

# LAMINATIONS AND GROUPS OF HOMEOMORPHISMS OF THE CIRCLE

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**ABSTRACT.** If  $M$  is an atoroidal 3-manifold with a taut foliation, Thurston showed that  $\pi_1(M)$  acts on a circle. Here, we show that some other classes of essential laminations also give rise to actions on circles. In particular, we show this for tight essential laminations with solid torus guts. We also show that pseudo-Anosov flows induce actions on circles. In all cases, these actions can be made into faithful ones, so  $\pi_1(M)$  is isomorphic to a subgroup of  $\text{Homeo}(S^1)$ . In addition, we show that the fundamental group of the Weeks manifold has no faithful action on  $S^1$ . As a corollary, the Weeks manifold does not admit a tight essential lamination, a pseudo-Anosov flow, or a taut foliation. Finally, we give a proof of Thurston's universal circle theorem for taut foliations based on a new, purely topological, proof of the Leaf Pocket Theorem.

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## 1. INTRODUCTION

Let  $M$  be an atoroidal 3-manifold with a taut foliation  $\mathcal{F}$ . The foliation  $\mathcal{F}$  gives rise to actions of  $\pi_1(M)$  on 1-manifolds in two ways: the action on the space of leaves in the universal cover, and the action on a universal circle. These actions are very useful for understanding  $\mathcal{F}$ . If  $M$  is an atoroidal 3-manifold with an essential lamination, one still has an action of  $\pi_1(M)$  on the space of leaves in the universal cover, but this space is just an order tree, not a 1-manifold. Our main goal here is to give analogues of the second kind of action: we construct actions of fundamental groups of 3-manifolds on circles arising from certain kinds of essential laminations. Before giving precise statements, we'll discuss the case of taut foliations.

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Let  $\widetilde{M}$  denote the universal cover of  $M$ , and  $\widetilde{\mathcal{F}}$  the foliation of  $\widetilde{M}$  covering a taut foliation  $\mathcal{F}$  of  $M$ . First, consider the leaf space  $L$  of  $\widetilde{\mathcal{F}}$ . The space  $L$  is simply connected and locally homeomorphic to  $\mathbb{R}$ , but is typically non-Hausdorff. We will still refer to  $L$  as a 1-manifold. There is a natural action of  $\pi_1(M)$  on  $L$ , and this action has no global fixed point. In fact, after changing  $\mathcal{F}$  slightly by a monotone equivalence, we can ensure that the homomorphism  $\pi_1(M) \rightarrow \text{Homeo}(L)$  is injective (see Theorem 7.10). Thus  $\pi_1(M)$  is isomorphic to a group of homeomorphisms of a 1-manifold.

Second, Thurston has shown that the foliation  $\mathcal{F}$  gives rise to another action of  $\pi_1(M)$  on a 1-manifold, namely on a *universal circle*. This circle,  $S^1_{\text{univ}}$ , is constructed by collating the circles at infinity of the leaves of  $\widetilde{\mathcal{F}}$ . As in the previous case, the action on a universal circle is faithful (Theorem 6.3). Thurston's paper [Thu] on this topic is largely unwritten, so we provide a complete proof here. The main technical tool for proving the existence of a universal circle is Thurston's *Leaf Pocket Theorem*. This theorem says, very roughly, that the leaves of  $\mathcal{F}$  come together in many directions. Thurston's proof of this theorem is analytic, and uses theorems of L. Garnett on the existence and quality of harmonic transverse measures for foliations. We give a new and purely topological proof of a slight variation of the Leaf Pocket Theorem, Theorem 5.2, which applies more generally to essential laminations. In Section 6, we use Theorem 5.2 to prove the existence of universal circles.

Now we will outline our results for essential laminations. For background and more detailed definitions see Section 2. An essential lamination  $\Lambda$  of a 3-manifold  $M$  is *tight* if the leaf space of  $\widetilde{\Lambda}$  is Hausdorff. Equivalently, leaves of  $\widetilde{\Lambda}$  are uniformly properly embedded in  $\widetilde{M}$ . The complementary regions of  $\Lambda$  can be partitioned into *interstitial regions* and *guts*. The interstitial regions are (typically noncompact)  $I$ -bundles whose  $\partial I$ -bundles lie on  $\Lambda$ , and the compact guts make up the rest of the complement.

Our main result is

**3.2. Theorem.** *Let  $M$  be an atoroidal 3-manifold containing an essential lamination  $\Lambda$ . If  $\Lambda$  is tight and all the gut regions are solid tori, then  $\pi_1(M)$  has a faithful action on  $S^1$ .*

Here's an outline of the proof. We can assume that  $\Lambda$  has non-empty guts (i.e. is *genuine*), as otherwise it can be blown down to a taut foliation which gives rise to an action on a universal circle. The first step in the proof is the following lemma, which is of independent interest:

**4.1. Filling Lemma.** *Let  $M$  be a 3-manifold and  $\Lambda$  a genuine lamination all of whose guts are solid tori. Then there is a genuine lamination  $\overline{\Lambda}$  containing  $\Lambda$ , whose complementary regions are all ideal polygon bundles over  $S^1$ . Moreover, if  $\Lambda$  is tight, so is  $\overline{\Lambda}$ .*

A lamination such as  $\overline{\Lambda}$  whose complementary regions are all ideal polygon bundles is said to be *very full*. A good example to keep in mind of a very full lamination is the suspension of the stable lamination of a pseudo-Anosov surface homeomorphism in the corresponding surface bundle.

The Filling Lemma 4.1 reduces the proof of Theorem 3.2 to the case where  $\Lambda$  is tight and very full, which is handled in Theorem 3.1. The basic idea is to flatten out  $\tilde{\Lambda}$  into a geodesic lamination  $\lambda$  of  $\mathbb{H}^2$ , taking care so that  $\lambda$  inherits a group action of  $\pi_1(M)$  which is equivariant with respect to the map  $\tilde{\Lambda} \rightarrow \lambda$ . The action of  $\pi_1(M)$  on  $\lambda$  induces an action on the circle at infinity of  $\mathbb{H}^2$ , which is the desired circle action.

