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Porosities of Mandelbrot percolation

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Abstract We study porosities in the Mandelbrot percolation process using a notion of porosity that is based on the construction geometry. We show that, almost surely at almost all points with respect to the natural measure, the construction based mean porosities of the set and the natural measure exist and are equal to each other for all parameter values outside of a countable exceptional set. As a corollary, we obtain that, almost surely at almost all points, the regular lower porosities of the set and the natural measure are equal to zero, whereas the regular upper porosities obtain their maximum values.

Keywords Random sets, porosity, mean porosity

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1 Introduction

The porosity of a set describes the sizes of holes in the set. The concept dates back to the 1920's when Denjoy introduced a notion which he called index (see [7]). In today's terminology, this index is called the upper porosity (see Definition 3.1). The term porosity was introduced by Dolženko in [8]. Intuitively, if the upper porosity of a set equals α, then, in the set, there are holes of relative size α at arbitrarily small distances. On the other hand, the lower porosity (see Definition 3.1) guarantees the existence of holes of certain relative size at all sufficiently small distances. The upper porosity turned out to be useful in order to describe properties of exceptional sets, for example, for measuring sizes of sets where certain functions are non-differentiable. For more details about the upper porosity, we refer to an article of Zajíček [37]. Mattila [28] utilised the lower porosity to find upper bounds for Hausdorff dimensions of set, and Salli [33] verified the corresponding results for packing and box counting dimensions.

It turns out that upper porosity cannot be used to estimate the dimension of a set (see [29, Section 4.12]). An observation that there are sets which are not lower porous but nevertheless contain so many holes that their dimension is smaller than the dimension of the ambient space, leads to the concept of mean porosity of a set introduced by Koskela and Rohde [25] in order to study the boundary behaviour of conformal and quasiconformal mappings. Mean porosity guarantees that certain percentage of scales, that is, distances which are integer powers of some fixed number, contain holes of fixed relative size. Koskela and Rohde showed that, if a subset of the m-dimensional Euclidean space is mean porous, then its Hausdorff and packing dimensions are smaller than m. For a modification of their definition, see Definition 3.6.

The lower porosity of a measure was introduced by Eckmann and E. and M. Järvenpää in [11], the upper one by Mera and Morán in [31] and the mean porosity by Beliaev and Smirnov in [3]. The relations between porosities and dimensions of sets and measures have been investigated, for example, in [3,4, 16–19,21,35]. For further information on this subject, we refer to a survey by Shmerkin [34]. Porosity has also been used for studying the conical densities of measures (see [22,23]).

Note that sets with same dimension may have different porosities. In [20], E. and M. Järvenpää and Mauldin and, in [36], Urbański characterised deterministic iterated function systems whose attractors have positive porosity. Porosities of random recursive constructions were studied in [20]. Particularly interesting random constructions are those in which the copies of the seed set are glued together in such a way that there are no holes left. Thus, the corresponding deterministic system would be non-porous and the essential question is whether the randomness in the construction makes the set or measure porous. A classical example is the Mandelbrot percolation process (also known as the fractal percolation) introduced by Mandelbrot in 1974 in [27].

In the Mandelbrot percolation process on $\mathbb{R}^m$, one fixes a natural number $k \geq 2$, starts with a unit cube $J = [0,1]^m$, divides it into k^m closed k-adic subcubes with side length k^{-1} and let each one of them "survive" or "die" independently with probabilities p and $1-p$, respectively. The cubes with side length k^{-1} are called the first level subcubes. This procedure is repeated recursively: Each surviving i^{th} level subcube C with side length k^{-i} is divided into k^m closed k-adic subcubes with side length k^{-i-1} and each one of them survives or dies with probability p and $1-p$, respectively, independently of all the cubes on the current and on the previous levels that survived. This defines a random collection of the $(i+1)^{\text{th}}$ level subcubes with side length $k^{-(i+1)}$. The union of the i^{th} level subcubes that survived, as i approaches infinity, forms a decreasing sequence of compact sets whose limit is the limit set of the Mandelbrot percolation process, denoted by K_ω, where $\omega \in \Omega$ and Ω corresponds to all possible options for subcube survival. There exists a natural probability measure P on Ω describing this process. The limiting set K_ω is empty with positive probability but, according to [30, Theorem 1.1] (see also [24]), the Hausdorff dimension of K_ω is P-almost surely equal to

$$d = \frac{\log(k^m p)}{\log k} = m + \frac{\log p}{\log k} \tag{1.1}$$

provided that $K_\omega \neq \emptyset$. Using this number d, one may construct a martingale, whose almost sure limit induces a finite Borel measure ν_ω supported on K_ω with dimension equal to d. In the next section, we describe the main steps in the construction of the measure ν_ω.

In [20], it was shown that, P-almost surely, the points with minimum porosity as well as those with maximum porosity are dense in the limit set. However, the question about porosity of typical points and that of the natural measure remained open. Later, it turned out that, for typical points, the lower porosity equals 0 and the upper one is equal to $\frac{1}{2}$ as conjectured in [20]. Indeed, this is a corollary of the results of Chen et al. in [6] dealing with estimates on the dimensions of sets of exceptional points regarding the porosity. The proofs in [6] rely on the extinction of random subtrees of a rooted k^m-ary tree $\mathcal{T}$ (similarly to the Galton-Watson process) that satisfy certain properties and make use of the geometric structures. The main idea of the present paper is based on the ergodicity (or, the strong law of large numbers) for the indicator functions of the holes, with a small role of the construction geometry.

In order to state our main results, we need to describe some steps in the percolation process more formally. Let $\mathcal{T}$ be a rooted k^m-ary tree. The limit set K_ω can be represented as a projection of the boundary of a subtree of $\mathcal{T}$ into the unit cube. Indeed, the initial cube $J = J_\emptyset$ of the percolation process corresponds to the root of $\mathcal{T}$, indexed by the empty sequence $\emptyset$. The vertices of $\mathcal{T}$, whose distance to the root is 1, are associated with the first level subcubes with side length k^{-1} and are indexed by the alphabet $I = \{1, \dots, k^m\}$, the vertices with distance 2 to the root are associated with the second level subcubes and indexed by two letter long sequences from I, etc. In this way, the set of vertices of $\mathcal{T}$ is associated with $I^* = \cup_{i=0}^{\infty} I^i$ and the boundary $\partial\mathcal{T}$ of $\mathcal{T}$ can be identified with $I^{\mathbb{N}}$. For all $i \in \mathbb{N}$ and $\sigma \in I^i$, we use the notation J_σ for the unique closed subcube of $J_\emptyset$ with side length k^{-i} coded by σ. The natural projection $x\colon I^{\mathbb{N}} \to J_\emptyset$ maps an infinite code word $\sigma \in I^{\mathbb{N}}$ to the intersection of the nested sequence of the subcubes corresponding to all finite initial subsequences of σ, and this intersection is a singleton. The probability space Ω is the set of all functions $\omega\colon I^* \to \{c, n\}$ with c referring for "choosing" and n for "neglecting." The probability measure P on the space Ω is generated by the Bernoulli measures with choosing probability p.

On $I^{\mathbb{N}}$, there exists a random measure μ_ω whose pushforward via the natural projection map x into K_ω is ν_ω. The precise definition of μ_ω is adduced in Section 2. Finally, we define a probability measure Q on $I^{\mathbb{N}} \times \Omega$, describing typical points of the limit set K_ω for typical ω, by integrating μ_ω with respect to P so that, for every Borel set $B \subset I^{\mathbb{N}} \times \Omega$, we have

$$Q(B) = \int \mu_\omega(B_\omega) dP(\omega), \tag{1.2}$$

where $B_\omega = \{\eta \in I^{\mathbb{N}} \mid (\eta, \omega) \in B\}$. The detailed construction of the measure Q (for example, the measurability issues) the interested reader can find in the paper by Graf, Mauldin and Williams [14, Section 1]. We note that the definition of μ_ω in [14,30] differs by the factor $(\operatorname{diam} J)^d$ compared to the convention we have chosen in this paper. The operation of integrating measures, in general, is described by Elliott in [10] as regular conditional measure.

In this paper, we consider mean porosities tied to the geometry of the construction. By the construction based mean porosity, we mean, for a given point of the limit set, the average proportion of levels of the tree $\mathcal{T}$ at which we can observe a cube-shaped hole of the given relative size inside the construction cube. We prove that the construction based mean porosities of the natural measure and of the limit set exist and are equal to each other Q-almost surely for all parameter values outside of a countable set (see Theorem 4.11). We also show that the construction based mean porosities are continuous as a function of the parameter outside this exceptional set (see Theorem 4.5). Unlike the upper and lower porosities, the construction based mean porosities of the set and the natural measure at typical points are non-trivial. Indeed, we prove that the construction based mean porosities of the set and the natural measure are positive and less than one Q-almost surely for all non-trivial parameter values (see Corollaries 4.8 and 4.13). In the sequel, we omit the adjective "construction based" when referring to porosities, if this is clear from the context. As an application of our results, we solve the conjecture of [20] completely (and give a new proof for the part solved in [6]) by showing that, almost surely for almost all points with respect to the natural measure ν_ω, the regular (based on Euclidean balls) lower porosities of the limit set and of the measure are equal to the minimum value of 0, whereas the regular upper porosities of the set and of the measure ν_ω attain their maximum values of $\frac{1}{2}$ and 1, respectively (see Corollary 4.14).

The article is organised as follows. In Section 2, we go over the definition of the limit set K_ω of the Mandelbrot percolation process and adduce some basic facts about the natural measure ν_ω. In Section 3, we define porosities and construction based mean porosities and describe some of their properties. Finally, in Section 4, we prove our results about construction based mean porosities of the limit set and of the natural measure in the Mandelbrot percolation process.

2 Preliminaries

For $k, m \in \mathbb{N}$ with $k \geq 2$, we consider the alphabet $I = \{1, \ldots, k^m\}$ and the set of all finite sequences of the alphabet $I^* = \bigcup_{i=0}^{\infty} I^i$, where $I^0 = \emptyset$. An element $\sigma \in I^i$ is called a word and its length is $|\sigma| = i$. For all $\sigma \in I^i$ and $\sigma' \in I^j$, we denote by $\sigma * \sigma'$ the element of I^{i+j} whose first i coordinates are those of σ and the last j coordinates are those of σ'. For all $i \in \mathbb{N}$ and $\sigma \in I^* \cup I^{\mathbb{N}}$, denote by $\sigma|_i$ the word in I^i formed by the first i elements of σ. For $\sigma \in I^*$ and $\tau \in I^* \cup I^{\mathbb{N}}$, we write $\sigma \prec \tau$ if the sequence τ starts with σ.

