.

Theorem 6.2 Let λ, μ, p, q, r and θ be real numbers. If $1 \leq p, q \leq r \leq \infty$,

$$Q\left(\frac{1}{p} - \frac{1}{r}\right) < \lambda \quad \text{and} \quad \mu < Q\left(\frac{1}{q} - \frac{1}{r}\right), \quad (6.10)$$

then we have the following inequalities:

- (i) $\|f\|_{\dot{B}_{r,1}^0(G)} \leq C \|f\|_{\dot{B}_{p,\infty}^\lambda(G)}^\theta \|f\|_{\dot{B}_{q,\infty}^\mu(G)}^{1-\theta}, \quad f \in \dot{B}_{p,\infty}^\lambda(G) \cap \dot{B}_{q,\infty}^\mu(G),$
- (ii) $\|f\|_{B_{r,1}^0(G)} \leq C \|f\|_{B_{p,\infty}^\lambda(G)}^\theta \|f\|_{B_{q,\infty}^\mu(G)}^{1-\theta}, \quad f \in B_{p,\infty}^\lambda(G) \cap B_{q,\infty}^\mu(G),$
- (iii) $\|f\|_{L^r(G)} \leq C \|f\|_{H^{\lambda,p}(G)}^\theta \|f\|_{H^{\mu,q}(G)}^{1-\theta}, \quad f \in H^{\lambda,p}(G) \cap H^{\mu,q}(G),$

where

$$\theta\left(\lambda - Q\left(\frac{1}{p} - \frac{1}{r}\right)\right) + (1-\theta)\left(\mu - Q\left(\frac{1}{q} - \frac{1}{r}\right)\right) = 0. \quad (6.11)$$

Proof In order to prove (i), we consider $f \in \dot{B}_{p,\infty}^\lambda(G) \cap \dot{B}_{q,\infty}^\mu(G)$ such that $f \not\equiv 0$ almost everywhere, with λ, μ, p and q satisfying (6.10). If ψ_{-1} is some smooth function supported in $[-1, \frac{1}{2}]$, then by using the fact that the corresponding Littlewood-Paley decomposition satisfies

$$\text{supp}(\psi_k) \cap \text{supp}(\psi_j) = \emptyset, \quad |j - k| \geq 2, \quad j, k \geq -1, \quad (6.12)$$

we can write

$$\begin{aligned} \|f\|_{\dot{B}_{r,1}^0(G)} &= \sum_{l=0}^{\infty} \|\psi_l(\mathcal{R}^{\frac{1}{v}})f\|_{L^r(G)} = \sum_{l=0}^{\infty} \left\| \sum_{k=0}^{\infty} \psi_k(\mathcal{R})\psi_l(\mathcal{R}^{\frac{1}{v}})f \right\|_{L^r(G)} \\ &\leq \sum_{l=0}^{\infty} \sum_{k=-1}^{\infty} \|\psi_k(\mathcal{R})\psi_l(\mathcal{R}^{\frac{1}{v}})f\|_{L^r(G)} = \sum_{l=0}^{\infty} \sum_{k=l-1}^{l+1} \|\psi_k(\mathcal{R})\psi_l(\mathcal{R}^{\frac{1}{v}})f\|_{L^r(G)} \\ &= \sum_{l=0}^{\infty} \sum_{k=l-1}^{l+1} \|\psi_l(\mathcal{R}^{\frac{1}{v}})f * \mathcal{F}_G^{-1}[\psi_k(\pi[\mathcal{R}])]\|_{L^r(G)}. \end{aligned}$$

If we use the Young inequality, for $\frac{1}{r} + 1 = \frac{1}{m} + \frac{1}{s}$, we have

$$\|\psi_l(\mathcal{R}^{\frac{1}{v}})f * \mathcal{F}_G^{-1}[\psi_k(\pi[\mathcal{R}])]\|_{L^r(G)} \leq \|\mathcal{F}_G[\psi_k(\pi[\mathcal{R}])]\|_{L^m(G)} \|\psi_l(\mathcal{R}^{\frac{1}{v}})f\|_{L^s(G)}. \quad (6.13)$$