We can lessen the requirement that  $M$  be tight somewhat, when the non-Hausdorff behavior of the lamination is not too bad. See Section 3.4 and Theorem 3.8. In particular Theorem 3.8 applies to the stable or unstable lamination of a pseudo-Anosov flow. Thus (Corollary 3.9) fundamental groups of 3-manifolds with pseudo-Anosov flows also act faithfully on the circle.

**1.1. Nonexistence results.** Given the results above, it is natural to ask the question: Which atoroidal 3-manifolds have faithful circle actions? We show that not every 3-manifold has such an action. In particular, for the closed hyperbolic 3-manifold with smallest known volume, we have:

**9.2. Theorem.** *The fundamental group of the Weeks manifold does not act faithfully on the circle.*

As a corollary of this and the above theorems on the existence of circle actions, it follows that the Weeks manifold does not admit a tight essential lamination with solid torus guts, a pseudo-Anosov flow, or a taut foliation. Using Agol's volume estimates for manifolds with tight laminations [Agol], one can show that any tight lamination in the Weeks manifold would have solid torus guts. So we have the stronger conclusion that the Weeks manifold contains no tight essential lamination (Corollary 9.4). We conjecture that the Weeks manifold contains no essential lamination at all, but this remains open. Also, the fundamental group of the Weeks manifold appears to be the first known example of a rank-1 lattice which cannot act faithfully on the circle. In contrast, Witte has shown that there are many higher rank lattices with this property [Wit].

Examples of hyperbolic manifolds without taut foliations were already known from the breakthrough work of Roberts, Shareshian, and Stein [RSS]. In fact, they constructed infinitely many such examples by looking at certain Dehn fillings of punctured torus bundles with negative trace. They showed the non-existence of taut foliations by proving that the fundamental groups of these manifolds can't act on a simply-connected 1-manifold without a global fixed point. Given the multitude of non-Hausdorff 1-manifolds, it is remarkable that such a proof can be made to work. One of our original motivations for this paper was to reprove non-existence of taut foliations by studying only actions on Hausdorff 1-manifolds, in particular actions on  $S^1$ .

The technique used for proving Theorem 9.2 is this. Let  $G$  be the fundamental group of the Weeks manifold. First, we pass to a finite index subgroup  $N$  where the action on  $S^1$  lifts to an action on  $\mathbb{R}$ . A faithful action of  $N$  on  $\mathbb{R}$  is equivalent to a left-invariant total order on  $N$  (see Section 7). The non-existence of such an order is then shown by considering various possibilities for which elements of  $N$  satisfy  $g > 1$ . For some 3-manifold groups this can be done algorithmically as discussed in Section 8. While the Weeks manifold was the only case where we managed to show that the

fundamental group doesn't act faithfully on the circle, we found many examples in the Hodgson-Weeks census whose fundamental groups can't act faithfully on  $\mathbb{R}$  (see Section 10).

**1.2. Further Questions.** Because essential laminations are very common, these results show that many 3-manifold groups are subgroups of  $\text{Homeo}(S^1)$ . So a natural question is: Suppose that  $\pi_1(M)$  is a subgroup of  $\text{Homeo}(S^1)$ . What does this tell us about the algebraic properties of  $\pi_1(M)$ ? Of course, a general finitely generated subgroup of  $\text{Homeo}(S^1)$  can be quite strange, e.g it can contain Thompson's infinite simple group. However, the actions we construct have additional special properties; for instance, they preserve the endpoints of a special kind of geodesic lamination of  $\mathbb{H}^2$ . So one can hope that the existence of such an action implies something interesting about  $\pi_1(M)$ .

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## 2. BACKGROUND ON ESSENTIAL LAMINATIONS

In this section, we summarize the definitions and some of the basic results about taut foliations and essential laminations (for more detail see the references in [Gab]). Throughout,  $M$  will be a closed 3-manifold. We'll begin with taut foliations:

**2.1. Definition.** *Let  $\mathcal{F}$  be a foliation of  $M$  by surfaces. The foliation  $\mathcal{F}$  is taut if there is a loop in  $M$  transverse to  $\mathcal{F}$  which intersects every leaf.*

We will use  $\tilde{\mathcal{F}}$  to denote the induced foliation of the universal cover of  $M$ . The *leaf space*  $L$  of  $\tilde{\mathcal{F}}$  is a simply-connected, not necessarily Hausdorff, 1-manifold. Some basic properties of manifolds with taut foliations are summarized in the following theorem:

**2.2. Theorem (Novikov, Rosenberg, Reeb).** *Let  $M$  be a 3-manifold with a taut foliation  $\mathcal{F}$ . Then either  $M$  is finitely covered by  $S^2 \times S^1$ , and  $\mathcal{F}$  is finitely covered by the product foliation by spheres, or  $\tilde{M} = \mathbb{R}^3$  foliated by planes. In particular, every leaf of  $\mathcal{F}$  is incompressible, and every loop transverse to  $\mathcal{F}$  is homotopically essential.*

A codimension one *lamination* of  $M$  is a foliation of a closed subset, i.e. a closed union of complete embedded surfaces—the *leaves* of the lamination—which intersect small open charts of  $M$  in products of the form  $\mathbb{R}^2 \times K$ , where  $K$  is any closed subset of an interval, and each leaf locally intersects the open set as a horizontal slice  $\mathbb{R}^2 \times \text{point}$ . Globally, of course, a leaf might intersect a product chart in infinitely many horizontal slices. Typically,  $K$  consists of a union of intervals, isolated points, and Cantor sets. Generalizing the notion of tautness for foliations is:

**2.3. Definition.** *A lamination  $\Lambda$  is essential if each complementary region is irreducible and has boundary which is incompressible and end-incompressible. Moreover, we require that no leaf of  $\Lambda$  is a 2-sphere, or a torus leaf bounding a Reeb component.*

Here a leaf is end-compressible if it has an end-compressing monogon—that is, a monogon  $D$  properly embedded in a complementary region  $C$  which is not homotopic (rel boundary) into  $\partial C$ . An end-compressing monogon is an obstruction to finding a metric with respect to which leaves of  $\Lambda$  are minimal surfaces; excluding such monogons ensures that essential laminations share many properties in common with taut foliations. In particular, manifolds with essential laminations have universal cover  $\mathbb{R}^3$  and any tight transversal is homotopically essential (see [GO]).

