The Schrödinger-HJW Theorem

K. A. Kirkpatrick

New Mexico Highlands University, Las Vegas, New Mexico 87701*

A concise presentation of Schrödinger's ancilla theorem (1936 Proc. Camb. Phil. Soc. **32**, 446) and its several recent rediscoveries.

1. Introduction

We re-present a theorem which Schrödinger proved in 1936. He commented that this theorem was one "for which I claim no priority but the permission of deducing it in the following section, for it is certainly not *well* known." His comment was amusingly prescient: The theorem was rediscovered by Jaynes in 1957 (whose work was extended by Hadjisavvas (1981)), rediscovered by Hughston, Jozsa, and Wootters (HJW) in 1993 (this last an expansion of a 1989 partial rediscovery by Gisin); in 1999, Mermin simplified a portion of HJW's proof — and it would appear none of these were aware of Schrödinger's work. Furthering the irony, Mermin commented that this is "a pertinent theorem which deserves to be more widely known."

But not only more widely known; this theorem deserves treatment in terms of physically relevant ancillae (following Mermin) rather than formal transformations by orthonormalcolumn matrices, and it deserves a complete statement and a concise proof in one place. Here is our attempt at such a presentation.

2. Preliminaries

Throughout, $\mathcal{H}^{\mathcal{S}}$ and $\mathcal{H}^{\mathcal{M}}$ are Hilbert spaces with dimensions $n_{\mathcal{S}}$ and $n_{\mathcal{M}}$, respectively, Tr_{\mathcal{M}} {·} is the trace over $\mathcal{H}^{\mathcal{M}}$ of an operator on $\mathcal{H}^{\mathcal{S}} \otimes \mathcal{H}^{\mathcal{M}}$, $|\Psi^{\mathcal{S}\mathcal{M}}\rangle$ is a vector in $\mathcal{H}^{\mathcal{S}} \otimes \mathcal{H}^{\mathcal{M}}$, and $\rho^{\mathcal{S}} = \operatorname{Tr}_{\mathcal{M}} \{|\Psi^{\mathcal{S}\mathcal{M}}\rangle\langle\Psi^{\mathcal{S}\mathcal{M}}|\}$. The dimension of the support of $\rho^{\mathcal{S}}$ is $n_{\rho}, n_{\rho} \leq n_{\mathcal{S}}$.

Lemma. $|\chi\rangle$ and $|\phi\rangle$ are vectors in $\mathcal{H}^{\mathcal{S}} \otimes \mathcal{H}^{\mathcal{M}}$. If $\operatorname{Tr}_{\mathcal{M}} \{|\chi\rangle\langle\chi|\} = \operatorname{Tr}_{\mathcal{M}} \{|\phi\rangle\langle\phi|\}$, then there exists a unitary transformation \mathbf{U} on $\mathcal{H}^{\mathcal{M}}$ such that $|\chi\rangle = (\mathbf{1}^{\mathcal{S}} \otimes \mathbf{U})|\phi\rangle$.

Proof: The operator $\mathbf{X} \equiv \operatorname{Tr}_{\mathcal{M}} \{ |\chi\rangle\langle\chi| \}$ is positive and Hermitian, so its eigenvalues are non-negative and its eigenkets $\{ |p_j\rangle \}_{j=1}^{n_{\mathcal{S}}}$ are an orthonormal basis of $\mathcal{H}^{\mathcal{S}}$; thus

$$\mathbf{X} = \sum_{s=1}^{n} w_s | p_s \rangle \langle p_s |, \quad \text{with} \quad \begin{array}{l} w_j > 0, \ 1 \le j \le n \le n_S, \\ w_j = 0, \ j > n. \end{array}$$
(1)

For $\{ |\mu_j\rangle \}$ any basis of $\mathcal{H}^{\mathcal{M}}$, $|\chi\rangle = \sum_{s=1}^{n_{\mathcal{S}}} \sum_{t=1}^{n_{\mathcal{M}}} \psi_{st} |p_s \mu_t\rangle$; setting $|\eta_s\rangle = \sum_{t=1}^{n_{\mathcal{M}}} \psi_{st} |\mu_t\rangle$, we have $|\chi\rangle = \sum_{s=1}^{n_{\mathcal{S}}} |p_s \eta_s\rangle$. Then

$$\mathbf{X} = \sum_{s}^{n_{\mathcal{S}}} |p_{s}\rangle \langle p_{s'} | \langle \eta_{s'} | \eta_{s} \rangle = \sum_{s=1}^{n} w_{s} |p_{s}\rangle \langle p_{s} |, \qquad (2)$$