By the action of the dilations D_t on G we have for $k \geq 1$, and $r = 2^{-\frac{k}{v}}$, that

$$\begin{aligned} \|\mathcal{F}_G^{-1}[\psi_k(\pi[\mathcal{R}])]\|_{L^m(G)}^m &= \int_G |\mathcal{F}_G^{-1}(\psi_k(\pi(\mathcal{R}))) (x)|^m dx \\ &= \int_G |\mathcal{F}_G^{-1}[\psi_0(2^{-k}\pi(\mathcal{R}))](x)|^m dx \\ &= \int_G |\mathcal{F}_G^{-1}[\psi_0(r^v\pi[\mathcal{R}])](x)|^m dx \\ &= \int_G r^{-Qm} |\mathcal{F}_G^{-1}[\psi_0(\pi[\mathcal{R}])](D_{r^{-1}}x)|^m dx \\ &= \int_G r^{-Qm+Q} |\mathcal{F}_G^{-1}[\psi_0(\pi[\mathcal{R}])](x)|^m dx. \end{aligned}$$

Hence we obtain

$$\|\mathcal{F}_G^{-1}[\psi_k(\pi[\mathcal{R}])]\|_{L^m(G)} = 2^{kQ(1-\frac{1}{m})} \|\mathcal{F}_G^{-1}[\psi_0(\pi[\mathcal{R}])]\|_{L^m(G)} \lesssim 2^{kQ(1-\frac{1}{m})}.$$

If $l \geq 2$, then by using (6.13) for $s = p, q$ simultaneously, we obtain

$$\begin{aligned}
 \|f\|_{\dot{B}_{r,1}^0(G)} &\leq \sum_{j=0}^{\infty} \sum_{k=j-1}^{j+1} \|\psi_j(\mathcal{R})f * \mathcal{F}_G^{-1}[\psi_k(\pi[\mathcal{R}])]\|_{L^r(G)} \\
 &\lesssim \sum_{j \geq l} \sum_{k=j-1}^{j+1} 2^{kQ(\frac{1}{p}-\frac{1}{r})} \|\psi_j((1+\mathcal{R})^{1/\nu})f\|_{L^p(G)} \\
 &\quad + \sum_{j < l} \sum_{k=j-1}^{j+1} 2^{kQ(\frac{1}{q}-\frac{1}{r})} \|\psi_j(\mathcal{R})f\|_{L^q(G)} \\
 &\lesssim \sum_{j \geq l} 2^{jQ(\frac{1}{p}-\frac{1}{r})} \|\psi_j(\mathcal{R})f\|_{L^p(G)} + \sum_{j < l} 2^{jQ(\frac{1}{q}-\frac{1}{r})} \|\psi_j(\mathcal{R})f\|_{L^q(G)} \\
 &\lesssim \sum_{j \geq l} 2^{jQ(\frac{1}{p}-\frac{1}{r}-\frac{\lambda}{Q})} 2^{j\lambda} \|\psi_j(\mathcal{R})f\|_{L^p(G)} + \sum_{j < l} 2^{jQ(\frac{1}{q}-\frac{1}{r}-\frac{\mu}{Q})} 2^{j\mu} \|\psi_j(\mathcal{R})f\|_{L^q(G)} \\
 &\lesssim \sum_{j \geq l} 2^{jQ(\frac{1}{p}-\frac{1}{r}-\frac{\lambda}{Q})} \|f\|_{\dot{B}_{p,\infty}^\lambda} + \sum_{j < l} 2^{jQ(\frac{1}{q}-\frac{1}{r}-\frac{\mu}{Q})} \|f\|_{\dot{B}_{q,\infty}^\mu} \\
 &\lesssim 2^{lQ(\frac{1}{p}-\frac{1}{r}-\frac{\lambda}{Q})} \|f\|_{\dot{B}_{p,\infty}^\lambda} + 2^{lQ(\frac{1}{q}-\frac{1}{r}-\frac{\mu}{Q})} \|f\|_{\dot{B}_{q,\infty}^\mu},
 \end{aligned}$$

the last inequality due to (6.10). If we put $A := \|f\|_{\dot{B}_{p,\infty}^\lambda} \|f\|_{\dot{B}_{q,\infty}^\mu}^{-1}$, then