For an essential lamination, we can define the leaf space  $L$  of  $\tilde{\Lambda}$ , but this is now just an order tree, not a 1-manifold; the vertices of  $L$  are the set of non-boundary leaves of  $\tilde{\Lambda}$  together with the set of closed complementary regions. If this leaf space is Hausdorff, we say  $\Lambda$  is *tight*. This is equivalent to the leaves of  $\tilde{\Lambda}$  being uniformly properly embedded in  $\tilde{M}$ . That is, there is a proper increasing function  $f : \mathbb{R}^+ \rightarrow \mathbb{R}^+$  such that if  $p$  and  $q$  are distance  $t$  apart in the path metric on  $\tilde{\Lambda}$ , then  $p$  and  $q$  are at least distance  $f(t)$  apart in  $\tilde{M}$ . When  $\Lambda$  is a foliation, tightness is equivalent to the leaf space being  $\mathbb{R}$ .

Regardless of whether  $\Lambda$  is tight, we have the following simple fact:

**2.4. Lemma.** *Let  $\Lambda$  be an essential lamination. Then there is an  $\epsilon > 0$  such that every leaf  $\lambda$  of  $\tilde{\Lambda}$  is quasi-isometrically embedded in its  $\epsilon$ -neighborhood. Such an  $\epsilon$  is called a separation constant for  $\Lambda$ .*

*Proof.* By compactness of  $M$ , we can pick an  $\epsilon$  so that every ball of radius  $2\epsilon$  in  $\tilde{\Lambda}$  is contained in a product chart. Here, by “product chart” we also require that every transverse arc without backtracking and with endpoints in  $\Lambda$  is tight, that is, can’t be homotoped rel endpoints into a leaf. Note that in  $\tilde{M}$  any leaf  $\lambda$  intersects a product chart at most once, as otherwise we can build a tight transversal loop to  $\tilde{\Lambda}$  by starting with a transversal in the chart and an arc in  $\lambda$ .

Now consider the  $\epsilon$ -neighborhood  $N_\epsilon$  of a leaf  $\lambda$  in  $\tilde{\Lambda}$ . We can cover the larger neighborhood  $N_{2\epsilon}$  by a product charts. As noted above,  $\lambda$  intersects each of these charts only once. Because  $M$  is compact, each of these charts has bounded geometry. Therefore,  $\lambda$  is quasi-isometrically embedded in  $N_\epsilon$ .  $\square$

**2.5. Complementary regions.** Let  $C$  be a complementary region to an essential lamination  $\Lambda$ . We will consider decompositions of  $C$  along a union of properly embedded annuli into two kinds of pieces, *guts* and *interstitial regions*. By definition, gut regions are compact and interstitial regions are I-bundles over (typically non-compact) surfaces. The annuli of the decomposition are called the *interstitial annuli*, and we require that no two of them are isotopic in a gut region. By convention, if  $C$  is itself an I-bundle over a closed surface, the only allowed decomposition is the trivial one with all of  $C$  being the interstitial region.

Such a partition of  $C$  into guts and interstices need not be unique: if  $R$  is an interstitial region  $R = \Sigma \times I$  over a surface  $\Sigma$ , and if  $\Sigma_0 \subset \Sigma$  is a compact subsurface

of  $\Sigma$  (possibly with boundary), then the subset  $\Sigma_0 \times I$  can be included into the gut. Conversely, an  $I$ -bundle over a compact surface which is contained in the gut can be included into the interstitial region. However, there exist several canonical (up to isotopy) partitions of  $C$  into guts and interstices. One such canonical partition has the property that every interstitial region is an  $I$ -bundle over a non-compact surface (see [GO] or [GK3]).

Often, it's useful to make the distinction between essential laminations and genuine ones:

**2.6. Definition.** *An essential lamination is genuine if some complementary region is not an  $I$ -bundle. Equivalently, there is a decomposition of the complementary regions with non-empty guts.*

An essential lamination which is not genuine can be filled in to a foliation by adding leaves in the complementary  $I$ -bundles.

One way to think of the qualities essential and genuine for a lamination is in terms of the proper essential surfaces (with boundary) contained in a complementary region. A lamination is essential if the complement admits no essential sphere, disk or monogon; that is, it contains no essential surface of positive Euler characteristic. An essential lamination is genuine if some complementary region contains an essential surface of *negative* Euler characteristic.

The following is an example of a genuine lamination which is a good example to think about throughout this paper.