whence $\langle \eta_{s'} | \eta_s \rangle = w_s \delta_{ss'}$. Hence the $\{ |b_j\rangle \equiv |\eta_j\rangle/\sqrt{w_j} \}_{j=1}^n$ are orthonormal and we may write $|\chi\rangle = \sum_{s=1}^n \sqrt{w_s} | p_s b_s \rangle$. A similar argument leads to $|\phi\rangle = \sum_{j=1}^n \sqrt{w_j} | p_j c_j \rangle$, the $\{ |c_j\rangle \}$ orthonormal. Extend the sets $\{ |b_j\rangle \}$ and $\{ |c_j\rangle \}$ to orthonormal bases of $\mathcal{H}^{\mathcal{M}}$ and set $\mathbf{U} = \sum_{s=1}^{n_{\mathcal{M}}} |b_s\rangle\langle c_s |$; clearly, \mathbf{U} is unitary and $\mathbf{1}^{\mathcal{S}} \otimes \mathbf{U}$ performs the desired transformation.

^{*}E-mail: kirkpatrick@physics.nmhu.edu

Definition (ρ^{S} -ensemble). Given a positive, Hermitian, unit trace operator ρ^{S} , a ρ^{S} -ensemble of order n ($n \ge n_{\rho}$) is a set $\left\{ \left(|\phi_{j}\rangle \in \mathcal{H}^{S}, w_{j} > 0 \right) \right\}_{j=1}^{n}$, with $\sum_{s=1}^{n} w_{s} = 1$ and the $\{ |\phi_{j}\rangle \}_{j=1}^{n}$ noncollinear, such that $\rho^{S} = \sum_{s=1}^{n} w_{s} |\phi_{s}\rangle\langle\phi_{s}|$. We will call a ρ^{S} -ensemble linearly independent if the $\{ |\phi_{j}\rangle \}_{j=1}^{n}$ are linearly independent.

Definition (Ancilla). A set $\{ |b_j\rangle \in \mathcal{H}^{\mathcal{M}} \}_{j=1}^n$ is an ancilla of the $\rho^{\mathcal{S}}$ -ensemble $\{ (|\phi_j\rangle \in \mathcal{H}^{\mathcal{S}}, w_j > 0) \}_{j=1}^n$ iff it is an orthonormal set and $|\Psi^{\mathcal{S}\mathcal{M}}\rangle = \sum_{s=1}^n \phi_s |\phi_s b_s\rangle$, $|\phi_j|^2 = w_j$.

Definition (U-map). The $\rho^{\mathcal{S}}$ -ensemble $\{(|\psi_k\rangle, v_k)\}_{k=1}^m$ is U-mapped to the $\rho^{\mathcal{S}}$ -ensemble $\{(|\phi_j\rangle, w_j)\}_{j=1}^n$ iff there exists a set

$$\left\{ u_{jk} \in \mathcal{C} \mid \sum_{t=1}^{n} u_{tk} u_{tk'}^* = \delta_{kk'} \right\}_{(j,k)=(1,1)}^{(n,m)}$$
(3)

such that

$$\left\{ \sqrt{w_j} \, | \, \phi_j \, \rangle = \sum_{t=1}^m \, u_{jt} \sqrt{v_t} \, | \, \psi_t \, \rangle \right\}_{j=1}^n \,. \tag{4}$$

A unitary transformation **U** on $\mathcal{H}^{\mathcal{M}}$ generates this U-map iff $\{u_{jk} = \langle b_j | \mathbf{U} | b_k \rangle \}_{(j,k)=(1,1)}^{(n,m)}$, with $\{|b_j\rangle \in \mathcal{H}^{\mathcal{M}}\}_{j=1}^{\max(n,m)}$ the ancilla of the greater-order $\rho^{\mathcal{S}}$ -ensemble.

3. The Theorem

Theorem (Schrödinger-HJW). The state of $S \oplus \mathcal{M}$ is $|\Psi^{S\mathcal{M}}\rangle$; $\rho^{S} = \operatorname{Tr}_{\mathcal{M}} \{ |\Psi^{S\mathcal{M}}\rangle \langle \Psi^{S\mathcal{M}}| \}$. Then

(a). Every ρ^{S} -ensemble of order $n \leq n_{\mathcal{M}}$ has a corresponding ancilla in $\mathcal{H}^{\mathcal{M}}$. If the ρ^{S} -ensemble is linearly independent, the ancilla is unique.

