# The $A$-fibered Burnside Ring as $A$-Fibered Biset Functor in Characteristic Zero* 

Robert Boltje ${ }^{\dagger}$ and Deniz Yılmaz<br>Department of Mathematics<br>University of California<br>Santa Cruz, CA 95064<br>U.S.A.<br>boltje@ucsc.edu and deyilmaz@ucsc.edu

September 7, 2019
revised August 12, 2020


#### Abstract

Let $A$ be an abelian group such that $G^{*}:=\operatorname{Hom}(G, A)$ is finite for all finite groups $G$, and let $\mathbb{K}$ be a field of characteristic zero which is a splitting field for $G^{*}$ for all finite groups $G$. In this paper we prove fundamental properties of the $A$-fibered Burnside ring functor $B_{\mathbb{K}}^{A}$ as an $A$-fibered biset functor over $\mathbb{K}$. This includes a description of the composition factors of $B_{\mathbb{K}}^{A}$ and the lattice of subfunctors of $B_{\mathbb{K}}^{A}$ in terms of what we call $B^{A}$-pairs and a poset structure on their isomorphism classes. Unfortunately, we are not able to classify $B^{A}$-pairs. The results of the paper extend results of Coşkun and Yılmaz for the $A$-fibered Burnside ring functor restricted to $p$-groups and results of Bouc in the case that $A$ is trivial, i.e., the case of the Burnside ring functor as a biset functor over fields of characteristic zero. In the latter case, $B^{A}$-pairs become Bouc's $B$-groups which are also not known in general.


## 1 Introduction

Let $A$ be a finite group and let $k$ be a commutative ring. An $A$-fibered biset functor $F$ over $k$ is, informally speaking, a functor that assigns to each finite group $G$ a $k$-module $F(G)$ together with maps $\operatorname{res}_{H}^{G}: F(G) \rightarrow F(H)$ and $\operatorname{ind}_{H}^{G}: F(H) \rightarrow F(G)$, whenever $H \leqslant G$, called restriction and induction, maps $\inf _{G / N}^{G}: F(G / N) \rightarrow F(G)$ and $\operatorname{def}_{G / N}^{G}: F(G) \rightarrow F(G / N)$, whenever $N$ is a normal subgroup of $G$, called inflation and deflation, and maps iso ${ }_{f}: F(G) \rightarrow F(H)$, whenever $f: G \rightarrow H$ is an isomorphism. Moreover, the abelian group $G^{*}:=\operatorname{Hom}(G, A)$ acts $k$-linearly on $F(G)$ for every finite group $G$. These operations satisfy natural relations. Standard examples are various representation rings of $K G$-modules, for a field $K$ and $A=K^{\times}$. In this case, $G^{*}$ is the group of one-dimensional $K G$-modules acting by multiplication on these representation rings. In [BC18] the simple $A$-fibered biset functors were parametrized. If $A$ is the trivial group then one

[^0]obtains the well-established theory of biset functors, see Bc10 as special case. $A$-fibered biset functors over $k$ can also be interpreted as the modules over the Green biset functor $B_{k}^{A}$, where $B_{k}^{A}(G)$ is the $A$-fibered Burnside ring of $G$ over $k$ (also called the $K$-monomial Burnside ring of $G$ over $k$, when $A=K^{\times}$for a field $K$ ), as introduced by Dress in D71. Another natural example of $A$-fibered biset functors (without deflation) is the unit group functor $G \mapsto B^{A}(G)^{\times}$. This structure was established in a recent paper by Bouc and Mutlu, see BM19] and generalizes the biset functor structure on the unit group $B(G)^{\times}$of the Burnside ring.

Representation rings carry more structure when viewed as $A$-fibered biset functors compared to the biset functor structure. One of the goals is to understand their composition factors as such functors. By various induction theorems (e.g. by Brauer, Artin, Conlon) some of these representation rings can be viewed as quotient functors of the functor $B_{k}^{A}$ for appropriate $A$ and $k$. Thus, it is natural to first investigate the lattice of subfunctors of $B_{k}^{A}$ and its composition factors. This is the objective of this paper, where $k$ is a field of characteristic zero containing sufficiently many roots of unity. Coşkun and Yılmaz achieved this already in CY19 for the same functor category restricted to finite $p$-groups for fixed $p$, and for $A$ being a cyclic $p$-group. The choice of $A$ being cyclic allows them to use results on primitive idempotents and species of $B_{k}^{A}(G)$ from [Ba04, which were based on embedding $A$ into $k^{\times}$, for $k$ a field that is large enough. Our choice of $A$ is more general (using only that $G^{*}$ is finite for all finite groups $G$ ), thereby requiring a more complicated parametrizing set for the species and primitive idempotents of $B_{k}^{A}(G)$. This way the roles of $A$ and $k$ are kept as separate as possible while still implying that $B_{k}^{A}(G)$ is split semisimple.

The paper is arranged as follows. In Section 2 we recall basic facts about the $A$-fibered Burnside ring $B_{k}^{A}(G)$, about fibered biset functors, and the definition of $B_{k}^{A}$ as fibered biset functor. In Section 3 we parametrize the set of primitive idempotents of $B_{\mathbb{K}}^{A}(G)$ over a field $\mathbb{K}$ of characteristic 0 which contains enough roots of unity in relation to the finite element orders of $A$. We also derive an explicit formula for these idempotents, using results from BRV19 on the $-_{+}$-construction. We take advantage of the fact that the Green biset functor $B_{\mathbb{K}}^{A}$ arises as the $-_{+}$-construction of the Green biset functor $G \mapsto \mathbb{K} G^{*}$. Interestingly, the idempotent formula we derive in Theorem 3.2 is different from the one given by Barker in Ba04 when specializing to the more restrictive cases of $A$ considered there. It is used as a crucial tool in the following sections. In Section 4 we provide formulas for the action of inductions, restrictions, inflations, deflations, isomorphisms and twists by $\phi \in G^{*}$ on these idempotents. Crucial among those is the action of $\operatorname{def}_{G / N}^{G}$, which maps a primitive idempotent of $B_{\mathbb{K}}^{A}(G)$ to a scalar multiple of a primitive idempotent of $B_{\mathbb{K}}^{A}(G / N)$. After establishing three technical lemmas in Section [5] this mysterious scalar is studied in more depth in Section 6. The vanishing of this scalar is a condition that leads to the notion of a $B^{A}$-pair $(G, \Phi)$, where $G$ is a finite group and $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$, in Section 7. There, we also study particular subfunctors $E_{(G, \Phi)}$ of $B_{\mathbb{K}}^{A}$. In Section 8 we show that every subfunctor of $B_{\mathbb{K}}^{A}$ is a sum of some of the functors $E_{(G, \Phi)}$ and that the subfunctors of $B_{\mathbb{K}}^{A}$ are in bijection with the set of subsets of isomorphism classes of $B^{A}$-pairs that are closed from above with respect to a natural partial order $\preccurlyeq$, see Theorem 8.8, In Section 9 we show that the composition factors of $B_{\mathbb{K}}^{A}$ are parametrized by isomorphism classes of $B^{A}$-pairs. For a given $B^{A}$-pair, we also determine the parameter (a quadruple) of the corresponding composition factor in terms of the parametrization of all simple $A$-fibered biset functors over $\mathbb{K}$ given in BC18. Finally, in Section 10, we consider the special case that $A$ is a subgroup of $\mathbb{K}^{\times}$. In this case, a natural isomorphism $G / O(G) \xrightarrow{\sim} \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$for a normal subgroup $O(G)$ of $G$ depending on $A$, allows to reinterpret the set of $B^{A}$-pairs and makes our results compatible with the language and setup in [Ba04] and CY19.

The approach in this paper follows closely the blueprint in Bc10, Section 5] for the case $A=\{1\}$.

However, the presence of the fiber group $A$ requires additional ideas and techniques to achieve the analogous results. The main technical problem is that a transitive $A$-fibered biset with stabilizer pair $(U, \phi)$, does in general not factor through the group $q(U) \cong p_{i}(U) / k_{i}(U), i=1,2$, since $\phi$ is in general non-trivial when restricted to $k_{1}(U) \times k_{2}(U)$.
1.1 Notation For a finite group $G$ we denote $\operatorname{by} \exp (G)$ the exponent of $G$. If $X$ is a left $G$-set, we write $x={ }_{G} y$ if two elements $x$ and $y$ of $X$ belong to the same $G$-orbit. For $x \in X$, we denote by $G_{x}$ or $\operatorname{stab}_{G}(x)$ the stabilizer of $x$ in $G$. By $X^{G}$ we denote the set of $G$-fixed points in $X$ and by $[G \backslash X]$ a set of representatives of the $G$-orbits of $X$. For subgroups $H$ and $K$ of $G$, we denote by $[H \backslash G / K]$ a set of representatives of the $(H, K)$-double cosets of $G$.

For an abelian group $A$, we denote by tor $(A)$ its subgroup of elements of finite order, and, for a ring $R$, we denote by $R^{\times}$its group of units.

## 2 Prerequisites on the $A$-fibered biset functor $B_{k}^{A}$

Throughout this paper, we fix a multiplicatively written abelian group $A$. For any finite group $G$ we set $G^{*}:=\operatorname{Hom}(G, A)$, which we view as abelian group with point-wise multiplication. We will freely use the language of bisets and biset functors as developed in [Bc10, Chapters 2 and 3]. Throughout, $G, H$, and $K$ denote finite groups, and $k$ denotes a commutative ring.
2.1 (a) The $A$-fibered Burnside ring $B^{A}(G)$ of $G$ is defined as the Grothendieck group of the category of $A$-fibered left $G$-sets, see [BC18, 1.1 and 1.7]. It has a standard $\mathbb{Z}$-basis consisting of the $G$-orbits $[U, \phi]_{G}$ of pairs $(U, \phi)$, where $U \leqslant G$ and $\phi \in U^{*}$. The set $\mathcal{M}(G)=\mathcal{M}^{A}(G)$ of such pairs has a natural $G$-poset structure, see [BC18, 1.2]. The multiplication in $B^{A}(G)$ is given by

$$
\begin{equation*}
[U, \phi]_{G} \cdot[V, \psi]_{G}=\sum_{g \in[U \backslash G / V]}\left[U \cap{ }^{g} V,\left.\left.\phi\right|_{U \cap{ }^{g} V} \cdot\left({ }^{g} \psi\right)\right|_{U \cap{ }^{g} V}\right]_{G} \tag{1}
\end{equation*}
$$

and $[G, 1]_{G}$ is the identity. We set $B_{k}^{A}(G):=k \otimes_{\mathbb{Z}} B^{A}(G)$. If $A$ is the trivial group, we recover the Burnside ring $B(G)$ of $G$.
(b) We further set $B^{A}(G, H):=B^{A}(G \times H)$. This group can also be considered as the Grothendieck group of $A$-fibered $(G, H)$-bisets, see [BC18, 1.1 and 1.7]. The standard basis elements of $B^{A}(G, H)$ will be denoted by $\left[\frac{G \times H}{U, \phi}\right]$, for $(U, \phi) \in \mathcal{M}(G \times H)$. The tensor product of $A$-fibered bisets induces a bilinear map

$$
\begin{equation*}
-_{H}-: B^{A}(G, H) \times B^{A}(H, K) \rightarrow B^{A}(G, K) \tag{2}
\end{equation*}
$$

which behaves associatively. Its evaluation on standard basis elements is given by the formula (see [BC18, Corollary 2.5])

$$
\begin{equation*}
\left[\frac{G \times H}{U, \phi}\right]_{H}\left[\frac{H \times K}{V, \psi}\right]=\sum_{\substack{\left.t \in\left[p_{2}(U) \backslash H / p_{1}(V)\right] \\ \phi_{2}\right|_{H_{t}}=\left.{ }^{t} \psi_{1}\right|_{H_{t}}}}\left[\frac{G \times K}{U *^{(t, 1)} V, \phi *{ }^{(t, 1)} \psi}\right] \tag{3}
\end{equation*}
$$

where $H_{t}:=k_{2}(U) \cap^{t} k_{1}(V)$. See [BC18, 1.2] for the definition of $p_{i}(U)$ and $k_{i}(U)$, for $i=1,2$, and [BC18, 2.3] for the definition of $U *{ }^{(t, 1)} V$ and $\phi *{ }^{(t, 1)} \psi$. Recall from the latter that $\phi_{i} \in k_{i}(U)^{*}$,
$i=1,2$, are defined as the unique elements satisfying $\left.\phi\right|_{k_{1}(U) \times k_{2}(U)}=\phi_{1} \times \phi_{2}^{-1}$, with $\phi_{2}^{-1}$ denoting the inverse of $\phi_{2}$ in the group $k_{2}(U)^{*}$.
(c) Recall from [CY19, 3.1] that there exists a ring homomorphism $\Delta: B^{A}(G) \rightarrow B^{A}(G, G)$, $[U, \phi]_{G} \mapsto\left[\frac{G \times G}{\Delta(U), \Delta(\phi)}\right]$, where $\Delta(U):=\{(u, u) \mid u \in U\}$ and $\Delta(\phi)(u, u):=\phi(u)$. Here, $B^{A}(G, G)$ is considered as ring via the multiplication $\cdot{ }_{G}$ from (2).
2.2 (a) The bilinear maps in (2) are used as the definition of composition in the $k$-linear category $\mathcal{C}_{k}^{A}$, whose objects are the finite groups and whose morphism set from $H$ to $G$ is $B_{k}^{A}(G, H)$. An $A$-fibered biset functor over $k$ is a $k$-linear functor $F: \mathcal{C}_{k}^{A} \rightarrow{ }_{k}$ Mod. Together with natural transformations, these functors form an abelian category $\mathcal{F}_{k}^{A}$. If $A$ is the trivial group, one recovers the biset category $\mathcal{C}_{k}$ and the category of biset functors $\mathcal{F}_{k}$ over $k$, see [Bc10, Sections 3.1 and 3.2].
(b) The association $G \mapsto B_{k}^{A}(G)$, together with the bilinear maps in (2) applied to $K=1$ (and using the canonical identifications of $H \cong H \times\{1\}$ and $G \cong G \times\{1\}$ ) defines the $A$-fibered biset functor $B_{k}^{A} \in \mathcal{F}_{k}^{A}$, which is the main object of study in this paper.
(c) If $f: A^{\prime} \rightarrow A$ is a homomorphism between abelian groups, one obtains induced maps $\operatorname{Hom}\left(G, A^{\prime}\right) \rightarrow \operatorname{Hom}(G, A), \mathcal{M}^{A^{\prime}}(G) \rightarrow \mathcal{M}^{A}(G)$, ring homomorphisms $B^{A^{\prime}}(G) \rightarrow B^{A}(G)$, and $k$-linear functors $\mathcal{C}_{k}^{A^{\prime}} \rightarrow \mathcal{C}_{k}^{A}$ and $\mathcal{F}_{k}^{A} \rightarrow \mathcal{F}_{k}^{A^{\prime}}$. In particular, when $A^{\prime}$ is the trivial group, we obtain embeddings $B(G, H) \subseteq B^{A}(G, H), \quad[(G \times H) / U] \mapsto\left[\frac{G \times H}{U, 1}\right]$, and $\mathcal{C}_{k} \subseteq \mathcal{C}_{k}^{A}$. This way, we can view the elementary bisets $\operatorname{res}_{H}^{G}, \operatorname{ind}_{H}^{G}, \inf _{G / N}^{G}, \operatorname{def}_{G / N}^{G}, \operatorname{iso} f$, for $H \leqslant G, N \unlhd G, f: G \xrightarrow{\sim} G^{\prime}$, as elements in $B^{A}(G, H), B^{A}(H, G), B^{A}(G, G / N), B^{A}(G / N, G), B^{A}\left(G^{\prime}, G\right)$, respectively. One can verify quickly with (3) for $K=\{1\}$, that their operations under the $A$-fibered biset functor structure of $B_{k}^{A}$ in (b) on standard basis elements are given by

$$
\begin{gathered}
\operatorname{res}_{H}^{G}\left([U, \phi]_{G}\right)=\sum_{g \in[H \backslash G / U]}\left[H \cap{ }^{g} U,\left.\left({ }^{g} \phi\right)\right|_{H \cap{ }^{g} U}\right]_{H}, \quad \operatorname{ind}_{H}^{G}\left([V, \psi]_{H}\right)=[V, \psi]_{G} \\
\inf _{G / N}^{G}\left([U / N, \phi]_{G / N}\right)=[U, \phi \circ \nu]_{G}
\end{gathered}
$$

where $\nu: U \rightarrow U / N$ is the natural epimorphism,

$$
\operatorname{def}_{G / N}^{G}\left([U, \phi]_{G}\right)= \begin{cases}{[U N / N, \tilde{\phi}]_{G / N},} & \text { if } U \cap N \leqslant \operatorname{ker}(\phi) \\ 0 & \text { otherwise }\end{cases}
$$

where $\tilde{\phi}(u N):=\phi(u)$ for $u \in U$, and

$$
\operatorname{iso}_{f}\left([U, \phi]_{G}\right)=\left[f(U),\left.\phi \circ f^{-1}\right|_{f(U)}\right]_{G^{\prime}}
$$

In particular, for $g \in G$ and $H \leqslant G$, and $(U, \phi) \in \mathcal{M}(H)$, we have

$$
g_{[ }[U, \phi]_{H}:=\operatorname{iso}_{c_{g}}\left([U, \phi]_{H}\right)=\left[{ }^{g} U,{ }^{g} \phi\right]{ }_{g}{ }_{H}
$$

where $c_{g}: H \xrightarrow{\sim} g H^{-1}$ is the conjugation map. For $\lambda \in G^{*}$ and $(U, \phi) \in \mathcal{M}(G)$, one additionally has

$$
\operatorname{tw}_{\lambda}\left([U, \phi]_{G}\right)=\left[U,\left.\lambda\right|_{U} \cdot \phi\right]_{G}
$$

the twist by $\lambda$, coming from the application of $\mathrm{tw}_{\lambda}:=\Delta\left([G, \lambda]_{G}\right)=\left[\frac{G \times G}{\Delta(G), \Delta(\lambda)}\right] \in B^{A}(G, G)$. It is easily verified that $\operatorname{res}_{H}^{G}, \inf _{G / N}^{G}$ and iso ${ }_{f}$ are ring homomorphisms, and that $B_{k}^{A}(G)$ is a $k G^{*}$-algebra via tw.