**2.7. Example.** Let  $M$  be a surface bundle over the circle with fiber  $\Sigma$  and pseudo-Anosov monodromy  $\phi : \Sigma \rightarrow \Sigma$ . There are a pair of geodesic laminations of  $\Sigma$  invariant under  $\phi$ , the stable and unstable laminations  $\lambda^\pm$ . Since these are invariant, they suspend in the mapping torus  $M$  to a pair of laminations  $\Lambda^\pm$  of  $M$  transverse to the surface fibration. The complementary regions of the geodesic laminations  $\lambda^\pm$  of  $\Sigma$  suspend to complementary regions of  $\Lambda^\pm$ . The complementary regions in  $\Sigma$  are finite sided ideal polygons, each with at least 3 sides. These suspend to ideal polygon bundles over  $S^1$ . Such complementary regions decompose into compact neutered ideal polygon bundles over  $S^1$  (the guts) and cusps  $S^1 \times \mathbb{R}^+ \times I$  (the interstitial regions). In particular, these laminations are genuine. In  $\widetilde{M}$ , the pair  $(\widetilde{M}, \widetilde{\Lambda}^\pm)$  is a product  $(\widetilde{\Sigma}, \widetilde{\lambda}^\pm) \times \mathbb{R}$ . As  $\lambda^\pm$  is a geodesic lamination, the leaf space of  $\widetilde{\lambda}^\pm$  is Hausdorff. Thus the leaf space of  $\widetilde{\Lambda}^\pm$  is also Hausdorff, and  $\Lambda^\pm$  is tight.

**2.8. Tangential geometry.** The tangential geometry of an essential lamination is controlled by the following theorem of Candel [Can1], which can be thought of as a uniformization theorem for Riemann surface laminations:

**2.9. Theorem (Candel).** *Let  $\Lambda$  be a Riemann surface lamination such that for every transverse measure  $\mu$  we have:*

$$\chi(\mu) < 0.$$

*Then there is a continuously varying leafwise metric on  $\Lambda$  where the leaves are locally isometric to  $\mathbb{H}^2$ .*

A lamination satisfying the conclusion of Candel's Theorem is said to have *hyperbolic leaves*. For an essential lamination  $\Lambda$  of an atoroidal 3-manifold, every transverse measure has negative Euler characteristic, and so  $\Lambda$  has hyperbolic leaves.

Now we restrict attention to a taut foliation  $\mathcal{F}$ . Candel's Theorem is very useful in understanding the following bundle:

**2.10. Definition.** *Let  $\mathcal{F}$  be a taut foliation. The circle bundle at infinity,  $E_\infty$ , is the circle bundle over  $L$  whose fiber over a leaf  $\lambda$  is the circle  $S_\infty^1(\lambda)$ .*

While the bundle  $E_\infty$  is easy to understand as a set, its topology requires some discussion. We topologize it as follows. For each transversal  $\tau$  to  $\tilde{\mathcal{F}}$ , the projection of  $\tau$  to  $L$  is an embedding, so it makes sense to talk about the restriction  $E_\infty|_\tau$ . Consider the restriction to  $\tau$  of the unit tangent bundle of  $\mathcal{F}$ , denoted  $UT\tilde{\mathcal{F}}|_\tau$ . For a point  $p \in \lambda$ , every vector  $v \in UT_p\tilde{\mathcal{F}}$  determines a point  $e(v) \in S_\infty^1(\lambda)$  by taking the endpoint of the geodesic ray in  $\lambda$  starting at  $p$  in direction  $v$ . Thus there is a bijection

$$e_\tau: UT\tilde{\mathcal{F}}|_\tau \rightarrow E_\infty|_\tau.$$

We topologize  $E_\infty$  by declaring that this map is a homeomorphism for each  $\tau$ .

We verify that this is well-defined. Suppose  $\tau_1$  and  $\tau_2$  are two transversals with the same projection to  $L$ . For each leaf  $\lambda$ , the map

$$e_{\tau_2}^{-1} \circ e_{\tau_1}: UT_{\tau_1 \cap \lambda} \rightarrow UT_{\tau_2 \cap \lambda}$$

is determined by the geometry of a compact disk containing the points  $\tau_1 \cap \lambda$  and  $\tau_2 \cap \lambda$ . In particular, since the geometry of the leaves  $\lambda_t$  intersecting the  $\tau_i$  varies continuously *on compact subsets*, the map

$$e_{\tau_2}^{-1} \circ e_{\tau_1}: UT\tilde{\mathcal{F}}|_{\tau_1} \rightarrow UT\tilde{\mathcal{F}}|_{\tau_2}$$

is a homeomorphism. Thus the topology of  $E_\infty$  is well-defined.

This may seem like a tedious verification, but there actually is a subtle point here. This construction uses in an essential way the full power of Candel's Theorem. If we merely knew that the leaves of  $\tilde{\mathcal{F}}$  were uniformly coarsely quasi-isometric to  $\mathbb{H}^2$ , the map  $e_{\tau_2}^{-1} \circ e_{\tau_1}$  between unit tangent circles would depend on the *global* geometry of the leaf  $\lambda$ , and not merely on the local geometry. But in general, the leaves  $\lambda$  do not vary continuously as subsets of  $\tilde{M}$  in any reasonable sense. This is the very subtlety that makes Candel's theorem nontrivial. There is an alternate construction of the topology of  $E_\infty$  without invoking Candel's theorem, but it is more involved, and uses the Leaf Pocket Theorem 5.2.

### 3. TIGHT LAMINATIONS

In this section, we prove that certain kinds of tight genuine laminations give rise to faithful actions on the circle. Recall that a genuine lamination is very full if the complementary regions are all finite-sided ideal polygon bundles over  $S^1$ . Our basic result here is:

**3.1. Theorem.** *Let  $M$  be an orientable atoroidal 3-manifold containing a very full essential lamination  $\Lambda$  which is tight. Then there is a faithful representation*

$$\rho: \pi_1(M) \rightarrow \text{Homeo}^+(S^1).$$

Moreover, the image of  $\rho$  preserves a dense lamination of  $S^1$ .

Here, a lamination  $\lambda$  of  $S^1$  is essentially just a geodesic lamination of  $\mathbb{H}^2$ . Intrinsically,  $\lambda$  is a closed, symmetric subset of  $\{S^1 \times S^1 - \text{diagonal}\}$  so that no two pairs of points  $(p_1, p_2)$  and  $(q_1, q_2)$  in  $\lambda$  are linked on  $S^1$ .