$$\begin{aligned}
 2^{lQ(\frac{1}{p}-\frac{1}{r}-\frac{\lambda}{Q})} \|f\|_{\dot{B}_{p,\infty}^\lambda} + 2^{lQ(\frac{1}{q}-\frac{1}{r}-\frac{\mu}{Q})} \|f\|_{\dot{B}_{q,\infty}^\mu} \\
 = (2^{lQ(\frac{1}{p}-\frac{1}{r}-\frac{\lambda}{Q})} A^{1-\theta} + 2^{lQ(\frac{1}{q}-\frac{1}{r}-\frac{\mu}{Q})} A^{-\theta}) \|f\|_{\dot{B}_{p,\infty}^\lambda}^\theta \|f\|_{\dot{B}_{q,\infty}^\mu}^{1-\theta}.
 \end{aligned}$$

Let us define the positive parameter $\sigma = (\lambda - \frac{Q}{p} + \frac{Q}{r}) - (\mu - \frac{Q}{q} + \frac{Q}{r})$, and assume that l satisfies

$$2^l \leq A^{\frac{1}{\sigma}} < 2^{\frac{l+1}{\sigma}}. \quad (6.14)$$

We can assume (6.14) if we take $l = [\frac{\nu}{\sigma} \log_2(A)]$, where $[\cdot]$ denotes the integer part function on the real numbers. Then we have

$$\begin{aligned}
 2^{lQ(\frac{1}{p}-\frac{1}{r}-\frac{\lambda}{Q})} A^{1-\theta} + 2^{lQ(\frac{1}{q}-\frac{1}{r}-\frac{\mu}{Q})} A^{-\theta} \\
 \lesssim A^{\frac{1}{\sigma}(\frac{Q}{p}-\frac{Q}{r}-\lambda)+\frac{1}{\sigma}(\lambda-\frac{Q}{p}+\frac{Q}{r})} + A^{\frac{1}{\sigma}(\frac{Q}{q}-\frac{Q}{r}-\mu)+\frac{1}{\sigma}(\mu-\frac{Q}{q}+\frac{Q}{r})} = 2,
 \end{aligned}$$

where in the last estimate we have used that

$$1 - \theta = \frac{1}{\sigma}(\lambda - \frac{Q}{p} + \frac{Q}{r}) \quad \text{and} \quad -\theta = \frac{1}{\sigma}(\mu - \frac{Q}{q} + \frac{Q}{r}). \quad (6.15)$$

Hence we obtain

$$\|f\|_{\dot{B}_{r,1}^0(G)} \lesssim \|f\|_{\dot{B}_{p,\infty}^\lambda(G)}^\theta \|f\|_{\dot{B}_{q,\infty}^\mu(G)}^{1-\theta},$$

which shows the estimate (i). The inequality in (ii) can be proved in a similar way. Finally, we have (iii) if we use (i) together with the embeddings $\dot{B}_{r,1}^0(G) \hookrightarrow L^r(G)$ and $H^{\rho,r}(G) \hookrightarrow \dot{B}_{r,\infty}^\rho(G)$ proved in Theorem 5.2.

7 Localisation of Besov Spaces on Graded Lie Groups

In this section we prove local embedding properties of Besov spaces $B_{p,q}^r(G)$ with the ones defined in a local way on \mathbb{R}^n . First we recall the notion of Besov spaces on \mathbb{R}^n . For $x, h \in \mathbb{R}^n$ and $f \in L^p(\mathbb{R}^n)$, let us denote

$$\Delta_h^m f(x) := \sum_{k=0}^m C_m^k (-1)^{m-k} f(x + kh) \quad (7.1)$$

and

$$\omega_p^m(t, f) := \sup_{|h| \leq t} \|\Delta_h^m f\|_{L^p(\mathbb{R}^n)}. \quad (7.2)$$

Then, by following [27] for $r > 0$ and $1 \leq p, q \leq \infty$, the Euclidean Besov space $B_{p,q}^r(\mathbb{R}^n)$ can be considered endowed with the norm

$$\|f\|_{B_{p,q}^r(\mathbb{R}^n)} = \|f\|_{L^p(\mathbb{R}^n)} + \sum_{m=0}^n \left(\int_0^\infty (t^{-r} \omega_p^m(t, f))^q dt \right)^{\frac{1}{q}} \quad (7.3)$$

for $q < \infty$, and with an obvious modification in the case $q = \infty$. As the reviewer of this paper noticed, the formula in (7.3) in the Euclidean case appeared first in the fundamental work due to Besov [4].

