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The Butzer-Kozakiewicz article on Riemann derivatives of 1954 and its influence

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Abstract
The article on Riemann derivatives by P.L. Butzer and W. Kozakiewicz of 1954 was the basis to generalizations of the classical scalar-valued derivatives to Taylor, Peano, and Riemann derivatives in the setting of semigroup theory. The present paper gives an overview of the 1954 article, describes its influence, and integrates it into the literature on related problems. It also describes the state of the mathematics department at McGill University where the article was written.

Keywords Generalized derivatives · Riemann derivatives · Difference equations · Semigroups of linear operators

Mathematics Subject Classification (2020) 26A24 · 47D03

1 Overview over the Butzer-Kozakiewicz paper

Let us first give a short overview of the main results of Butzer-Kozakiewicz [1]. For an arbitrary real-valued function f the central difference of order $s \in \mathbb{N}$ with respect to the increment $h > 0$ is defined by

$$\Delta^1_{2h} f(x) = f(x+h) - f(x-h), \quad \Delta^s_{2h} f(x) = \Delta^1_{2h}[\Delta^{s-1}_{2h} f(x)].$$

A tribute to Maurice Dodson, a unique and long-standing friend of the authors (A short biography of Maurice Dodson can be found as Appendix C)

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The differences can be represented as

$$\Delta_{2h}^{s} f(x) = \sum_{j=0}^{s} (-1)^j \binom{s}{j} f[x + (s - 2j)h].$$

If the limit of the difference quotient

$$\lim_{h \to 0} (2h)^{-s} \Delta_{2h}^{s} f(x)$$

exists at the point x and is finite, it is called the sth Riemann derivative of f at the point x. If the ordinary derivative $f^{(s)}$ exists at x, then the above limit is equal to $f^{(s)}(x)$, i.e., the Riemann derivative exists and equals the ordinary derivative.

The motivation of the paper is the fundamental uniqueness theorem for trigonometric series (see, e.g., [2, p. 274]) which the author (P.L.B.) learned to know in a course on Fourier series given at Loyola College, Montréal, in 1947. It states

A. *If f is continuous in $[a, b]$ and has at every point of this interval a finite second Riemann derivative g, with $g \in L(a, b)$, then*

$$f(x) = \int_a^x dt_1 \int_a^{t_1} g(t_2)\, dt_2 + c_0 + c_1 x, \quad a \le x \le b,$$

where c_0 and c_1 are constants.

A first major result is an elementary but apparently powerful method, a theorem which contains a well-known proposition of Brouwer [3] and Popoviciu [4], which states

B. *If $f(x)$ is continuous for $a < x < b$ and*

$$\Delta_{2h}^{s} f(x) = 0, \quad a < x - sh < x + sh < b,$$

then f is a polynomial of degree at most $(s - 1)$ in (a, b).

As to further notations, let $L(a, b)$ be the space of functions Lebesgue integrable over (a, b) equipped with the norm $\|f\| = \int_a^b |f(u)|\, du$. A sequence (f_n) in $L(a, b)$ is said to be convergent in the mean to $f \in L(a, b)$, if $\lim_{n\to\infty} \|f_n - f\| = 0$. If

$$\begin{cases} \|f_n\| \le M \text{ for all } n, \\ \lim_{n\to\infty} \displaystyle\int_a^x f_n(t)\, dt = \int_a^x f(t)\, dt \text{ for all } x \in [a, b], \end{cases} \tag{1}$$

then (f_n) is said to be weakly convergent[1] to f. Obviously, convergence in the mean implies weak convergence of f_n to f in $L(a, b)$.

[1] See the remarks on weak convergence at the end of Sect. 2.

Let $L\{a,b\}$ be the class of functions integrable over every closed subinterval of (a,b).

The integral means of order $s \in \mathbb{N}$ (or repeated averages) play a major role in the proofs of the following results, they being defined for $f \in L\{a,b\}$ and $h > 0$ by

$$A_h^1 f(x) = \frac{1}{2h}\int_{-h}^{h} f(x+t)\,dt = \frac{1}{2h}\int_{x-h}^{x+h} f(t)\,dt, \quad A_h^s f(x) = A_h^1[A_h^{s-1} f(x)]. \tag{2}$$

Their properties are listed in [1, Lemmas 1–5]. Let us record that

$$\lim_{h\to 0+} A_h^s f(x) = f(x) \tag{3}$$

pointwise a. e. on (a,b) and in the norm of $L(\alpha,\beta)$ for every subinterval $[\alpha,\beta] \subset (a,b)$, and

$$[A_h^s f(x)]^{(s)} = (2h)^{-s}\Delta_{2h}^s f(x) = A_h^s f^{(s)}(x), \tag{4}$$

where the first equality holds for all $f \in L\{a,b\}$ and the latter provided $f^{(s)}$ exists.