The hypothesis of very full is much stronger than just having solid torus guts; the former requires in addition that the interstitial regions have as little topology as possible. An typical example of a lamination with solid torus guts which is not very full is depicted in Figure 4.3. However, the Filling Lemma 4.1 shows that a lamination with solid torus guts can be included as a sublamination of a very full lamination. Moreover, this process preserves tightness. So as an immediate consequence of the Filling Lemma and Theorem 3.1 we have:

**3.2. Theorem.** *Let  $M$  be an orientable atoroidal 3-manifold, and  $\Lambda$  a tight lamination with solid torus guts. Then  $\pi_1(M)$  has a faithful representation into  $\text{Homeo}^+(S^1)$ .*

While the restriction of solid torus guts might initially seem quite strong, in fact many constructions of essential laminations yield examples of this type. Moreover, for non-Haken manifolds, the gut regions are always handlebodies of some sort [Bri1].

Now let's prove Theorem 3.1.

*Proof of Theorem 3.1.* For any essential lamination  $\Lambda$ , the universal cover  $\tilde{M}$  is topologically  $\mathbb{R}^3$ , and the lifted lamination  $\tilde{\Lambda}$  is topologically a product of a lamination  $\lambda$  of  $\mathbb{R}^2$  with  $\mathbb{R}$  [GK1]. Moreover, the leaves of  $\lambda$  are proper. Thus  $\lambda$  is quite close to being a geodesic lamination of  $\mathbb{H}^2$ . As our  $\Lambda$  is very full, the complementary regions of  $\lambda$  are all finite-sided ideal polygons. In essence, we'll construct an action of  $\pi_1(M)$  on  $(\mathbb{R}^2, \lambda)$  so that the map  $(\mathbb{R}^3, \tilde{\Lambda}) \rightarrow (\mathbb{R}^2, \lambda)$  is equivariant, and from this we'll get the required action on a circle. As  $\Lambda$  is very full, the complementary regions of  $\tilde{\Lambda}$  are all products  $P_i \times \mathbb{R}$  for finite sided ideal polygons  $P_i$ . When turning  $(\mathbb{R}^3, \tilde{\Lambda})$  into a product, there are essentially two possible ways of flattening each  $P_i \times \mathbb{R}$  into a complementary region of  $(\mathbb{R}^2, \lambda)$ . The key to constructing the desired action on  $S^1$  is to do this flattening in a consistent, equivariant way.

So let's begin the actual construction, which involves fitting the  $P_i$  together as ideal polygons in the unit disk. First, downstairs in  $M$ , choose an orientation on the core curve of each complementary region of  $\Lambda$ . Lifting upstairs, this defines a (dual) orientation on each  $P_i$  which is preserved by the action of  $\pi_1(M)$ . As with any essential lamination, there is a natural leaf space  $L$  associated to  $\tilde{\Lambda}$ . The leaf space  $L$  is an order tree associated to  $\tilde{\Lambda}$  whose vertices are the set of non-boundary leaves of  $\tilde{\Lambda}$  together with the set of closed complementary regions. This order tree can be canonically included in an  $\mathbb{R}$ -order tree (see e.g. [GK2]); this makes the explanation of the construction easier to follow, so we suppose this has been done. As  $\Lambda$  is tight,  $L$  is Hausdorff.

We will insert the  $P_i$  as ideal polygons in the unit disk  $D$  via a map  $\pi : P_i \rightarrow D$  which satisfies the following conditions:

1. If  $i \neq j$ , then the closed polygons  $\pi(P_i)$  and  $\pi(P_j)$  are disjoint.

2. The circular ordering on the vertices of  $\pi(P_i)$  induced by  $\partial D$  agrees with that arising from the orientation on  $P_i$ .
3. Suppose  $P_i$  is joined to  $P_j$  by an embedded arc in  $L$  joining a side  $e_i$  of  $P_i$  to a side  $e_j$  of  $P_j$ . Then the corresponding edges  $\pi(e_i)$  and  $\pi(e_j)$  separate the interiors of  $\pi(P_i)$  and  $\pi(P_j)$  in  $D$ .

Here's how to construct  $\pi$ . Because the  $\mathbb{R}$ -order tree  $L$  is Hausdorff, it is the union of a countable number of finite simplicial trees  $L_j$  [GK2, §3]. Here,  $L_j$  is subdivided and extended to get  $L_{j+1}$ . Consider  $L_j$  as a simplicial complex with no edges of valence two. Each interior vertex of  $L_j$  corresponds to some complementary region of  $\tilde{\Lambda}$ , and thus to some  $P_i$ . The edges of  $L_j$  coming into a vertex correspond to the edges of the associated  $P_i$ . Thus, the orientation on  $P_i$  induces a cyclic ordering of the edges around each vertex of  $L_j$ . There is a unique embedding  $\pi$  of  $L_j$  into the interior of  $D$  which respects these cyclic orderings (i.e. at each vertex  $v$  of  $L_j$  the ordering of edges agrees with the clockwise ordering of the image edges about  $\pi(v)$ ). For each vertex  $v$  in  $L_j$ , place the corresponding  $P_i$  as an ideal polygon in  $D$  so that it contains  $\pi(v)$ . The resulting polygons satisfy (1-3), and moreover their placement was unique up to homeomorphism of  $D$ . Inducting up the  $L_j$  gives us the required map  $\pi$ .

Now, the closure

$$K = \overline{\bigcup_i \pi(P_i)}$$

might not be all of  $D$ , but the components of  $D - K$  are easy to understand. Any component  $C$  of  $D - K$  is a polygon whose sides alternate between arcs of  $\partial D$  and geodesics in  $D$  which are limits of edges in  $\pi(\cup_i \partial P_i)$ . Moreover, we claim  $C$  must be a quadrilateral. To see this, pick one  $P_i$  in each component of  $D - C$  and join them by a finite simplicial tree  $L'$  in  $L$ . Embedding  $L'$  in  $D$ , we see that if there were more than two components to  $D - C$ , then  $L'$  would have to have a vertex in  $C$ .