By considering the property $(I - \mathcal{L})^{\frac{r}{2}}(B_{p,q}^r(\mathbb{R}^n)) = B_{p,q}^{r-\alpha}(\mathbb{R}^n)$, where \mathcal{L} is the Laplace operator on \mathbb{R}^n , for $r < 0$, we can consider on $B_{p,q}^r(\mathbb{R}^n)$, $1 \leq p, q \leq \infty$, the norm

$$\|f\|_{B_{p,q}^r(\mathbb{R}^n)} = \|(I - \mathcal{L})^{-\frac{s}{2}} f\|_{B_{p,q}^{s+r}(\mathbb{R}^n)}, \quad (7.4)$$

where s is a fixed real satisfying $s + r > 0$. It is a known fact that the definitions of Besov spaces on \mathbb{R}^n by using the functional (7.2) are equivalent to those using Littlewood-Paley partitions for the Laplacian on \mathbb{R}^n , in a analogous way as we have defined Besov space on graded Lie groups by using Rockland operators. It can be obtained if in particular in Definition 5.1 we put $G = \mathbb{R}^n$ and $\mathcal{R} = \Delta_x$, the positive Laplacian over \mathbb{R}^n . If we denote for a graded Lie group G the localisation space by

$$B_{p,q}^r(G, loc) = \{f \in \mathcal{D}'(G) : \phi \cdot f \in B_{p,q}^r(G), \text{ for all } \phi \in C_0^\infty(G)\} \quad (7.5)$$

we have the following result.

Proposition 7.1 *If $B_{p,q}^r(G, loc)$ denotes the local Besov space defined above, then for all $r \in \mathbb{R}$, $1 < p < \infty$ and $0 < q \leq \infty$ we have*

$$B_{p,q}^{\frac{r}{v_1}}(G, loc) \subset B_{p,q}^r(\mathbb{R}^n, loc) \subset B_{p,q}^{\frac{r}{v_n}}(G, loc), \quad (7.6)$$

where v_1 and v_n are respectively the smallest and the largest weights of the dilations.

Proof It was proved in [9, Theorem 4.4.24] that the following embedding of local Sobolev spaces holds:

$$H^{\frac{s}{v_1}, p}(G, loc) \subset H^{s,p}(\mathbb{R}^n, loc) \subset H^{\frac{s}{v_n}, p}(G, loc), \quad (7.7)$$

for all $s \in \mathbb{R}$. Thus, the result now follows by using real interpolation in the sense of Theorem 6.1.

Remark 7.1 A number of embeddings results describing relations between subelliptic and classical functional spaces were given by Rothschild and Stein [24]. As for a comparison with the norm in (7.3) we refer the reader to Pesenson [21].

8 Fourier Multipliers and Spectral Multipliers

In this section we give results for the boundedness of spectral and of Fourier multipliers in Besov spaces on graded Lie groups.

8.1 Negative Results for Left-Invariant Operators

There are some restrictions on indices for Besov spaces on which left-invariant operators may be bounded.

Theorem 8.1 *Let G be a graded Lie group and let T be a linear left-invariant operator bounded from $B_{p,q}^r(G)$ (respectively, $\dot{B}_{p,q}^r(G)$) into $B_{\tilde{p},\tilde{q}}^{\tilde{r}}(G)$, (respectively, $\dot{B}_{\tilde{p},\tilde{q}}^{\tilde{r}}(G)$), for $1 \leq p, \tilde{p} < \infty$, $-\infty < r, \tilde{r} < \infty$, and $0 < q, \tilde{q} \leq \infty$. If $1 \leq \tilde{p} < p < \infty$, then $T = 0$.*