Using the above notation, one obtains a canonical decomposition of a standard basis element of $B^{A}(G, H)$ into elementary bisets and a standard basis element for smaller groups.
2.3 Theorem ( $\left[\overline{B C 18}\right.$, Proposition 2.8]) Let $(U, \phi) \in \mathcal{M}(G \times H)$ and set $P:=p_{1}(U), Q:=p_{2}(U)$, $K:=\operatorname{ker}\left(\phi_{1}\right)$, and $L:=\operatorname{ker}\left(\phi_{2}\right)$. Then $K \unlhd P, L \unlhd Q, K \times L \unlhd U$, and

$$
\left[\frac{G \times H}{U, \phi}\right]=\operatorname{ind}_{P}^{G} \cdot \inf _{P / K}^{P} \underset{P / K}{ } \cdot\left[\frac{P / K \times Q / L}{U /(K \times L), \bar{\phi}}\right]_{Q / L} \operatorname{def}_{Q / L}^{Q} \cdot \operatorname{res}_{L}^{H}
$$

where $\bar{\phi} \in(U /(K \times L))^{*}$ is induced by $\phi$ and $U /(K \times L)$ is viewed as subgroup of $P / K \times Q / L$ via the canonical isomorphism $(P \times Q) /(K \times L) \cong P / K \times Q / L$.
2.4 Remark Let $\mathcal{D}_{k}$ be the subcategory of $\mathcal{C}_{k}$ with the same objects as $\mathcal{C}_{k}$, but with morphism sets generated by all elementary bisets, excluding inductions. In other words, $\operatorname{Hom}_{\mathcal{D}_{k}}(H, G) \subseteq$ $B_{k}(G, H)$ is the free $k$-module generated by all standard basis elements $[(G \times H) / U]$ with $p_{1}(U)=$ $G$. Mapping $G$ to the group algebra $k G^{*}$ defines a Green biset functor $F$ on $\mathcal{D}_{k}$ over $k$ in the sense of Romero's reformulation [R11, Definición 3.2.7, Lema 4.2.3] of Bouc's original definition Bc10, Definition 8.5.1], with restriction, inflation and isomorphisms defined as usual, viewing Hom $(-, A)$ as contravariant functor, and deflation defined by

$$
\operatorname{def}_{G / N}^{G}(\phi):=\left\{\begin{array}{l}
\bar{\phi}, \text { if }\left.\phi\right|_{N}=1 \\
0, \text { otherwise }
\end{array}\right.
$$

whenever $N$ is a normal subgroup of $G$ and $\phi \in G^{*}$. Here, $\bar{\phi} \in(G / N)^{*}$ is induced by $\phi$. In fact, it is straightforward to check that all the relations in [Bc10, 1.1.3] that do not involve inductions are satisfied. Thus, we are in the situation of BRV19, Theorem 7.3(a),(b)] and obtain via the $-_{+-}$ construction a Green biset functor $F_{+}$on $\left(\mathcal{D}_{k}\right)_{+}=\mathcal{C}_{k}$. The Green biset functor $F_{+}$is isomorphic to the Green biset functor $B_{k}^{A}$ on $\mathcal{C}_{k}$. This follows from BRV19, Theorem 4.7(c)] and by comparing the explicit formulas for the elementary biset operations in 2.2 (c) with the explicit formulas in [BRV19, Remark 4.8]. We will use this point of view repeatedly in Sections 3 and 4

## 3 Primitive idempotents of $B_{\mathbb{K}}^{A}(G)$

Throughout this section we assume that $G$ is a finite group such that $H^{*}=\operatorname{Hom}(H, A)$ is a finite abelian group for every $H \leqslant G$. This is equivalent to $\operatorname{tor}_{\exp (G)}(A)$ being finite. Moreover, we assume that $\mathbb{K}$ is a splitting field of characteristic zero for all $H^{*}, H \leqslant G$. Note that this holds if and only if $\mathbb{K}$ has a root of unity of order $\exp \left(\operatorname{tor}_{\exp (G)}(A)\right)$. Also note that in this case $S^{*}$ is finite and $\mathbb{K}$ is a splitting field for $S^{*}$, for each subquotient $S$ of $G$.

We define $\mathcal{X}(G)$ as the set of all pairs $(H, \Phi)$ with $H \leqslant G$ and $\Phi \in \operatorname{Hom}\left(H^{*}, \mathbb{K}^{\times}\right)$and note that $G$ acts on $\mathcal{X}(G)$ by conjugation: ${ }^{g}(H, \Phi):=\left({ }^{g} H,{ }^{g} \Phi\right)$, with ${ }^{g} \Phi(\phi):=\Phi\left({ }^{g^{-1}} \phi\right)$, for $g \in G$, $(H, \Phi) \in \mathcal{X}(G)$, and $\phi \in H^{*}$. The assumptions on $\mathbb{K}$ imply that, for any $H \leqslant G$,

$$
\begin{equation*}
\mathbb{K} H^{*} \rightarrow \prod_{\Phi \in \operatorname{Hom}\left(H^{*}, \mathbb{K}^{\times}\right)} \mathbb{K}, \quad a \mapsto\left(s_{\Phi}^{H}(a)\right)_{\Phi} \tag{4}
\end{equation*}
$$

is an isomorphism of $\mathbb{K}$-algebras. Here, we $\mathbb{K}$-linearly extended $\Phi$ to a $\mathbb{K}$-algebra homomorphism

$$
s_{\Phi}^{H}: \mathbb{K} H^{*} \rightarrow \mathbb{K}
$$

The first orthogonality relation implies that, for $\Psi \in \operatorname{Hom}\left(H^{*}, \mathbb{K}^{\times}\right)$, the element

$$
\begin{equation*}
e_{\Psi}^{H}:=\frac{1}{\left|H^{*}\right|} \sum_{\phi \in H^{*}} \Psi\left(\phi^{-1}\right) \phi \in \mathbb{K} H^{*} \tag{5}
\end{equation*}
$$

is the primitive idempotent of $\mathbb{K} H^{*}$ which is mapped under the isomorphism in (4) to the primitive idempotent $\epsilon_{\Psi}^{H} \in \prod_{\Phi} \mathbb{K}$ whose $\Phi$-component is $\delta_{\Phi, \Psi}$.

For any $H \leqslant G$ we consider the map

$$
\pi_{H}: B_{\mathbb{K}}^{A}(H) \rightarrow \mathbb{K} H^{*}, \quad[U, \phi]_{H} \mapsto \begin{cases}\phi & \text { if } U=H \\ 0 & \text { otherwise }\end{cases}
$$

It is easily seen by the multiplication formula in (11) that $\pi_{H}$ is a $\mathbb{K}$-algebra homomorphism and we obtain for every $(H, \Phi) \in \mathcal{X}(G)$, a $\mathbb{K}$-algebra homomorphism

$$
s_{(H, \Phi)}^{G}:=s_{\Phi}^{H} \circ \pi_{H} \circ \operatorname{res}_{H}^{G}: B_{\mathbb{K}}^{A}(G) \rightarrow B_{\mathbb{K}}^{A}(H) \rightarrow \mathbb{K} H^{*} \rightarrow \mathbb{K} .
$$

3.1 Theorem The map

$$
\begin{equation*}
B_{\mathbb{K}}^{A}(G) \rightarrow\left(\prod_{(H, \Phi) \in \mathcal{X}(G)} \mathbb{K}\right)^{G}, \quad x \mapsto\left(s_{(H, \Phi)}^{G}(x)\right)_{(H, \Phi)} \tag{6}
\end{equation*}
$$

is a $\mathbb{K}$-algebra isomorphism. Here, $G$ acts on $\prod_{(H, \Phi) \in \mathcal{X}(G)} \mathbb{K}$ by permuting the coordinates according to the $G$-action on $\mathcal{X}(G)$. In particular, every $\mathbb{K}$-algebra homomorphism $B_{G}^{A}(G) \rightarrow \mathbb{K}$ is of the form $s_{(H, \Phi)}^{G}$ for some $(H, \Phi) \in \mathcal{X}(G)$. For $(H, \Phi),(K, \Psi) \in \mathcal{X}(G)$ one has $s_{(H, \Phi)}^{G}=s_{(K, \Psi)}^{G}$ if and only if $(H, \Phi)={ }_{G}(K, \Psi)$.

Proof By Theorem BRV19, Theorems 6.1 and 7.3(c)] and using Remark[2.4, the mark morphism

$$
\begin{equation*}
m_{G}: B_{\mathbb{K}}^{A}(G) \rightarrow\left(\prod_{H \leqslant G} \mathbb{K} H^{*}\right)^{G}, \quad x \mapsto\left(\pi_{H}\left(\operatorname{res}_{H}^{G}(x)\right)\right)_{H \leqslant G}, \tag{7}
\end{equation*}
$$

is a homomorphism of $\mathbb{K}$-algebras and by Theorem BRV19, Corollary 6.4] it is an isomorphism, since $|G|$ is invertible in $\mathbb{K}$. Here, $G$ acts on $\prod_{H \leqslant G} \mathbb{K} H^{*}$ by ${ }^{g}\left(\left(a_{H}\right)_{H \leqslant G}\right):=\left({ }^{g} a_{g^{-1} H g}\right)_{H \leqslant G}$. Using the $\mathbb{K}$-algebra isomorphisms from (4), we obtain a $G$-equivariant $\mathbb{K}$-algebra isomorphism

$$
\prod_{H \leqslant G} \mathbb{K} H^{*} \rightarrow \prod_{(H, \Phi) \in \mathcal{X}(G)} \mathbb{K}
$$

Applying the functor of $G$-fixed points to this isomorphism and composing it with the isomorphism in (7), we obtain the isomorphism in (6). The remaining assertions follow immediately.

Clearly, for each $(H, \Phi) \in \mathcal{X}(G)$, we obtain a primitive idempotent $\epsilon_{(H, \Phi)}^{G}$ of the right hand side of the isomorphism (6). More precisely, $\epsilon_{(H, \Phi)}^{G}$ has entries equal to 1 at indices labelled by the $G$-conjugates of $(H, \Phi)$ and entries equal to 0 everywhere else. We denote the idempotent of $B_{\mathbb{K}}^{A}(G)$ corresponding to $\epsilon_{(H, \Phi)}^{G}$ by $e_{(H, \Phi)}^{G} \in B_{\mathbb{K}}^{A}(G)$. If $(H, \Phi)$ runs through a set of representatives of the
$G$-orbits of $\mathcal{X}(G)$ then $e_{(H, \Phi)}^{G}$ runs through the set of primitive idempotents of $B_{\mathbb{K}}^{A}(G)$, without repetition. Thus, we have

$$
s_{(H, \Phi)}^{G}\left(e_{(K, \Psi)}^{G}\right)= \begin{cases}1, & \text { if }(H, \Phi)=_{G}(K, \Psi), \quad \text { and } \quad x \cdot e_{(H, \Phi)}^{G}=s_{(H, \Phi)}^{G}(x) e_{(H, \Phi)}^{G},  \tag{8}\\ 0, & \text { otherwise },\end{cases}
$$

for any $(H, \Phi),(K, \Psi) \in \mathcal{X}(G)$ and any $x \in B_{\mathbb{K}}^{A}(G)$.
The following theorem gives an explicit formula for $e_{(H, \Phi)}^{G}$. A different formula for particular choices of $A$ was given by Barker in [Ba04, Theorem 5.2]. For any $H \leqslant G$ and $a=\sum_{\phi \in H^{*}} a_{\phi} \phi \in$ $\mathbb{K} H^{*}$ we will use the notation $[H, a]_{G}:=\sum_{\phi \in H^{*}} a_{\phi}[H, \phi]_{G} \in B_{\mathbb{K}}^{A}(G)$. Moreover, $N_{G}(H, \Phi)$ denotes the stabilizer of $(H, \Phi)$ under $G$-conjugation.
3.2 Theorem For $(H, \Phi) \in \mathcal{X}(G)$ one has

$$
\begin{align*}
e_{(H, \Phi)}^{G} & =\frac{1}{\left|N_{G}(H, \Phi)\right|} \sum_{K \leqslant H}|K| \mu(K, H)\left[K, \operatorname{res}_{K}^{H}\left(e_{\Phi}^{H}\right)\right]_{G}  \tag{9}\\
& =\frac{1}{\left|N_{G}(H, \Phi)\right|} \sum_{\substack{K \leqslant\left. H \\
\Phi\right|_{K^{\perp}=1}}}|K| \mu(K, H)\left[K, \operatorname{res}_{K}^{H}\left(e_{\Phi}^{H}\right)\right]_{G}  \tag{10}\\
& =\frac{1}{\left|N_{G}(H, \Phi)\right| \cdot\left|H^{*}\right|} \sum_{\substack{K \leqslant\left. H \\
\Phi\right|_{K^{\perp}=1}}} \sum_{\phi \in H^{*}}|K| \mu(K, H) \Phi\left(\phi^{-1}\right)\left[K,\left.\phi\right|_{K}\right]_{G} \in B_{\mathbb{K}}^{A}(G) \tag{11}
\end{align*}
$$

where $K^{\perp}:=\left\{\phi \in H^{*}|\phi|_{K}=1\right\} \leqslant H^{*}$ and $\mu$ is the Möbius function on the poset of all subgroups of $G$.