If we quotient out  $\partial D$  by collapsing the arcs in  $\partial D - K$  to points, we get a circle because  $\tilde{\Lambda}$  has no isolated leaves. Let  $\phi : \partial D \rightarrow S^1$  be this monotone quotient map. The closure of the edges of the  $\pi(\partial P_i)$  is a geodesic lamination  $\lambda_K$  of  $D$ . The map  $\phi$  gives a map on pairs of endpoints of geodesics of  $\lambda_K$  which preserves unlinking, so there is a well-defined image lamination  $\lambda = \phi_*(\lambda_K)$  in  $S^1$ . The endpoints of leaves of  $\lambda$  are dense in  $S^1$ . The action of  $\pi_1(M)$  on  $\bigcup_i P_i$  induces an action on  $\lambda$ . Going back to our original construction of  $\pi$ , it is not hard to see that this extends to a continuous action on  $S^1$  which preserves orientation.

To finish the proof, we need show this action is faithful. Consider a nontrivial element  $\alpha \in \pi_1(M)$  which acts trivially on  $S^1$ . Then the action of  $\alpha$  on  $\tilde{M}$  would stabilize every complementary region. So  $\alpha$  defines the core curve of every complementary region. It follows that  $\alpha$  is central in  $\pi_1(M)$ , and that  $\pi_1(M)$  contains a  $\mathbb{Z} + \mathbb{Z}$ . This violates our assumption that  $M$  is atoroidal. Thus we have constructed the required faithful action on a circle.  $\square$

**3.3. Flips.** If there are  $n$  complementary regions to  $\Lambda$ , the above construction gives  $2^n$  representations which are pairwise non-conjugate in  $\text{Homeo}^+(S^1)$  (they fall into only  $2^{n-1}$  conjugacy classes in the full group  $\text{Homeo}(S^1)$ ).

Abstractly, this reflects an interesting operation on representations of groups

$$\sigma : G \rightarrow \text{Homeo}^+(S^1)$$

whose image preserves a lamination  $\lambda$  of  $S^1$ . Given an orbit class of complementary regions to  $\lambda$ , we get a new representation as follows. Let  $P$  be a complementary region of  $\lambda$ . We can “cut”  $D$  along the edges of each translate  $\sigma(\gamma)(P)$  into pieces which are  $\sigma(\gamma)(P)$  itself, and the pieces on the other sides of the boundary edges of  $\sigma(\gamma)(P)$ . Then we can reverse the orientation of  $\sigma(\gamma)(P)$  and glue the other pieces back so that each pair of sides is glued in the same way as before, but the relative orientations are now switched. Do this in some order for each  $\gamma \in G$ , to get a new lamination  $\lambda^P$  on which  $G$  acts. This action extends in the obvious way to a representation  $\sigma^P : G \rightarrow \text{Homeo}^+(S^1)$ . Note that  $(\sigma^P)^P = \sigma$  and  $(\lambda^P)^P = \lambda$  for any  $P$ . We call this operation *the flip of  $\sigma$  along  $P$* .

If  $\lambda$  only has one orbit class of complementary region, this operation merely reverses the orientation of  $S^1$ , but in general it seems somewhat mysterious. It might be interesting to analyze this operation for some familiar  $G$ . For instance, one could look at  $G = \pi_1(M)$  for  $M$  a hyperbolic 3-manifold fibering over  $\Sigma$ , and  $\lambda$  the lift of the stable lamination of the monodromy of the fibering.

The moral of these examples is that if a group  $G$  acts on a ( $\mathbb{R}$ -, order-) tree  $T$  and there is at least one way of embedding  $T$  in the plane in a  $G$ -equivariant way, there are usually many such ways.

In the sequel, we will be interested in obtaining representations of  $G = \pi_1(M)$  in  $\text{Homeo}(S^1)$  with Euler class  $e \in H^2(G)$  equal to 0 (see Section 7). We suspect the flip operation might be a useful tool in this regard. It is not hard to see that if  $\lambda$  has  $n$  equivalence classes of complementary polygons, there is an expression of  $e$  as a sum

$$e = \sum_{i=1}^n e_i$$

such that the Euler class of  $\sigma^{P_j}$  is

$$e_{P_j} = \left( \sum_{i=1}^n e_i \right) - 2e_j.$$

**3.4. Laminations and cataclysms.** The rest of this section is devoted to generalizations of Theorem 3.2 where we partially relax the requirement that the lamination be tight. To start, we need to get a handle on what a non-tight lamination looks like.

If  $\Lambda$  is not tight, by definition the leaf space  $L$  of  $\tilde{\Lambda}$  is a non-Hausdorff order tree. Recall that two leaves  $\mu$  and  $\lambda$  in  $\tilde{\Lambda}$  are comparable if they can be jointed by a tight transversal, and incomparable otherwise. A sequence  $\{\mu_i\}$  of leaves in  $L$  is *monotone ordered* if all of the  $\{\mu_i\}$  lie in an ordered segment  $I \subset L$ , so that the  $\mu_i$  form an increasing sequence in  $I$ . We define a *cataclysm* to be a collection of incomparable leaves  $\{\lambda_j\}$  of  $\tilde{\Lambda}$ , called the *limit leaves*, for which there is a monotone ordered sequence  $\mu_i$ , called the *approximating sequence*, which converges on compact subsets in the Hausdorff topology to the union of the  $\lambda_j$ ; that is, for all compact subsets

$K \subset \widetilde{M}$ ,

$$\lim_{i \rightarrow \infty} (\mu_i \cap K) = \left( \bigcup_j \lambda_j \right) \cap K$$

in the Hausdorff topology on closed subsets of  $K$ .