Proof Let $|\cdot|$ be a homogeneous norm on G . It is known (see [8, Lemma 3.2.5]) that

$$\lim_{|h| \rightarrow \infty} \|f + \tau_h f\|_{L^p(G)} = 2^{\frac{1}{p}} \|f\|_{L^p(G)},$$

where τ_h is defined by $\tau_h f(x) = f(hx)$, $x, h \in G$. First, we will prove the case where $0 < \tilde{q} < \infty$ in the inhomogeneous case. By the boundedness of T we have $\|Tf\|_{B_{\tilde{p},\tilde{q}}^{\tilde{r}}(G)} \leq \|T\| \|f\|_{B_{p,q}^r(G)}$, where $\|T\| = \|T\|_{B(r,p,q;\tilde{r},\tilde{p},\tilde{q})}$ is the usual operator norm. So, for every $h \in G$ we have

$$\|T(f + \tau_h f)\|_{B_{\tilde{p},\tilde{q}}^{\tilde{r}}(G)} \leq C \|f + \tau_h f\|_{B_{p,q}^r(G)}.$$

Now, we compute both sides as $|h| \rightarrow \infty$. We observe that

$$\begin{aligned} \|T(f + \tau_h f)\|_{B_{\tilde{p},\tilde{q}}^{\tilde{r}}(G)} &= \left(\sum_{l=0}^{\infty} 2^{l\tilde{r}\tilde{q}} \|\psi_l((I + \mathcal{R})^{\frac{1}{\tilde{p}}}) T(f + \tau_h f)\|_{L^{\tilde{p}}(G)}^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}} \\ &= \left(\sum_{l=0}^{\infty} 2^{l\tilde{r}\tilde{q}} \|\psi_l((I + \mathcal{R})^{\frac{1}{\tilde{p}}}) Tf + \psi_l((I + \mathcal{R})^{\frac{1}{\tilde{p}}}) T\tau_h f\|_{L^{\tilde{p}}(G)}^{\tilde{q}} \right)^{\frac{1}{\tilde{q}}}. \end{aligned}$$

Because, T and $\psi_l(\mathcal{R}^{\frac{1}{\tilde{p}}})$, $l \in \mathbb{N}_0$, are left-invariant, we obtain

$$\begin{aligned} \lim_{|h| \rightarrow \infty} \|\psi_l((I + \mathcal{R})^{\frac{1}{\tilde{p}}}) Tf + \psi_l(I + \mathcal{R}) T\tau_h f\|_{L^{\tilde{p}}(G)} \\ = \lim_{|h| \rightarrow \infty} \|\psi_l((I + \mathcal{R})^{\frac{1}{\tilde{p}}}) Tf + \tau_h \psi_l((I + \mathcal{R})^{\frac{1}{\tilde{p}}}) Tf\|_{L^{\tilde{p}}(G)} \\ = 2^{\frac{1}{\tilde{p}}} \|\psi_l((I + \mathcal{R})^{\frac{1}{\tilde{p}}}) Tf\|_{L^{\tilde{p}}(G)}. \end{aligned}$$

Hence

$$\lim_{|h| \rightarrow \infty} \|T(f + \tau_h f)\|_{B_{\tilde{p},\tilde{q}}^{\tilde{r}}(G)} = 2^{\frac{1}{\tilde{p}}} \|Tf\|_{B_{\tilde{p},\tilde{q}}^{\tilde{r}}(G)}.$$

With a similar proof we obtain

$$\lim_{|h| \rightarrow \infty} \|f\|_{B_{p,q}^r(G)} = 2^{\frac{1}{p}} \|f\|_{B_{p,q}^r(G)}.$$

Hence

$$2^{\frac{1}{\tilde{p}}} \|Tf\|_{B_{\tilde{p},\tilde{q}}^{\tilde{r}}(G)} \leq 2^{\frac{1}{\tilde{p}}} \|T\| \|f\|_{B_{p,q}^r(G)}.$$

The last inequality implies that $\|T\| \leq 2^{\frac{1}{p}-\frac{1}{\tilde{p}}} \|T\|$. Thus, if $p > \tilde{p}$ then T is the null operator. The proof for $\tilde{q} = \infty$ is analogous.