Let $\Omega = \{c, n\}^{I^*}$ be the set of functions $\omega \colon I^* \to \{c, n\}$ equipped with the topology induced by the metric $\rho(\omega, \omega') = k^{-|\omega \wedge \omega'|}$, where

$$|\omega \wedge \omega'| = \min\{j \in \mathbb{N} \mid \exists \sigma \in I^j \text{ with } \omega(\sigma) \neq \omega'(\sigma)\}.$$

Each $\omega \in \Omega$ can be thought of as a code that tells us which cubes we choose (c) and which we neglect (n). The image of $\eta \in I^{\mathbb{N}}$ under the natural projection from $I^{\mathbb{N}}$ to $[0,1]^m$ is denoted by $x(\eta)$, that is,

$$x(\eta) = \bigcap_{i=0}^{\infty} J_{\eta|_i},$$

where $\eta|_0 = \emptyset$ and $J_{\eta|_i} \subset [0,1]^m$ is the k-adic cube with side length k^{-i} as described in the Introduction. If $\omega(\sigma) = n$ for $\sigma \in I^i$ we define $J_\sigma(\omega) = \emptyset$, and if $\omega(\sigma) = c$ we set $J_\sigma(\omega) = J_\sigma$. For all $\omega \in \Omega$, we define

$$K_\omega = \bigcap_{i=0}^{\infty} \bigcup_{\sigma \in I^i} J_\sigma(\omega).$$

Fix $0 \leq p \leq 1$. We make the above construction random by defining a probability measure P on Ω by setting $P = (p\delta_c + (1-p)\delta_n)^{I^*}$, where δ_s is the Dirac measure at s for $s \in \{c, n\}$, that is, every cube J_σ is chosen with respect to the measure $p\delta_c + (1-p)\delta_n$ independently of every other cube J_τ. We shortly explain how our definition is related to the one described in the fifth paragraph of the Introduction. Observe that if $\omega(\sigma) = n$ for some $\sigma \in I^*$, then the value of ω on $\sigma * \tau$ for $\tau \in I^*$ plays no role in the definition of K_ω since $J_\sigma(\omega) = \emptyset$. Define an equivalence relation $\sim$ on Ω by setting that $\omega \sim \omega'$ if, for all $\eta \in I^{\mathbb{N}}$ with $x(\eta) \notin K_\omega$, we have $\min\{i \mid \omega(\eta|_i) = n\} = \min\{i \mid \omega'(\eta|_i) = n\}$ and similarly for all $\eta \in I^{\mathbb{N}}$ with $x(\eta) \notin K_{\omega'}$. Let $\widetilde{\Omega} = \Omega / \sim$ and let $\widetilde{P}$ be the probability measure on $\widetilde{\Omega}$ induced by P. Then $(\widetilde{\Omega}, \widetilde{P})$ is the probability space defining the Mandelbrot percolation as described in the fifth paragraph of the Introduction, that is, for any finite set of codes σ_i of finite length such that $\sigma_{i_1} \not\prec \sigma_{i_2}$ and $\sigma_{i_2} \not\prec \sigma_{i_2}$ for all $i_1 \neq i_2$, if J_{σ_i} is chosen then $J_{\sigma_i * j}$, $j = 1, \ldots, k^m$, are chosen independently with probability p for all different squares J_{σ_i}. Both (Ω, P) and $(\widetilde{\Omega}, \widetilde{P})$ are used in the literature to model the Mandelbrot percolation but, in the sequel, we use only the space (Ω, P).

It is a well-known result in the theory of branching processes that if the expectation of the number of chosen cubes of side length k^{-1} does not exceed one, then the limit set K_ω is P-almost surely empty and, otherwise, $K_\omega \neq \emptyset$ with positive probability (see [1, Theorem 1]). In our case, this expectation equals $k^m p$ and, thus,

$$P(K_\omega \neq \emptyset) > 0 \iff k^{-m} < p \leq 1. \tag{2.1}$$

For all $\sigma, \tau \in I^*$, we define the random variable

$$l_{\sigma,\tau}(\omega) = k^{-|\tau|} \mathbb{1}_{\{\omega(\sigma*(\tau|_j))=c \text{ for all } j=1,\ldots,|\tau|\}}$$

on Ω. In the case $\sigma = \emptyset$, we write simply l_τ for $l_{\emptyset,\tau}$. For all $\sigma \in I^*$, the martingale $\{\sum_{\tau \in I^j} l^d_{\sigma,\tau}\}_{j\in\mathbb{N}}$ is L^2-bounded according to [30, Theorem 2.1], where d is as in (1.1). Therefore, it converges almost surely to a finite limit, which we denote by $X_\sigma(\omega)$. By the choice of d, we have that

$$E_P\Big[\sum_{\tau \in I^j} l^d_{\sigma,\tau}\Big] = 1 \text{ for all } j \in \mathbb{N} \text{ and } E_P[X_\sigma] = 1. \tag{2.2}$$

Further,

$$X_\sigma(\omega) = \sum_{\tau \in I^j} l_{\sigma,\tau}(\omega)^d X_{\sigma*\tau}(\omega) \tag{2.3}$$

whenever X_σ is defined. For all $\sigma, \tau \in I^*$, the random variables X_σ and X_τ are identically distributed, and if $\sigma \not\prec \tau$ and $\tau \not\prec \sigma$, they are independent. For P-almost all $\omega \in \Omega$, we define a non-trivial finite random Borel measure μ_ω on $I^{\mathbb{N}}$ by setting $\mu_\omega([\sigma]) = l_\sigma(\omega)^d X_\sigma(\omega)$ for cylinder sets $[\sigma] = \{\tau \in I^{\mathbb{N}} : \sigma \prec \tau\}$ and extending μ_ω naturally to all Borel sets. (Recall that the cylinder sets generate the topology of $I^{\mathbb{N}}$.) Let ν_ω be the projection of μ_ω under the projection map $x : I^{\mathbb{N}} \to K_\omega$. By construction, we have the following properties (for a detailed proof, see [30, Theorem 3.2]).

Property 2.1 For P-almost all $\omega \in \Omega$, we have that

$$\nu_\omega(J_\sigma) = l_\sigma(\omega)^d X_\sigma(\omega) \text{ for all } \sigma \in I^* \text{ and}$$
$$\sum_{\substack{\tau \in I^j \\ J_\tau \cap B \neq \emptyset}} l_\tau(\omega)^d X_\tau(\omega) \searrow \nu_\omega(B) \text{ as } j \to \infty \text{ for all Borel sets } B \subset K_\omega.$$

By (1.2) and Property 2.1, expectations with respect to the measures P and Q are connected in the following way (see also [14, (1.16)]).

Property 2.2 If $j \in \mathbb{N}$ and $Y : I^{\mathbb{N}} \times \Omega \to \mathbb{R}$ is a random variable such that $Y(\eta, \omega) = Y(\eta', \omega)$ provided that $\eta|_j = \eta'|_j$, then

$$E_Q[Y] = E_P\Big[\sum_{\sigma \in I^j} l^d_\sigma X_\sigma Y(\sigma, \cdot)\Big].$$

For all $l \in \mathbb{N} \cup \{0\}$, define random variables X_l on $I^{\mathbb{N}} \times \Omega$ by $X_l(\eta, \omega) = X_{\eta|_l}(\omega)$. Then the random variables X_l, $l \in \mathbb{N} \cup \{0\}$, have the same distribution. By the definitions of μ_ω and Q and Property 2.2, we have that

$$\begin{aligned} &Q(X_l = 0) = 0 \text{ and} \\ &\quad E_Q[X_l] = E_P[\sum_{\sigma \in I^l} l_\sigma^d X_\sigma^2] = E_P[X_0^2] < \infty \text{ for all } l \in \mathbb{N} \cup \{0\}, \end{aligned} \tag{2.4}$$

since X_0 has a finite second moment by [30, Theorem 2.1].

3 Porosities

In this section, we define porosities and construction based mean porosities of sets and measures and prove some basic properties for them.

Definition 3.1 Let $A \subset \mathbb{R}^m$, $x \in \mathbb{R}^m$ and $r > 0$. The local porosity of A at x at distance r is

$$\begin{aligned} \text{por}(A, x, r) = \sup\{\alpha \geq 0 \mid &\text{there is } z \in \mathbb{R}^n \text{ such that} \\ &B(z, \alpha r) \subset B(x, r) \setminus A\}, \end{aligned}$$

where the open ball centred at x and with radius r is denoted by $B(x, r)$. The lower and upper porosities of A at x are defined as

$$\underline{\text{por}}(A, x) = \liminf_{r \to 0} \text{por}(A, x, r) \text{ and } \overline{\text{por}}(A, x) = \limsup_{r \to 0} \text{por}(A, x, r),$$

respectively. If $\underline{\text{por}}(A, x) = \overline{\text{por}}(A, x)$, the common value, denoted by $\text{por}(A, x)$, is called the porosity of A at x.

Definition 3.2 The lower and upper porosities of a Radon measure μ on $\mathbb{R}^m$ at a point $x \in \mathbb{R}^m$ are defined by

$$\begin{aligned} \underline{\text{por}}(\mu, x) &= \lim_{\varepsilon \to 0} \liminf_{r \to 0} \text{por}(\mu, x, r, \varepsilon) \text{ and} \\ \overline{\text{por}}(\mu, x) &= \lim_{\varepsilon \to 0} \limsup_{r \to 0} \text{por}(\mu, x, r, \varepsilon), \end{aligned}$$

respectively, where for all $r, \varepsilon > 0$,

$$\begin{aligned} \text{por}(\mu, x, r, \varepsilon) = \sup\{\alpha \geq 0 \mid &\text{ there is } z \in \mathbb{R}^m \text{ such that} \\ &B(z, \alpha r) \subset B(x, r) \text{ and } \mu(B(z, \alpha r)) \leq \varepsilon \mu(B(x, r))\}. \end{aligned}$$

If the upper and lower porosities agree, the common value is called the porosity of μ at x and denoted by $\text{por}(\mu, x)$.

Remark 3.3 (a) In some sources, the condition $B(z,\alpha r)\subset B(x,r)\setminus A$ in Definition 3.1 is replaced by the condition $B(z,\alpha r)\cap A=\emptyset$, leading to the definition

$$\widetilde{\operatorname{por}}(A,x,r)=\sup\{\alpha\geq 0\mid \text{there is } z\in B(x,r) \text{ such that } B(z,\alpha r)\cap A=\emptyset\}.$$

It is not difficult to see that

$$\widetilde{\operatorname{por}}(A,x)=\frac{\operatorname{por}(A,x)}{1-\operatorname{por}(A,x)},$$

which is valid both for the lower and upper porosity. Indeed, this follows from two geometric observations. First, $B(z,\alpha r)\cap A=\emptyset$ with $z\in\partial B(x,r)$ if and only if $B(z,\alpha r)\subset B(x,(1+\alpha)r)\setminus A$ with $\partial B(z,\alpha r)\cap\partial B(x,(1+\alpha)r)\neq\emptyset$, where the boundary of a set B is denoted by ∂B. Second, at local minima and maxima of the function $r\mapsto\operatorname{por}(A,x,r)$, we have $\partial B(z,\alpha r)\cap\partial B(x,(1+\alpha)r)\neq\emptyset$, and at local minima and maxima of the function $r\mapsto\widetilde{\operatorname{por}}(A,x,r)$, we have $z\in\partial B(x,r)$.