Proof We use the inversion formula of the $\mathbb{K}$-algebra isomorphism (7) from BRV19, Proposition 6.3] and obtain

$$
\begin{equation*}
e_{(H, \Phi)}^{G}=\frac{1}{|G|} \sum_{L \leqslant K \leqslant G}|L| \mu(L, K)\left[L, \operatorname{res}_{L}^{K}\left(a_{K}\right)\right]_{G} \tag{12}
\end{equation*}
$$

with $a_{K} \in \mathbb{K} K^{*}, K \leqslant G$, given by $a_{H}=\sum_{x \in\left[N_{G}(H) / N_{G}(H, \Phi)\right]}{ }^{x} e_{\Phi}^{H}, a^{g}{ }_{H}={ }^{g} a_{H}$, for any $g \in G$, and $a_{K}=0$ for all $K$ not $G$-conjugate to $H$. Thus, in the above sum, for $K$ it suffices to consider only subgroups that are $G$-conjugate to $H$. We obtain

$$
\begin{equation*}
e_{(H, \Phi)}^{G}=\frac{1}{|G|} \sum_{x \in\left[G / N_{G}(H)\right]} \sum_{L \leqslant{ }^{x} H}|L| \mu\left(L,{ }^{x} H\right)\left[L, \operatorname{res}_{L}^{{ }^{x} H}\left({ }^{x} a_{H}\right)\right]_{G} \tag{13}
\end{equation*}
$$

Replacing $L$ with ${ }^{x} K$, for $K \leqslant H$, we see that the sum over $L$ is independent of $x$ and we obtain

$$
\begin{equation*}
e_{(H, \Phi)}^{G}=\frac{\left[G: N_{G}(H)\right]}{|G|} \sum_{K \leqslant H}|K| \mu(K, H)\left[K, \operatorname{res}_{K}^{H}\left(a_{H}\right)\right]_{G} \tag{14}
\end{equation*}
$$

Substituting $a_{H}=\sum_{g \in\left[N_{G}(H) / N_{G}(H, \Phi)\right]} g_{\Phi} e_{\Phi}^{H}$ and using the same argument as above, we obtain the formula in (9). In order to prove the formula in (10) it suffices to show that $\operatorname{res}_{K}^{H}\left(e_{\Phi}^{H}\right)=0$ if $\left.\Phi\right|_{K^{\perp}} \neq 1$. Substituting the formula (5) for $e_{\Phi}^{H}$, we obtain

$$
\begin{equation*}
\operatorname{res}_{K}^{H}\left(e_{\Phi}^{H}\right)=\left.\frac{1}{\left|H^{*}\right|} \sum_{\phi \in H^{*}} \Phi\left(\phi^{-1}\right) \phi\right|_{K} \tag{15}
\end{equation*}
$$

Note that $K^{\perp}=\operatorname{ker}\left(\operatorname{res}_{K}^{H}: H^{*} \rightarrow K^{*}\right)$ and choose for every $\psi \in \operatorname{im}\left(\operatorname{res}_{K}^{H}\right) \leqslant K^{*}$ an element $\tilde{\psi} \in H^{*}$ with $\left.\tilde{\psi}\right|_{K}=\psi$. Then the right hand side in (15) is equal to

$$
\frac{1}{\left|H^{*}\right|} \sum_{\psi \in \operatorname{im}\left(\operatorname{res}_{K}^{H}\right)} \sum_{\lambda \in K^{\perp}} \Phi\left(\tilde{\psi}^{-1} \lambda^{-1}\right) \psi=\frac{1}{\left|H^{*}\right|} \sum_{\psi \in \operatorname{im}_{\left(\operatorname{res}_{K}^{H}\right)}} \Phi\left(\tilde{\psi}^{-1}\right)\left(\sum_{\lambda \in K^{\perp}} \Phi\left(\lambda^{-1}\right)\right) \psi
$$

and it suffices to show that $\left.\sum_{\lambda \in K^{\perp}} \Phi\right|_{K^{\perp}}\left(\lambda^{-1}\right)=0$. But

$$
\begin{equation*}
\left.\sum_{\lambda \in K^{\perp}} \Phi\right|_{K^{\perp}}\left(\lambda^{-1}\right)=\left[K^{\perp}: K^{\perp} \cap \operatorname{ker}(\Phi)\right] \sum_{\bar{\lambda} \in K^{\perp} /\left(K^{\perp} \cap \operatorname{ker}(\Phi)\right)} \overline{\left.\Phi\right|_{K^{\perp}}}\left(\bar{\lambda}^{-1}\right), \tag{16}
\end{equation*}
$$

with an injective homomorphism from $\overline{\left.\Phi\right|_{K^{\perp}}}: K^{\perp} /\left(K^{\perp} \cap \operatorname{ker}(\Phi)\right) \rightarrow \mathbb{K}^{\times}$. It follows that $K^{\perp} /\left(K^{\perp} \cap\right.$ $\operatorname{ker}(\Phi))$ is cyclic, say of order $n$. Our assumption on $\mathbb{K}$ implies that $\mathbb{K}$ has a primitive $n$-th root of unity. Moreover, since $\left.\Phi\right|_{K^{\perp}} \neq 1$, we have $n>1$. Thus the sum on the right hand side of (16) is equal to the sum of all $n$-th roots of unity in $\mathbb{K}$, which is 0 . This proves Equation (10). Formula (11) is now immediate after substituting the formula for $e_{\Phi}^{H}$.
3.3 Remark If $A^{\prime}$ is the trivial subgroup of $A$ we have $B_{\mathbb{K}}^{A^{\prime}}(G)=B_{\mathbb{K}}(G)$, the Burnside algebra over $\mathbb{K}$. Using the functoriality properties in $2.2(\mathrm{c})$, we obtain a commutative diagram

of $\mathbb{K}$-algebra homomorphisms, where the left horizontal maps are the mark isomorphisms $m_{G}$ from (17) given by $\left(\pi_{H} \circ \operatorname{res}_{H}^{G}\right)$, the right top horizontal map is the identity, the right bottom horizontal map is the product of the isomorphisms from (4), the middle vertical map is the product of the unique $\mathbb{K}$-algebra homomorphisms $\mathbb{K} \rightarrow \mathbb{K} H^{*}$, and the right vertical map is induced by the $G$ equivariant map $\mathcal{X}(G) \rightarrow\{H \leqslant G\},(H, \Phi) \mapsto H$, between the indexing sets. We denote the primitive idempotents of $B_{\mathbb{K}}(G)$ by $e_{H}^{G}$, for any $H \leqslant G$. Thus, by the above commutative diagram,

$$
\begin{equation*}
e_{G}^{G}=\sum_{\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)} e_{(G, \Phi)}^{G} \tag{17}
\end{equation*}
$$

3.4 Lemma For any $x \in B_{\mathbb{K}}^{A}(G)$ one has $e_{G}^{G} \cdot x=0$ if and only if $\pi_{G}(x)=0$.

Proof Since $m_{G}: B_{\mathbb{K}}^{A}(G) \rightarrow \prod_{H \leqslant G} \mathbb{K}$ is injective and multiplicative, one has $e_{G}^{G} \cdot x=0$ if and only if $m_{G}\left(e_{G}^{G}\right) \cdot m_{G}(x)=0$. But $m_{G}\left(e_{G}^{G}\right)$ has entry 1 in the $G$-component and entry 0 everywhere else. Thus, $e_{G}^{G} \cdot x=0$ if and only if the entry of $m_{G}(x)$ in the $G$-component is equal to 0 . But this entry equals $\pi_{G}(x)$.

## 4 Elementary operations on primitive idempotents

Throughout this section we assume as in Section 3 that $G$ is a finite group such that $S^{*}=$ $\operatorname{Hom}(S, A)$ is finite for all subquotients $S$ of $G$, and that $\mathbb{K}$ is a field of characteristic 0 which is a splitting field of $S^{*}$ for all subquotients $S$ of $G$.

In this section we will establish formulas for elementary fibered biset operations on the primitive idempotents of $B_{\mathbb{K}}^{A}(S)$ for subquotients $S$ of $G$. These formulas will be used in later sections.
4.1 Proposition Let $H \leqslant G$.
(a) For $(L, \Psi) \in \mathcal{X}(H)$ one has $s_{(L, \Psi)}^{H} \circ \operatorname{res}_{H}^{G}=s_{(L, \Psi)}^{G}: B_{\mathbb{K}}^{A}(G) \rightarrow \mathbb{K}$.
(b) For $(K, \Phi) \in \mathcal{X}(G)$ one has

$$
\operatorname{res}_{H}^{G}\left(e_{(K, \Phi)}^{G}\right)=\sum_{\substack{(L, \Psi) \in[H \backslash \mathcal{X}(H)] \\(L, \Psi)={ }_{G}(K, \Phi)}} e_{(L, \Psi)}^{H}
$$

(c) For $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$and $H<G$ one has $\operatorname{res}_{H}^{G}\left(e_{(G, \Phi)}^{G}\right)=0$.

Proof (a) We use the point of view from Remark 2.4. By [BRV19, Equation (13) and Theorem 6.1] the left square in

is commutative, where the left horizontal maps are the mark homomorphisms from (7), the right horizontal maps are given by the isomorphisms in (4), and the middle and right vertical maps are the canonical projections. Since the right hand square commutes as well, following up with the projection onto the $(L, \Psi)$-component of $\prod_{(L, \Psi) \in \mathcal{X}(H)} \mathbb{K}$, yields the result.
(b) Since $\operatorname{res}_{H}^{G}: B_{\mathbb{K}}^{A}(G) \rightarrow B_{\mathbb{K}}^{A}(H)$ is a $\mathbb{K}$-algebra homomorphism, $\operatorname{res}_{H}^{G}\left(e_{(K, \Phi)}\right)$ is the sum of certain primitive idempotents $e_{(L, \Psi)}^{H},(L, \Psi) \in[H \backslash \mathcal{X}(H)]$. Moreover, $e_{(L, \Psi)}^{H}$ occurs in this sum if and only if $s_{(L, \Psi)}^{H}\left(\operatorname{res}_{H}^{G}\left(e_{(K, \Phi)}^{G}\right)\right) \neq 0$. The result follows now immediately from (a).
(c) This follows immediately from Part (b).
4.2 Proposition Let $N \unlhd G$.
(a) For $(H, \Phi) \in \mathcal{X}(G)$ one has $s_{(H, \Phi)}^{G} \circ \inf _{G / N}^{G}=s_{\left(H N / N, \Phi_{N}\right)}^{G / N}$, where $\Phi_{N}:=\Phi \circ \nu^{*} \circ \alpha^{*} \in$ $\operatorname{Hom}\left((H N / N)^{*}, \mathbb{K}^{\times}\right)$with $\alpha: H /(H \cap N) \xrightarrow{\sim} H N / N$ denoting the canonical isomorphism and $\nu: H \rightarrow H /(H \cap N)$ denoting the canonical epimorphism.
(b) For $(U / N, \Psi) \in \mathcal{X}(G / N)$ with $N \leqslant U \leqslant G$, one has

$$
\inf _{G / N}^{G}\left(e_{(U / N, \Psi)}^{G / N}\right)=\sum_{\substack{(H, \Phi) \in[G \backslash \mathcal{X}(G)] \\\left(H N / N, \Phi_{N}\right)=G / N(U / N, \Psi)}} e_{(H, \Phi)}^{G} .
$$

Proof (a) We use again the point of view from Remark 2.4. By BRV19, Equation (12)] applied to $D:=\{(g, g N) \mid g \in G\} \leqslant G \times G / N$ and [BRV19, Theorem 6.1] the left square in

is commutative, where the left horizontal maps are the mark homomorphisms from (7), the right horizontal maps are given by the isomorphisms in (4), the middle vertical homomorphism maps the family $\left(a_{U / N}\right)_{N \leqslant U \leqslant G}$ to $\left(\inf _{H /(H \cap N)}^{H}\left(\alpha^{*}\left(a_{H N / N}\right)\right)\right)_{H \leqslant G}$ with $\alpha: H /(H \cap N) \xrightarrow{\sim} H N / N$ denoting the canonical isomorphism, and the right vertical homomorphism maps the family $\left(a_{(U / N, \Psi)}\right)_{(U / N, \Psi) \in \mathcal{X}(G / N)}$ to the family $\left(a_{\left(H N / N, \Phi_{N}\right)}\right)_{(H, \Phi) \in \mathcal{X}(G)}$. Since the right hand square commutes as well (note that $\inf _{H /(H \cap N)}^{H}: \mathbb{K}(H /(H \cap N))^{*} \rightarrow \mathbb{K} H^{*}$ is the $\mathbb{K}$-linear extension of $\nu^{*}$ from (a)), following up with the projection onto the $(H, \Phi)$-component of $\prod_{(H, \Phi) \in \mathcal{X}(G)} \mathbb{K}$, yields the result.
(b) Since $\inf _{G / N}^{G}: B_{\mathbb{K}}^{A}(G / N) \rightarrow B_{\mathbb{K}}^{A}(G)$ is a $\mathbb{K}$-algebra homomorphism, $\inf _{G / N}^{G}\left(e_{(U / N, \Psi)}^{G / N}\right)$ is the sum of certain primitive idempotents $e_{(H, \Phi)}^{G},(H, \Phi) \in[G \backslash \mathcal{X}(G)]$. Moreover, $e_{(H, \Phi)}^{G}$ occurs in this sum if and only if $s_{(H, \Phi)}^{G}\left(\inf _{G / N}^{G}\left(e_{(U / N, \Psi)}^{G / N}\right)\right) \neq 0$. Part (a) now implies the result.

### 4.3 Proposition Let $N \unlhd G$.

(a) For all $(H, \Phi) \in \mathcal{X}(G)$, there exists $m_{(H, \Phi)}^{G, N} \in \mathbb{K}$ such that

$$
\begin{equation*}
\operatorname{def}_{G / N}^{G}\left(e_{(H, \Phi)}^{G}\right)=m_{(H, \Phi)}^{G, N} \cdot e_{\left(H N / N, \Phi_{N}\right)}^{G} \tag{18}
\end{equation*}
$$

with $\Phi_{N}$ defined as in Proposition4.2(a).
(b) For all $\Phi \in G^{*}$ one has

$$
\begin{equation*}
m_{(G, \Phi)}^{N}:=m_{(G, \Phi)}^{G, N}=\frac{\left|(G / N)^{*}\right|}{|G| \cdot\left|G^{*}\right|} \sum_{\substack{K \leqslant G \\ K N=\left.G \\ \Phi\right|_{K^{\perp}=1}}}|K| \cdot\left|K^{\perp}\right| \cdot \mu(K, G) \in \mathbb{Q} \tag{19}
\end{equation*}
$$

Proof (a) For any $x \in B_{\mathbb{K}}^{A}(G / N)$ we have

$$
\begin{aligned}
x \cdot \operatorname{def}_{G / N}^{G}\left(e_{(H, \Phi)}^{G}\right) & =\operatorname{def}_{G / N}^{G}\left(\inf _{G / N}^{G}(x) \cdot e_{(H, \Phi)}^{G}\right)=\operatorname{def}_{G / N}^{G}\left(s_{(H, \Phi)}^{G}\left(\inf _{G / N}^{G}(x)\right) \cdot e_{(H, \Phi)}^{G}\right) \\
& =\operatorname{def}_{G / N}^{G}\left(s_{\left(H N / N, \Phi_{N}\right)}^{G / N}(x) \cdot e_{(H, \Phi)}^{G}\right)=s_{\left(H N / N, \Phi_{N}\right)}^{G / N}(x) \cdot \operatorname{def}_{G / N}^{G}\left(e_{(H, \Phi)}^{G}\right)
\end{aligned}
$$

In fact, the first equation follows from the Green biset functor axioms (see BRV19, Definition 7.2(a)] and [R11, Definición 3.2.7, Lema 4.2.3]), the second from (8), and the third from Proposition $4.2(\mathrm{a})$. Choosing $x=e_{\left(H N / N, \Phi_{N}\right)}^{G / N}$, and reading the above equations backward, we obtain

$$
\operatorname{def}_{G / N}^{G}\left(e_{(H, \Phi)}^{G}\right)=\operatorname{def}_{G / N}^{G}\left(e_{(H, \Phi)}^{G}\right) \cdot e_{\left(H N / N, \Phi_{N}\right)}^{G}
$$

Now, (8) implies the result with $m_{(H, \Phi)}^{G, N}=s_{\left(H N / N, \Phi_{N}\right)}^{G / N}\left(\operatorname{def}_{G / N}^{G}\left(e_{(H, \Phi)}^{G}\right)\right)$.
(b) Substituting the explicit idempotent formula (11) for $e_{(G, \Phi)}^{G}$ and using the explicit formula for $\operatorname{def}_{G / N}^{G}: B_{\mathbb{K}}^{G}(G) \rightarrow B_{\mathbb{K}}^{A}(G / N)$ from $2.2(\mathrm{c})$, the left hand side of (18) is equal to

$$
\frac{1}{|G| \cdot\left|G^{*}\right|} \sum_{\substack{K \leqslant\left. G \\ \Phi\right|_{K} ^{\perp}=1}} \sum_{\substack{\phi \in G^{*} \\ \phi \mid K \cap N \\ K \cap 1}}|K| \mu(K, G) \Phi\left(\phi^{-1}\right)\left[K N / N, \widetilde{\left.\phi\right|_{K}}\right]_{G / N}
$$

where $\widetilde{\left.\phi\right|_{K}}(k N):=\phi(k)$ for $k \in K$. Moreover, using the explicit formula (11) for $e_{\left(G / N, \Phi_{N}\right)}^{G / N}$, the right hand side of (18) is equal to

$$
\frac{m_{(G, \Phi)}^{N}}{|G / N| \cdot\left|(G / N)^{*}\right|} \sum_{\substack{U / N \leqslant G /\left.N \\ \Phi_{N}\right|_{(U / N)^{\perp}=1}}} \sum_{\psi \in(G / N)^{*}}|U / N| \mu(U / N, G / N) \Phi_{N}\left(\psi^{-1}\right)\left[U / N,\left.\psi\right|_{U / N}\right]_{G / N}
$$

Next we compare the coefficients at the standard basis element $[G / N, 1]_{G / N}$ of $B_{\mathbb{K}}^{A}(G / N)$ on both sides. On the left hand side, we only have to sum over those $K \leqslant G$ with $K N=G$ and those $\phi \in G^{*}$ with $\widetilde{\left.\phi\right|_{K}}=1$. By the definition of $\widetilde{\left.\phi\right|_{K}}$, this is equivalent to $\phi \in K^{\perp}$. But then $\left.\Phi\right|_{K^{\perp}}=1$ implies that $\Phi\left(\phi^{-1}\right)=1$ for all such $\phi$. Thus, the coefficient of $[G / N, 1]_{G / N}$ on the left hand side of (18) is equal to

$$
\frac{1}{|G| \cdot\left|G^{*}\right|} \sum_{\substack{K \leqslant\left. G \\ \Phi\right|_{K} ^{\perp}=1 \\ K N=G}}|K|\left|K^{\perp}\right| \mu(K, G)
$$

On the right hand side of (18) only the summands with $U=G$ and $\psi=1$ contribute to the coefficient of $[G / N, 1]_{G / N}$ and this coefficient evaluates to $m_{(G, \Phi)}^{N} /\left|(G / N)^{*}\right|$. The result follows.
4.4 Proposition Let $H$ be a subgroup of $G$ and let $(K, \Psi) \in \mathcal{X}(H)$. Then

$$
\operatorname{ind}_{H}^{G}\left(e_{(K, \Psi)}^{H}\right)=\frac{\left|N_{G}(K, \Psi)\right|}{\left|N_{H}(K, \Psi)\right|} \cdot e_{(K, \Psi)}^{G}
$$

Proof This is an immediate consequence of the explicit formula in (11), since $\operatorname{ind}_{H}^{G}\left([L, \phi]_{H}\right)=$ $[L, \phi]_{G}$ for any $(L, \phi) \in \mathcal{M}(H)$.
4.5 Proposition (a) For every isomorphism $f: G \xrightarrow{\sim} G^{\prime}$ and $(H, \Phi) \in \mathcal{X}(G)$ one has iso ${ }_{f}\left(e_{(H, \Phi)}^{G}\right)=$ $e_{\left(f(H), \Phi \circ\left(\left.f\right|_{H}\right)^{*}\right)}^{G^{\prime}}$.
(b) For every $g \in G, H \leqslant G$, and $(K, \Psi) \in \mathcal{X}(H)$ one has $\left.{ }^{g} e_{(K, \Psi)}^{H}=e_{\left({ }^{g} K, ~\right.}^{g^{g}}{ }^{g} \Psi\right)$.
(c) For every $(H, \Phi) \in \mathcal{X}(G)$ and $\alpha \in G^{*}$ one has $\operatorname{tw}_{\alpha}\left(e_{(H, \Phi)}^{G}\right)=\Phi\left(\left.\alpha\right|_{H}\right) e_{(H, \Phi)}^{G}$ and $\Delta\left(e_{(H, \Phi)}^{G}\right) \cdot{ }_{G}$ $\mathrm{tw}_{\alpha}=\Phi\left(\left.\alpha\right|_{H}\right) \Delta\left(e_{(H, \Phi)}^{G}\right)$, with $\Delta$ as in 2.1 (c).
Proof All three parts follow immediately from the explicit formulas for the three operations in 2.2(c), the explicit idempotent formula (11), and the formula in (3).