8.2 Fourier Multipliers on Besov Spaces

Throughout this subsection we consider (right) homogeneous and inhomogeneous Besov spaces. In order to introduce our main result of this subsection we consider the following remark on the commutativity of operators with spectral measures.

Remark 8.2 Let R be a self-adjoint operator with spectral measure $E(\lambda)_{\lambda>0}$. Then, the spectral theorem gives $R = \int \lambda dE_\lambda$, and by the Stone's formula we have the following integral representation for every spectral projection $E(\lambda)$, (see Theorem 7.17 of [28])

$$E(\lambda) = \lim_{\delta \rightarrow 0^+} \lim_{\varepsilon \rightarrow 0^+} \int_{-\infty}^{\lambda+\delta} ([t - i\varepsilon - R]^{-1} - [t + i\varepsilon - R]^{-1}) dt. \quad (8.1)$$

If a closed operator T commutes with R , then T commutes with its resolvent operator $(z - R)^{-1}$ and hence with its spectral measure $(E(\lambda))_{\lambda>0}$. Now, if f is a bounded continuous function on $[0, \infty)$ and

$$f(R) = \int f(\lambda) dE(\lambda), \quad (8.2)$$

then we can write

$$f(R) = \lim_{\|P\| \rightarrow 0^+} \sum_{i=1, \lambda_i \in P}^{\infty} [f(\lambda_i)] [E_{\lambda_i} - E_{\lambda_{i-1}}], \quad (8.3)$$

where in the limit above, $P = \{0 = \lambda_0 < \lambda_1 < \lambda_2 < \dots\}$ denotes a partition of $[0, \infty)$. So, if T commutes with R , then it also commutes with every bounded continuous function of R defined by the functional calculus.

Now we present the following theorem on Fourier multipliers in Besov spaces where we establish a connection between L^p boundedness and Besov continuity of Fourier multipliers.

Theorem 8.3 Let G be a graded Lie group. Let $\sigma = \{\sigma(\pi) : \pi \in \widehat{G}\}$ be a μ -measurable field of operators in $L^2(\widehat{G})$. Let us assume that the corresponding operator $T = T_\sigma$, given by

$$T_\sigma u(x) = \int_{\widehat{G}} \text{Tr}(\pi(x)\sigma(\pi)\widehat{u}(\pi)) d\mu(\pi),$$

is a bounded operator from $L^{p_1}(G)$ into $L^{p_2}(G)$, $1 \leq p_i \leq \infty$. Then T is a bounded operator from the (right) Besov space $\dot{B}_{p_1,q}^r(G)$ into the (right) Besov space $\dot{B}_{p_2,q}^r(G)$, for all $-\infty < r < \infty$ and $0 < q \leq \infty$. Moreover, T is also a bounded operator from the (right) Besov space $B_{p,q}^r(G)$ into the (right) Besov space $B_{p,q}^r(G)$.

Proof For $f \in \mathcal{S}(G)$ we have $\mathcal{F}_G(Tf)(\pi) = \sigma(\pi)\widehat{f}(\pi) = \mathcal{F}_G(f * (\mathcal{F}_G^{-1}\sigma))(\pi)$. If \mathcal{R} is a right invariant positive Rockland operator, then for every $a \in \mathbb{C}$ (see Proposition 4.4.30 of [9])

$$\mathcal{R}^a T f = \mathcal{R}^a (f * \mathcal{F}_G^{-1}\sigma) = (\mathcal{R}^a f) * \mathcal{F}_G^{-1}\sigma = T(\mathcal{R}^a f), \quad (8.4)$$

in particular T commutes with \mathcal{R} . Since T commutes with \mathcal{R} , it commutes with its spectral measures, and with every bounded function of \mathcal{R} defined by the functional calculus (see, Remark 8.2). So,

$$\begin{aligned}\|Tf\|_{\dot{B}_{p_2,q}^r(G)}^q &= \sum_{l \in \mathbb{N}_0} 2^{lqr} \|\psi_l(\mathcal{R}^{\frac{1}{v}})Tf\|_{L^{p_2}}^q \\ &= \sum_{l \in \mathbb{N}_0} 2^{lqr} \|T\psi_l(\mathcal{R}^{\frac{1}{v}})f\|_{L^{p_2}}^q \\ &\leq \sum_{l \in \mathbb{N}_0} 2^{lqr} \|T\|_{\mathcal{L}(L^{p_1}(G), L^{p_2}(G))}^q \|\psi_l(\mathcal{R}^{\frac{1}{v}})f\|_{L^{p_1}}^q \\ &= \|T\|_{\mathcal{L}(L^{p_1}(G), L^{p_2}(G))}^q \|f\|_{\dot{B}_{p_1,q}^r(G)}^q.\end{aligned}$$