(b) Unlike the dimension, which is the same for equivalent metrics, the porosity is sensitive to the choice of a metric. For example, defining cube-porosities by using cubes instead of balls in the definition, there is no formula to convert porosities to cube-porosities or vice versa. It is easy to construct a set such that the cube-porosity attains its maximum value (at some point) but the porosity does not. Take, for example, the union of the x- and y-axes in the plane. However, the lower porosity is positive, if and only if the lower cube-porosity is positive.

(c) In general metric spaces, in addition to $B(z,\alpha r)\subset B(x,r)\setminus A$, it is sometimes useful to require that the empty ball $B(z,\alpha r)$ is inside the reference ball $B(x,r)$ also algebraically, that is, $d(x,z)+\alpha r\leq r$. For further discussion about this matter, see [32].

The lower and upper porosities give the relative sizes of the smallest and largest holes, respectively. Taking into considerations the frequency of scales where the holes appear, leads to the notion of mean porosity. We proceed by giving a definition which is adapted to the Mandelbrot percolation process. We will use the maximum metric ϱ, that is, $\varrho(x,y)=\max_{i\in\{1,\dots,m\}}\{|x_i-y_i|\}$, and denote by $B_\varrho(y,r)$ the open ball centred at y and with radius r with respect to this metric. Recall that the balls in the maximum metric are cubes whose faces are parallel to the coordinate planes.

Definition 3.4 Let $A\subset\mathbb{R}^m$, μ be a Radon measure on $\mathbb{R}^m$, $x\in\mathbb{R}^m$, $\alpha\in[0,1]$ and $\varepsilon>0$. For $j\in\mathbb{N}$, we say that A has an α-hole at scale j near x if there is a point $z\in Q_j^k(x)$ such that

$$B_\varrho(z,\tfrac{1}{2}\alpha k^{-j})\subset Q_j^k(x)\setminus A.$$

Here $Q_j^k(x)$ is the half-open k-adic cube of side length k^{-j} containing x and $B_\varrho(z,\frac{1}{2}\alpha k^{-j})$ is called an α-hole. We say that μ has an (α,ε)-hole at scale j near x if there is a point $z\in Q_j^k(x)$ such that

$$B_\varrho(z,\tfrac{1}{2}\alpha k^{-j})\subset Q_j^k(x) \text{ and } \mu(B_\varrho(z,\tfrac{1}{2}\alpha k^{-j}))\leq\varepsilon\mu(Q_j^k(x)).$$

Remark 3.5 Note that, unlike in Definition 3.1, we have divided the radius of the ball in the complement of A as well as that with small measure by 2 and, therefore, α may attain values between 0 and 1. The reason for this is that the point x may be arbitrarily close to the boundary of $Q_j^k(x)$ and, if the whole cube $Q_j^k(x)$ is empty, it is natural to say that there is a hole of relative size 1.

Definition 3.6 Let $\alpha \in [0,1]$. The lower α-mean porosity of a set $A \subset \mathbb{R}^m$ at a point $x \in \mathbb{R}^m$ is

$$\underline{\kappa}(A,x,\alpha) = \liminf_{i\to\infty} \frac{N_i(A,x,\alpha)}{i}$$

and the upper α-mean porosity is

$$\overline{\kappa}(A,x,\alpha) = \limsup_{i\to\infty} \frac{N_i(A,x,\alpha)}{i},$$

where

$$N_i(A,x,\alpha) = \operatorname{card}\{j \in \mathbb{N} \mid j \le i \text{ and } A \text{ has an } \alpha\text{-hole at scale } j \text{ near } x\}.$$

In the case the limit exists, it is called the α-mean porosity and denoted by $\kappa(A,x,\alpha)$. The lower α-mean porosity of a Radon measure μ on $\mathbb{R}^m$ at $x \in \mathbb{R}^m$ is

$$\underline{\kappa}(\mu,x,\alpha) = \lim_{\varepsilon\to 0} \liminf_{i\to\infty} \frac{\widetilde{N}_i(\mu,x,\alpha,\varepsilon)}{i}$$

and the upper one is

$$\overline{\kappa}(\mu,x,\alpha) = \lim_{\varepsilon\to 0} \limsup_{i\to\infty} \frac{\widetilde{N}_i(\mu,x,\alpha,\varepsilon)}{i},$$

where

$$\widetilde{N}_i(\mu,x,\alpha,\varepsilon) = \operatorname{card}\{j \in \mathbb{N} \mid j \le i \text{ and } \mu \text{ has an } (\alpha,\varepsilon)\text{-hole at scale } j \text{ near } x\}.$$

If the lower and upper mean porosities coincide, the common value, denoted by $\kappa(\mu,x,\alpha)$, is called the α-mean porosity of μ.

Remark 3.7 Mean porosity is highly sensitive to the choice of parameters. The definition is given in terms of k-adic cubes. For the Mandelbrot percolation, this is natural. For general sets, fixing an integer $h > 1$, a natural choice is to say that A has an α-hole at scale j near x, if there is $z \in \mathbb{R}^m$ such that $B(z,\alpha h^{-j} r_0) \subset B(x,h^{-j} r_0) \setminus A$ for some (or for all) $h^{-1} < r_0 \le 1$. However, the choice of r_0 and h matters as will be shown in Example 3.8 below. Shmerkin proposed in [35] the following base and starting scale independent notion of lower mean porosity of a measure (which can be adapted for sets and upper porosity as well): a measure μ is lower (α,κ)-mean porous at a point $x\in\mathbb{R}^m$ if

$$\liminf_{\rho\to 1}(\log\tfrac{1}{\rho})^{-1}\int_\rho^1 \mathbb{1}_{\{r\mid \mathrm{por}(\mu,x,r,\varepsilon)\geq\alpha\}}r^{-1}\,dr\geq\kappa \text{ for all } \varepsilon>0.$$

The disadvantage of this definition is that it is more complicated to calculate than the discrete version. To avoid these problems, one option is to aim at qualitative results concerning all parameter values, as our approach will show.

Next, we give a simple example demonstrating the dependence of mean porosity on the starting scale and the base of scales.

Example 3.8 Fix an integer $h>1$. In this example, we use a modification of Definitions 3.4 and 3.6, where $A\subset\mathbb{R}^m$ has an α-hole at scale j near x if there exists $z\in\mathbb{R}^m$ such that $B(z,\alpha h^{-j})\subset B(x,h^{-j})\setminus A$. Let $x\in\mathbb{R}^2$. We define a set $A\subset\mathbb{R}^2$ as follows. For all $i\in\mathbb{N}\cup\{0\}$, consider the half-open annulus $D(i)=\{y\in\mathbb{R}^2\mid h^{-i-1}<|y-x|\leq h^{-i}\}$. Let $A=\bigcup_{i=0}^\infty D(3i+1)\cup D(3i+2)$, that is, we choose two annuli out of every three successive ones and leave the third one empty. In this case, $\kappa(A,x,\frac12(1-h^{-1}))=\frac13$. If we replaced h by h^3 in the definition of scales, we would conclude that $\kappa(A,x,\frac12(1-h^{-1}))=1$. (Note that the lower and upper porosities are equal.) If we define A by starting with the two filled annuli, that is, $A=\bigcup_{i=0}^\infty D(3i)\cup D(3i+1)$, then $\kappa(A,x,\frac12(1-h^{-1}))=\frac13$ using scales determined by h and $\kappa(A,x,\frac12(1-h^{-1}))=0$ if scales are determined by powers of h^3. By mixing these construction in a suitable way, one easily finds an example where $\kappa(A,x,\frac12(1-h^{-1}))=\frac13$ for scales given by h, but $\underline{\kappa}(A,x,\frac12(1-h^{-1}))=0$ and $\overline{\kappa}(A,x,\frac12(1-h^{-1}))=1$ if the scales are determined by h^3.

We finish this section with some measurability results. For that we need some notation.

Definition 3.9 For all $j\in\mathbb{N}$ and $\alpha\in[0,1]$, define a function $\chi_j^\alpha\colon I^{\mathbb{N}}\times\Omega\to\{0,1\}$ by setting $\chi_j^\alpha(\eta,\omega)=1$, if and only if K_ω has an α-hole at scale j near $x(\eta)$. Define a function $\overline{\chi}_j^\alpha\colon I^{\mathbb{N}}\times\Omega\to\{0,1\}$ in the same way except that the α-hole is a closed ball instead of an open one. For all $\alpha\in(0,1)$, $\varepsilon>0$ and $j\in\mathbb{N}$, define a function $\chi_j^{\alpha,\epsilon}\colon I^{\mathbb{N}}\times\Omega\to\{0,1\}$ by setting $\chi_j^{\alpha,\epsilon}(\eta,\omega)=1$, if and only if ν_ω has an (α,ϵ)-hole at scale j near $x(\eta)$. Finally, define a function $\overline{\chi}_j^{\alpha,\epsilon}\colon I^{\mathbb{N}}\times\Omega\to\{0,1\}$ by setting $\overline{\chi}_j^{\alpha,\epsilon}(\eta,\omega)=1$, if and only if there exists $z\in Q_j^k(x(\eta))$ such that $\nu_\omega(\overline{B}_\varrho(z,\frac12\alpha k^{-j}))<\varepsilon\nu_\omega(Q_j^k(x(\eta)))$. Here the closed ball in metric ϱ centred at $z\in\mathbb{R}^m$ with radius $r>0$ is denoted by $\overline{B}_\varrho(z,r)$.

Lemma 3.10 *The maps*

$$\begin{aligned}&(\eta,\omega)\mapsto\underline{\kappa}(K_\omega,x(\eta),\alpha),\\&(\eta,\omega)\mapsto\overline{\kappa}(K_\omega,x(\eta),\alpha),\\&(\eta,\omega)\mapsto\underline{\kappa}(\nu_\omega,x(\eta),\alpha)\ \textit{and}\\&(\eta,\omega)\mapsto\overline{\kappa}(\nu_\omega,x(\eta),\alpha)\end{aligned}$$

are Borel measurable for all $\alpha\in[0,1]$.