Theorem 1 *For $f, g \in L\{a,b\}$ the following assertions are equivalent:*
(i) There exists a polynomial P_{s-1} of degree not exceeding $s-1$ such that

$$f(x) = \int_c^x dt_1 \int_c^{t_1} dt_2 \ldots \int_c^{t_{s-1}} g(t_s)\,dt_s + P_{s-1}(x) \tag{5}$$

a. e. in (a,b) with $a < c < b$.

(ii) There holds

$$(2h)^{-s}\Delta_{2h}^s f(x) = A_h^s g(x) \tag{6}$$

for almost every x satisfying $a < x - sh < x + sh < b$.

(iii) There exists a null sequence (h_n) of positive reals such that $(2h_n)^{-s}\Delta_{2h_n}^s f(x)$ converges for $n \to \infty$ weakly to g on every subinterval $[\alpha,\beta] \subset (a,b)$.

(iv) There exists a null sequence (h_n) of positive reals such that $(2h_n)^{-s}\Delta_{2h_n}^s f(x)$ converges for $n \to \infty$ in the mean of $L(\alpha,\beta)$ to g for every interval $[\alpha,\beta] \subset (a,b)$.

Short sketch of the proof We omit the details regarding the ranges of x and h. Put

$$\mathfrak{I}_c^s g(x) = \int_c^x dt_1 \int_c^{t_1} dt_2 \ldots \int_c^{t_{s-1}} g(t_s)\,dt_s,$$
$$\tau_n(x) := (2h_n)^{-s}\Delta_{2h_n}^s f(x), \quad \sigma_n(x) := \tau_n(x) - g(x).$$

(i)⇒(iv): One has

$$\int_\alpha^\beta |\tau_n(x) - g(x)|dx \le \int_\alpha^\beta |\tau_n(x) - A_{h_n}^s g(x)|dx + \int_\alpha^\beta |A_{h_n}^s g(x) - g(x)|dx.$$

Now the first term on the right is zero by (4), and the second tends to zero for $n \to \infty$ by (3).
(iv)⇒(iii): This is obvious since strong convergence implies weak convergence.
(iii)⇒(ii): It follows from the definition of weak convergence that $A_h^1 \sigma_n(x)$ converges dominatedly to zero for $n \to \infty$. Repeating this argument we finally find that

$$\lim_{n\to\infty} A_h^s \sigma_n(x) = 0. \tag{7}$$

On the other hand, by (4) we deduce

$$\lim_{n\to\infty} (2h_n)^{-s} \Delta_{2h_n}^s A_h^s f(x) = [A_h^s f(x)]^{(s)} = (2h)^{-s} \Delta_{2h}^s f(x) \quad a.e. \tag{8}$$

Moreover, applying the operator A_h^s to both sides of the definition of τ_n and noting that $\Delta_{2h_n}^s$ commutes with A_h^s, we find that

$$A_h^s \sigma_n(x) = (2h_n)^{-s} \Delta_{2h_n}^s A_h^s f(x) - A_h^s g(x). \tag{9}$$

Combining now (9), (8) and (7) yields (ii).
(ii)⇒(i): First assume $g = 0$. Applying the operator A_t^{s+1} to both sides of (ii) yields $(2h)^{-s} \Delta_{2h}^s A_t^{s+1} f(x) = 0$, and for $h \to 0+$ we find that $[A_t^{s+1} f(x)]^{(s)} = 0$. This means that $P_{s-1}(x; t) := A_t^{s+1} f(x)$ is a polynomial of degree $s - 1$ at most, and letting $t \to 0+$ it follows by (3) that f is also a polynomial of degree $s - 1$ at most.
Now let $g \in L\{a, b\}$ be arbitrary, and consider the function $F(x) := f(x) - \mathfrak{I}_c^s g(x)$. Then by (4),

$$\Delta_{2h}^s F(x) := \Delta_{2h}^s f(x) - \Delta_{2h}^s \mathfrak{I}_c^s g(x) = \Delta_{2h}^s f(x) - (2h)^s A_{2h}^s g(x) = 0.$$

The assertion now follows from the case $g = 0$. This completes the sketch. □

For functions defined on $\mathbb{R}$, thus those belonging to C_{loc} or L_{loc}^1, results corresponding to those of Theorem 1 can be found in [5, pp. 383–388].
Let us now return to Riemann derivatives and the motivation. The foregoing theorem can indeed be expressed more directly in terms of Riemann derivatives in the following though weaker form; it follows directly from Theorem 1 (ii) ⇒ (i) by Lebesgue's dominated convergence theorem.