(b). Every orthonormal basis of $\mathcal{H}^{\mathcal{M}}$ contains exactly one ancilla corresponding to exactly one ρ^{S} -ensemble.

(c). Given any two ρ^{S} -ensembles, there exist unitary transformations on $\mathcal{H}^{\mathcal{M}}$ which generate a U-map from one to the other; if each ρ^{S} -ensemble is linearly independent, the transformation is unique.

(d). Every unitary transformation on $\mathcal{H}^{\mathcal{M}}$ generates a U-map of every $\rho^{\mathcal{S}}$ -ensemble to another.

(e). Every vector in the support of ρ^{S} appears as an element of at least one ρ^{S} -ensemble. Proof:

(a). Given the $\rho^{\mathcal{S}}$ -ensemble $\left\{ \left(|\phi_{j}\rangle, w_{j} \right) \right\}_{j=1}^{n}$, use an arbitrary orthonormal set $\left\{ |d_{j}\rangle \in \mathcal{H}^{\mathcal{M}} \right\}_{j=1}^{n}$ to construct the vector $|\Psi'\rangle = \sum_{s=1}^{n} \phi_{s} |\phi_{s} d_{s}\rangle$. By the Lemma, there exists a unitary transform **U** on $\mathcal{H}^{\mathcal{M}}$ such that

$$\left|\Psi^{\mathcal{SM}}\right\rangle = \left(\mathbf{1}^{\mathcal{S}} \otimes \mathbf{U}\right) \left|\Psi'\right\rangle = \sum_{s=1}^{n} \phi_{s} \left|\phi_{s}\right\rangle \otimes \mathbf{U} \left|d_{s}\right\rangle; \tag{5}$$

the ancilla is the orthonormal set $\{ \mathbf{U} | d_j \rangle \}_{j=1}^n$. Clearly, if the $\{ | \phi_j \rangle \}$ are linearly independent, that ancilla is unique.

(b). $\{ |b_k\rangle \}$ is an arbitrary orthonormal basis of $\mathcal{H}^{\mathcal{M}}$; expand $|\Psi^{S\mathcal{M}}\rangle$ in terms of it and any basis $\{ |p_j\rangle \}$ of \mathcal{H}^{S} : $|\Psi^{S\mathcal{M}}\rangle = \sum_{j=1}^{n_S} \sum_{k=1}^{n_M} \alpha_{jk} |p_j b_k\rangle$. Define $\phi_k |\phi_k\rangle \equiv \sum_{j=1}^{n_S} \alpha_{jk} |p_j\rangle$ and $w_k \equiv |\phi_k|^2$, and re-order the index k so $w_k > 0$, $k \le n \le n_{\mathcal{M}}$, $w_k = 0$, k > n; then

$$\Psi^{S\mathcal{M}} \rangle = \sum_{j=1}^{n} \phi_j | \phi_j b_j \rangle.$$
(6)

Thus $\{ |b_j\rangle \}_{j=1}^n$ is an ancilla, and $\{ (|\phi_j\rangle, w_j) \}_{j=1}^n$ is the corresponding $\rho^{\mathcal{S}}$ -ensemble. Suppose the set $\{ |b_j\rangle \}_{j=1}^{n_{\mathcal{M}}}$ contained two ancillae, say $\{ |b_j\rangle | j \in \mathcal{A} \}$ and $\{ |b_k\rangle | k \in \mathcal{B} \}$, with $\mathcal{A}, \mathcal{B} \subset \{ 1 \dots n_{\mathcal{M}} \}$:

$$\left|\Psi^{\mathcal{SM}}\right\rangle = \sum_{s\in\mathcal{A}} \phi_s \left|\phi_s \, b_s\right\rangle = \sum_{t\in\mathcal{B}} \psi_t \left|\psi_t \, b_t\right\rangle. \tag{7}$$

The orthonormality of the $\{ |b_j\rangle \}$ guarantees $\mathcal{A} = \mathcal{B}$, hence the uniqueness of the ancilla within $\{ |b_j\rangle \}$, and it guarantees $|\phi_j\rangle = |\psi_j\rangle$ and $\phi_j = \psi_j$ for all j in \mathcal{A} , hence the uniqueness of the $\rho^{\mathcal{S}}$ -ensemble corresponding to that ancilla.