For any sequence  $p_i \in \mu_i$  which converges to some  $p \in \lambda_j$ , the sequence of pointed metric spaces  $(\mu_i, p)$  converges to  $(\lambda_j, p)$ . In particular, if  $p_i$  and  $q_i$  are two sequences with  $p_i, q_i \in \mu_i$  where  $p_i \rightarrow p \in \lambda_j$  and  $q_i \rightarrow q \in \lambda_k$  for  $\lambda_j \neq \lambda_k$ , then the leafwise distance from  $p_i$  to  $q_i$  in  $\mu_i$  goes to infinity.

We define an equivalence relation on cataclysms as follows: two cataclysms  $C = (\{\lambda_j\}, \{\mu_i\})$  and  $C' = (\{\lambda'_j\}, \{\mu'_i\})$  are *equivalent* if  $\{\lambda_j\}$  and  $\{\lambda'_j\}$  are equal as *sets*, and if for some  $N$ , the union

$$\{\mu_j\}_{j \geq N} \cup \{\mu'_j\}_{j' \geq N}$$

can be totally ordered in  $L$ .

The equivalence class of a cataclysm is determined by the collection  $\{\lambda_j\}$  of leaves, since for any  $p \in \lambda_j$  and any  $p_i \rightarrow p$  contained in  $\widetilde{\Lambda}$  on the side of  $\lambda_i$  which contains the other  $\lambda_j$ , the leaves  $\mu_i$  with  $p_i \in \mu_i$  are an approximating sequence, unique up to equivalence. It follows that the stabilizer of a cataclysm acts by permutations on the set of limit leaves, and by isometry on their union, thought of as a subset of  $\widetilde{M}$ .

The following condition is natural for trying to build circle actions:

**3.5. Definition.** *A lamination  $\Lambda$  has orderable cataclysms if for each equivalence class of cataclysm  $[C]$ , there is an ordering on the set of limit leaves  $\{\lambda_i\}$  of  $[C]$  which is invariant under the action of the stabilizer of  $[C]$  in  $\pi_1(M)$ .*

We will show that orderable cataclysms can replace tight as a hypothesis in Theorem 3.2, but let's first give an example of this condition.

**3.6. Example.** Let  $X$  be a pseudo-Anosov flow, and denote the two dimensional stable and unstable singular foliations  $\mathcal{F}^s$  and  $\mathcal{F}^u$ . For ease of notation, we will focus on  $\mathcal{F}^u$  but of course similar statements are true for  $\mathcal{F}^s$ . The foliation  $\mathcal{F}^u$  lifts to a singular foliation  $\widetilde{\mathcal{F}}^u$  of  $\widetilde{M}$  invariant under the lifted flow  $\widetilde{X}$ . Each nonsingular leaf of  $\widetilde{\mathcal{F}}^u$  is topologically a plane foliated by flow lines of  $\widetilde{X}$ . If  $\lambda$  is a nonsingular leaf of  $\widetilde{\mathcal{F}}^u$ , the flow along  $\lambda$  by  $\widetilde{X}$  compresses the flowlines together. Thus the holonomy along this foliation is defined in the positive direction for all time. In particular, it is easy to see from this that  $\lambda$  is isometric to  $\mathbb{H}^2$ . The foliation  $\lambda \cap \widetilde{X}$  is a foliation of  $\mathbb{H}^2$  by geodesics asymptotic to a unique point in  $S_\infty^1$ . A singular leaf  $\lambda$  of  $\widetilde{\mathcal{F}}^u$  has the following structure. Let  $h_1, \dots, h_n$  be copies of  $\mathbb{H}^2$  foliated by geodesics asymptotic to  $p_i \in S_\infty^1(h_i)$ , and let  $\gamma_i \subset h_i$  be one of these geodesics. Then  $\gamma_i$  separates  $h_i$  into two sides,  $h_i^\pm$ . We obtain a quotient space

$$l = \bigcup_i h_i$$

by gluing  $h_i^+$  to  $h_{i+1}^-$  via an isometry taking  $\gamma_i$  to  $\gamma_{i+1}$ , where the indices  $i$  are taken modulo  $n$ . Then  $\lambda = l$ , and the  $h_i$  are called the *faces* of the singular leaf  $\lambda$ . The image of the  $\gamma_i$  in  $l$  is a singular orbit of  $\widetilde{X}$  and covers a circular singular orbit in  $M$ .

The singular foliation  $\mathcal{F}^u$  can be split open into a lamination  $\Lambda^u$  whose leaves are exactly the nonsingular leaves of the  $\tilde{\mathcal{F}}^u$  together with one leaf for every face of a singular leaf. If  $X$  is pseudo-Anosov but not Anosov, then  $\Lambda^u$  is a very full genuine lamination, and if  $X$  is Anosov then  $\Lambda^u = \mathcal{F}^u$ . The foliation of the leaves of  $\mathcal{F}^u$  by flow lines of  $X$  gives rise to foliations of the leaves of  $\Lambda^u$ . Consider a cataclysm  $C$  with limit leaves  $\lambda_i$  and approximating sequence  $\mu_j$ . The foliations of the  $\mu_j$  by flowlines of  $\tilde{X}$  converge on compact sets to the foliations of the  $\lambda_i$ . So if we pick points  $p_{ij} \in \mu_j$  so that  $p_{ij}$  is very close to  $\lambda_i$ , there is a natural order on the  $p_{ij}$  coming from the order on the leaf space of the  $\tilde{X}$  foliation of  $\mu_j$ , which is  $\mathbb{R}$ . This determines an order on the indices  $i$ . As  $j \rightarrow \infty$ , the order on the indices  $i$  is eventually constant, and therefore determines a natural order on the set  $\{\lambda_i\}$ , which is invariant under the stabilizer in  $\pi_1(M)$  of  $[C]$ . Thus  $\Lambda^u$  has orderable cataclysms.

This natural order structure on the cataclysms of pseudo-Anosov flows was observed by Fenley in [Fen], where he gives examples of pseudo-Anosov flows where  $\Lambda^u$  is not tight.