## 5 Three lemmata

Throughout this section, $G$ and $H$ denote finite groups and $\mathbb{K}$ a field of characteristic 0 such that $S^{*}$ is finite and $\mathbb{K}$ is a splitting field of $S^{*}$ for all subquotients $S$ of $G$ and $H$.

For $U \leqslant G \times H$ we set $q(U):=U /\left(k_{1}(U) \times k_{2}(U)\right.$. Thus, $q(U) \cong p_{i}(U) / k_{i}(U)$ for $i=1,2$.
5.1 Lemma Let $G$ and $H$ be finite groups and let $k$ be a commutative ring. For $(U, \phi) \in \mathcal{M}(G \times H)$ with $p_{1}(U)=G$ and $p_{2}(U)=H$ the following are equivalent:
(i) There exists $\alpha \in G^{*}$ such that $\left.\alpha\right|_{k_{1}(U)}=\phi_{1}$.
(ii) There exists $\beta \in H^{*}$ such that $\left.\beta\right|_{k_{2}(U)}=\phi_{2}$.
(iii) There exists $\psi \in(G \times H)^{*}$ such that $\left.\psi\right|_{U}=\phi$.
(iv) In the category $\mathcal{C}^{A}$, the morphism $\left[\frac{G \times H}{U, \phi}\right]$ factors through $q(U)$.
(v) In the category $\mathcal{C}_{k}^{A}$, the morphism $\left[\frac{G \times H}{U, \phi}\right]$ factors through $q(U)$.

Proof Clearly, (iii) implies (i) and (ii).
We next show that (i) implies (iii). Let $\alpha \in G^{*}$ be an extension of $\phi_{1}$. Since $\phi \in U^{*}$ and $\alpha \times 1 \in(G \times\{1\})^{*}$ coincide on $U \cap(G \times\{1\})=k_{1}(U) \times\{1\}$ and since $G \times H=(G \times\{1\}) U$ with $G \times\{1\}$ normal in $G \times H$, the function $\psi: G \times H \rightarrow A,(g, 1) u \mapsto \alpha(g) \phi(u)$ is well-defined and extends $\phi$. It is also a homomorphism, since $G \times\{1\}$ is normal in $G \times H$ and $\alpha \times 1$ is $U$-stable.

Similarly one proves that (ii) implies (iii).
Next we show that (iii) and implies (iv). Let $\psi=\alpha \times \beta \in(G \times H)^{*}$ extend $\phi \in U^{*}$. By (3), we have

$$
\left[\frac{G \times H}{U, \phi}\right]=\operatorname{tw}_{\alpha} \dot{G}\left[\frac{G \times H}{U, 1}\right]_{H} \operatorname{tw}_{\beta}
$$

and the morphism $\left[\frac{G \times H}{U, 1}\right]$ factors through $G / k_{1}(U)$ by Theorem 2.3,
Clearly, (iv) implies (v).
Finally, we show that (v) implies (iii). Assume that $\left[\frac{G \times H}{U, \phi}\right]$ factors in $\mathcal{C}_{k}^{A}$ through $K:=q(U)$ with $K \cong G / k_{1}(U)$. Then there exist $(V, \Psi) \in \mathcal{M}(G \times K)$ and $(W, \rho) \in \mathcal{M}(K \times H)$ such that $\left[\frac{G \times H}{U, \phi}\right]$ occurs with nonzero coefficient in $\left[\frac{G \times K}{V, \psi}\right] \cdot K_{K}\left[\frac{K \times H}{W, \rho}\right]$. By the formula in (3), this implies that there exists $t \in K$ and $(g, h) \in G \times H$ such that $\left.\psi_{2}\right|_{K_{t}}=\left.\rho_{1}\right|_{K_{t}}$ with $K_{t}:=k_{2}(V) \cap{ }^{t} k_{1}(W)$ and $(U, \phi)={ }^{(g, h)}\left(V *{ }^{(t, 1)} W, \psi *{ }^{(t, 1)} \rho\right)$. Replacing $(V, \psi)$ with ${ }^{(g, 1)}(V, \psi)$ and $(W, \rho)$ with ${ }^{(t, h)}(W, \rho)$, we may assume that there exist $(V, \psi) \in \mathcal{M}(G \times K)$ and $(W, \rho) \in \mathcal{M}(K \times H)$ with $\left.\psi_{2}\right|_{k_{2}(V) \cap k_{1}(W)}=$ $\left.\rho_{1}\right|_{k_{2}(V) \cap k_{1}(W)}$ and $(V * W, \psi * \rho)=(U, \phi)$. By [Bc10, 2.3.22.2] one has

$$
k_{1}(V) \leqslant k_{1}(V * W)=k_{1}(U) \leqslant p_{1}(U) \leqslant p_{1}(V)
$$

Since $p_{1}(U)=G$, this implies that $p_{1}(V)=G$. Moreover, since $p_{1}(U) / k_{1}(U)$ is isomorphic to a subquotient of $p_{1}(V) / k_{1}(V) \cong p_{2}(V) / k_{2}(V)$, which in turn is a subquotient of $K \cong p_{1}(U) / k_{1}(U)$, we otain that $k_{1}(V)=k_{1}(U), p_{2}(V)=K$ and $k_{2}(V)=\{1\}$. Since $\psi_{2}=1$, it extends trivially to $K=p_{2}(V)$. By the first part of the proof (note that $p_{1}(V)=G$ and $p_{2}(V)=K$ ) we also obtain that $\psi$ extends to $\alpha \times 1 \in(G \times K)^{*}$ for some $\alpha \in G^{*}$. Similarly one shows that $\rho$ extends to $1 \times \beta \in(K \times H)^{*}$ for some $\beta \in H^{*}$. But then $\phi=\psi * \rho$ is the restriction of $(\alpha \times 1) *(1 \times \beta)=\alpha \times \beta$ and (iii) holds.
5.2 Lemma $\operatorname{Let}(U, \phi) \in \mathcal{M}(G \times H)$.
(a) If $\Phi \in \operatorname{Hom}\left(H^{*}, \mathbb{K}^{\times}\right)$satisfies $\left[\frac{G \times H}{U, \phi}\right] \cdot H e_{(H, \Phi)}^{H} \neq 0$ then $p_{2}(U)=H$ and $\phi_{2}$ extends to $H^{*}$.
(b) If $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$satisfies $e_{(G, \Phi)}^{G} \cdot G\left[\frac{G \times H}{U, \phi}\right] \neq 0$ then $p_{1}(U)=G$ and $\phi_{1}$ extends to $G^{*}$.

Proof We only prove Part (a). Part (b) is proved similarly.
Since $\left[\frac{G \times H}{U, \phi}\right]=\left[\frac{G \times p_{2}(U)}{U, \phi}\right] \cdot p_{2}(U) \operatorname{res}_{p_{2}(U)}^{H}$, Proposition4.1(c) implies that $p_{2}(U)=H$.
We will show that $\phi_{2}$ extends to $H$ by induction on $|G|$. If $|G|=1$ then $\phi_{1}$ is trivial, thus extends to $G$, and Lemma 5.1 implies that $\phi_{2}$ extends to $H$. From now on assume that $|G|>1$. We distinguish two cases.

Case 1: $\pi_{G}\left(\left[\frac{G \times H}{U, \phi}\right] \cdot{ }_{H} e_{(H, \Phi)}^{H}\right)=0$. By Lemma3.4, this implies that $e_{G}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right] \cdot{ }_{H} e_{(H, \Phi)}^{H}\right)=$ 0 and therefore, using the primitive idempotents $e_{K}^{G}$ of the Burnside ring (see Remark 3.3) and CY19, Proposition 2], we have

$$
\begin{aligned}
0 & \neq\left[\frac{G \times H}{U, \phi}\right]_{H} e_{(H, \Phi)}^{H}=\left(1-e_{G}^{G}\right) \cdot\left(\left[\frac{G \times H}{U, \phi}\right]_{H} e_{(H, \Phi)}^{H}\right) \\
& \left.=\sum_{\substack{K \in[G \backslash \mathcal{S}(G)] \\
K<G}} e_{K}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right]_{H}^{H} e_{(H, \Phi)}^{H}\right)=\sum_{\substack{K \in[G \backslash \mathcal{S}(G)] \\
K<G}} \Delta\left(e_{K}^{G}\right)\right)_{G}\left(\left[\frac{G \times H}{U, \phi}\right] ._{H}^{H} e_{(H, \Phi)}^{H}\right) .
\end{aligned}
$$

Moreover, using the explicit formula for $e_{K}^{G}$ (in the special case that $A$ is trivial), we have

$$
\begin{aligned}
0 & \neq\left[\frac{G \times H}{U, \phi}\right]_{H} e_{(H, \Phi)}^{H} \in \sum_{K<G} \mathbb{K} \cdot\left[\frac{G \times G}{\Delta(K), 1}\right] \cdot\left(\left[\frac{G \times H}{U, \phi}\right] \cdot e_{(H, \Phi)}^{H}\right) \\
& =\sum_{K<G} \mathbb{K} \cdot\left(\left[\frac{G \times G}{\Delta(K), 1}\right]_{G}\left[\frac{G \times H}{U, \phi}\right]\right) \dot{H}_{H}^{H} e_{(H, \Phi)}^{H}=\sum_{K<G} \mathbb{K} \cdot\left[\frac{G \times H}{\Delta(K) * U, 1 * \phi}\right] \dot{H}_{(H, \Phi)}^{H} .
\end{aligned}
$$

Therefore, there exists $K<G$ such that $\left[\frac{G \times H}{\Delta(K) * U, 1 * \phi}\right] \cdot H_{H} e_{(H, \Phi)}^{H} \neq 0$. Note that $p_{1}(\Delta(K) * U) \leqslant K$, so that we can decompose

$$
\left[\frac{G \times H}{\Delta(K) * U, 1 * \phi}\right]=\operatorname{ind}_{K}^{G} \cdot\left[\frac{K \times H}{\Delta(K) * U, 1 * \phi}\right]
$$

Thus,

$$
\begin{equation*}
\left[\frac{K \times H}{\Delta(K) * U, 1 * \phi}\right]_{H} e_{(H, \Phi)}^{H} \neq 0 . \tag{20}
\end{equation*}
$$

By the first part of the proof this implies that $p_{2}(\Delta(K) * U)=H$. Moreover, it is straightforward to verify that $k_{2}(\Delta(K) * U)=k_{2}(U)$ and that $(1 * \phi)_{2}=\phi_{2} \in k_{2}(U)^{*}$. Since $K<G$, the inductive hypothesis applied to (20) yields that $\phi_{2}$ extends to $H$.

Case 2: $\pi_{G}\left(\left[\frac{G \times H}{U, \phi}\right] \cdot{ }_{H} e_{(H, \Phi)}^{H}\right) \neq 0$. The explicit formula for $e_{(H, \Phi)}^{H}$ in (11) implies that

$$
\sum_{\substack{K \leqslant\left. H \\ \Phi\right|_{K^{\perp}}=1}} \sum_{\psi \in H^{*}}|K| \mu(K, H) \Phi\left(\psi^{-1}\right) \pi_{G}\left(\left[\frac{G \times H}{U, \phi}\right]_{H}\left[K,\left.\psi\right|_{K}\right]_{H}\right) \neq 0
$$

Thus, there exists $K \leqslant H$ and $\psi \in H^{*}$ such that $\pi_{G}\left(\left[\frac{G \times H}{U, \phi}\right] \cdot_{H}\left[K,\left.\psi\right|_{K}\right]_{H}\right) \neq 0$. This implies that $U * K=G$ and $\left.\phi_{2}\right|_{k_{2}(U) \cap K}=\left.\psi\right|_{k_{2}(U) \cap K}$. Since $\phi_{2}$ is stable under $p_{2}(U)=H$ and $\phi_{2}$ and $\psi$ coincide on $k_{2}(U) \cap K$, there exists an extension $\beta \in\left(k_{2}(U) K\right)^{*}$ of $\phi_{2}$ and $\psi$. Moreover, $U * K=G$ implies $k_{2}(U) K=H$. In fact, if $h \in H=p_{2}(U)$ then there exists $g \in G$ with $(g, h) \in U$. Since $g \in G=U * K$, there exists $k \in K$ such that $(g, k) \in U$. But then $h k^{-1} \in k_{2}(U)$ and $h \in k_{2}(U) K$. This completes the proof of the lemma.