Theorem 2 *Let $f \in L\{a, b\}$. If*
(i) there exists a null sequence (h_n) of positive numbers such that

$$\lim_{n\to\infty} (2h_n)^{-s} \Delta_{2h_n}^s f(x) = g(x) \quad a.e.\ in\ (a, b),$$

(ii) there exists a function $\tau \in L\{a, b\}$ such that

$$\sup_{n\in\mathbb{N}} \left|(2h_n)^{-s} \Delta_{2h_n}^s f(x)\right| \le \tau(x), \quad a < x - sh_n < x + sh_n < b,$$

then there exists a polynomial P_{s-1} of degree not exceeding $s-1$ such that

$$f(x) = \int_c^x dt_1 \int_c^{t_1} dt_2 \ldots \int_c^{t_{s-1}} g(t_s)\, dt_s + P_{s-1}(x) \quad a.e.\ in\ (a,b),$$

where $a < c < b$.

Theorem 2 may be considered as a certain extension of assertion A from the second to higher order Riemann derivatives, but with the additional condition (ii). The direct generalization of assertion A to higher-order Riemann derivatives would be the following:

C. *If f is continuous on (a,b) and if $f^{(s-2)}$ exists everywhere on (a,b), and f has a finite sth Riemann derivative $g \in L\{a,b\}$, then for $a < x < b$*

$$f(x) = \int_c^x dt_1 \int_c^{t_1} dt_2 \ldots \int_c^{t_{s-1}} g(t_s)\, dt_s + P_{s-1}(x),$$

where $a < c < b$.

Whereas for $s = 2$ the result is classical, for $s = 3,\ 4$ it is due to Verblunsky [6, 7] and Saks [8]. Our conjecture in 1954 was that it would also be valid for $s \geq 5$. The Indian expert in the broad area of various higher order derivatives, Satya Narayan Mukhopadhyay, Burdwan University, India, managed to give a partial answer to conjecture C, the mathematical formulation being quite involved that we leave it to the reader; see [9, 10]. In his recent book Mukhopadhyay [11, Section 2.23] proves the result for $s = 3$ and 4 in the setting of his general approach to symmetric Riemann (and de la Vallée Poussin) derivatives.

That one must assume the existence of $f^{(s-2)}$ for $s \geq 3$ even in case $g(x) = 0$ can be seen from the counter-example $f(x) = |x|x^{(s-3)}$. In fact, the first $s-3$ ordinary derivatives exist, but $f^{(s-2)}(0)$ does not at $x = 0$, while the Riemann sth derivative is everywhere zero.

The importance of assertion C lies in the fact that it is used in proving the result that if a trigonometric series converges, except in an enumerable set to a finite square-integrable function g, then it is the Fourier series of g (see, e.g., [2, p. 274]).

In 1954, the authors believed that no previous attention had been given to results of the type showing a relation between the class of functions defined on a finite interval (a,b) having an sth order Riemann derivative and polynomials of degree s.

Dutta and Mukhopadhyay [9] succeeded in establishing all results of [1] in the setting of Peano derivatives; in fact, since they used the same notation it is easy to follow up their careful arguments. At the same time, they extended the results of the latter to a wider class of functions and even by replacing polynomials of degree $s-1$ by s-convex functions. This procedure was already indicated in the last section of [1] and has been used by most authors working in the broad area of Riemann derivatives not later than Verblunsky [6, 7] and Saks [8], one following [1] being Kassimatis

[12]. The wider class is expressed by the scale of $C_r P$-integrals defined by Burkill [13], upper and lower Riemann derivatives also being essential. Observe that since a function is both s-convex and s-concave if and only if it is a polynomial of degree $s-1$, these results contain the more classical ones. They finally apply their results to trigonometric series. Here Riemann's method of summation comes into play; see, e.g., [5, p. 54].

In a main result of his article Kemperman [14, pp. 81, 82] makes "essential use of a special case of a result of Butzer and Kozakiewicz" in his study of generalized convex functions.

As recorded in [1], every one of the theorems given there is valid not only for the forward and backward differences but also for the above associated Peano difference.

2 Influence of the paper

The Butzer-Kozakiewicz paper [1] deals with the basic question: What are the necessary and sufficient conditions in order that a given integrable function f be a.e. equal to an indefinite repeated integral of another function g. A main result, namely Theorem 1, gives this condition in terms of a difference quotient, which leads to the characterization in terms of the Riemann derivative in Theorem 2

The paper is the basis to Sections 2.1.2 to 2.2.3 of [15, pp. 92–111], dealing with Taylor, Peano, and Riemann operators in the setting of semigroup theory, they being generalizations of the classical scalar-valued derivatives. These sections are largely taken from [16].