Proof Note that $\overline{\chi}_j^\alpha(\cdot,\omega)$ is locally constant for all $\omega \in \Omega$, that is, its value depends only on $\eta|_j$. Further, suppose that $\overline{\chi}_j^\alpha(\eta,\omega) = 1$. Then K_ω has a closed α-hole $\mathcal{H}$ at scale j near $x(\eta)$. Since $\mathcal{H}$ and K_ω are closed, their distance is positive. So there exists a finite set $T \subset I^*$ such that $\omega(\tau) = n$ for all $\tau \in T$ and

$$\mathcal{H} \subset \bigcup_{\tau \in T} J_\tau.$$

If $\omega' \in \Omega$ is close to ω, then $\omega'(\tau) = n$ for all $\tau \in T$, which implies that $\overline{\chi}_j^\alpha(\eta,\omega') = 1$. We conclude that $\overline{\chi}_j^\alpha$ is continuous at (η,ω). Trivially, $\overline{\chi}_j^\alpha$ is lower semi-continuous at those points where $\overline{\chi}_j^\alpha(\eta,\omega) = 0$. Therefore, $\overline{\chi}_j^\alpha$ is lower semi-continuous.

Let α_i be a strictly increasing sequence approaching α. We claim that

$$\lim_{i\to\infty} \overline{\chi}_j^{\alpha_i}(\eta,\omega) = \chi_j^\alpha(\eta,\omega) \tag{3.1}$$

for all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$. Indeed, obviously $\chi_j^\alpha(\eta,\omega) \le \overline{\chi}_j^{\alpha_i}(\eta,\omega)$ for all $i \in \mathbb{N}$, and the sequence $(\overline{\chi}_j^{\alpha_i}(\eta,\omega))_{i\in\mathbb{N}}$ is decreasing. Thus, it is enough to study the case $\lim_{i\to\infty} \overline{\chi}_j^{\alpha_i}(\eta,\omega) = 1$. Let $(\overline{B}_\varrho(z_i, \frac{1}{2}\alpha_i k^{-j}))_{i\in\mathbb{N}}$ be a corresponding sequence of closed holes. In this case, one may find a convergent subsequence of $(z_i)_{i\in\mathbb{N}}$ converging to $z \in \mathbb{R}^d$ and $B_\varrho(z, \frac{1}{2}\alpha k^{-j}) \subset Q_j^k(x(\eta)) \setminus K_\omega$, completing the proof of (3.1). As a limit of semi-continuous functions, χ_j^α is Borel measurable. Now $N_i(K_\omega, x(\eta), \alpha) = \sum_{j=1}^i \chi_j^\alpha(\eta,\omega)$, implying that the map $(\eta,\omega) \mapsto \underline{\kappa}(K_\omega, x(\eta), \alpha)$ (as well as the upper mean porosity) is Borel measurable.

By construction, the map $\omega \mapsto X_\tau(\omega)$ is Borel measurable for all $\tau \in I^*$. Therefore, $\omega \mapsto \nu_\omega(B)$ is a Borel map for all Borel sets $B \subset \mathbb{R}^m$ by Property 2.1. In particular, the map $(\eta,\omega) \mapsto \nu_\omega\big(\overline{B}_\varrho(z, \frac{1}{2}\alpha k^{-j})\big) - \varepsilon\nu_\omega\big(Q_j^k(x(\eta))\big)$ is Borel measurable for all $z \in \mathbb{R}^m$, $\alpha \in [0,1]$, $\varepsilon > 0$ and $j \in \mathbb{N}$. Let $(z_i)_{i\in\mathbb{N}}$ be a dense set in $[0,1]^m$. Let $s > 0$, $\alpha \in [0,1]$ and $j \in \mathbb{N}$. Suppose that there exists $z \in Q_j^k(x(\eta))$ such that $\nu_\omega(\overline{B}_\varrho(z, \frac{1}{2}\alpha k^{-j})) < s$. Since the map $x \mapsto \nu_\omega(\overline{B}_\varrho(x,r))$ is upper semicontinuous, there exists $z_i \in Q_j^k(x(\eta))$ such that $\nu_\omega(\overline{B}_\varrho(z_i, \frac{1}{2}\alpha k^{-j})) < s$. Thus, $\overline{\chi}_j^{\alpha,\varepsilon}$ is Borel measurable. Further, $\chi_j^{\alpha,\varepsilon}(\eta,\omega) = 1$ if and only if there exist an increasing sequence $(\alpha_i)_{i\in\mathbb{N}}$ tending to α and a decreasing sequence $(\varepsilon_i)_{i\in\mathbb{N}}$ tending to ε such that $\overline{\chi}_j^{\alpha_i,\varepsilon_i}(\eta,\omega) = 1$. Therefore, $\chi_j^{\alpha,\varepsilon}$ is Borel measurable, and the claim follows as in the case of mean porosities of sets.

Remark 3.11 (a) Note that, for all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$, the function $\alpha \mapsto \chi_0^\alpha(\eta,\omega)$ is decreasing and, thus, the lower and upper mean porosity functions are also decreasing as functions of α.

(b) Later, we will need modifications of the functions χ_j^α defined in the proof of Lemma 3.10. Their Borel measurability can be proven analogously to that of χ_j^α.

4 Results

In this section, we state and prove our results concerning mean porosities of Mandelbrot percolation and its natural measure. To prove the existence of mean porosity and to compare the mean porosities of the limit set and the construction measure, we need a tool to establish the validity of the strong law of large numbers for certain sequences of random variables. We will use [15, Theorem 1] (see also [26, Corollary 11]), which we state (in a simplified form) for the convenience of the reader. Denote the covariance and the variance of random variables by Cov and Var, respectively.

Theorem 4.1 *Let $\{Y_n\}_{n\in\mathbb{N}}$ be a sequence of square-integrable random variables and suppose that there exists a sequence of constants $(\rho_k)_{k\in\mathbb{N}}$ such that*

$$\sup_{n\in\mathbb{N}} |\operatorname{Cov}(Y_n, Y_{n+k})| \le \rho_k$$

for all $k \in \mathbb{N}$. Assume that

$$\sum_{n=1}^{\infty} \frac{\operatorname{Var}(Y_n)\log^2 n}{n^2} < \infty \text{ and } \sum_{k=1}^{\infty} \rho_k < \infty.$$

Then $\{Y_n\}_{n\in\mathbb{N}}$ satisfies the strong law of large numbers, i.e., almost surely

$$\lim_{k\to\infty} \frac{\sum_{n=1}^{k}(Y_n - E[Y_n])}{k} = 0.$$

We will apply Theorem 4.1 to stationary sequences of random variables which are indicator functions of events with equal probabilities. In this setup, all conditions of the theorem will be satisfied if

$$\sum_{j=1}^{\infty} \operatorname{Cov}(Y_0, Y_j) < \infty \tag{4.1}$$

and, in particular, if Y_i and Y_j are uncorrelated once $|i-j|$ is greater than some fixed integer.

For all $\alpha \in [0,1]$ and $j, r \in \mathbb{N}$, define $\chi^{\alpha}_{j,r} \colon I^{\mathbb{N}} \times \Omega \to \{0,1\}$ similarly to χ^{α}_j with the exception that the whole hole is assumed to be in $J_{\eta|_j} \setminus J_{\eta|_{j+r}}$. Observe that $\chi^{\alpha}_{j,r}(\eta,\omega) = \chi^{\alpha}_{j,r}(\eta',\omega)$ provided that $\eta|_{j+r} = \eta'|_{j+r}$. Therefore, for any $\tau \in I^{j+r}$, we may define $\chi^{\alpha}_{j,r}(\tau,\omega) = \chi^{\alpha}_{j,r}(\eta,\omega)$, where $\eta|_{j+r} = \tau$. Note that, given $l_{\tau|_j} \neq 0$, the value of the function $\chi^{\alpha}_{j,r}(\tau,\cdot)$ depends only on the restriction of ω to $[\tau|_j]^* \setminus [\tau|_{j+r}]^*$, where $[\sigma]^* = \{\eta \in I^* \mid \sigma \prec \eta\}$ for all $\sigma \in I^*$. Further, given $l_{\tau|_j} \neq 0$, the distribution of $\chi^{\alpha}_{j,r}(\tau|_j * \cdot, \cdot)$ is independent of $\tau|_j$.

The following "uncorrelation" lemma will be used several times later.

Lemma 4.2 *Let $f, g\colon I^{\mathbb{N}} \times \Omega \to \mathbb{R}$ be Borel functions. Assume that there are $a, b, c \in \mathbb{N}$ with $a < b \le c$ such that, for all $\eta, \eta' \in I^{\mathbb{N}}$, we have that $f(\eta, \cdot) = f(\eta', \cdot)$ provided $\eta|_b = \eta'|_b$, given $l_{\eta|_b} \neq 0$, $f(\eta|_b, \cdot)$ depends only on $[\eta|_a]^* \setminus [\eta|_b]^*$ and, given $l_{\eta|_c} \neq 0$, $g(\eta, \cdot)$ depends only on $[\eta|_c]^*$ and the distribution of $g(\eta|_c * \cdot, \cdot)$ is independent of $\eta|_c$. Then*

$$E_Q[fg] = E_Q[f]E_Q[g].$$

Proof Since the distribution of $g(\eta|_c * \cdot, \cdot)$ is independent of $\eta|_c$ given $l_{\eta|_c} \neq 0$, we have that $E_Q[g(\eta|_c * \cdot, \cdot) \mid l_{\eta|_c} \neq 0] = C_g$ is independent of $\eta|_c$. By the definition of μ_ω, we have, for all $\sigma \in I^*$, that $(\mu_\omega)|_{[\sigma]} = l_\sigma(\omega)^d \mu_{(\omega|_{[\sigma]^*})}$, where $\mu_{(\omega|_{[\sigma]^*})}$ is the conditional measure given $l_\sigma \neq 0$. For all $A \subset I^*$, we denote by $P|_A$ the restriction of P to $\{c, n\}^A$. Recalling (2.2), we conclude that

$$\begin{aligned}
E_Q[g] &= \sum_{\eta \in I^c} \int l_\eta(\omega|_{I^* \setminus [\eta]^*})^d \\
&\quad \times \iint g(\eta * \tau, \omega|_{[\eta]^*})\, d\mu_{(\omega|_{[\eta]^*})}(\tau) dP|_{[\eta]^*}(\omega|_{[\eta]^*}) dP|_{I^* \setminus [\eta]^*}(\omega|_{I^* \setminus [\eta]^*}) \\
&= C_g \sum_{\eta \in I^c} \int l_\eta(\omega|_{I^* \setminus [\eta]^*})^d \, dP|_{I^* \setminus [\eta]^*}(\omega|_{I^* \setminus [\eta]^*}) \\
&= C_g \int \sum_{\eta \in I^c} l_\eta(\omega)^d \, dP(\omega) = C_g
\end{aligned}$$

and, further,

$$\begin{aligned}
E_Q[fg] &= \sum_{\sigma \in I^b} \int l_\sigma(\omega|_{I^* \setminus [\sigma]^*})^d f(\sigma, \omega|_{[\sigma|_a]^* \setminus [\sigma]^*}) \sum_{\eta \in I^{c-b}} \int l_{\sigma,\eta}(\omega|_{[\sigma]^* \setminus [\sigma * \eta]^*})^d \\
&\quad \times \iint g(\sigma * \eta * \tau, \omega|_{[\sigma*\eta]^*})\, d\mu_{(\omega|_{[\sigma*\eta]^*})}(\tau) dP|_{[\sigma*\eta]^*}(\omega|_{[\sigma*\eta]^*}) \\
&\quad \times dP|_{[\sigma]^* \setminus [\eta]^*}(\omega|_{[\sigma]^* \setminus [\eta]^*}) dP|_{I^* \setminus [\sigma]^*}(\omega|_{I^* \setminus [\sigma]^*}) \\
&= E_Q[g] \sum_{\sigma \in I^b} \int l_\sigma(\omega|_{I^* \setminus [\sigma]^*})^d f(\sigma, \omega|_{[\sigma|_a]^* \setminus [\sigma]*}) \, dP|_{I^* \setminus [\sigma]^*}(\omega|_{I^* \setminus [\sigma]^*}) \\
&= E_Q[g] \sum_{\sigma \in I^b} \int l_\sigma(\omega|_{I^* \setminus [\sigma]^*})^d f(\sigma, \omega|_{[\sigma|_a]^* \setminus [\sigma]*}) \iint 1 \, d\mu_{(\omega|_{[\sigma]^*})}(\tau) \\
&\quad \times dP|_{[\sigma]^*}(\omega|_{[\sigma]^*}) dP|_{I^* \setminus [\sigma]^*}(\omega|_{I^* \setminus [\sigma]^*}) \\
&= E_Q[g] E_Q[f].
\end{aligned}$$

Next we prove a lemma which gives lower and upper bounds for mean porosities at typical points.