(c). Given two $\rho^{\mathcal{S}}$ -ensembles $\left\{ \left(| \phi_j \rangle, w_j \right) \right\}_{j=1}^m$ and $\left\{ \left(| \psi_k \rangle, v_k \right) \right\}_{k=1}^n$, (a) guarantees ancillae $\left\{ | b_j \rangle \in \mathcal{H}^{\mathcal{M}} \right\}_{j=1}^m$ and $\left\{ | c_k \rangle \in \mathcal{H}^{\mathcal{M}} \right\}_{k=1}^n$, respectively, with

$$\left|\Psi^{\mathcal{SM}}\right\rangle = \sum_{s=1}^{m} \sqrt{w_s} \left|\phi_s b_s\right\rangle = \sum_{t=1}^{n} \sqrt{v_t} \left|\psi_t c_t\right\rangle \tag{8}$$

(phases are absorbed into the ancillary kets). Extend $\{|b_j\rangle\}$ and $\{|c_k\rangle\}$ to orthonormal bases of $\mathcal{H}^{\mathcal{M}}$; the unitary operator $\mathbf{U} = \sum_{t=1}^{n_{\mathcal{M}}} |c_t\rangle\langle b_t|$ transforms $|c_k\rangle = \mathbf{U} |b_k\rangle = \sum_{s=1}^{n_{\mathcal{M}}} u_{sk} |b_s\rangle$, where $u_{jk} = \langle b_j | \mathbf{U} | b_k \rangle = \langle b_j | c_k \rangle$, so

$$\left|\Psi^{\mathcal{SM}}\right\rangle = \sum_{s=1}^{n_{\mathcal{M}}} \left(\sum_{t=1}^{n} \sqrt{v_t} \, u_{st} \, | \, \psi_t \, \rangle\right) | \, b_s \, \rangle; \tag{9}$$

Equating (8) and (9), we obtain the U-generated U-map

$$\sum_{t=1}^{n} \sqrt{v_t} u_{jt} | \psi_t \rangle = \begin{cases} \sqrt{w_j} | \phi_j \rangle & j \in \{1 \cdots m\} \\ 0 & \text{otherwise.} \end{cases}$$
(10)

If each ρ^{S} -ensemble is linearly independent, each ancilla is unique, so the transformation between them is unique (modulo element-label permutation).

(d). Given any ρ^{S} -ensemble, (a) guarantees an ancilla; this may be extended to an orthonormal basis of $\mathcal{H}^{\mathcal{M}}$. Any unitary transformation maps this to another orthonormal basis; by (b), this second basis contains a single ancilla correlated to a single ρ^{S} -ensemble. By Eq. (10), this unitary transformation provides the U-mapping between these two ρ^{S} -ensembles.

(e). $|\xi\rangle$ is an arbitrary vector in the support of ρ^S ; we will construct a ρ^S -ensemble with $|\xi\rangle$ as its first element. The Schmidt expression of $|\Psi^{S\mathcal{M}}\rangle$, $\sum_{s=1}^{n}\psi_s|p_s a_s\rangle$, gives a basis for the expansion our arbitrary vector: $|\xi\rangle = \sum_{s=1}^{n}\gamma_s|p_s\rangle$, with $\sum_{s=1}^{n}|\gamma_s|^2 = 1$. For any $n_{\mathcal{M}} \times n_{\mathcal{M}}$ unitary matrix u_{jk} we have

$$\left|\Psi^{\mathcal{SM}}\right\rangle = \sum_{t} \phi_{t} \left|\phi_{t} b_{t}\right\rangle, \quad \text{where } \phi_{j} \left|\phi_{j}\right\rangle = \sum_{s=1}^{n} \psi_{s} u_{sj} \left|p_{s}\right\rangle \text{ and } \left|b_{k}\right\rangle = \sum_{s} u_{sk}^{*} \left|a_{s}\right\rangle. \tag{11}$$

Let us find u_{jk} such that the first element of this $\rho^{\mathcal{S}}$ -ensemble is our arbitrary vector, that is, $|\phi_1\rangle = |\xi\rangle$. Then

$$|\phi_1\rangle = \sum_{s=1}^n \gamma_s |p_s\rangle = \frac{1}{\phi_1} \sum_{s=1}^n \psi_s u_{s1} |p_s\rangle, \qquad (12)$$

whence $u_{s1} = \phi_1 \gamma_s / \psi_s$. Unitarity requires $\sum_s |u_{s1}|^2 = 1 = \phi_1 \sum_s |\gamma_s / \psi_s|^2$. Thus we have

$$|b_1\rangle = \sum_{s} u_{s1}^* |a_s\rangle, \text{ with } u_{j1} = (\gamma_j/\psi_j)/\sqrt{\sum_{s=1}^n |\gamma_s/\psi_s|^2}.$$
 (13)

Arbitrarily complete the orthonormal set $\{ | b_k \rangle \}_{k=1}^{n_{\mathcal{M}}}$, which, by (b), must contain an ancilla of a $\rho^{\mathcal{S}}$ -ensemble containing $| \xi \rangle$.