Let $\mathcal{E}(X)$ the algebra of all bounded linear transformations of a Banach space X into itself, and $\mathcal{T} = \{T(t); 0 \le t < \infty\}$ be a semi-group of class $(\mathcal{C}_0)$ in $\mathcal{E}(X)$, i.e., $T(t) : X \to X$ satisfies the semigroup equation

$$T(t_1 + t_2) = T(t_1)T(t_2), \quad T(0) = \mathrm{Id}_X,$$

for all real $t_1, t_2 \ge 0$ together with the (C_0)-property

$$\operatorname*{s-lim}_{t\to 0+} T(t)f = T(0)f = f \quad (f \in X),$$

where s-lim denotes the limit in the norm topology of X.

The (infinitesimal) generator A of $\mathcal{T}$ is defined by the strong limit

$$Af = T'(0)f = \operatorname*{s-lim}_{t\to 0+} t^{-1}[T(t)f - f]$$

on $\mathcal{D}(A)$, the domain of A, the set of all $f \in X$ for which this limit exits. The operator A is closed with $\overline{\mathcal{D}(A)} = X$. The powers A^r for $r \in \mathbb{N}$ are defined inductively by $A^r := A(A^{(r-1)})$ with $A^0 = \mathrm{Id}_X$, their domains being $D(A^r)$.

The paper [16] (see also [15, Cor. 2.2.17]) deals with the so-called fundamental saturation theorem for a family of operators, namely,

D. *Let X be a reflexive Banach space and $\mathcal{T} = \{T(t); 0 \le t < \infty\}$ be a semi-group of class $(\mathcal{C}_0)$ in $\mathcal{E}(X)$ with infinitesimal generator A. The family of operators*

$$C_t^r f := T(t)f - \sum_{k=0}^{r-1} \frac{t^k}{k!} A^k f \quad (t > 0).$$

defined on $D(A^{r-1})$ is saturated with order $\mathcal{O}(t^r)$ for $t \to 0+$, i.e.,

$$\|C_t^r f - f\| = o(t^r) \text{ for } t \to 0+ \implies C_t^r f = f \text{ for all } t > 0,$$

and

$$\|C_t^r f - f\| = \mathcal{O}(t^r) \text{ for } t \to 0+ \iff f \in D(A^r).$$

The method of proof in [16] is that of [1]. Indeed, the operators

$$B_t^s f = \frac{t^s}{s!}\Big\{T(t)f - \sum_{k=0}^{s-1} \frac{t^k}{k!} A^k f\Big\} = \frac{t^s}{s!} C_t^s f \quad (t > 0)$$

here take the place of the sth integral means $A_t^s f$ of (2). In fact, in case $\mathcal{T}$ is the translation semigroup, i.e., $T(t)f(x) = f(x+t)$, then the operators

$$B_t^s f(x) = \frac{t^s}{s!}\Big\{f(x+t) - \sum_{k=0}^{s-1} \frac{t^k}{k!} f^{(k)}(x)\Big\} \quad (t > 0)$$

tend to $f^{(s)}(x)$ for $t \to 0+$, which presents exactly the sth Peano (or Taylor) derivative of f; see [17].

One of the most important applications of semi-group theory is defining fractional powers of the infinitesimal generator. It concerns derivatives of fractional orders in the semi-group setting, e.g., by Riemann or Peano derivatives; see [15, Section 2.2.1]. A large literature is attached to this broad problem, very early papers are [18–20].

In her remarkable and understandable paper [21] Westphal develops a systematic approach to fractional powers of order $\alpha > 0$ of infinitesimal generators based of Laplace transforms in the setting of the operational calculus of L. Schwartz. She applies her new fractional calculus to Marchaud-type representations of fractional powers $(-A)^\alpha$ as well as to Liouville-Grünwald-Butzer-Westphal representations [22]. In [23] the equivalence of the Weyl with the Marchaud fractional derivative on $\mathbb{R}$ together with their connection to Riesz derivatives are treated. For U. Westphal see also [24, Section 13.2].

A basic application of those fractional powers is PDEs. One first example is Cauchy's problem for linear partial differential equations containing derivatives of fractional order, which includes as a particular case a boundary-value problem of the heat conduction equation. For the state of the art in 1966 see, e.g., the discussion in [15, pp. 153–156]. A monumental study in this respect is the three-volume treatise by Lions and Magenes [25].