Lemma 4.3 *For all $\alpha \in (0, 1)$, we have*

$$E_Q[\overline{\chi}_0^\alpha] \le \underline{\kappa}(K_\omega, x(\eta), \alpha) \le \overline{\kappa}(K_\omega, x(\eta), \alpha) \le E_Q[\chi_0^\alpha]$$

for Q-almost all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$.

Proof Note that, for every $\alpha \in (0,1)$ and $r \in \mathbb{N}$ with $k^{-r} < \alpha$, we have

$$\chi_{j,r}^{\alpha}(\eta,\omega) \le \chi_{j}^{\alpha}(\eta,\omega) \le \chi_{j,r}^{\alpha-k^{-r}}(\eta,\omega) \tag{4.2}$$

for all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$ satisfying $x(\eta) \in K_\omega$. Recall that ν_ω is supported on K_ω for P-almost all $\omega \in \Omega$. Observe that $\chi_{i,r}^{\alpha}$ and $\chi_{j,r}^{\alpha}$ satisfy the assumptions of Lemma 4.2 provided $j - i \ge r$ with $f = \chi_{i,r}^{\alpha}$, $g = \chi_{j,r}^{\alpha}$, $a = i$, $b = i + r$ and $c = j$. Thus, the assumptions of Theorem 4.1 are valid and, for all $r \in \mathbb{N}$ and Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$, we conclude that (note that $E_Q[\chi_{j,r}^{\alpha}] = E_Q[\chi_{0,r}^{\alpha}]$ for all $j \in \mathbb{N}$)

$$\begin{aligned} E_Q[\chi_{0,r}^{\alpha}] &= \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{j,r}^{\alpha}(\eta,\omega) \le \underline{\kappa}(K_\omega, x(\eta), \alpha) \\ &\le \overline{\kappa}(K_\omega, x(\eta), \alpha) \le \lim_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} \chi_{j,r}^{\alpha-k^{-r}}(\eta,\omega) = E_Q[\chi_{0,r}^{\alpha-k^{-r}}]. \end{aligned}$$

Observe that, for all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$ satisfying $x(\eta) \in K_\omega$ holds the inequality $\lim_{r\to\infty} \chi_{0,r}^{\alpha}(\eta,\omega) \ge \overline{\chi}_0^{\alpha}(\eta,\omega)$, since the distance between a closed α-hole and K_ω is positive. Further, the inequality

$$\chi_{0,r}^{\alpha-k^{-r}} \le \overline{\chi}_0^{\alpha-2k^{-r}} \tag{4.3}$$

is always valid if $2k^{-r} \le \alpha$, and $\lim_{r\to\infty} \overline{\chi}_0^{\alpha-2k^{-r}} = \chi_0^{\alpha}$ by (3.1). Hence,

$$E_Q[\overline{\chi}_0^{\alpha}] \le \underline{\kappa}(K_\omega, x(\eta), \alpha) \le \overline{\kappa}(K_\omega, x(\eta), \alpha) \le E_Q[\chi_0^{\alpha}]$$

for Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$.

In fact, the upper bound we have found in Lemma 4.3 is an exact equality.

Proposition 4.4 *For all $\alpha \in (0,1)$, we have that*

$$\overline{\kappa}(K_\omega, x(\eta), \alpha) = E_Q[\chi_0^{\alpha}]$$

for Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$.

Proof We start by proving that, for all $\alpha \in (0,1)$,

$$\lim_{j\to\infty} \mathrm{Cov}(\chi_0^{\alpha}, \chi_j^{\alpha}) = 0.$$

Let $\alpha \in (0,1)$. By (4.2), we have the following estimate:

$$\begin{aligned} \mathrm{Cov}(\chi_0^{\alpha}, \chi_j^{\alpha}) &= E_Q[\chi_0^{\alpha}\chi_j^{\alpha}] - E_Q[\chi_0^{\alpha}]E_Q[\chi_j^{\alpha}] \\ &\le E_Q[\chi_{0,j}^{\alpha-k^{-j}}\chi_j^{\alpha}] - E_Q[\chi_0^{\alpha}]E_Q[\chi_j^{\alpha}] \end{aligned} \tag{4.4}$$

for all $j \in \mathbb{N}$ with $2k^{-j} < \alpha$. The functions $\chi_{0,j}^{\alpha-k^{-j}}$ and χ_j^α satisfy the assumptions of Lemma 4.2 with $a = 0$ and $b = c = j$, giving $E_Q[\chi_{0,j}^{\alpha-k^{-j}}\chi_j^\alpha] = E_Q[\chi_{0,j}^{\alpha-k^{-j}}]E_Q[\chi_j^\alpha]$. Hence, by (4.4) and (4.3),

$$\mathrm{Cov}(\chi_0^\alpha, \chi_j^\alpha) \le E_Q[\chi_j^\alpha]E_Q[\chi_{0,j}^{\alpha-k^{-j}} - \chi_0^\alpha] \le E_Q[\chi_0^\alpha]E_Q[\overline{\chi}_0^{\alpha-2k^{-j}} - \chi_0^\alpha].$$

Since $\lim_{j\to\infty}(\alpha - 2k^{-j}) = \alpha$, the equality (3.1) and the dominated convergence theorem imply that $\lim_{j\to\infty} E_Q[\overline{\chi}_0^{\alpha-2k^{-j}} - \chi_0^\alpha] = 0$. Now, by Bernstein's theorem [2] (see also [13, p. 265]), the sequence $\frac{1}{n}N_n(A, x, \alpha) = \frac{1}{n}\sum_{i=1}^n \chi_i^\alpha$ converges in probability to $E_Q[\chi_0^\alpha]$. Once we have the convergence in probability, we can find a subsequence converging almost surely and, therefore, the upper bound in Lemma 4.3 is attained.

Define

$$D = \{\alpha \in (0,1) \mid \beta \mapsto E_Q[\chi_0^\beta] \text{ is discontinuous at } \beta = \alpha\}.$$

Since $\beta \mapsto E_Q[\chi_0^\beta]$ is decreasing, the set D is countable.

Theorem 4.5 *For Q-almost all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$, we have that*

$$\kappa(K_\omega, x(\eta), \alpha) = E_Q[\chi_0^\alpha]$$

for all $\alpha \in (0,1) \setminus D$. In particular, for Q-almost all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$, the function $\alpha \mapsto \kappa(K_\omega, x(\eta), \alpha)$ is defined and continuous at all $\alpha \in (0,1) \setminus D$.

Proof Since $\chi_0^{\alpha'} \le \overline{\chi}_0^\alpha \le \chi_0^\alpha$ for all $\alpha' > \alpha$, we have that $E_Q[\overline{\chi}_0^\alpha] = E_Q[\chi_0^\alpha]$ for all $\alpha \in (0,1) \setminus D$. Lemma 4.3 implies that, for all $\alpha \in (0,1) \setminus D$, there exists a Borel set $B_\alpha \subset I^{\mathbb{N}} \times \Omega$ such that $\kappa(K_\omega, x(\eta), \alpha) = E_Q[\chi_0^\alpha]$ for all $(\eta, \omega) \in B_\alpha$ and $Q(B_\alpha) = 1$. Let $(\alpha_i)_{i\in\mathbb{N}}$ be a dense set in $(0,1)$. Since the functions $\alpha \mapsto \underline{\kappa}(K_\omega, x(\eta), \alpha)$ and $\alpha \mapsto \overline{\kappa}(K_\omega, x(\eta), \alpha)$ are decreasing, we have, for all $(\eta, \omega) \in \bigcap_{i=1}^\infty B_{\alpha_i}$, that $\kappa(K_\omega, x(\eta), \alpha) = E_Q[\chi_0^\alpha]$ for all $\alpha \in (0,1) \setminus D$. Since $Q(\bigcap_{i=1}^\infty B_{\alpha_i}) = 1$, the proof is complete.

Proposition 4.6 *Suppose that $p > k^{-m+1}$. Then the set D is non-empty.*

Proof Since $E_Q[\chi_0^{\alpha'}] \le E_Q[\overline{\chi}_0^\alpha] \le E_Q[\chi_0^\alpha]$ for all $\alpha' > \alpha$, it is enough to show that there exists $\alpha \in (0,1)$ such that $E_Q[\overline{\chi}_0^\alpha] < E_Q[\chi_0^\alpha]$. This, in turn, follows if

$$Q(\{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid \overline{\chi}_0^\alpha(\eta, \omega) = 0 \text{ and } \chi_0^\alpha(\eta, \omega) = 1\}) > 0, \tag{4.5}$$

since $\overline{\chi}_0^\alpha \le \chi_0^\alpha$. A pair (η, ω) belongs to the set defined in (4.5), if $\alpha = k^{-r}$ for some $r \in \mathbb{N}$, one construction cube J_σ at level r is neglected, K_ω intersects all $(m-1)$-dimensional faces of J_σ and there is no hole of size α other than J_σ. Next we make this idea precise.