4. Overview of the several treatments

Schrödinger (1936) established our parts (a), (d), and (e).

Jaynes (1957, p. 173) attempted to establish our (c) and (d). He required, for a particular matrix **A** and a transformation **T**, the equality $\mathbf{AA}^{\dagger} = \mathbf{ATT}^{\dagger}\mathbf{A}^{\dagger}$, which he incorrectly interpreted to require¹ the unitarity of **T**. Thus Jaynes failed to establish his (correct) claim of isomorphism between the group of unitary transformations and the group of U-mappings of ρ^{S} -ensembles. Hadjisavvas (1981), extending Jaynes, elegantly obtained our (c), (d), and (e) for infinite dimensions. It appears that none of the later authors were aware of either Jaynes' or Hadjisavvas's results.

Gisin (1989) established a weak (and unphysical — see below) version of our (a).

HJW establish our (c) and (d) in their theorem's parts (b) and (a), respectively, and our (a) in the "application" in their Section 3.3. (The term " ρ^{S} -ensemble" is due to HJW; Jaynes called it an "array".)

The "GHJW Theorem" presented by Mermin (1999) is essentially HJW's Section 3.3 (i.e., our (a)). Our proofs of the lemma and of part (a) of the theorem are quite similar to the development in the appendix of Mermin (1999). (In personal correspondence, David Mermin gives credit for what I've formalized as the Lemma to a conversation with Chris Fuchs.)

Our (b) ("every complete variable of \mathcal{M} determines exactly one ρ^{S} -ensemble") seems to have its first explicit statement here.

We see that this theorem is primarily Schrödinger's — not only by priority, but by having established the major portion, parts (a), (d), and (e). Hadjisavvas and HJW added part (c). It is fairly well-known by the initials "HJW"; taking all this into account, and letting the "H" work a little harder, we arrive at "Schrödinger-HJW" as a priority-recognizing name.

5. Comments and Discussion

The U-map (orthogonal-columns matrix transformation) approach of Schrödinger and HJW, though it successfully generates all ρ^{S} -ensembles, has no obvious physical significance. Mermin, by treating the ancillary system \mathcal{M} as a real physical system with $|\Psi^{S\mathcal{M}}\rangle$ as the actual quantum-mechanical state of $S \oplus \mathcal{M}$ (and utilizing Fuchs' "lemma"), was able to obtain an arbitrary ρ^{S} -ensemble's ancilla (HJW's Section 3.3) without using the U-maps of HJW's theorem. This makes it possible to express the HJW theorem in terms of unitary transformations between observables in the ancillary system (our (c) and (d)). Instead of the mathematical formality of the U-maps, we have the physical significance of the correlation of each ancillary variable with a ρ^{S} -ensemble.

In Gisin's treatment, $|\Psi^{S\mathcal{M}}\rangle$ is arbitrarily imposed: The state vector of the joint system is constructed in the proof (as was the $|\Psi'\rangle$ of our proof), not assumed given (as was the $|\Psi^{S\mathcal{M}}\rangle$ of our theorem). However, it is necessary that the two systems were prepared *ab initio* in a pure joint state (i.e., $|\Psi^{S\mathcal{M}}\rangle$) which reduces to the specified mixture; Gisin's "steering" theorem, as presented, is not physical. Schrödinger and HJW make this clear; Mermin does also, in the body of his paper. However, in his appendix, Mermin states that, for any mixed state of Alice's system, it is always "possible to provide Bob with a system of his own for which the joint Alice-Bob system has a pure state" which reduces to Alice's mixture. As expressed, this statement may mislead: For example, if Alice's system is mixed because it is already entangled in a pure joint state with Carol's system, then no such system may be provided to Bob (except by stealing Carol's). That is, it is not possible to introduce the ancillary system *post facto* — it and the system must be initialized together.