For $U \leqslant G \times H$ we denote by $\eta_{U}: p_{2}(U) / k_{2}(U) \xrightarrow{\sim} p_{1}(U) / k_{1}(U)$ the isomorphism induced by $U$ (see [Bc10, Lemma 2.3.25]).
5.3 Lemma Let $(U, \phi) \in \mathcal{M}(G \times H), \Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$and $\Psi \in \operatorname{Hom}\left(H^{*}, \mathbb{K}^{\times}\right)$. Then

$$
e_{(G, \Phi)}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right]_{H} \cdot e_{(H, \Psi)}^{H}\right)=0
$$

unless $p_{1}(U)=G, p_{2}(U)=H$, $\phi$ has an extension $\alpha \times \beta \in(G \times H)^{*}, \Phi_{k_{1}(U)}=\Psi_{k_{2}(U)} \circ \eta_{U}^{*}$, and $m_{(H, \Phi)}^{k_{2}(U)} \neq 0$, in which case one has

$$
e_{(G, \Phi)}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right]_{H} e_{(H, \Psi)}^{H}\right)=\Phi(\alpha) \Psi(\beta) m_{(H, \Psi)}^{k_{2}(U)} \cdot e_{(G, \Phi)}^{G} .
$$

Proof Assume that

$$
\begin{equation*}
e_{(G, \Phi)}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right] . e_{(H, \Psi)}^{H}\right) \neq 0 \tag{21}
\end{equation*}
$$

By Lemma5.2, $p_{2}(U)=H$ and $\phi_{2}$ extends to $H$. Assume $p_{1}(U)<G$. Then $\left[\frac{G \times H}{U, \phi}\right] \cdot e_{(H, \Psi)}^{H}$ is in the $\mathbb{K}$-span of elements of the form $[U * L, \psi]_{G}$, with $L \leqslant H$ and $\psi \in(U * L)^{*}$. Since $U * L \leqslant p_{1}(U)<G$ we obtain $m_{G}\left(e_{(G, \Phi)} \cdot[U * L, \psi]_{G}\right)=m_{G}\left(e_{(G, \Phi)}\right) \cdot m_{G}\left([U * L, \psi]_{G}\right)=0$, because the first factor vanishes in the components indexed by proper subgroups of $G$ and the second factor vanishes in the $G$-component. By the injectivity of $m_{G}$ this implies that the element in (21) vanishes, a contradiction. Thus, $p_{1}(U)=G$. By Lemma 5.1, $\phi$ has an extension $\alpha \times \beta \in(G \times H)^{*}$. Further,

$$
\begin{aligned}
e_{(G, \Phi)}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right] \dot{H} e_{(H, \Psi)}^{H}\right) & =e_{(G, \Phi)}^{G} \cdot\left(\operatorname{tw}_{\alpha} \cdot\left[\frac{G \times H}{U, 1}\right] \cdot \operatorname{tw}_{\beta} \dot{H}_{(H, \Psi)}^{H}\right) \\
& =\Delta\left(e_{(G, \Phi)}^{G}\right) \dot{G}^{\operatorname{tw}_{\alpha}} \cdot\left[\frac{G \times H}{U, 1}\right] \cdot \operatorname{tw}_{\beta} \dot{H}_{H}^{H} e_{(H, \Psi)}^{H},
\end{aligned}
$$

where the last equation follows from CY19, Proposition 2]. By Proposition 4.5(c) we have tw ${ }_{\beta} \cdot{ }_{H}$ $e_{(H, \Psi)}^{H}=\operatorname{tw}_{\beta}\left(e_{(H, \Psi)}^{H}\right)=\Psi(\beta) \cdot e_{(H, \Psi)}^{H}$ and $\Delta\left(e_{(G, \Phi)}^{G}\right) \cdot G \operatorname{tw}_{\alpha}=\Phi(\alpha) \cdot \Delta\left(e_{(G, \Phi)}^{G}\right)$. Thus, using again CY19, Proposition 2] and Theorem 2.3,

$$
\begin{aligned}
& e_{(G, \Phi)}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right] \cdot e_{H}^{H}\right)=\Phi(\alpha) \Psi(\beta) \cdot \Delta\left(e_{(G, \Phi)}^{G}\right) \dot{G}_{G}\left[\frac{G \times H}{U, 1}\right]_{H} e_{(H, \Psi)}^{H} \\
& =\Phi(\alpha) \Psi(\beta) e_{(G, \Phi)} \cdot\left(\inf _{G / k_{1}(U)}^{G / k_{1}(U)}\left[\frac{G \times H}{\bar{U}, 1}\right]_{H / k_{2}(U)} \operatorname{def}_{H / k_{2}(U)}^{H} \cdot e_{(H, \Psi)}^{H}\right) \\
& =\Phi(\alpha) \Psi(\beta) e_{(G, \Phi)}^{H} \cdot\left(\inf _{G / k_{1}(U)}^{G / k_{1}(U)} \dot{\operatorname{iso}}_{\eta_{U}} \underset{H / k_{2}(U)}{ } \operatorname{def}_{H / k_{2}(U)}^{H} \cdot e_{(H, \Psi)}^{H}\right) .
\end{aligned}
$$

Using Propositions 4.3(a), 4.5(a) and 4.2(b), we obtain

$$
\begin{equation*}
e_{(G, \Phi)}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right]_{H} e_{(H, \Psi)}^{H}\right)=\Phi(\alpha) \Psi(\beta) m_{(H, \Psi)}^{k_{2}(U)} \sum_{(K, \Theta)} e_{(G, \Phi)}^{G} \cdot e_{(K, \Theta)}^{G} \tag{22}
\end{equation*}
$$

where the sum runs over those $(K, \Theta) \in[G \backslash \mathcal{X}(G)]$ satisfying $\left(K k_{1}(U) / k_{1}(U), \Theta_{k_{1}(U)}\right)==_{G / k_{1}(U)}$ $\left(G / k_{1}(U), \Psi_{k_{2}(U)} \circ \eta_{U}^{*}\right)$. Since the term in (22) is nonzero, one of these pairs $(K, \Theta)$ must be $G$-conjugate, and then also equal, to $(G, \Phi)$. This implies the result.

## 6 The constant $m_{(G, \Phi)}^{N}$

Throughout this section, $G$ and $H$ denote finite groups and $\mathbb{K}$ denotes a field of characteristic 0 such that for any subquotients $S$ of $G$ and $T$ of $H$ the groups $S^{*}$ and $T^{*}$ are finite and $\mathbb{K}$ is a splitting field for $S^{*}$ and $T^{*}$.

In this section we prove the crucial Proposition 6.4 stating that $m_{(G, \Phi)}^{M}=m_{(G, \Phi)}^{N}$ if $\left(G / M, \Phi_{M}\right) \cong\left(G / N, \Phi_{N}\right)$ (see Definition 7.3(b)).
6.1 Proposition Let $N$ and $M$ be normal subgroups of $G$ with $N \leqslant M$ and let $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$. Then

$$
m_{(G, \Phi)}^{M}=m_{(G, \Phi)}^{N} \cdot m_{\left(G / N, \Phi_{N}\right)}^{M / N}
$$

Proof This follows immediately from Proposition 4.3(a) and applying iso ${ }_{f} \circ \operatorname{def}_{(G / N) /(M / N)}^{G / N} \circ$ $\operatorname{def}_{G / N}^{G}=\operatorname{def}_{G / M}^{G}$ to $e_{(G, \Phi)}^{G}$, where $f:(G / N) /(M / N) \rightarrow G / M$ is the canonical isomorphism.
6.2 Lemma Let $f_{1}, f_{2}: G \rightarrow H$ be group homomorphisms, let $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$, and let $K \leqslant G$ be such that $\left.\Phi\right|_{K^{\perp}}=1$ and $\left.f_{1}\right|_{K}=\left.f_{2}\right|_{K}$. Then $\Phi \circ f_{1}^{*}=\Phi \circ f_{2}^{*} \in \operatorname{Hom}\left(H^{*}, \mathbb{K}^{\times}\right)$.

Proof Let $\lambda \in H^{*}$. Then $\left(\Phi \circ f_{1}^{*}\right)(\lambda)=\left(\Phi \circ f_{2}^{*}\right)(\lambda)$ if and only if $\Phi\left(\lambda \circ f_{1}\right)=\Phi\left(\lambda \circ f_{2}\right)$ which in turn is equivalent to $\Phi\left(\left(\lambda \circ f_{1}\right) \cdot\left(\lambda \circ f_{2}\right)^{-1}\right)=1$. But, since $\left.f_{1}\right|_{K}=\left.f_{2}\right|_{K}$, we have $\left(\lambda \circ f_{1}\right) \cdot\left(\lambda \circ f_{2}\right)^{-1} \in K^{\perp}$ and since $\left.\Phi\right|_{K^{\perp}}=1$ the result follows.
6.3 Proposition For $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$and normal subgroups $M$ and $N$ of $G$ one has

$$
m_{(G, \Phi)}^{M}=\frac{1}{|G| \cdot\left|G^{*}\right|} \sum_{\substack{K \leqslant G \\ K N=K M=\left.G \\ \Phi\right|_{K^{\perp}=1}}}|K| \mu(K, G)\left|\Sigma_{M, N}^{K}\right| m_{\left(G / N, \Phi_{N}\right)}^{(K \cap M) N / N}
$$

where $\Sigma_{M, N}^{K}$ is the set of elements $\phi \in G^{*}$ such that $\left.\phi\right|_{K \cap M \cap N}=1$ and such that there exists $\psi \in((G / M) \times(G / N))^{*}$ with $\psi(k M, k N)=\phi(k)$ for all $k \in K$.

Proof Consider the element

$$
v:=e_{\left(G / M, \Phi_{M}\right)}^{G / M} \cdot\left(\operatorname{def}_{G / M}^{G} \dot{G}{ }_{G} \Delta\left(e_{(G, \Phi)}^{G}\right) \cdot \inf _{G / N}^{G} \underset{G / N}{ } e_{\left(G / N, \Phi_{N}\right)}^{G / N}\right) \in B_{\mathbb{K}}^{A}(G / M)
$$

Then, on the one hand,

$$
\begin{equation*}
v=m_{(G, \Phi)}^{M} e_{\left(G / M, \Phi_{M}\right)}^{G / M} \in B_{\mathbb{K}}^{A}(G / M) \tag{23}
\end{equation*}
$$

In fact, by [Y19, Proposition 2] and Proposition4.2(b),

$$
\Delta\left(e_{(G, \Phi)}^{G}\right) \cdot \dot{\inf }_{G / N}^{G} \dot{G / N}^{e^{\left.G / N / N, \Phi_{N}\right)}}=e_{(G, \Phi)}^{G} \cdot\left(\inf _{G / N}^{G} \dot{G / N}^{G} e_{\left(G / N, \Phi_{N}\right)}^{G / N}\right)=e_{(G, \Phi)}^{G},
$$

and then Proposition4.3(a) implies (23). On the other hand, using the formula in (11) for $e_{(G, \Phi)}^{G}$ we obtain

$$
\begin{equation*}
v=\frac{1}{|G| \cdot\left|G^{*}\right|} \sum_{\substack{K \leqslant\left. G \\ \Phi\right|_{K^{\perp}}=1}} \sum_{\phi \in G^{*}}|K| \mu(K, G) \Phi\left(\phi^{-1}\right) e_{\left(G / M, \Phi_{M}\right)}^{G / M} \cdot\left(x_{K, \phi}^{G / N} e_{\left(G / N, \Phi_{N}\right)}^{G / N}\right) \tag{24}
\end{equation*}
$$

with

$$
x_{K, \phi}:=\operatorname{def}_{G / M}^{G} \dot{G}^{*} \Delta\left(\left[K,\left.\phi\right|_{K}\right]_{G}\right) \dot{\inf }_{G / N}^{G}= \begin{cases}{\left[\frac{G / M \times G / N}{\Delta_{M, N}^{K}, \phi}\right],} & \text { if }\left.\phi\right|_{K \cap M \cap N}=1 \\ 0, & \text { otherwise }\end{cases}
$$

for $K \leqslant G$ with $\left.\Phi\right|_{K^{\perp}}=1$, by the formula in (3), where, in the first case, $\Delta_{M, N}^{K}:=\{(k M, k N) \mid$ $k \in K\}$ and $\bar{\phi}\left((k M, k N):=\phi(k)\right.$ for $k \in K$. Note that $A_{K}:=k_{1}\left(\Delta_{M, N}^{K}\right)=(K \cap N) M / M$, $B_{K}:=k_{2}\left(\Delta_{M, N}^{K}\right)=(K \cap M) N / N, p_{1}\left(\Delta_{M, N}^{K}\right)=K M / M$, and $p_{2}\left(\Delta_{M, N}^{K}\right)=K N / N$. Lemma 5.3 implies that, if the term $e_{\left(G / M, \Phi_{M}\right)}^{G / M} \cdot\left(x_{K, \phi} \dot{G / N} e_{\left(G / N, \Phi_{N}\right)}^{G / N}\right)$ in (24) is nonzero then $K M=G=K N$, $\left.\phi\right|_{K \cap M \cap N}=1$, and $\bar{\phi}$ extends to some $\psi=\alpha \times \beta \in((G / M) \times(G / N))^{*}$. Then the formula in Lemma 5.3 yields

$$
\begin{align*}
& v=\frac{1}{|G| \cdot\left|G^{*}\right|} \sum_{\substack{K \leqslant G \\
\text { KM } \\
\Phi=G=K N}} \sum_{\substack{\Phi_{K}=1}}|K| \mu(K, G) \Phi\left(\Sigma_{M, N}^{K}\right. \\
& \quad \cdot e_{\left(G / M, \Phi_{M}\right)}^{G / M} \tag{25}
\end{align*}
$$

where, for $K$ and $\phi$ as above, $\alpha_{K, \phi} \in(G / M)^{*}$ and $\beta_{K, \psi} \in(G / N)^{*}$ are chosen such that $\alpha_{K, \phi} \times \beta_{K, \phi}$ is an extension of $\bar{\phi}$. Denoting the inflations of $\alpha_{K, \psi}$ and $\beta_{K, \psi}$ to $G$ by $\tilde{\alpha}_{K, \psi} \in$ and $\tilde{\beta}_{K, \psi}$, we have

$$
\Phi\left(\phi^{-1}\right) \Phi_{M}\left(\alpha_{K, \phi}\right) \Phi_{N}\left(\beta_{K, \phi}\right)=\Phi\left(\phi^{-1} \tilde{\alpha}_{K, \phi} \tilde{\beta}_{K, \phi}\right)=1
$$

since $\tilde{\alpha}_{K, \phi}(k) \tilde{\beta}_{K, \phi}(k)=\alpha_{K, \phi}(k M) \beta_{K, \phi}(k N)=\bar{\phi}(k M, k N)=\phi(k)$ for all $k \in K$ and $\left.\Phi\right|_{K^{\perp}}=1$. Thus,

$$
\begin{equation*}
v=\frac{1}{|G| \cdot\left|G^{*}\right|} \sum_{\substack{K \leqslant G \\ K=G=K N \\ \Phi| |_{K} \perp=1}}\left|\Sigma_{M, N}^{K}\right||K| \mu(K, G) m_{\left(G / N, \Phi_{N}\right)}^{B_{K}} e_{\left(G / M, \Phi_{M}\right)}^{G / M} \tag{26}
\end{equation*}
$$

Comparing Equations (23) and (26), the formula for $m_{(G, \Phi)}^{M}$ follows.
6.4 Proposition Let $M$ and $N$ be normal subgroups of $G$ such that there exists an isomorphism $f: G / N \xrightarrow{\sim} G / M$ and let $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$be such that $\Phi_{N} \circ f^{*}=\Phi_{M} \in \operatorname{Hom}\left((G / M)^{*}, \mathbb{K}^{\times}\right)$. Then one has $m_{(G, \Phi)}^{M}=m_{(G, \Phi)}^{N}$.
Proof We proceed by induction on $|G|$. If $|G|=1$ the result is clearly true. So assume that $|G|>1$. If $M=N$ is the trivial subgroup of $G$ then the result holds for trivial reasons. So assume that $M$ and $N$ are not trivial. By Proposition 6.3 one has

$$
m_{(G, \Phi)}^{M}=\frac{1}{|G| \cdot\left|G^{*}\right|} \sum_{\substack{K \leqslant G \\ K N=K M=\left.G \\ \Phi\right|_{K} \pm=1}}|K| \mu(K, G)\left|\Sigma_{M, N}^{K}\right| m_{\left(G / N, \Phi_{N}\right)}^{(K \cap M) N / N}
$$

and

$$
m_{(G, \Phi)}^{N}=\frac{1}{|G| \cdot\left|G^{*}\right|} \sum_{\substack{K \leqslant G \\ K N=K M=\left.G \\ \Phi\right|_{K} \perp=1}}|K| \mu(K, G)\left|\Sigma_{N, M}^{K}\right| m_{\left(G / M, \Phi_{M}\right)}^{(K \cap N) M / M}
$$

We will show that these sums coincide by comparing them summand by summand. Since $\Sigma_{M, N}^{K}=$ $\Sigma_{N, M}^{K}$, it suffices to show that $m_{\left(G / N, \Phi_{N}\right)}^{(K \cap M) N / N}=m_{\left(G / M, \Phi_{M}\right)}^{(K \cap N) M / M}$. By Proposition 4.3(b), for any $X \unlhd$ $G / N$ we have

$$
\left.m_{\left(G / N, \Phi_{N}\right)}^{X}=\frac{\left|((G / N) / X)^{*}\right|}{|G / N| \cdot\left|(G / N)^{*}\right|} \sum_{\substack{L \leqslant G / N \\ L X=G /\left.N \\\left(\Phi_{N}\right)\right|_{L} \perp=1}}|L|\left|L^{\perp}\right| \right\rvert\, \mu(L, G / N)
$$

and

$$
m_{\left(G / M, \Phi_{M}\right)}^{f(X)}=\frac{\left|((G / M) / f(X))^{*}\right|}{|G / M| \cdot\left|(G / M)^{*}\right|} \sum_{\substack{K \leqslant G / N \\ K f(X)=G /\left.N \\ \Phi_{M}\right|_{K}=1}}|K|\left|K^{\perp}\right| \mu(K, G / M)
$$