Let $\alpha = k^{-r}$ for some $r \in \mathbb{N}$. Fix $\sigma \in I^r$ and set $\Sigma_\sigma = \{\sigma' \in I^r \mid \dim(J_{\sigma'} \cap J_\sigma) = m-1\}$. Let $A = \{\omega \in \Omega \mid \omega(\sigma) = n \text{ and } \omega(\sigma') = c \text{ for all } \sigma' \in I^r \setminus \{\sigma\}\}$. Clearly, A is a Borel set and $P(A) > 0$. For all $\sigma' \in \Sigma_\sigma$, define

$F_{\sigma'} = J_{\sigma'} \cap J_\sigma$ and $A_{\sigma'} = \{\omega \in \Omega \mid K_\omega \cap \mathrm{Int}^{\mathrm{m\text{-}1}} F_{\sigma'} \neq \emptyset\}$, where $\mathrm{Int}^{\mathrm{m\text{-}1}}$ refers to the interior when $F_{\sigma'}$ is viewed as a $(m-1)$-dimensional set. Given A, $K_\omega \cap F_{\sigma'}$ defines an $(m-1)$-dimensional Mandelbrot percolation process. Since $p > k^{-m+1}$, it follows from (2.1) (with m replaced by $m-1$) that $P(K_\omega \cap F_{\sigma'} \neq \emptyset \mid A) > 0$. Since $P(K_\omega \subset \partial J_\emptyset \mid K_\omega \neq \emptyset) = 0$ in any dimension, we have that $P(A_{\sigma'} \mid A) > 0$. Given A, the events $A_{\sigma'}$, $\sigma' \in \Sigma_\sigma$, are clearly independent. Let $\widetilde{A} = \{\omega \in \Omega \mid K_\omega \cap \mathrm{Int} J_{\tilde\sigma} \neq \emptyset$ for all $\tilde\sigma \in I^{r+1}$ with $\sigma \not\prec \tilde\sigma\}$. Then $P(\widetilde{A} \mid A) > 0$ and the events $\widetilde{A}$ and $\cap_{\sigma' \in \Sigma_\sigma} A_{\sigma'}$ are positively correlated. Therefore, $A_0 = A \cap \widetilde{A} \cap \cap_{\sigma' \in \Sigma_\sigma} A_{\sigma'}$ is a Borel set and $P(A_0) > 0$. For all $\omega \in A_0$ and for all $\eta \in I^{\mathbb{N}}$, we have that $\chi_0^\alpha(\eta,\omega) = 1$ and $\overline{\chi}_0^\alpha(\eta,\omega) = 0$. This implies inequality (4.5).

Remark 4.7 A similar construction as in the proof of Proposition 4.6 can be done for any positive $\alpha = \sum_{j=1}^n q_j k^{-r_j} < 1$, where $r_j \in \mathbb{N}$ and $q_j \in \mathbb{Z}$, that is, for any hole which is a finite union of construction squares. We do not know whether $\kappa(K_\omega, x(\eta), \alpha)$ exists for $\alpha \in D$. If $p \leq k^{-m+1}$, we have that $K_\omega \cap F = \emptyset$ almost surely for all $(m-1)$-dimensional faces F of construction cubes. Thus the above proof does not apply. We do not know whether $D = \emptyset$ in this case.

Corollary 4.8 *For P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$, we have that*

$$0 < \underline{\kappa}(K_\omega, x, \alpha) \leq \overline{\kappa}(K_\omega, x, \alpha) < 1$$

for all $\alpha \in (0,1)$, $\kappa(K_\omega, x, 0) = 1$ and $\kappa(K_\omega, x, 1) = 0$.

Proof Since $0 < E_Q[\chi_0^\alpha] < 1$ for all $\alpha \in (0,1)$ and the functions $\alpha \mapsto \underline{\kappa}(K_\omega, x(\eta), \alpha)$ and $\alpha \mapsto \overline{\kappa}(K_\omega, x(\eta), \alpha)$ are decreasing, the first claim follows from Theorem 4.5. The claim $\kappa(K_\omega, x, 0) = 1$ is obvious. Finally, if $\overline{\kappa}(K_\omega, x, 1) > 0$, the set K_ω has a 1-hole near x at scale j for some $j \in \mathbb{N}$. Hence, x should be on the boundary of the hole and $J_{\eta|_j}$ which, in turn, implies that K_ω has a 1-hole near x at all scales larger than j. Thus $\kappa(K_\omega, x, \alpha) = 1$ for all $\alpha \leq 1$ which is a contradiction with the first claim.

To study the mean porosities of the natural measure, we need some auxiliary results.

Proposition 4.9 *For all $s > 0$, the sequence $\{\mathbb{1}_{\{X_j \leq s\}}\}_{j \in \mathbb{N}}$ satisfies the strong law of large numbers.*

Proof Since the sequence $(X_j)_{j \in \mathbb{N}}$ is stationary, we only have to check that the series (4.1) converges with $Y_j = \mathbb{1}_{\{X_j \leq s\}}$. Since $X_0 - k^{-jd} X_j$ and X_j satisfy the assumptions of Lemma 4.2 with $a = 0$ and $b = c = j$, also $\mathbb{1}_{\{X_0 - k^{-jd} X_j \leq s\}}$ and $\mathbb{1}_{\{X_j \leq s\}}$ satisfy them for all $s > 0$. Using the fact that X_j and X_0 have the same distribution, we can make the following estimate:

$$\begin{aligned}
&\mathrm{Cov}(\mathbb{1}_{\{X_0\le s\}}, \mathbb{1}_{\{X_j\le s\}}) \\
&\quad = Q(X_0 \le s \text{ and } X_j \le s) - Q(X_0\le s)Q(X_j\le s) \\
&\quad \le Q(X_0 - k^{-jd}X_j \le s \text{ and } X_j \le s) - Q(X_0\le s)Q(X_j\le s) \\
&\quad = Q(X_j\le s)\big(Q(X_0 - k^{-jd}X_j \le s) - Q(X_0\le s)\big) \\
&\quad = Q(X_0\le s)Q(s < X_0 \le s + k^{-jd}X_j) \\
&\quad \le Q(X_0\le s)\big(Q(s < X_0 \le s + k^{-\frac{1}{2}jd}) + Q(X_0 > k^{\frac{1}{2}jd})\big).
\end{aligned}$$

By a result of Dubuc and Seneta [9] (see also [1, Theorem II.5.2]), the distribution of X_0 has a continuous P-density $q(x)$ on $(0,+\infty)$. From Property 2.2, we obtain

$$\begin{aligned}
Q(s < X_0 \le s + k^{-\frac{1}{2}jd}) &= E_Q\big[\mathbb{1}_{\{s<X_0\le s+k^{-\frac{1}{2}jd}\}}\big] \\
&= E_P\big[X_0 \mathbb{1}_{\{s<X_0\le s+k^{-\frac{1}{2}jd}\}}\big] \\
&\le (s + k^{-\frac{1}{2}d})P(s < X_0 \le s + k^{-\frac{1}{2}jd}) \\
&\le (s + k^{-\frac{1}{2}d})k^{-\frac{1}{2}jd} \max_{x\in[s,s+k^{-\frac{1}{2}d}]} q(x).
\end{aligned}$$

Therefore, by Markov's inequality,

$$\sum_{j=1}^{\infty} \mathrm{Cov}(\mathbb{1}_{\{X_0\le s\}}, \mathbb{1}_{\{X_j\le s\}}) \le$$

$$Q(X_0\le s)\big((s + k^{-\frac{1}{2}d}) \max_{x\in[s,s+k^{-\frac{1}{2}d}]} q(x) + E_Q(X_0)\big) \sum_{j=1}^{\infty} k^{-\frac{1}{2}jd} < \infty.$$

For all $\alpha \in (0,1)$, $\varepsilon, \delta > 0$ and $j \in \mathbb{N}$, define a function $H_j^{\alpha,\varepsilon,\delta}\colon I^{\mathbb{N}} \times \Omega \to \{0,1\}$ by setting $H_j^{\alpha,\varepsilon,\delta}(\eta,\omega) = 1$, if and only if ν_ω has an (α,ϵ)-hole at scale j near $x(\eta)$ but K_ω does not have an $(\alpha-\delta)$-hole at scale j near $x(\eta)$.

Lemma 4.10 *Let $\alpha \in (0,1)$. For all $\delta > 0$, there exists $\varepsilon_0 > 0$ such that, for Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$ holds the inequality*

$$\limsup_{n\to\infty} \frac{1}{n} \sum_{j=1}^{n} H_j^{\alpha,\varepsilon,\delta}(\eta,\omega) \le \delta$$

for all $0 < \varepsilon \le \varepsilon_0$.

Proof Fix $0 < \delta < \alpha$. Let $r \in \mathbb{N}$ be the smallest integer such that $2k^{-r} < \delta$. Let $\varepsilon > 0$. Assume that $H_j^{\alpha,\varepsilon,\delta}(\eta,\omega) = 1$ and denote by $\mathcal{H}$ the (α,ε)-hole at scale j near $x(\eta)$. Considering the relative positions of $\mathcal{H}$ and $J_{\eta|_{j+r}}$, we will argue that we arrive at the following possibilities:

(i) If $J_{\eta|_{j+r}} \subset \mathcal{H}$, we have $\nu_\omega(J_{\eta|_{j+r}}) \le \varepsilon\nu_\omega(J_{\eta|_j})$.

(ii) In the case $J_{\eta|_{j+r}} \not\subset \mathcal{H}$, there exists $\tau \in I^{j+r}$ such that $\tau \neq \eta|_{j+r}$, $J_\tau \subset \mathcal{H}$, $K_\omega \cap J_\tau \neq \emptyset$ and $\nu_\omega(J_\tau) \leq \varepsilon \nu_\omega(J_{\eta|_j})$.

Suppose that (i) is not valid. Since $H_j^{\alpha,\varepsilon,\delta}(\eta,\omega) = 1$, the set K_ω does not have an $(\alpha-\delta)$-hole at scale j near $x(\eta)$. Observe that

$$\mathcal{H} \setminus \bigcup_{\substack{\sigma \in I^{j+r} \\ J_\sigma \not\subset \mathcal{H}}} J_\sigma$$

contains a cube with side length $(\alpha-\delta)k^{-j}$ since $2k^{-r} < \delta$. Since $J_{\eta|_{j+r}} \not\subset \mathcal{H}$, there exists $\tau \in I^{j+r}$ as in (ii).