The Butzer-Kozakiewicz paper is the foundation to the Peano, Riemann, and Taylor derivatives, in the scalar-valued form. An alternative, fully different approach to this question is the Fourier transform method, the basis of which is an elegant, complete, and systematic theory.

First results in this direction, namely that the Riemann derivatives of all orders are equal to the Peano derivatives, provided they exist, can be found in [17]; see also [26]. The complete theory for strong as well as weak derivatives in $C_{2\pi}$, $C(\mathbb{R})$, $L^p_{2\pi}$ and $L^p(\mathbb{R})$ for $1 \le p < \infty$ is treated in Section 5.1.4 and in the full Chapter 10 of [5]. The present authors are not aware of the fact whether such results have been (or can be) established for functions defined on a finite interval.

Let us return to the work of Mukhopadhyay, already mentioned in connection with assertion C. He treats the matter using neither semi-group nor Fourier transform theory. In the short introduction to his book *Higher Order Derivatives* [11] he devotes four lines to "Riemann derivatives of higher order [which] are the special study by P.L. Butzer and W. Kozakiewicz ..." Here he cites his paper [9] as well as 16 further papers by authors from four countries including [27]. As to the contents of his book, he studies a variety of derivatives of integral order for real-valued functions, thus Peano, de la Vallée-Poussin, Cesàro, Borel, Abel, and Laplace derivatives. Concerning the relations among them, he concentrates on the question of whether they are equal, provided they exist finitely pointwise.

As to the definition of weak convergence. The usual definition reads:

A sequence (f_n) in $L(a,b)$ is said to be convergent weakly to $f \in L(a,b)$, if

$$\lim_{n\to\infty} \int_a^b f_n(u)\varphi(u)\,du = \int_a^b f(u)\varphi(u)\,du \textit{ for all } \varphi \in L^\infty(a,b). \tag{10}$$

This definition of weak convergence is different from that given in (1) (cf. [1]). Indeed, the conditions of (1) are necessary for weak convergence in the sense of (10), but not sufficient; see [28, p. 82] in this respect.

Using this notion of weak convergence the classical definition of a weak Riemann derivative reads:

If for $f \in L(a,b)$ there exists $D^s_w f \in L(a,b)$ such that

$$\lim_{h\to 0} \int_a^b (2h)^{-s}\Delta^s_{2h} f(u)\varphi(u)\,du = \int_a^b D^s_w f(u)\varphi(u)\,du \textit{ for all } \varphi \in L^\infty(a,b), \tag{11}$$

then $D^s_w f$ is called the weak derivative of f of order s.

It is clear that if

$$\lim_{h\to 0} \|(2\,h)^{-s}\Delta^s_{2\,h} f - D^s_s f\| = 0,$$

thus if the strong Riemann derivative $D^s_s f$ exists, so does the weak one and $D^s_s f = D^s_w f$.

Now, since (1) is necessary for (10), the existence of $D^s_w f$ on some interval $[\alpha,\beta]$ in the sense of (11) implies the assertion of Theorem 1 (iii). This in turn implies Theorem 1 (iv), i.e., the strong Riemann derivative $D^s_s f$ exists on $[\alpha,\beta]$. This means that Theorem 1 remains valid if the weak convergence in Theorem 1 (iii) is understood in the sense of (10).

The particular case $g = 0$ of Theorem 1 can be slightly generalized as follows. For simplicity, we assume $a = -\infty$ and $b = \infty$.

Corollary 3 *Let $f \in L\{-\infty, \infty\}$ and $s \in \mathbb{N}$. If for every fixed $h > 0$ the relation*

$$\Delta_{2h}^{s} f \in \mathcal{P}_n \tag{12}$$

holds in the sense that the left-hand side equals for almost every $x \in \mathbb{R}$ a function in $\mathcal{P}_n$, the space of all polynomials not exceeding n, then there exists a polynomial $p_{n+s} \in \mathcal{P}_{n+s}$ such that $f(x) = p_{n+s}$ almost everywhere in $(-\infty, \infty)$.

Conversely, each $q_{n+s} \in \mathcal{P}_{n+s}$ satisfies relation (12) for all $x \in \mathbb{R}$.

The proof is an easy consequence of the fact that the difference operator Δ_{2h}^{1} decreases the degree of a polynomial by at least one. Hence, by iteration, $(\Delta_{2h}^{n+1} p_n)(x) = 0$ for each $p_n \in \mathcal{P}_n$. So, it follows from (12) that

$$(\Delta_{2h}^{n+s+1} f)(x) = (\Delta_{2h}^{n+1}\big(\Delta_{2h}^{s} f)\big)(x) = 0 \tag{13}$$

for almost every $x \in \mathbb{R}$, which yields the assertion by Theorem 1 with $g = 0$.