Note that the summand for $L$ in the first sum is equal to the summand for $K=f(L)$ in the second sum. Thus, with $X=(K \cap M) N / N$, we obtain

$$
\begin{equation*}
m_{\left(G / N, \Phi_{N}\right)}^{(K \cap M M) N}=m_{\left(G / M, \Phi_{M}\right)}^{f((K \cap M) N / N)} \tag{27}
\end{equation*}
$$

for any $K \leqslant G$ with $K M=G=K N$ and $\left.\Phi\right|_{K^{\perp}}=1$. It now suffices to find an isomorphism $f_{1}:(G / M) / f((K \cap M) N / N) \xrightarrow{\sim}(G / M) /((K \cap N) M / M)$ such that $\left(\Phi_{M}\right)_{f((K \cap M) N / N)} \circ$ $f_{1}^{*}=\left(\Phi_{M}\right)_{(K \cap N) M / M}$, since then, by induction, we have $m_{\left(G / M, \Phi_{M}\right)}^{f((K \cap M) N / N)}=m_{\left(G / M, \Phi_{M}\right)}^{(K \cap N) M / M}$ and together with Equation (27) this implies that desired equation. We define $f_{1}:=\eta \circ \bar{f}^{-1}$, where $\bar{f}:(G / N) /((K \cap M) N / N) \xrightarrow{\sim}(G / M) / f((K \cap M) N / N)$ is induced by $f$ and $\eta:=$ $\eta_{\Delta_{M, N}^{K}}:(G / N) /((K \cap M) N / N) \xrightarrow{\sim}(G / M) /((K \cap N) M / M)$ is induced by $\Delta_{M, N}^{K}:=\{(k M, k N) \mid$ $k \in K\}$, see [Bc10, Lemma 2.3.25]. It remains to prove that

$$
\begin{equation*}
\Phi \circ \nu_{M}^{*} \circ \nu_{f((K \cap M) N / N)}^{*} \circ f_{1}^{*}=\Phi \circ \nu_{M}^{*} \circ \nu_{((K \cap N) M / M)}^{*}, \tag{28}
\end{equation*}
$$

where $\nu_{M}: G \rightarrow G / M, \nu_{f(K \cap M) N / N}: G / M \rightarrow(G / M) / f((K \cap M) N / N)$, and $\nu_{(K \cap N) M / M}: G / M \rightarrow$ $(G / M) /((K \cap N) M / M)$ denote the natural epimorphisms. But $f_{1}^{*}=\left(\bar{f}^{-1}\right)^{*} \circ \eta^{*}, \nu_{f((K \cap M) N / N)}^{*} \circ$ $\left(\bar{f}^{-1}\right)^{*}=\left(f^{-1}\right)^{*} \circ \nu_{(K \cap N) M / M}^{*}$, and $\Phi \circ \nu_{M}^{*} \circ\left(f^{-1}\right)^{*}=\Phi_{M} \circ\left(f^{-1}\right)^{*}=\Phi_{N}=\Phi \circ \nu_{N}^{*}$. Thus, the left
hand side of Equation (28) is equal to $\Phi \circ \nu_{N}^{*} \circ \nu_{(K \cap M) N / N^{*}}^{*} \circ \eta^{*}$. By Lemma 6.2 and since $\left.\Phi\right|_{K^{\perp}}=1$, it now suffices to show that

$$
\left.\left(\eta \circ \nu_{(K \cap M) N / N} \circ \nu_{N}\right)\right|_{K}=\left.\left(\nu_{(K \cap M) N / N} \circ \nu_{N}\right)\right|_{K} .
$$

But this follows from $(k M, k N) \in \Delta_{M, N}^{K}$ for $k \in K$, and the proof is complete.

## $7 \quad B^{A}$-pairs and the subfunctors $E_{(G, \Phi)}$ of $B_{\mathbb{K}}^{A}$

Throughout this section we assume that $G^{*}$ is finite for all finite groups $G$, and we assume that $\mathbb{K}$ is a field of characteristic 0 which is a splitting field of $\mathbb{K} G^{*}$ for all finite groups $G$. The latter is equivalent to requiring that, for any torsion element $a$ of $A$, the field $\mathbb{K}$ has a root of unity whose order equals the order of $a$.

In this section we introduce the important subfunctors $E_{(G, \Phi)}$ of $B_{\mathbb{K}}^{A}$ and study their properties.
7.1 Definition For any finite group $G$ and $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$we denote by $E_{(G, \Phi)}$ the subfunctor of $B_{\mathbb{K}}^{A}$ generated by $e_{(G, \Phi)}^{G}$. In other words, for each finite group $H$, one has

$$
E_{(G, \Phi)}(H)=\left\{x \cdot{ }_{G} e_{(G, \Phi)}^{G} \mid x \in B_{\mathbb{K}}^{A}(H, G)\right\}
$$

7.2 Proposition For any finite group $G$ and $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$, the following are equivalent:
(i) If $H$ is a finite group with $E_{(G, \Phi)}(H) \neq\{0\}$ then $|G| \leqslant|H|$.
(ii) If $H$ is a finite group with $E_{(G, \Phi)}(H) \neq\{0\}$ then $G$ is isomorphic to a subquotient of $H$.
(iii) For all $\{1\} \neq N \unlhd G$ one has $m_{(G, \Phi)}^{N}=0$.
(iv) For all $\{1\} \neq N \unlhd G$ one has $\operatorname{def}_{G / N}^{G}\left(e_{(G, \Phi)}^{G}\right)=0$.

Proof That (ii) implies (i) and that (i) implies (iv) follows from the definitions. Moreover, that (iii) and (iv) are equivalent follows from Proposition 4.3 (a). So, it suffices to prove that (iv) implies (ii).

Assume that (iv) holds and let $H$ be a finite group with $E_{(G, \Phi)}(H) \neq\{0\}$. By the definition of $E_{(G, \Phi)}$ this implies that there exists $(U, \phi) \in \mathcal{M}(H \times G)$ such that $\left[\frac{H \times G}{U, \phi}\right] \cdot{ }_{G} e_{(G, \Phi)}^{G} \neq 0$. The canonical decomposition of $\left[\frac{H \times G}{U, \phi}\right]$ from Theorem 2.3 implies that

$$
\left[\frac{(P / K) \times(Q / L)}{\bar{U}, \bar{\phi}}\right]_{p_{2}(U) / \operatorname{ker}\left(\phi_{2}\right)} \operatorname{def}_{Q / L}^{Q} \cdot \operatorname{res}_{Q}^{G} \cdot e_{(G, \Phi)}^{G} \neq 0
$$

with $P:=p_{1}(U), K:=\operatorname{ker}\left(\phi_{1}\right), Q:=p_{2}(U), L:=\operatorname{ker}\left(\phi_{2}\right), \bar{U}$ corresponding to $U /(K \times L)$ via the canonical isomorphism $(P \times Q) /(K \times L) \cong(P / K) \times(Q / L)$, and $\bar{\phi} \in(\bar{U})^{*}$ induced by $\phi$. Proposition4.1(c) implies that $G=Q$, and then Proposition 4.3(a) implies that $L=\{1\}$. Thus, $p_{1}(\bar{U})=P / K, p_{2}(\bar{U})=G$, and

$$
\left[\frac{P / K \times G}{\bar{U}, \bar{\phi}}\right]_{G} e_{(G, \Phi)}^{G} \neq 0
$$

Lemma 5.2 implies that $(\bar{\phi})_{2}$ extends to $G$ and Lemma 5.1 implies that $\bar{\phi}$ extends to some $\alpha \times \beta \in$ $((P / K) \times G)^{*}$ with $\alpha \in(P / K)^{*}$ and $\beta \in G^{*}$, since $p_{1}(\bar{U})=P / K$ and $p_{2}(\bar{U})=G$. Further, we have

$$
0 \neq\left[\frac{(P / K) \times G}{\bar{U}, \bar{\phi}}\right]_{G} e_{(G, \Phi)}^{G}=\operatorname{tw}_{\alpha} \underset{P / K}{ }\left[\frac{(P / K) \times G}{\bar{U}, 1}\right] \dot{G} \operatorname{tw}_{\beta} \dot{G}_{(G, \Phi)}^{G}
$$

with $\operatorname{tw}_{\beta} \dot{G}_{G} e_{(G, \Phi)}^{G}=\Phi(\beta) e_{(G, \Phi)}^{G}$ by Proposition4.5(c). Thus, using the canonical decomposition of $\left[\frac{(P / K) \times G}{U, 1}\right]$ as in Theorem 2.3 we obtain $\operatorname{def}_{G / k_{2}(\bar{U})}^{G} \cdot{ }_{G} e_{(G, \Phi)}^{G} \neq 0$. Proposition 4.3(a) implies that $k_{2}(\bar{U})=\{1\}$. But this implies that $G \cong G /\{1\} \cong p_{2}(\bar{U}) / k_{2}(\bar{U}) \cong p_{1}(\bar{U}) / k_{1}(\bar{U})$ is isomorphic to a subquotient of $p_{1}(\bar{U})$, which is isomorphic to a subquotient of $H$.

Note that by the formula for $m_{(G, \Phi)}^{N}$ in Proposition 4.3, the condition in Proposition 7.2 (iii) is independent of the choice of $\mathbb{K}$ as long as $\mathbb{K}$ has enough roots of unity.
7.3 Definition Let $G$ and $H$ be finite groups and let $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$and $\Psi \in \operatorname{Hom}\left(H^{*}, \mathbb{K}^{\times}\right)$.
(a) The pair $(G, \Phi)$ is called a $B^{A}$-pair if the equivalent conditions in Proposition 7.2 are satisfied.
(b) We call $(G, \Phi)$ and $(H, \Psi)$ isomorphic and write $(G, \Phi) \cong(H, \Psi)$ if there exists an isomor$\operatorname{phism} f: H \xrightarrow{\sim} G$ such that $\Psi \circ f^{*}=\Phi$. Note that if $(G, \Phi) \cong(H, \Psi)$ then $E_{(G, \Phi)}=E_{(H, \Psi)}$. In fact, if $f: H \xrightarrow{\sim} G$ satisfies $\Psi \circ f^{*}=\Phi$ then $\operatorname{iso}_{f}\left(e_{(H, \Psi)}^{H}\right)=e_{(G, \Phi)}^{G}$, by Proposition 4.5(a).
(c) We write $(H, \Psi) \preccurlyeq(G, \Phi)$ if there exists a normal subgroup $N$ of $G$ such that $(H, \Psi) \cong$ $\left(G / N, \Phi_{N}\right)$. The relation $\preccurlyeq$ is reflexive and transitive. Moreover, if $(H, \Psi) \preccurlyeq(G, \Phi)$ and $(G, \Phi) \preccurlyeq$ $(H, \Psi)$ then $(G, \Phi) \cong(H, \Psi)$. Thus, $\preccurlyeq$ induces a partial order on the set of isomorphism classes $[G, \Phi]$ of pairs $(G, \Phi)$, where $G$ is a finite group and $\Phi \in \operatorname{Hom}\left(G, \mathbb{K}^{\times}\right)$. We denote this relation again by $\preccurlyeq$. This partial order restricts to a partial order on the set $\mathcal{B}^{A}$ of isomorphism classes of $B^{A}$-pairs.
7.4 Proposition Let $G$ and $H$ be finite groups and let $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$and $\Psi \in \operatorname{Hom}\left(H^{*}, \mathbb{K}^{\times}\right)$.
(a) If $(H, \Psi) \preccurlyeq(G, \Phi)$ then $E_{(G, \Phi)} \subseteq E_{(H, \Psi)}$.
(b) If $(H, \Psi)$ is a $B^{A}$-pair and $E_{(G, \Phi)} \subseteq E_{(H, \Psi)}$ then $(H, \Psi) \preccurlyeq(G, \Phi)$.

Proof (a) Let $N \unlhd G$ and let $f: H \xrightarrow{\sim} G / N$ be an isomorphism with $\Psi \circ f^{*}=\Phi_{N}$. Then, by Proposition 4.5(a), Proposition 4.2(b) and CY19, Proposition 2], we have

$$
e_{(G, \Phi)}^{G}=e_{(G, \Phi)}^{G} \cdot\left(\inf _{G / N}^{G} \underset{G / N}{\operatorname{iso}_{f}} \dot{H}_{(H, \Psi)}^{H}\right) \in E_{(H, \Psi)}(G),
$$

so that $E_{(G, \Phi)} \subseteq E_{(H, \Psi)}$.
(b) Since $E_{(G, \Phi)} \subseteq E_{(H, \Psi)}$, we have $e_{(G, \Phi)}^{G} \in E_{(H, \Psi)}(G)$ and there exists $(U, \phi) \in \mathcal{M}(G \times H)$ such that

$$
\begin{equation*}
0 \neq e_{(G, \Phi)}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right]_{H} e_{(H, \Psi)}^{H}\right) \tag{29}
\end{equation*}
$$

Lemma 5.3 implies that $p_{1}(U)=G, p_{2}(U)=H, \phi$ extends to some $\alpha \times \beta \in(G \times H)^{*}, \Phi_{k_{1}(U)}=$ $\Psi_{k_{2}(U)} \circ \eta_{U}^{*}$, and $m_{(H, \Psi)}^{k_{2}(U)} \neq 0$. Since $(H, \Psi)$ is a $B^{A}$-pair, we obtain $k_{2}(U)=\{1\}$ and $\Phi_{k_{1}(U)}=\Psi \circ \eta_{U}^{*}$. Thus, $\eta_{U}$ is an isomorphism $H \xrightarrow{\sim} G / k_{1}(U)$ with $\Phi_{k_{1}(U)}=\Psi \circ \eta_{U}^{*}$, so that $(H, \Psi) \preccurlyeq(G, \Phi)$.

## 8 Subfunctors of $B_{\mathbb{K}}^{A}$

We keep the assumptions on $A$ and $\mathbb{K}$ from Section 7. In this section we prove one of our main results, Theorem 8.8, which describes the lattice of subfunctors of $B_{\mathbb{K}}^{A}$ in terms of the poset $\left(\mathcal{B}^{A}, \preccurlyeq\right)$.
8.1 Definition Let $k$ be a commutative ring and $F \in \mathcal{F}_{k}^{A}$. A finite group $G$ is called a minimal group for $F$ if $G$ is a group of minimal order with $F(G) \neq\{0\}$.

For any finite group $G$, the group $\operatorname{Aut}(G)$ acts on $\mathcal{X}(G)$ via ${ }^{f}(K, \Psi):=\left(f(K), \Psi \circ\left(\left.f\right|_{K}\right)^{*}\right)$. We will denote by $\hat{\mathcal{X}}(G) \subseteq \mathcal{X}(G)$ the set of those pairs $(K, \Psi)$ with $K=G$. Note that $\hat{\mathcal{X}}(G)$ is Aut $(G)$-invariant and that $G$ acts trivially by conjugation on $\hat{\mathcal{X}}(G)$, so that $\hat{\mathcal{X}}(G)$ can be viewed as an $\operatorname{Out}(G)$-set.
8.2 Proposition Let $F$ be a subfunctor of $B_{\mathbb{K}}^{A}$ in $\mathcal{F}_{\mathbb{K}}^{A}$.
(a) For each finite group $G$ one has

$$
F(G)=\bigoplus_{(K, \Psi) \in\left[G \backslash \mathcal{X}_{F}(G)\right]} \mathbb{K} \cdot e_{(K, \Psi)}^{G}
$$

where $\mathcal{X}_{F}(G):=\left\{(K, \Psi) \in \mathcal{X}(G) \mid e_{(K, \Psi)}^{G} \in F(G)\right\}$.
(b) For any finite group $G$, the set $\mathcal{X}_{F}(G)$ is invariant under the action of $\operatorname{Aut}(G)$.
(c) Suppose that $H$ is a minimal group for $F$. Then $\mathcal{X}_{F}(H)$ contains only elements from $\hat{\mathcal{X}}(H)$. Moreover, each $(H, \Psi) \in \mathcal{X}_{F}(H)$ is a $B^{A}$-pair and one has $E_{(H, \Psi)} \subseteq F$.