Thus $\|Tf\|_{\dot{B}_{p_2,q}^r(G)}^q \leq \|T\|_{\mathcal{L}(L^{p_1}(G), L^{p_2}(G))}^q \|f\|_{\dot{B}_{p_1,q}^r(G)}^q$. The proof for the inhomogeneous case is similar. So we end the proof.

We end this section with applications of Theorem 8.3 to some examples for the Fourier multipliers bounded on L^p and (right) Besov spaces. For notations and terminologies we follow [9].

Example 8.4 Let $T : \mathcal{S}(G) \rightarrow \mathcal{S}(G)$, G be a graded Lie group of homogeneous dimension Q . If T is left-invariant and homogeneous of degree v with

$$-Q < \operatorname{Re}(v) < 0, \quad (8.5)$$

and such that the right convolution kernel of T is continuous away from the origin, then $T : L^p(G) \rightarrow L^q(G)$ is a bounded operator for $1 < p, q < \infty$ and

$$\frac{1}{q} - \frac{1}{p} = \frac{\operatorname{Re}(v)}{Q}. \quad (8.6)$$

(c.f. Proposition 3.2.8 of [9, p. 138]). By Theorem 8.3, T is a bounded operator from the right Besov space $B_{p,s}^r(G)$ into the right Besov space $B_{q,s}^r(G)$ with p and q satisfying (8.6), $r \in \mathbb{R}$ and $0 < s \leq \infty$.

Example 8.5 Let $T : L^2(G) \rightarrow L^2(G)$ be a bounded and left-invariant operator. Let us assume that its distributional kernel coincides on $G \setminus \{0\}$ with a continuously differentiable function k with

$$\begin{aligned}\int_{|x| \geq \frac{1}{2}} |k(x)| dx &\leq A < \infty, \quad \sup_{0 < |x| \leq 1} |x|^Q |k(x)| \leq A, \\ \sup_{0 < |x| \leq 1} |x|^{Q+v_j} |X_j k(x)| &\leq A, \quad j = 1, 2, \dots,\end{aligned}$$

for some homogeneous quasi-norm $|\cdot|$ on G and for some $A > 0$. Then T is weak type $(1,1)$ and bounded on $L^p(G)$, $1 < p < \infty$, (c.f. [9, p. 145]). By using Theorem 8.3 we obtain the boundedness of T on the right Besov space $B_{p,q}^r(G)$, $0 < q \leq \infty$ and $r \in \mathbb{R}$.

Example 8.6 Let G be a graded Lie group. Let $\sigma \in L^2(\widehat{G})$. If

$$\|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}} \|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}} < \infty$$

with $s > \frac{Q}{2}$, then the corresponding multiplier T_σ extends to a bounded operator on $L^p(G)$ for all $1 < p < \infty$. By Theorem 2.4 we have

$$\|T_\sigma\|_{\mathcal{L}(L^p(G))} \leq C \max\{\|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}}, \|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}}\}. \quad (8.7)$$

This is the Hörmander-Mihlin Theorem presented in [8]. By Theorem 8.3, we obtain the boundedness of T_σ on the right Besov space $B_{p,q}^r(G)$ and by observing the proof of such theorem we conclude that

$$\|T_\sigma\|_{\mathcal{L}(B_{p,q}^r(G))} \leq C \max\{\|\sigma\|_{H^s, l.u., L, \eta, \mathcal{R}}, \|\sigma\|_{H^s, l.u., R, \eta, \mathcal{R}}\}. \quad (8.8)$$

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Declarations

Conflicts of interest On behalf of all authors, the corresponding author states that there is no conflict of interest.

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