The converse is obvious, again by the degree decreasing property of the difference operator.

For an entirely different approach to the case $s = 1$ of this result see [29, Lemma 2.5]. This lemma inspired us to deduce the above result from Theorem 1.

Appendix A: Paul Butzer's reminiscences of the mathematics department at McGill University from 1945 to 1959 and the spirit of the collaboration with W. Kozakiewicz

In 1951, when Professor Wacław Kozakiewicz came to McGill University, Montréal, he was highly respected in Canada as an expert, versatile mathematician, not only in his research area, mathematical statistics but also in probability and real variable theory. His foregoing stations had been the Dominion Bureau of Statistics (= DBS), Ottawa), the University of Sasketchevan (Saskatoon), and the Université de Montréal; see [30, 31] and Appendix B.

When McGills' Principal Dr. Frank James and Dr. Herbert Tate McGills Chairman of Mathematics (from ca. 1935 to 1964?) decided to create a graduate school in 1945, their chief aim was to attract first-class mathematicians of senior rank. The first to come was Hans Zassenhaus from the University of Hamburg, who was appointed Peter Redpath Professor of Mathematics in 1949 (my brother Karl Butzer received his BSc in mathematics in 1954, Dr. Zassenhaus being the advisor); cf. [32]. The mathematical analyst Charles Fox, from Birkbeck College, London, since 1949 at McGill, who was promoted to professor in 1956, was a pleasant colleague. He raised the question whether there exist processes which simultaneously approximate and interpolate a given function. Such a process is the Fejèr-Hermite interpolation process, not aware to approximation theorists in North America at the time.

While Wacław came in 1951, Philip R. Wallace, a doctoral student of Leopold Infeld at Toronto, was a theoretical physicist and one of the bright lights of the mathematics department already since 1946.

After Dr. Tate had heard that Dr. Zassenhaus had offered me a position in his research group in spring 1952 (I had received my PhD in mathematics, minor physics, at the University of Toronto in 1951, Dr. George Lorentz being the advisor), he counter-offered with a regular lectureship (with promotion to Assistant Professor in 1953). During my 3 years at McGill I gave graduate courses on the theory of divergent series (Dr. Loyd Williams, who had taught it regularly, passed this course on to me before his retirement 1954) and Lebesgue integration (Jean Maranda being one of my good students; he received his PhD under Dr. Zassenhaus).

Until 1945, mathematics had been almost wholly a service department, mainly for engineering, with only seven faculty members. Dr. Zassenhaus had alone nine PhD students during his 10 years at McGill, seven until 1954. The first in 1950 was Joachim Lambek (1922–2014), who was born in Leipzig and spent 2 years in a Canadian internment camp. He was the first PhD in mathematics granted at McGill, even the first in the Province of Quebec. (The total number of students and PhDs in mathematics was not available to me for the period.)

All in all, the establishment of a graduate school in mathematics was a full success. For the year 1964, it is known to have some 40 newcomer students in the graduate school per year, half of them masters, the other half being doctors. According to the "QS The World University Ranking," released June 2023, McGill ranks 30 (Toronto 21).

This was the state of mathematics when Wacław and I wrote our joint paper in late fall of 1952. At the time Wacław was a well-established mathematician in Canada, a product of the unique generation of mathematicians born in Poland between the two World Wars, with nine papers to his credit, in the broad area of probability theory (the first of 1933), while I had just two on approximation theory; he was 41, I just 24, a youngster in the Canadian mathematical community. We worked in Wacław's home in Montréal-Westmount, while his spouse kindly prepared tea or hot chocolate with sandwiches or cakes. She was a refined, reserved, but friendly lady, who was always concerned that Wacław may excerpt himself too much; he seemed to have health problems. They had a son, John (Christopher), who often wanted to play with us while we were working.

Wacław, who was a polite, contemplative, very helpful, and kind person, never made me feel like a youngster during our many sessions. We tackled a problem, new for us two, in close cooperation.

In contrast, Dr. Lorentz[2] (who claimed that through his mother, who was a member of the large Prince Chergodaev family, he was a descendant of Genghis Khan the fearsome Mongol warrior of the 13th century) had suggested a thesis topic, namely generalizing a theorem of A. O. Gelfond in a complex function theory setting. I saw no way and have not seen such a generalization since. There was no constructive

[2] For an orbituary of G. G. Lorentz see [33].

help on his part—just criticism. I thought of my own topics; the approach of one of them, on linear combinations of Bernstein polynomials, was applied by some dozens of authors to a variety of approximation processes, in particular under the heading "linear combinations"; see [34]. The papers [35–37] were chapters of my Toronto dissertation. [38] was completed shortly after them. It is quite often cited in the more recent literature.