To Schrödinger, the central point was that "*in general* a sophisticated experimenter can, by a suitable device which does *not* involve measuring non-commuting variables, produce a non-vanishing probability of driving the system into any state he chooses" by means

¹ This equality is satisfied by $\mathbf{TT}^{\dagger} = \mathbf{1} + \mathbf{X}$ with $\mathbf{AXA}^{\dagger} = 0$.

of measurement on the entangled ancilla. Unfortunately, Schrödinger's phrase "driving the system into any state he chooses" tends to overwhelm the modifier thereof, "a non-vanishing probability," so Schrödinger's point tends to be misstated; for example, Jammer (1974, p. 221) summarizes Schrödinger's statement "an experimenter can indeed steer a far-away system, without interfering with it at all, into any state out of an infinity of possible states...," leaving an incorrect impression of deterministic control of the outcome.

The Schrödinger-HJW Theorem gives a complete catalog of potential correlations between the ρ^{S} -ensembles of S and the disjoint (orthogonal) sets of states of \mathcal{M} , in the case that $S \oplus \mathcal{M}$ was prepared in a pure state. This catalog allows the design of an experiment involving a measurement on \mathcal{M} which makes exactly one ρ^{S} -ensemble "visible" — that is, allows every occurrence of S to be assigned a particular state of the ρ -ensemble without "disturbing" S. To be more precise, the measurement² of an ancilla (say, an observable B of \mathcal{M} with eigenkets $\{ |b_j\rangle \}$, the ancilla of the ρ^{S} -ensemble $\{ (|\phi_j\rangle, w_j) \}$) changes the state of $S \oplus \mathcal{M}$ from the pure state $|\Psi^{S\mathcal{M}}\rangle = \sum_s \sqrt{w_s} e^{i\theta_s} |\phi_s b_s\rangle$ to the mixture $\rho^{S \oplus \mathcal{M}} =$ $\sum_s w_s |\phi_s b_s\rangle \langle \phi_s b_s|$. The correlation of this particular ancilla with this particular ρ^{S} ensemble is unique³ (Kirkpatrick, 2002) — none of the other putative correlations in the Schrödinger-HJW theorem survive the measurement of the ancilla.

Acknowledgments

Thanks to Richard Jozsa for pointing out the Jaynes and Hadjisavvas articles, and to David Mermin for generously sharing his recollections regarding the development of the proof he published.

References

Gisin, N. (1989). "Stochastic quantum dynamics and relativity," *Helvetica Physica Acta* **62**, 363–371.

Hadjisavvas, N. (1981). "Properties of mixtures on non-orthogonal states," Lett. Math. Phys. 5, 327–332.

Hughston, L. P., Jozsa, R., and Wootters, W. K. (1993). "A complete classification of quantum ensembles having a given density matrix," *Phys. Lett. A* **183**, 14–18.

Jammer, M. (1974). The Philosophy of Quantum Mechanics. Wiley, New York.

Jaynes, E. T. (1957). "Information theory and statistical mechanics. II," Phys. Rev. 108(2), 171– 190.

Kirkpatrick, K. A. (2002). "Uniqueness of a convex sum of products of projectors," J. Math. Phys. 43(1), 684–686, eprint quant-ph/0104093.

Mermin, N. D. (1999). "What do these correlations know about reality? Nonlocality and the absurd," Found. Phys. 29(4), 571–587, eprint quant-ph/9807055.

Schrödinger, E. (1936). "Probability relations between separated systems," Proc. Camb. Phil. Soc. 32, 446–452.

² A "measurement" of **B** is an interaction of \mathcal{M} with \mathcal{X} (a system external to $\mathcal{S} \oplus \mathcal{M}$) such that $|b_j \xi\rangle \longrightarrow |b_j x_j\rangle$, the $\{|x_j\rangle \in \mathcal{H}^{\mathcal{X}}\}$ orthonormal; the state of $\mathcal{S} \oplus \mathcal{M}$ changes from pure to mixed as a direct result of this unitary transformation on $\mathcal{H}^{\mathcal{S}} \otimes \mathcal{H}^{\mathcal{M}} \otimes \mathcal{H}^{\mathcal{X}}$. So long as $\mathcal{S} \oplus \mathcal{M}$ remains isolated, not only do the ancilla and ρ -ensemble remain correlated, but also the $\{|b_j\rangle\}$ maintain their correlation with disjoint elements of the exterior (initially the $\{|x_j\rangle\}$), regardless of further external interactions with \mathcal{X} ; von Neumann's infinite regression of measurements is irrelevant to our discussion.

³ The correlation is unique even in the case of degenerate probabilities (e.g., EPR).