Next we estimate how often (i) or (ii) may happen for Q-typical $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$. We denote by $A_j^{1,\varepsilon}$ the event that $\nu_\omega(J_{\eta|_{j+r}}) \leq \varepsilon \nu_\omega(J_{\eta|_j})$, that is, according to Property 2.1,

$$A_j^{1,\varepsilon} = \big\{(\eta,\omega) \in I^{\mathbb{N}} \times \Omega \mid \\ X_{\eta|_{j+r}}(\omega) \leq \frac{\varepsilon}{1-\varepsilon} \sum_{\substack{\tau \in I^{j+r} \\ \eta|_j \prec \tau,\ \tau \neq \eta|_{j+r},\ l_\tau(\omega) \neq 0}} X_\tau(\omega)\big\}.$$

For all $s > 0$, let

$$A_{j,1}^{s,\varepsilon} = \big\{(\eta,\omega) \in I^N \times \Omega \mid X_{\eta|_{j+r}}(\omega) \leq \frac{\varepsilon s}{1-\varepsilon}\big\}$$
$$\text{and } A_{j,2}^{s} = \big\{(\eta,\omega) \in I^N \times \Omega \mid \sum_{\substack{\tau \in I^{j+r} \\ \eta|_j \prec \tau,\ \tau \neq \eta|_{j+r},\ l_\tau(\omega) \neq 0}} X_\tau(\omega) > s\big\}.$$

The set in the case (ii) is covered by

$$A_j^{2,\varepsilon} = \{(\eta,\omega) \in I^{\mathbb{N}} \times \Omega \mid \exists \tau \in I^{j+r} \text{ such that } \tau \succ \eta|_j, \\ \tau \neq \eta|_{j+r} \text{ and } 0 < k^{-rd} X_\tau(\omega) \leq \varepsilon X_{\eta|_j}(\omega)\}.$$

Recall that, for any $\tau \in I^*$, we have $P(X_\tau(\omega) > 0 \mid K_\omega \cap J_\tau \neq \emptyset) = 1$ by [30, Theorem 3.4]. For all $s > 0$, we define

$$A_{j,3}^{s} = \{(\eta,\omega) \in I^{\mathbb{N}} \times \Omega \mid X_{\eta|_j}(\omega) > s\} \text{ and}$$
$$A_{j,4}^{s,\varepsilon} = \{(\eta,\omega) \in I^{\mathbb{N}} \times \Omega \mid \exists \tau \in I^{j+r} \text{ such that } \tau \succ \eta|_j, \\ \tau \neq \eta|_{j+r} \text{ and } 0 < k^{-rd} X_\tau(\omega) \leq \varepsilon s\}.$$

Combining the above definitions, we conclude that

$$H_j^{\alpha,\varepsilon,\delta} \leq \mathbb{1}_{A_j^{1,\varepsilon}} + \mathbb{1}_{A_j^{2,\varepsilon}} \leq \mathbb{1}_{A_{j,1}^{s,\varepsilon}} + \mathbb{1}_{A_{j,2}^{s}} + \mathbb{1}_{A_{j,3}^{s}} + \mathbb{1}_{A_{j,4}^{s,\varepsilon}}$$

for all $s > 0$. We will fix an appropriate $s > 0$ at the end of this proof. By Proposition 4.9, the functions $\mathbb{1}_{A_{j,1}^{s,\varepsilon}}$ and $\mathbb{1}_{A_{j,3}^{s}}$ satisfy the strong law of large numbers. The functions $\mathbb{1}_{A_{i,2}^{s}}$ and $\mathbb{1}_{A_{j,2}^{s}}$ as well as $\mathbb{1}_{A_{i,4}^{s,\varepsilon}}$ and $\mathbb{1}_{A_{j,4}^{s,\varepsilon}}$ satisfy the assumptions of Lemma 4.2 as soon as $j-i \geq r$. Therefore, by Theorem 4.1, the functions $\mathbb{1}_{A^s_{j,2}}$ and $\mathbb{1}_{A^{s,\varepsilon}_{j,4}}$ also satisfy the strong law of large numbers. Hence, we obtain the estimate

$$\limsup_{n\to\infty} \frac{1}{n}\sum_{j=1}^{n} H_j^{\alpha,\varepsilon,\delta}(\eta,\omega) \leq Q(A^{s,\varepsilon}_{0,1}) + Q(A^s_{0,2}) + Q(A^s_{0,3}) + Q(A^{s,\varepsilon}_{0,4})$$

for Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$. Observe that the left hand side of the above inequality decreases as ε decreases for all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$. For all large enough s, the value of $Q(A^s_{0,2}) + Q(A^s_{0,3})$ is less than $\frac{1}{2}\delta$. Fix such an $0 < s < \infty$. According to (2.4), we have $Q(X_r = 0) = 0$. Therefore, for all ε small enough, we have $Q(A^{s,\varepsilon}_{0,1}) + Q(A^{s,\varepsilon}_{0,4}) < \frac{1}{2}\delta$, completing the proof.

Now we are ready to prove that the mean porosity of the natural measure equals that of the Mandelbrot percolation set.

Theorem 4.11 *For Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$ the following equality*

$$\kappa(K_\omega, x(\eta), \alpha) = \kappa(\nu_\omega, x(\eta), \alpha)$$

holds for all $\alpha \in (0,1) \setminus D$.

Proof For all $\alpha \in (0,1)$, $\varepsilon > 0$ and $j \in \mathbb{N}$, let χ_j^α and $\chi_j^{\alpha,\epsilon}$ be as in Definition 3.9 and $H_j^{\alpha,\varepsilon,\delta}$ as in Lemma 4.10. The inequalities

$$\underline{\kappa}(K_\omega, x(\eta), \alpha) \leq \underline{\kappa}(\nu_\omega, x(\eta), \alpha) \text{ and } \overline{\kappa}(K_\omega, x(\eta), \alpha) \leq \overline{\kappa}(\nu_\omega, x(\eta), \alpha)$$

are obvious for all $\alpha \in (0,1)$ and $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$ since $\chi_j^\alpha \leq \chi_j^{\alpha,\epsilon}$. Therefore, for Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$, we have $\kappa(K_\omega, x(\eta), \alpha) \leq \underline{\kappa}(\nu_\omega, x(\eta), \alpha)$ for all $\alpha \in (0,1) \setminus D$ by Theorem 4.5.

Let $0 < \delta < \alpha$ and $\varepsilon > 0$. Since $\chi_j^{\alpha,\varepsilon} \leq \chi_j^{\alpha-\delta} + H_j^{\alpha,\varepsilon,\delta}$ for all $j \in \mathbb{N}$, the following holds by Lemma 4.10, for Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$:

$$\begin{aligned}\overline{\kappa}(\nu_\omega, x(\eta), \alpha) &= \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_j^{\alpha,\varepsilon}(\eta,\omega)\\ &\leq \limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}\chi_j^{\alpha-\delta}(\eta,\omega) + \lim_{\varepsilon\to 0}\limsup_{n\to\infty}\frac{1}{n}\sum_{j=1}^{n}H_j^{\alpha,\varepsilon,\delta}(\eta,\omega)\\ &\leq \overline{\kappa}(K_\omega, x(\eta), \alpha-\delta) + \delta.\end{aligned}$$

Since $\alpha \mapsto \overline{\kappa}(K_\omega, x(\eta), \alpha)$ is continuous at all $\alpha \in (0,1)\setminus D$ by Theorem 4.5, we conclude that, for all $\alpha \in (0,1) \setminus D$, we have for Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$ that $\overline{\kappa}(\nu_\omega, x(\eta), \alpha) \leq \kappa(K_\omega, x(\eta), \alpha)$. As in the proof of Theorem 4.5, we see that the order of the quantifiers may be reversed.

Before stating a corollary of the previous theorem, we prove a lemma, which is well known, but for which we did not find a reference.

Lemma 4.12 *For all hyperplanes V the following holds:*

$$P\big(\nu_\omega(K_\omega \cap V) > 0\big) = 0.$$

Proof According to Property 2.1,

$$\nu_\omega(K_\omega \cap V) = \lim_{j\to\infty} \sum_{\substack{\tau\in I^j \\ J_\tau\cap V\neq\emptyset}} l_\tau(\omega)^d X_\tau(\omega),$$

and the above sequence decreases monotonically as j tends to infinity. Hence,

$$E_P[\nu_\omega(K_\omega \cap V)] \le \lim_{j\to\infty} E_P\Bigg[\sum_{\substack{\tau\in I^j \\ J_\tau\cap V\neq\emptyset}} l_\tau(\omega)^d X_\tau(\omega)\Bigg].$$

Note that, without the restriction $J_\tau \cap V \neq \emptyset$, the expectation on the right hand side equals 1. Since the restriction $J_\tau \cap V \neq \emptyset$ determines an exponentially decreasing proportion of indices as j tends to infinity and since the random variables $l_\tau^d X_\tau$ have the same distribution, the limit of the expectation equals 0.

Next corollary is the counterpart of Corollary 4.8 for mean porosities of the natural measure.

Corollary 4.13 *For P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$, we have the following inequalities*

$$0 < \underline{\kappa}(\nu_\omega, x, \alpha) \le \overline{\kappa}(\nu_\omega, x, \alpha) < 1$$

for all $\alpha \in (0,1)$, $\kappa(\nu_\omega, x, 0) = 1$ and $\kappa(\nu_\omega, x, 1) = 0$.

Proof The first claim follows from Corollary 4.8, Theorem 4.11 and the monotonicity of the functions $\alpha \mapsto \underline{\kappa}(\nu_\omega, x, \alpha)$ and $\alpha \mapsto \overline{\kappa}(\nu_\omega, x, \alpha)$. Since for all $x \in K_\omega$ holds the inequality $\underline{\kappa}(K_\omega, x, 0) \le \underline{\kappa}(\nu_\omega, x, 0)$, the second claim follows from Corollary 4.8. Note that $\chi_j^{1,\epsilon}(\eta,\omega) = 1$ only if $\nu_\omega(\partial J_{\eta|_j}) > 0$. Therefore, the last claim follows from Lemma 4.12.

The following corollary solves completely Conjecture 3.2 stated in [20].

Corollary 4.14 *For P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$, the following holds*

$$\underline{\mathrm{por}}(K_\omega, x) = \underline{\mathrm{por}}(\nu_\omega, x) = 0, \quad \overline{\mathrm{por}}(K_\omega, x) = \frac{1}{2} \text{ and } \overline{\mathrm{por}}(\nu_\omega, x) = 1.$$

Proof By Corollary 4.8, for Q-almost all $(\eta,\omega) \in I^{\mathbb{N}} \times \Omega$, we have that $\overline{\kappa}(K_\omega, x(\eta), \alpha) < 1$ for all $\alpha > 0$. Hence, for P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$, there are, for all $\alpha > 0$, arbitrarily large $i \in \mathbb{N}$ such that K_ω does not have an α-hole at scale i near x which is contained in $Q_i^k(x)$. Recall that Corollary 4.8 concerns the construction based mean porosities that are defined in terms of k-adic cubes (see Definitions 3.4 and 3.6). Therefore, we may only conclude that there is no $z \in \mathbb{R}^m$ such that

$$B(z, \frac{1}{2}\sqrt{m}\alpha k^{-i}) \subset (B(x, k^{-i}) \cap Q_i^k(x)) \setminus K_\omega.$$

Since the claim concerns the regular porosity (Definitions 3.1 and 3.2) where the holes are defined using Euclidean balls, we should prove that the above claim is true without the intersection with $Q_i^k(x)$ on the right hand side. We show that there are infinitely many $i \in \mathbb{N}$ such that this is indeed the case. The heuristic reason for this is that, for typical points, positive proportion of scales i are such that x is close to the centre of $Q_i^k(x)$ and this event is essentially independent of the non-existence of holes of certain size. Thus, both events appear with positive probability. Now we formalise this idea.