Proof (a) For all $a \in F(G)$ and $x \in B_{\mathbb{K}}^{A}(G)$, CY19, Proposition 2] implies $x \cdot a=\Delta(x) \cdot{ }_{G} a \in F(G)$. Thus, $F(G)$ is an ideal of $B_{\mathbb{K}}^{A}(G)$. Since the elements $e_{(K, \Theta)}^{G}$ with $(K, \Theta) \in[G \backslash \mathcal{X}(G)]$ form a $\mathbb{K}$-basis of $B_{\mathbb{K}}^{A}(G)$ consisting of pairwise orthogonal idempotents, the assertion in (a) follows.
(b) If $e_{(K, \Psi)}^{G} \in F(G)$ and $f \in \operatorname{Aut}(G)$ then $e_{\left(f(K), \Psi \circ\left(\left.f\right|_{K}\right)^{*}\right)}^{G}=\operatorname{iso}_{f}\left(e_{(K, \Psi)}^{G}\right) \in F(G)$.
(c) Assume that $H$ is a minimal group for $F$ and that $(K, \Psi) \in \mathcal{X}_{F}(H)$. Then $e_{(K, \Psi)}^{H} \in F(H)$ and

$$
0 \neq e_{(K, \Psi)}^{K}=e_{(K, \Psi)}^{K} \operatorname{res}_{K}^{H}\left(e_{(K, \Psi)}^{H}\right) \in F(K)
$$

by Proposition4.1(b). The minimality of $H$ implies $K=H$. By Proposition4.3(a), the minimality also implies that $m_{(H, \Psi)}^{N}=0$ for all $\{1\} \neq N \unlhd H$, since $e_{(H, \Psi)}^{H} \in F(H)$. Clearly, $E_{(H, \Psi)} \subseteq F$.
8.3 Definition Let $F$ be a subfunctor of $B_{\mathbb{K}}^{A}$ in $\mathcal{F}_{\mathbb{K}}^{A}$. If $H$ is a minimal group for $F$ and $\Psi \in \operatorname{Hom}\left(H, \mathbb{K}^{\times}\right)$is such that $(H, \Psi) \in \mathcal{X}_{F}(H)$ then we call $(H, \Psi)$ a minimal pair for $F$. By Proposition 8.2 (c), each minimal pair for $F$ is a $B^{A}$-pair.
8.4 Proposition Let $H$ be a finite group, $\Psi \in \operatorname{Hom}\left(H^{*}, \mathbb{K}^{*}\right)$, and let $(G, \Phi)$ be a minimal pair for $E_{(H, \Psi)}$. Then:
(a) $E_{(H, \Psi)}=E_{(G, \Phi)}$.
(b) There exists $N \unlhd H$ with $m_{(H, \Psi)}^{N} \neq 0$ and $\left(H / N, \Psi_{N}\right) \cong(G, \Phi)$. In particular $(G, \Phi) \preccurlyeq$ $(H, \Psi)$. Moreover, if also $N^{\prime} \unlhd H$ satisfies $\left(H / N^{\prime}, \Psi_{N^{\prime}}\right) \cong(G, \Phi)$ then $m_{(H, \Psi)}^{N^{\prime}}=m_{(H, \Psi)}^{N} \neq 0$.
(c) Up to isomorphism, $(G, \Phi)$ is the only minimal pair for $E_{(H, \Psi)}$.
(d) If $(H, \Psi)$ is a $B^{A}$-pair, then, up to isomorphism, $(H, \Psi)$ is the only minimal pair of $E_{(H, \Psi)}$. In particular,

$$
E_{(H, \Psi)}(H)=\bigoplus_{\substack{\left(H, \Psi^{\prime}\right) \in \hat{\mathcal{X}}(H) \\\left(H, \Psi^{\prime}\right)==_{\operatorname{Out}(H)}(H, \Psi)}} \mathbb{K} e_{\left(H, \Psi^{\prime}\right)}^{H}
$$

Proof (b) Since $(G, \Phi)$ is a minimal pair for $E_{(H, \Psi)}$, there exists $x \in B_{\mathbb{K}}^{A}(G, H)$ such that $e_{(G, \Phi)}^{G}=x \cdot{ }_{H} e_{(H, \Psi)}$. Multiplication with $e_{(G, \Phi)}^{G}$ yields $e_{(G, \Phi)}^{G}=e_{(G, \Phi)}^{G} \cdot\left(x \cdot{ }_{H} e_{(H, \Psi)}^{H}\right)$. Thus, there exists $(U, \phi) \in \mathcal{M}(G \times H)$ with

$$
e_{(G, \Phi)}^{G} \cdot\left(\left[\frac{G \times H}{U, \phi}\right]_{H} e_{(H, \Psi)}^{H}\right) \neq 0 .
$$

Lemma 5.3 implies that $p_{1}(U)=G, p_{2}(U)=H, \phi$ has an extension to $G \times H, \Phi_{k_{1}(U)}=\Psi_{k_{2}(U)} \circ \eta_{U}^{*}$, and $m_{(H, \Psi)}^{k_{2}(U)} \neq 0$. Since $\phi$ has an extension to $G \times H,\left[\frac{G \times H}{U, \phi}\right]$ factors through $q(U) \cong G / k_{1}(U)$ by Lemma 5.1. Since $G$ is a minimal group for $E_{(H, \Psi)}$ this implies $k_{1}(U)=\{1\}$. Set $N:=k_{2}(U) \unlhd H$. Then $\eta_{U}: H / N \xrightarrow{\sim} G$ satisfies $\Psi_{N} \circ \eta_{U}^{*}=\Phi$. If also $N^{\prime} \unlhd G$ satisfies $\left(H / N^{\prime}, \Psi_{N^{\prime}}\right) \cong(G, \Phi)$, then $\left(H / N, \Psi_{N^{\prime}}\right) \cong\left(H / N, \Psi_{N}\right)$ and we obtain $m_{(H, \Psi)}^{N^{\prime}}=m_{(H, \Psi)}^{N} \neq 0$ by Proposition 6.4,
(a) Note that $E_{(G, \Phi)} \subseteq E_{(H, \Psi)}$ by Proposition $8.2(\mathrm{c})$. The converse follows from Proposition 7.4(a) and Part (b).
(c) Assume that also $\left(G^{\prime}, \Phi^{\prime}\right)$ is a minimal pair for $E_{(H, \Phi)}=E_{(G, \Phi)}$. Then, by Part (b) applied to $\left(G^{\prime}, \Phi^{\prime}\right)$ and $(G, \Phi)$ in place of $(G, \Phi)$ and $(H, \Psi)$, we have $\left(G^{\prime}, \Phi^{\prime}\right) \preccurlyeq(G, \Phi)$. Since both $G$ and $G^{\prime}$ are minimal groups for $E_{(H, \Psi)}$, they have the same order. Thus, $\left(G^{\prime}, \Phi^{\prime}\right) \cong(G, \Phi)$.
(d) Now assume that $(H, \Psi)$ is a $B^{A}$-pair and let $(G, \Phi)$ be a minimal pair for $E_{(H, \Psi)}$. Then, by Part (b), there exists $N \unlhd G$ with $m_{(H, \Psi)}^{N} \neq 0$ and $\left(H / N, \Psi_{N}\right) \cong(G, \Phi)$. Since $(H, \Psi)$ is a $B^{A}$-pair, this implies $N=\{1\}$ and $(H, \Psi) \cong(G, \Phi)$.
8.5 Notation For any finite group $G$ and any $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$we denote by $\beta(G, \Phi)$ the class of all minimal pairs for $E_{(G, \Phi)}$. Thus $\beta(G, \Phi)$ is the isomorphism class $[H, \Psi]$ of a $B^{A}$-pair $(H, \Psi)$. Note that $\beta(G, \Phi) \preccurlyeq[G, \Phi]$ by Proposition 8.4(b).

The following proposition is not used in this paper, but of interest in its own right. It is the analogue of [Bc10, Theorem 5.4.11].
8.6 Proposition Let $G$ be a finite group and let $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$.
(a) If $(H, \Psi)$ is a $B^{A}$-pair with $(H, \Psi) \preccurlyeq(G, \Phi)$ then $[H, \Psi] \preccurlyeq \beta(G, \Phi)$.
(b) For any $N \unlhd G$ the following are equivalent:
(i) $m_{(G, \Phi)}^{N} \neq 0$.
(ii) $\beta(G, \Phi) \preccurlyeq\left[G / N, \Phi_{N}\right]$.
(iii) $\beta(G, \Phi)=\beta\left(G / N, \Phi_{N}\right)$.
(c) For any $N \unlhd G$ the following are equivalent:
(i) $\left[G / N, \Phi_{N}\right] \cong \beta(G, \Phi)$.
(ii) $\left(G / N, \Phi_{N}\right)$ is a $B^{A}$-pair and $m_{(G, \Phi)}^{N} \neq 0$.

Proof Let $(K, \Theta) \in \beta(G, \Phi)$. Thus, $E_{(G, \Phi)}=E_{(K, \Theta)}$ by Proposition 8.4(a) and $(K, \Theta)$ is a $B^{A}$-pair.
(a) Let $(H, \Psi)$ be as in the statement. Then $E_{(K, \Theta)}=E_{(G, \Phi)} \subseteq E_{(H, \Psi)}$ by Proposition 7.4 (a). Now Proposition 7.4(b) implies $(H, \Psi) \preccurlyeq(K, \Theta)$.
(b) $(\mathrm{i}) \Rightarrow\left(\right.$ ii) : Since $m_{(G, \Phi)}^{N} \neq 0$, Proposition 4.3(a) implies that

$$
e_{\left(G / N, \Phi_{N}\right)}^{G / N}=\left(m_{(G, \Phi)}^{N}\right)^{-1} \operatorname{def}_{G / N}^{G}\left(e_{(G, \Phi)}^{G}\right) \in E_{(G, \Phi)}(G / N)=E_{(K, \Theta)}(G / N)
$$

so that $E_{\left(G / N, \Phi_{N}\right)} \subseteq E_{(K, \Theta)}$. Proposition 7.4 (b) implies $(K, \Psi) \preccurlyeq\left(G / N, \Phi_{N}\right)$.
 $\beta\left(G / N, \Phi_{N}\right)$. Conversely, we have $\beta\left(G / N, \Phi_{N}\right) \preccurlyeq\left[G / N, \Phi_{N}\right] \preccurlyeq[G, \Phi]$ and Part (a) again implies $\beta\left(G / N, \Phi_{N}\right) \preccurlyeq \beta(G, \Phi)$.
(iii) $\Rightarrow$ (i): By Proposition 8.4(b) there exists $M \unlhd G$ such that $m_{(G, \Phi)}^{M} \neq 0$ and $\left[G / M, \Phi_{M}\right]=\beta(G, \Phi) . \quad$ Similarly, there exists $N \leqslant M^{\prime} \unlhd G$ such that $m_{\left(G / N, \Phi_{N}\right)}^{M^{\prime} / N} \neq 0$ and $\left[(G / N) /\left(M^{\prime} / N\right),\left(\Phi_{N}\right)_{M^{\prime}}\right]=\beta\left(G / N, \Phi_{N}\right)$. Since

$$
\left[G / M^{\prime}, \Phi_{M^{\prime}}\right]=\left[(G / N) /\left(M^{\prime} / N\right),\left(\Phi_{N}\right)_{M^{\prime}}\right]=\beta\left(G / N, \Phi_{N}\right)=\beta(G, \Phi)=\left[G / M, \Phi_{M}\right]
$$

Proposition 6.4 implies that $m_{(G, \Phi)}^{M^{\prime}}=m_{(G, \Phi)}^{M} \neq 0$. By Proposition 6.1 we have $m_{(G, \Phi)}^{M^{\prime}}=m_{(G, \Phi)}^{N}$. $m_{\left(G / N, \Phi_{N}\right)}^{M^{\prime}}$ which implies that $m_{(G, \Phi)}^{N} \neq 0$.
(c) This follows immediately from the equivalence between (i) and (iii) in Part (b), noting that $\beta\left(G / N, \Phi_{N}\right)=\left[G / N, \Phi_{N}\right]$ if $\left(G / N, \Phi_{N}\right)$ is a $B^{A}$-pair and that $\beta(G, \Phi)$ consists of $B^{A}$-pairs.
8.7 Definition A subset $\mathcal{Z}$ of the poset $\mathcal{B}^{A}$, ordered by the relation $\preccurlyeq$ (cf. Definition 7.3), is called closed if for every $[H, \Psi] \in \mathcal{Z}$ and $[G, \Phi] \in \mathcal{B}^{A}$ with $[H, \Psi] \preccurlyeq[G, \Phi]$ one has $[G, \Phi] \in \mathcal{Z}$.
8.8 Theorem Let $\mathcal{S}$ denote the set of subfunctors of $B_{\mathbb{K}}^{A}$ in $\mathcal{F}_{\mathbb{K}}^{A}$, ordered by inclusion of subfunctors, and let $\mathcal{T}$ denote the set of closed subsets of $\mathcal{B}^{\mathcal{A}}$, ordered by inclusion of subsets. The map

$$
\alpha: \mathcal{S} \rightarrow \mathcal{T}, \quad F \mapsto\left\{[H, \Psi] \in \mathcal{B}^{A} \mid E_{(H, \Psi)} \subseteq F\right\}
$$

is an isomorphism of posets with inverse given by

$$
\beta: \mathcal{T} \rightarrow \mathcal{S}, \quad \mathcal{Z} \mapsto \sum_{[H, \Psi] \in \mathcal{Z}} E_{(H, \Psi)}
$$

Proof Clearly, $\alpha$ and $\beta$ are order-preserving. Let $F \in \mathcal{S}$. By Proposition 8.2(a) we have

$$
F=\sum_{\substack{G \\(H, \Psi) \in \mathcal{X} \mathcal{X}_{F}(G)}}\left\langle e_{(H, \Psi)}^{G}\right\rangle
$$

where $G$ runs through a set of representatives of the isomorphism classes of finite groups and $\left\langle e_{(H, \Psi)}^{G}\right\rangle$ denotes the subfunctor of $B_{\mathbb{K}}^{A}$ generated by $e_{(H, \Psi)}^{G}$. For any finite group $G$ and any $(H, \Psi) \in$
$\mathcal{X}(G)$ one has $e_{(H, \Psi)}^{G} \in F(G)$ if and only if $e_{(H, \Psi)}^{H} \in F(H)$. In fact, $e_{(H, \Psi)}^{H}=e_{(H, \Psi)} \cdot \operatorname{res}_{H}^{G}\left(e_{(H, \Psi)}^{G}\right)$ by Proposition 4.1 and $e_{(H, \Psi)}^{G} \in \mathbb{K} \cdot \operatorname{ind}_{H}^{G}\left(e_{(H, \Psi)}^{H}\right)$ by Proposition 4.4. Thus

$$
F=\sum_{\substack{H \\(H, \Psi) \in \hat{\mathcal{X}}_{F}(H)}} E_{(H, \Psi)}
$$

where $H$ runs again through a set of representatives of the isomorphism classes of finite groups and $\hat{\mathcal{X}}_{F}(H)=\hat{\mathcal{X}}(H) \cap \mathcal{X}_{F}(H)$. By Propositions 8.2(c) and 8.4 (a), we obtain

$$
F=\sum_{\substack{[H, \Psi] \in \mathcal{B}^{A} \\(H, \Psi) \in \mathcal{X}_{F}(H)}} E_{(H, \Psi)}=\sum_{[H, \Psi] \in \alpha(F)} E_{(H, \Psi)}=\beta(\alpha(F)),
$$

since $(H, \Psi) \in \mathcal{X}_{F}(H)$ if and only if $E_{(H, \Psi)} \subseteq F$.
Let $\mathcal{Z}$ be a closed subset of $\mathcal{B}^{A}$. By definition of $\alpha$ and $\beta$ we have

$$
\alpha(\beta(\mathcal{Z}))=\left\{[H, \Psi] \in \mathcal{B}^{A} \quad \mid E_{(H, \Psi)} \subseteq \sum_{[G, \Phi] \in \mathcal{Z}} E_{(G, \Phi)}\right\}
$$

The inclusion $\mathcal{Z} \subseteq \alpha(\beta(\mathcal{Z}))$ is obvious. Conversely, assume that $[H, \Psi] \in \mathcal{B}^{A}$ satisfies $E_{(H, \Psi)} \subseteq$ $\sum_{[G, \Phi] \in \mathcal{Z}} E_{(G, \Phi)}$. Evaluation at $H$ and Proposition 8.2(a) imply that there exists $[G, \Phi] \in \mathcal{Z}$ with $e_{(H, \Psi)}^{H} \in E_{(G, \Phi)}(H)$, which implies $E_{(H, \Psi)} \subseteq E_{(G, \Phi)}$. Since $(G, \Phi)$ is a $B^{A}$-pair, Proposition 7.4(b) implies $[G, \Phi] \preccurlyeq[H, \Psi]$. Since $[G, \Phi] \in \mathcal{Z}$ and $\mathcal{Z}$ is closed we obtain $[H, \Psi] \in \mathcal{Z}$. Thus, $\alpha(\beta(\mathcal{Z})) \subseteq$ $\mathcal{Z}$, and the proof is complete.
8.9 Remark (a) If $(G, \Phi)$ is a $B^{A}$-pair, then the subfunctor $E_{(G, \Phi)}$ of $B_{\mathbb{K}}^{A}$ corresponds under the bijection in Theorem 8.8 to the subset $\mathcal{B}_{\succcurlyeq[G, \Phi]}^{A}:=\left\{[H, \Psi] \in \mathcal{B}^{A} \mid[G, \Phi] \preccurlyeq[H, \Psi]\right\}$. Clearly, $\mathcal{B}_{\succ[G, \Phi]}^{A}:=\left\{[H, \Psi] \in \mathcal{B}^{A} \mid[G, \Phi] \prec[H, \Psi]\right\}$ is the unique maximal closed subset of $\mathcal{B}_{\succcurlyeq[G, \Phi]}^{A}$.
(b) For every element $[G, \Phi] \in \mathcal{B}^{A}$ there exist only finitely many elements $[H, \Psi] \in \mathcal{B}^{A}$ with $[H, \Psi] \preccurlyeq[G, \Phi]$. Therefore, every non-empty subset of $\mathcal{B}^{A}$ has a minimal element.