Let me emphasize that the kind, helpful, and cooperative way Wacław worked with me became the model for my work with my many students I later had in Aachen.

Appendix B: A short biography of Wacław Kozakiewicz

Wacław Kozakiewicz was born in Warsaw on 23 January 1911, and baptized in the village Złotniki in the Jędrzejów district. He was the son of Jan Kozakiewicz (1882–1945) and Taida Maria Anna, née Trószyńska. His brother was Stefan Kozakiewicz, later professor of art history at Warsaw University.

From September 1920 he was a pupil at the Mikołaj-Rey Grammar School in Warsaw, where his father was mathematics and physics teacher and director; see https://www.rej.edu.pl/dyrektorzy_szkoly/. He received his school-leaving certificate (humanistic type) in May 1929.

In October 1929, he began his studies of mathematics at the University of Warsaw. At first he was interested in the theory of analytical functions, then in probability theory and mathematical statistics. He attended lectures by Wacław Sierpiński, Jan Łukasiewicz, Stanisław Saks, Otto Nikodým and Aleksander Rajchman, among others, but was mainly a student of Stefan Mazurkiewicz. He worked in the student Mathematical and Physical Circle and was its vice-president for 1 year. On 7 November 1933, he was awarded the Master's degree.

In the academic year 1933/34, he studied at the Faculty of Humanities of the University of Warsaw as part of the Pedagogical Year Study.

From November 1933 to October 1938 he was a senior assistant in the Department of Mathematical Statistics at the Faculty of Horticulture of the Warsaw University of Life Sciences (SGGW) in Warsaw, headed by the statistician Jerzy Spława-Neyman. He received his doctorate in mathematics from the University of Warsaw in 1936, his thesis being supervised by Stefan Mazurkiewicz.

At the onset of the war, he joined the Polish armed forces in France and served until the surrender of France in June 1940. As he could not leave France, he taught mathematics at the Polish lyceum for the next 4 years. When he was ordered to report for forced labor in Germany, he fled via Spain to Canada, where he arrived in October 1944.

There he first joined the staff of the Dominion Bureau of Statistics (DBS) in Ottawa, and in the autumn of 1945, he became an associate professor of mathematics at the University of the Province of Saskatchewan in Saskatoon, Canada, where he became a close friend of W. J. R. Crosby, who valued him most highly. In 1949, the Université de Montréal wanted to establish a program in mathematical statistics and they were therefore looking for a mathematician trained in the French tradition. They offered

Kozakiewicz this position, which he accepted. In 1951 he migrated to McGill University.

He died of a heart attack in Montréal on 8 March 1959, leaving his wife Naomi, née Pelletier (who died in her home on March 27, 2014, at the age of 102), a Canadian, and their son John. He had met Naomi at the DSB, they were married in the winter of 1947.

This biography is based on that of Krysko [30] and the obituary [31], which also contains a list of his publications.

Appendix C: A short biography of Maurice Dodson

Michael Maurice Dodson and his twin brother George Guy Dodson were born in Palmerston North, New Zealand on 1 January 1937. Maurice graduated from Auckland University in 1957 with a BSc, and in 1958 with an MSc in Maths (1st Class Honors) and the Mathematics Prize. He went to Cambridge in 1959 and graduated with a BA in Maths in 1962. He studied Number Theory under Harold Davenport and obtained his PhD in additive Number Theory in 1965. His brother Guy studied chemistry and went on to Oxford in 1961 to work with the eminent Professor Dorothy Hodgkin, who was awarded a Nobel Prize in chemistry in 1964.

Maurice joined the just 1-year-old University of York in 1964. His research interests broadened from Diophantine equations and approximation in number theory to catastrophe theory, and through a connection with X-ray crystallography, came to include Fourier series, harmonic analysis, and applications to sampling in 1985 when he first met Rowland Higgins. He also had become interested in chaos, biology, and dynamical systems. According to MathSciNet, Maurice was the author of 90 scientific publications in these fields.

As for sampling, Maurice focused on sampling in abstract spaces. In this regard, see his chapter “Abstract harmonic analysis and the sampling theorem” in [39, Chapter 10], written together with M. G. Beaty. He was an invited speaker at the SampTA conference held at Samsun, Turkey, 2005, with the lecture “The Whittaker-Kotel’nikov-Shannon sampling theorem in abstract harmonic spaces”; see [40]. He contributed to the workshop “Approximation Theory and Signal Analysis” in Lindau 2009 to celebrate Paul Butzer’s 80th year with “Approximating signals in the abstract” [41]. See also [42, Section 11.2] and [43, Sections 6.2 and 6.3].