Fix $\alpha \in (0, \frac{1}{4})$ and $r > 8$ large enough so that $2k^{-r} < \alpha$. Let $I' \subset I^r$ be the set of words such that, for all $\tau \in I'$, the ϱ-distance from all points of J_τ to the centre of $J_\emptyset$ is at most $\frac{1}{4}$. For all $i \in \mathbb{N}$, define $Y_i^\alpha \colon I^{\mathbb{N}} \times \Omega \to \{0, 1\}$ by setting $Y_i^\alpha(\eta, \omega) = 1$, if and only if $J_{\eta|_i}$ is chosen, $\eta|_{i+r}$ ends with a word from I' and K_ω does not have an $\frac{1}{2}\alpha$-hole at scale i near $x(\eta)$ (in the sense of Definition 3.4) which is completely inside $J_{\eta|_i} \setminus J_{\eta|_{i+r}}$. Note that if $x(\eta) \in K_\omega$, and K_ω has an α-hole at scale i near $x(\eta)$, then at least half of this hole is in $J_{\eta|_i} \setminus J_{\eta|_{i+r}}$. Thus, K_ω does not have an α-hole at scale i near $x(\eta)$ if $Y_i^\alpha(\eta, \omega) = 1$. The indicator functions of the events $\{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid Y_i^\alpha(\eta, \omega) = 1\}$ and $\{(\eta, \omega) \in I^{\mathbb{N}} \times \Omega \mid Y_j^\alpha(\eta, \omega) = 1\}$ satisfy the assumptions of Lemma 4.2 provided $j - i \geq r$. Therefore, by Theorem 4.1, the averages of the random variables $Y_i^\alpha(\eta, \omega)$ converge to $E_Q(Y_0^\alpha) > 0$ for Q-almost all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$. If $Y_i^\alpha(\eta, \omega) = 1$, then $B(x(\eta), \frac{1}{4}k^{-i}) \subset J_{\eta|_i}$ and there is no $z \in B(x(\eta), \frac{1}{4}k^{-i})$ such that $B(z, \frac{1}{2}\sqrt{m}\alpha k^{-i}) \subset B(x(\eta), \frac{1}{4}k^{-i}) \setminus K_\omega$. Therefore, $\operatorname{por}(K_\omega, x(\eta), \frac{1}{4}k^{-i}) \leq 2\sqrt{m}\alpha$. A similar argument shows that $\operatorname{por}(\nu_\omega, x(\eta), \frac{1}{4}k^{-i}) \leq 2\sqrt{m}\alpha$. Let $(\alpha_j)_{j \in \mathbb{N}}$ and $(\varepsilon_k)_{k \in \mathbb{N}}$ be sequences tending to 0. For Q-almost all $(\eta, \omega) \in I^{\mathbb{N}} \times \Omega$, we have for all $j, k \in \mathbb{N}$ that there are infinitely many scales $i \in \mathbb{N}$ such that

$$\operatorname{por}(K_\omega, x(\eta), \frac{1}{4}k^{-i}) < \alpha_j \text{ and } \operatorname{por}(\nu_\omega, x(\eta), \frac{1}{4}k^{-i}, \varepsilon_k) < \alpha_j.$$

Hence, we conclude that

$$\underline{\operatorname{por}}(K_\omega, x) = 0 = \underline{\operatorname{por}}(\nu_\omega, x)$$

for P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$.

The remaining claims are proven in a similar manner. To prove that the upper porosity $\overline{\operatorname{por}}(K_\omega, x) = \frac{1}{2}$, we need to show that typical points are infinitely often close to the centre of a $(m-1)$-dimensional face of a construction cube and, at the same time, the construction cube has a large hole. This can be formalised as follows: Let $\alpha = 1 - \delta$, where $\delta < \frac{1}{8}$. Choose $r \in \mathbb{N}$ such that $k^{-r} < \delta$. Let F be a $(m-1)$-dimensional face of $J_\emptyset$. Define $\widetilde{I} \subset I^r$ as the set of those $\tau \in I^r$ for which $J_\tau \cap F \neq \emptyset$ and the distance of all points of J_τ to the centre of F is less than $\frac{1}{4}$. For all $i \in \mathbb{N}$, define $\widetilde{Y}_i^\alpha \colon I^{\mathbb{N}} \times \Omega \to \{0,1\}$ by setting $\widetilde{Y}_i^\alpha(\eta,\omega) = 1$, if and only if $J_{\eta|_i}$ is chosen, $\eta|_{i+r}$ ends with a word from $\widetilde{I}$ and K_ω has an α-hole at scale i near $x(\eta)$ (in the sense of Definition 3.4) which is completely inside $J_{\eta|_i} \setminus J_{\eta|_{i+r}}$. If $\widetilde{Y}_i^\alpha(\eta,\omega) = 1$, we have that there is $z \in \mathbb{R}^m$ such that $B(z, \frac{1}{8}k^{-i}) \subset B(x(\eta), (\frac{1}{4}+\delta)k^{-i}) \setminus K_\omega$, giving $\overline{\mathrm{por}}(K_\omega, x(\eta), (\frac{1}{4}+\delta)k^{-i}) \geq \frac{1}{2(1+4\delta)}$. Since $E_Q[\widetilde{Y}_0^\alpha] > 0$, we conclude as above that

$$\overline{\mathrm{por}}(K_\omega, x) = \frac{1}{2}$$

for P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$.

Finally, to prove that $\overline{\mathrm{por}}(\nu_\omega, x) = 1$, we need to show that typical points x are infinitely often close to the centre of $Q_i^k(x)$ and, at the same time, $\nu_\omega(B(x,r))$ is small compared to $\nu_\omega(B(x,(1+\delta)r))$. More precisely, fix $\alpha = 1-\delta$ with $\delta < \frac{1}{8}$ and $r \in \mathbb{N}$ such that $\sqrt{m}k^{-r} < \delta$. Let I' be as above. For all $i \in \mathbb{N}$, define $\widehat{Y}_i^\alpha \colon I^{\mathbb{N}} \times \Omega \to \{0,1\}$ by setting $\widehat{Y}_i^\alpha(\eta,\omega) = 1$, if and only if $J_{\eta|_i}$ is chosen, $\eta|_{i+r}$ ends with a word from $\widetilde{I}$, $B(x(\eta), \frac{1}{4}k^{-i}) \cap K_\omega \subset J_{\eta|_{i+r}}$, there are $Ck^{r(m-1)}$ words $\tau \in I^r$ such that $J_{\eta|_i * \tau} \subset B(x(\eta), \frac{1}{4}(1+\delta)k^{-i}) \setminus B(x(\eta), \frac{1}{4}k^{-i})$ and the measures $\nu_\omega(J_{\eta|_i*\tau})$ are comparable to each other and to $\nu_\omega(J_{\eta|_{i+r}})$. Here the constant C depends only on m. If $\widehat{Y}_i^\alpha(\eta,\omega) = 1$ then $\overline{\mathrm{por}}(\nu_\omega, x(\eta), \frac{1}{4}(1+\delta)k^{-i}, \hat{c}k^{-r(m-1)}) \geq \frac{1}{1+4\delta}$, where $\hat{c}$ depends only on m. Since $E_Q[\widehat{Y}_i^\alpha] > 0$, we conclude as above that

$$\overline{\mathrm{por}}(\nu_\omega, x) = 1$$

for P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$.

Remark 4.15 (a) We can obtain some inequalities connecting regular mean porosities and our construction based mean porosities. We note that a square based hole of size α at scale i guarantees the presence of a Euclidean hole of size $\frac{1}{2}\alpha k^{-i}$ (recall the factor $\frac{1}{2}$ in Definition 3.4) in a ball of size $\sqrt{m}k^{-i}$. So if we define the regular lower mean porosity $\underline{p}(A,x,\alpha)$ using scales $\sqrt{m}k^{-i}$, we have, for all sets $A \subset \mathbb{R}^m$, for all points $x \in A$ and for all $\alpha \geq 0$, that

$$\underline{p}(A,x,\alpha) \geq \underline{\kappa}(A,x,2\sqrt{m}\alpha), \tag{4.6}$$

and similarly for the upper mean porosities. For an arbitrary set $A \subset \mathbb{R}^m$ and a point $x \in A$, there is no upper bound for $\underline{p}(A,x,\alpha)$ in terms of $\underline{\kappa}(A,x,\beta)$. Indeed, let $A = [0,1]^m$ and $x = 0$. Then $\overline{\kappa}(A,x,\alpha) = 0$ and $\underline{p}(A,x,\alpha) = 1$ for all $\alpha > 0$.

For typical points of K_ω, we may say more. Note that if $\alpha \geq \frac{1}{2\sqrt{m}}$, then inequality (4.6) is trivial. However, for any $0 < \alpha < \frac{1}{2}$, one may guarantee an existence of a Euclidean hole of size α in a ball of radius k^{-i} by covering a ball with construction cubes and noting that, with positive probability, all of these cubes are neglected. Thus, for P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in K_\omega$, we have $\underline{p}(K_\omega,x,\alpha) > 0$ for all $0 \leq \alpha < \frac{1}{2}$. One may also bound

$\overline{p}(K_\omega, x, \alpha)$ from above for typical points x. Namely, if x is close to the centre of a construction cube $Q_i^k(x)$, then there is no Euclidean hole of size α in $B(x, ck^{-i})$ if there is no cube-shaped hole of size $\frac{2}{\sqrt{m}}\alpha$ in $Q_i^k(x)$, where c depends on how close to the centre of $Q_i^k(x)$ the point x is. Since the events "x is close to the centre of $Q_i^k(x)$" and "$Q_i^k(x)$ does not contain a cube-shaped hole of size $\frac{2}{\sqrt{m}}\alpha$" can be estimated by independent events, there is positive probability that both of them happens. Thus, for P-almost all $\omega \in \Omega$ and for ν_ω-almost all $x \in k_\omega$, we have $\overline{p}(K_\omega, x, \alpha) < 1$ for all $0 < \alpha \leq \frac{1}{2}$. In order to get bounds which are close to the optimal ones, one should estimate the frequencies of scales when x is close to the boundaries of $Q_i^k(x)$ and there is a hole in a neighbour cube of $Q_i^k(x)$ close to the boundary of $Q_i^k(x)$.

(b) Similarly to the cube shaped holes, for an arbitrary convex seed set $J = \mathrm{Cl}(\mathrm{Int}(J)) \subset \mathbb{R}^d$, we can consider holes in the metric whose unit ball is shaped like J and then consider construction based mean porosities with holes in that metric (cf. Definition 3.4). We believe that, given that the fractal does not concentrate on the boundary of the seed set (cf. random strong open set condition in [5]), for a corresponding homogeneous random stochastically self-similar set with the seed set J, Theorems 4.5 and 4.11 and Corollary 4.14 will still hold.

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