## 9 Composition factors of $B_{\mathbb{K}}^{A}$

We keep the assumptions on $A$ and $\mathbb{K}$ from Section [7. In this section we show that the composition factors of $B_{\mathbb{K}}^{A}$ are parametrized by isomorphism classes of $B^{A}$-pairs.
9.1 Recall from [BC18, Section 9] that the simple $A$-fibered biset functors $S$ over $\mathbb{K}$ are parametrized by isomorphism classes of certain quadruples $(G, K, \kappa, V)$. Here, $G$ is a minimal group for $S,(K, \kappa) \in \mathcal{M}(G)$ is such that the idempotent $f_{(K, \kappa)} \in B_{\mathbb{K}}^{A}(G, G)$ (see [BC18, Subsection 4.3]) does not annihilate $S(G)$, and $V:=S(G)$ is an irreducible $\mathbb{K} \Gamma_{(G, K, \kappa)}$-module for the finite group $\Gamma_{(G, K, \kappa)}$ defined in BC18, 6.1(c)]. All we need for the analysis in the following theorem is that the idempotent $f_{(K, \kappa)}$ is in the $\mathbb{K}$-span of standard basis elements $\left[\frac{G \times G}{U, \phi}\right]$ with $(U, \phi) \in \mathcal{M}(G, G)$ such that $k_{2}(U) \geqslant K$, and that in the case $(K, \kappa)=(\{1\}, 1)$, the group $\Gamma_{(G,\{1\}, 1)}$ is the set of standard basis elements $\left[\frac{G \times G}{U, \phi}\right]$ of $B_{\mathbb{K}}^{A}(G, G)$ with $p_{1}(U)=G=p_{2}(U)$ and $k_{1}(U)=\{1\}=k_{2}(U)$. The
multiplication is given by $\cdot G$. Thus, in this case, $\left[\frac{G \times G}{U, \phi}\right]=\operatorname{tw}_{\alpha} \cdot G$ iso ${ }_{f}$, where $f:=\eta_{U} \in \operatorname{Aut}(G)$ is defined by $U=\left\{\left(\eta_{U}(g), g\right) \mid g \in G\right\}$, and $\alpha \in G^{*}$ is given by $\alpha(g)=\phi\left(\eta_{U}(g), g\right)$ for $g \in G$. Mapping $\left[\frac{G \times G}{U, \phi}\right]$ to the element $(\alpha, \bar{f}) \in G^{*} \rtimes \operatorname{Out}(G)$ defines an isomorphism. Here, Out $(G)$ acts on $G^{*}$ via $\bar{f}_{\alpha}:=\alpha \circ f^{*}$. Moreover, $\Gamma_{(G,\{1\}, 1)}$ acts on $S(G)$ by ${ }^{G}$.
9.2 Proposition $\operatorname{Let}(G, \Phi)$ be a $B^{A}$-pair. The subfunctor $E_{(G, \Phi)}$ of $B_{\mathbb{K}}^{A}$ has a unique maximal subfunctor $J_{(G, \Phi)}$, given by

$$
J_{(G, \Phi)}=\sum_{[H, \Psi] \in \mathcal{B}_{\succ[G, \Psi]}^{A}} E_{(H, \Psi)}
$$

The simple functor $S_{(G, \Phi)}:=E_{(G, \Phi)} / J_{(G, \Phi)}$ is isomorphic to $S_{\left(G,\{1\}, 1, V_{\Phi}\right)}$ where $V_{\Phi}$ is the irreducible $\mathbb{K}\left[G^{*} \rtimes \operatorname{Out}(G)\right]$-module

$$
V_{\Phi}:=\operatorname{Ind}_{G^{*} \rtimes \operatorname{Out}(G)_{\Phi}}^{G^{*} \rtimes \operatorname{Out}(G)}\left(\mathbb{K}_{\tilde{\Phi}}\right)
$$

with $\tilde{\Phi} \in \operatorname{Hom}\left(G^{*} \rtimes \operatorname{Out}(G)_{\Phi}, \mathbb{K}^{\times}\right)$defined by $\tilde{\Phi}(\phi, \bar{f}):=\Phi(\phi)$ for $\phi \in G^{*}$ and $f \in \operatorname{Aut}(G)$.
Proof By Remark 8.9(a) and Theorem 8.8, $J_{(G, \Phi)}$ is the unique maximal subfunctor of $E_{(G, \Phi)}$. Thus, the functor $S:=S_{(G, \Phi)}$ is a simple object in $\mathcal{F}_{\mathbb{K}}^{A}$. Moreover, $G$ is a minimal group for $S$, since $G$ is a minimal group for $E_{(G, \Phi)}$ and $E_{(H, \Psi)}(G)=\{0\}$ for all $[H, \Psi] \in \mathcal{B}_{\succ[G, \Phi]}^{A}$.

Let $(U, \phi) \in \mathcal{M}(G \times G)$ with $k_{2}(U) \neq\{1\}$. Then $\left[\frac{G \times G}{U, \phi}\right]$ factors through the group $q(U)$ which has smaller order than $G$. Thus, $\left[\frac{G \times G}{U, \phi}\right] \cdot{ }_{G} e_{\left(G, \Phi^{\prime}\right)}^{G}=0$ for all $\left(G, \Phi^{\prime}\right) \in \hat{\mathcal{X}}(G)$ with $\left(G^{\prime}, \Phi^{\prime}\right)==_{\operatorname{Out}(G)}$ $(G, \Phi)$. By Proposition $8.4(\mathrm{~d})$ this yields $\left[\frac{G \times G}{U, \phi}\right] \cdot{ }_{G} S(G)=\{0\}$. Thus, $f_{(K, \kappa)} \cdot G S(G)=0$ for all $(K, \kappa)$ with $|K|>1$.

This implies that $S$ is parametrized by the quadruple $(G,\{1\}, 1, V)$, with $V=S(G)$ viewed as $\mathbb{K} \Gamma_{(G,\{1\}, 1)}$-module. Since $S(G)$ is the $\mathbb{K}$-span of the idempotents $e_{\left(G, \Phi^{\prime}\right)}^{G}$, with $\left(G, \Phi^{\prime}\right)$ running through the $\operatorname{Out}(G)$-orbit of $(G, \Phi)$, and since $\operatorname{tw}_{\alpha} \cdot G \operatorname{iso}_{f} \cdot{ }_{G} e_{\left(G, \Phi^{\prime}\right)}^{G}=\Phi(\alpha) \cdot e_{\left(G, \Phi^{\prime} \circ f^{*}\right)}^{G}$ for all $\alpha \in G^{*}$ and $f \in \operatorname{Aut}(G)$ and $\left(G, \Phi^{\prime}\right)$, the $\mathbb{K} \Gamma_{(G,\{1\}, 1)^{*}}$ module $S(G)$ is monomial. The stabilizer of the one-dimensional subspace $\mathbb{K} e_{(G, \Phi)}^{G}$ is equal to $G^{*} \rtimes \operatorname{Out}(G)_{\Phi}$ and this group acts on $\mathbb{K} e_{(G, \Phi)}^{G}$ via $\tilde{\Phi}$. Thus, $S(G) \cong V_{\Phi}$ as $\mathbb{K} \Gamma_{(G,\{1\}, 1)}$-module and the proof is complete.
9.3 Theorem Let $F^{\prime} \subset F \subseteq B_{\mathbb{K}}^{A}$ be subfunctors in $\mathcal{F}_{\mathbb{K}}^{A}$ such that $F / F^{\prime}$ is simple. Then there exists a unique $[G, \Phi] \in \mathcal{B}^{A}$ such that $E_{(G, \Phi)} \subseteq F$ and $E_{(G, \Phi)} \nsubseteq F^{\prime}$. Moreover, $E_{(G, \Phi)}+F^{\prime}=F$, $E_{(G, \Phi)} \cap F^{\prime}=J_{(G, \Phi)}$, and $F / F^{\prime} \cong S_{(G, \Phi)}$.
Proof Since $\alpha\left(F^{\prime}\right)$ is a maximal subset of $\alpha(F)$ and both are closed, it follows from Theorem 8.8 and Remark 8.9 that $\alpha(F) \backslash \alpha\left(F^{\prime}\right)=\{[G, \Phi]\}$ for a unique $[G, \Phi] \in \mathcal{B}^{A}$. For any $[H, \Psi] \in \mathcal{B}^{A}$ one has $E_{(H, \Psi)} \subseteq F$ and $E_{(H, \Psi)} \nsubseteq F^{\prime}$ if and only if $[H, \Psi] \in \alpha(F)$ but $[H, \Psi] \notin \alpha\left(F^{\prime}\right)$. Thus, the first condition is equivalent to $[H, \Psi]=[G, \Phi]$. Further, we have $F^{\prime} \subset F^{\prime}+E_{(G, \Phi)} \subseteq F$ which implies $\left.F^{\prime}+E_{(G, \Phi}\right)=F$, since $F / F^{\prime}$ is simple. Thus, $0 \neq E_{(G, \Phi)} /\left(E_{(G, \Phi)} \cap F^{\prime}\right) \cong\left(E_{(G, \Phi)}+F^{\prime}\right) / F^{\prime}=F / F^{\prime}$, and by Proposition 9.2 we obtain $E_{(G, \Phi)} \cap F^{\prime}=J_{(G, \Phi)}$ so that $F / F^{\prime} \cong E_{(G, \Phi)} / J_{(G, \Phi)} \cong S_{(G, \Phi)}$.

## 10 The case $A \leqslant \mathbb{K}^{\times}$

In this section we assume that $A$ is a subgroup of the unit group of a field $\mathbb{K}$ of characteristic 0 . Then the assumptions on $A$ and $\mathbb{K}$ from the beginnings of Sections 319 are satisfied. This special case has been used for instance in the theory of canonical induction formulas, see Bo98. This assumption was also used in Ba04 and CY19. By double duality it allows to consider pairs $(G, g O(G))$ for a normal subgroup $O(G)$ of $G$ instead of pairs $(G, \Phi)$ with $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$. This section makes this translation precise and also translates previously defined features for pairs $(G, \Phi)$ to features for pairs $(G, g O(G))$.

For any finite group $G$ we have a homomorphism

$$
\zeta_{G}: G \rightarrow \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right), \quad g \mapsto \varepsilon_{g}, \quad \text { with } \varepsilon_{g}(\phi):=\phi(g),
$$

for $\phi \in G^{*}$. Note that $\zeta_{G}$ is functorial in $G$, i.e., if $f: G \rightarrow H$ is a group homomorphism then $\zeta_{H} \circ f=\operatorname{Hom}\left(f^{*}, \mathbb{K}^{\times}\right) \circ \zeta_{G}$. We set

$$
O^{A}(G):=O(G):=\operatorname{ker}\left(\zeta_{G}\right)=\bigcap_{\phi \in G^{*}} \operatorname{ker}(\phi),
$$

which is a normal subgroup of $G$ containing the commutator subgroup $[G, G]$ of $G$. Thus, we obtain an injective homomorphism $\bar{\zeta}_{G}: G / O(G) \rightarrow \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$.
10.1 Proposition Let $G$ be a finite group.
(a) The homomorphism $\zeta_{G}$ is surjective and $\bar{\zeta}_{G}: G / O(G) \xrightarrow{\sim} \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$is an isomorphism.
(b) The subgroup $O(G)$ is the smallest subgroup $[G, G] \leqslant M \leqslant G$ such that $A$ has an element of order $\exp (G / M)$.
(c) For any normal subgroup $N$ of $G$ one has $O(G / N)=O(G) N / N$.

Proof (a) Applying the functoriality with respect to the natural epimorphism $f: G \rightarrow G /[G, G]$, and using that $f^{*}$ is an isomorphism, it suffices to show the statement when $G$ is abelian. Since $\operatorname{Hom}\left(-^{*}, \mathbb{K}^{\times}\right)$preserves direct products of abelian groups, we are reduced to the case that $G$ is cyclic. Using again the functoriality with respect to the natural epimorphism onto the largest quotient of $G$ whose order occurs as an element order in $A$, we are reduced to the case that $G$ is cyclic of order $n$ and $A$ has an element of order $n$. In this case it is easy to see that $\zeta_{G}$ is injective and that $G$ and $\operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$have the same order.
(b) First note that if $M_{1}$ and $M_{2}$ have the stated property, then also $M_{1} \cap M_{2}$ has this property. In fact, $G /\left(M_{1} \cap M_{2}\right)$ is isomorphic to a subgroup of $G / M_{1} \times G / M_{2}$, whose exponent is equal to the order of an element in $A$. Here we use that if elements $a$ and $b$ in $A$ have orders $k$ and $l$ respectively, then $A$ has an element whose order is the least common multiple of $k$ and $l$. Thus, there exists a smallest subgroup $M$ with the stated property. Clearly, $\operatorname{ker}(\phi)$ has the property for every $\phi \in G^{*}$. Therefore, also $O(G)$ has the desired property. Conversely, if $M$ has the property, then by writing $G / M$ as a direct product of $n$ cyclic groups whose orders are achieved as element order in $A$, it is easy to construct elements $\phi_{1}, \ldots, \phi_{n} \in G^{*}$ such that $\bigcap_{i=1}^{n} \operatorname{ker}\left(\phi_{i}\right)=M$, implying that $O(G) \leqslant M$.
(c) Since the exponent of $G / O(G)$ is equal to the order of an element of $A$ also the exponent of $(G / N) /(O(G) N / N) \cong G /(O(G) N)$ is equal to the order of an element of $A$. Thus, $O(G / N) \leqslant$
$O(G) N / N$. Conversely,

$$
O(G)=\bigcap_{\phi \in G^{*}} \operatorname{ker}(\phi) \leqslant \bigcap_{\substack{\left.\phi \in G^{*} \\ \phi\right|_{N}=1}} \operatorname{ker}(\phi)
$$

and taking images in $G / N$ yields the reverse inclusion.
For any finite group $G$, Proposition 10.1(a) yields a bijection between the set of pairs of the form $(G, \Phi)$, with $\Phi \in \operatorname{Hom}\left(G^{*}, \mathbb{K}^{\times}\right)$and the set of pairs $(G, g O(G))$ with $g O(G) \in G / O(G)$. More precisely, we identify $(G, g O(G))$ with $\left(G, \varepsilon_{g}\right)$. The following proposition translates various relevant features of pairs $(G, \Phi)$ to features of the corresponding pairs $(G, g O(G))$. The proofs are straightforward and left to the reader.
10.2 Proposition Let $G$ and $H$ be finite groups.
(a) Let $g \in G$ and $h \in H$. Then $(G, g O(G)) \cong(H, h O(H))$ if and only if there exists an isomorphism $f: G \rightarrow H$ such that $f(g) O(H)=h O(H)$.
(b) Let $N$ be a normal subgroup of $G$, let $g \in G$, and set $\Phi:=\varepsilon_{g}$. Then $\Phi_{N}=\varepsilon_{g N}$.
(c) Let $g \in G$ and $h \in H$. Then $\left(H, \varepsilon_{h}\right) \preccurlyeq\left(G, \varepsilon_{g}\right)$ if and only if there exists a normal subgroup $N$ of $G$ and an isomorphism $f: H \xrightarrow{\sim} G / N$ with $f(h) \in g O(G) N$.
(d) Let $K \leqslant G$ and $g \in G$. Then $\left.\varepsilon_{g}\right|_{K^{\perp}}=1$ if and only if $g \in K O(G)$.

## References

[Ba04] L. Barker: Fibred permutation sets and the idempotents and units of monomial Burnside rings. J. Algebra 281 (2004), 535-566.
[Bo98] R. BoltJe: A general theory of canonical induction formulae. J. Algebra 206 (1998), 293-343.
[BC18] R. Boltje, O. Coşkun: Fibered biset functors. Adv. Math. 339 (2018), 540-598.
[BRV19] R. Boltje, G. Raggi-Cárdenas, L. Valero-Elizondo: The $-_{+}$and $-_{+}^{+}$constructions for biset functors. J. Algebra 523 (2019), 241-273.
[Bc10] S. Bouc: Biset functors for finite groups. Lecture Notes in Mathematics, 1990. SpringerVerlag, Berlin, 2010.
[BM19] S. Bouc, H. Mutlu: Monomial G-posets and their Lefschetz invariants. J. Algebra 527 (2019), 399-436.
[CY19] O. Coşkun, D. Yilmaz: Fibered p-biset functor structure of the fibered Burnside rings. Algebras and Representation Theory 22 (2019), 21-41.
[D71] A. Dress: The ring of monomial representations I. Structure theory. J. Algebra 18 (1971) 137-157.
[R11] N. Romero: Funtores de Mackey. PhD Thesis, UNAM Morelia, 2011.


[^0]:    *MR Subject Classification 19A22, 20C15; Keywords: Burnside ring, monomial Burnside ring, biset functors, fibered biset functors
    $\dagger$ Corresponding author