Maurice was invited to the British-Russian Workshop in Functional Analysis, held at the Euler International Mathematical Institute (associated with the Steklov Institute), St. Petersburg, from 13 to 17 October 1996. There were about 30 participants from different parts of the former Soviet Union and from England, Scotland, and Wales.

In 1988, the authors participated in Maurice’s “Symposium on Fourier Analysis, Interpolation and Signal Processing” in York. A large number of prominent mathematicians were present, including Walter Hayman from Imperial College, who had invited me (PLB) to a lecture at his university during my first trip to England and Scotland in 1973, together with my parents, a tour organized by Lionel Cooper. Jointly with Jim Clunie, Maurice organized an even larger follow-up conference in 1993.

During the 1988 symposium, Maurice suggested that the University of York and RWTH Aachen join forces in establishing the Aachen-York "Alcuin Symposia," a series of conferences on mathematics, history, electrical engineering, and on biochemistry. It was Johannes Erger, chairman of the external institute of the RWTH (see [24]) who succeeded in incorporating its core into the Erasmus Student Mobility program of the EU, a student exchange program that ultimately involved 22 universities from seven EU countries. Many students from Aachen and most of those going from the Lehrstuhl A für Mathematik, went to the University of York, especially because Maurice took care of them; see also [44]. In connection with the Erasmus program, he took part in workshops in Segovia (Spain), Orleans (France), Thessaloniki (Greece), and Assisi (Italy).

The first of the "Alcuin symposia," conducted by the authors, was held in Aachen in 1989, followed by one in York 1 year later and the symposium "Science and History in Western and Eastern Civilisation" in Aachen in 1991; the proceedings appeared in [45]. The Alcuin Symposia on biochemistry were held as joint workshops on insulin and related proteins between the groups of Guy Dodson in York and the groups of Dietrich Brandenburg and Axel Wollmer in Aachen. The last one took place in Aachen in 2000; for the proceedings see [46].

Maurice was due to retire in 2004, but the closure of the Department of Mathematics at the nearby University of Hull, announced in late in 2004, led the two Universities agreeing that the York Department would adapt and take on those whom it could. This was a very complicated and delicate operation at a time when York, like many universities in the UK, was also experiencing cuts. Moreover, the Head of Department was due for a year's leave. In view of his competence and international reputation, the Vice-Chancellor appealed to Maurice to postpone his retirement and serve as Head of Department for a year. The many problems of staff and student accommodation were managed by Maurice, the new program started on time and Maurice finally retired for good, now as Professor Emeritus in 2005.

In 1974, he married Haleh Afshar, an Iranian who became a professor of politics and women's studies at the University of York, was awarded an OBE for her work on women and Islam and was elevated to the House of Lords in 2008, as Baroness Haleh Afshar of Heslington. She died too early on 12 May 2022.

At a convocation ceremony at the University of York in July 1997, three persons were honored with a Doctorate from the University, the retired Archbishop of York, John Habgood, the renowned president of the European Commission, Jacques Delors, and I (PLB) myself; see [44]. Maurice proposed and organized in full the honorary doctorates for Christopher Zeeman[3] and myself, and together with his Number Theory

[3] Erik Christopher Zeeman was born in Japan on 4 February 1925. His family moved to England 1 year after his birth. He studied mathematics at Christ's College, Cambridge, and received an MA in 1950 and a PhD in 1954 (the latter under the supervision of Shaun Wylie) from the University of Cambridge. Zeeman was elected as a Fellow of the Royal Society in 1975 and was awarded the Society's Faraday Medal in 1988. He was the 63rd President of the London Mathematical Society in 1986–88. He was awarded the Senior Whitehead Prize of the Society in 1982. Between 1988 and 1994 he was the Professor of Geometry at Gresham College. He received a knighthood in the 1991 Birthday Honors for "mathematical excellence and service to British mathematics and mathematics education" (extracted from [47], see also https://warwick.ac.uk/newsandevents/knowledgecentre/science/maths-statistics/zeeman and https://mathshistory.st-andrews.ac.uk/Biographies/Zeeman/).

colleagues Richard Hall and Terence Jackson that for Paul Erdős. My deep thanks are due to Maurice for this special honor.

Rudolf and Paul always have fond memories of their visits to York, where they were often invited by Maurice to his cozy house and hosted by him and his wife Haleh at a huge wooden table with extraordinary warmth; see also [44]. Rudolf and Paul would like to express their profound gratitude to Maurice for his true friendship over the many years.

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Conflict of interest The authors declare no competing interests.

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