**3.7. Example.** Not all cataclysms are orderable. Here is a simple example. Start with a tight lamination  $\Lambda$  with a complementary region which is an ideal square bundle over the circle where the monodromy is the rotation through angle  $\pi$ . Then fill in that region with a saddle bundle over  $S^1$ . In the universal cover, the bundle of saddles lifts to  $\mathbb{R}$  leaves which limit in either direction to a pair of distinct leaves corresponding to opposite sides of the ideal square. Either end gives a cataclysm with two limit leaves. The stabilizer of the cataclysm corresponds to the monodromy of the core of the complementary region downstairs; in particular, it interchanges these two limit leaves, and this cataclysm is not orderable.

Now we'll show:

**3.8. Theorem.** *Let  $M$  be an orientable atoroidal 3-manifold containing a lamination  $\Lambda$  with solid torus guts and orderable cataclysms. Then  $\pi_1(M)$  has a faithful representation in  $\text{Homeo}^+(S^1)$ .*

*Proof.* By the Filling Lemma 4.1,  $\Lambda$  can be filled to a very full lamination. The proof that filling preserves tightness further shows that filling preserves the set of equivalence classes of cataclysms, and that no added leaf is a limit leaf of a cataclysm. In particular, the filled lamination has orderable cataclysms. So from now on, we'll assume that  $\Lambda$  is very full.

The proof is now morally the same as for Theorem 3.1, but this time it's easier to do things a little more abstractly. A circular order on a set  $S$  is an assignment to each triple of distinct elements  $(s_1, s_2, s_3)$  an orientation of clockwise or anti-clockwise, satisfying the obvious rules coming from triples of points on  $S^1$ . Here, we will show that the set of ends  $E$  of  $L$  has a natural circular order which is invariant under the action of  $\pi_1(M)$ . We will complete  $E$  with respect to this circular order to get the needed circle. (For a detailed definitions and basic properties of circular orders on sets and left-invariant circular orders on sets acted on by a group, see e.g. [Tar].)

By definition, a circular order on an order tree  $L$  consists of two things: a circular order on sets of segments with only a vertex in common, and a linear order on sets of

segments which differ only by a vertex (for details see [GK2]). For the leaf space  $L$  of our lamination  $\tilde{\Lambda}$ , we can define a circular order which is invariant under the action of  $\pi_1(M)$  as follows. A set of segments with only a vertex in common corresponds to the set of faces of a closed complementary region  $C = P_i \times \mathbb{R}$  of  $\tilde{\Lambda}$ . As before, if we fix orientations of the core curves of the complementary regions  $M - \Lambda$ , the faces of  $C$  have a natural circular order. A set of segments which differ only in a vertex corresponds to a cataclysm, and so has an order by assumption. In the cataclysm case, we can certainly choose our orderings to be  $\pi_1(M)$ -equivariant.

Thus  $L$  is an order tree with a  $\pi_1(M)$ -invariant circular order. Let  $E$  be the set of ends of  $L$ . If  $(e_1, e_2, e_3)$  are three distinct ends in  $E$ , we define their circular order as follows. Take any point  $p$  in  $L$  and consider rays  $r_i$  starting at  $p$  and ending at  $e_i$ . By looking at the (circularly ordered) subtree  $T = \cup r_i$ , we get a circular order of the ends of the  $r_i$ . This order is independent of the choice of point  $p$ , and so we have a natural circular order on  $E$  which is invariant under the action of  $\pi_1(M)$ . Up to homeomorphism, there is a unique way to embed  $E$  into a circle  $S^1$  so that the embedding respects the circular orderings and is continuous with respect to the topology on  $E$  induced by the order. The closure of the image of  $E$  may omit some gaps, which we can collapse to get the promised circle on which  $\pi_1(M)$  acts. As in the proof of Theorem 3.1, if some element  $\alpha \in \pi_1(M)$  acts trivially on this circle, it must fix every complementary domain; then  $\pi_1(M)$  has a nontrivial center, violating the assumption that  $M$  is atoroidal.  $\square$

**3.9. Corollary.** *Let  $M$  be an atoroidal 3-manifold which admits a pseudo-Anosov flow  $X$ . Then  $\pi_1(M)$  admits a faithful representation in  $\text{Homeo}(S^1)$ .*

*Proof.* If  $X$  is pseudo-Anosov but not Anosov, the singular unstable foliation can be split open to a very full genuine lamination. As explained in Example 3.6, these laminations have orderable cataclysms and so Theorem 3.8 applies. If  $X$  is Anosov, the unstable foliation is a taut foliation, and the circle action comes from Theorem 6.3.  $\square$

**3.10. Invariant laminations for cataclysms.** For the sake of completeness, we give a more precise description of the non-Hausdorff behavior of the leaf space at a cataclysm. The following theorem shows that the set of limit leaves of a cataclysm can be parameterized by a geodesic lamination of  $\mathbb{H}^2$  in a (topologically) canonical way.

**3.11. Theorem.** *Let  $\Lambda$  be an essential lamination, and let  $C$  be a cataclysm of  $\Lambda$ . Then we can associate a lamination  $L$  of  $\mathbb{H}^2$  to  $C$  in such a way that the complementary regions to  $L$  are in 1-1 correspondence with the limit leaves of  $C$ , and the stabilizer  $\Gamma \subset \pi_1(M)$  of  $[C]$  acts by homeomorphisms of  $\mathbb{H}^2$  which preserve  $L$ , and permute the complementary regions by the permutation action on the limit leaves of  $C$ .*

*Proof.* For each limit leaf  $\lambda_i$ , pick some point  $p_i$ , and for each approximating leaf  $\mu_j$  pick  $p_{ij}$  such that the  $p_{ij}$  limit to  $p_i$  as  $j \rightarrow \infty$ , for each  $i$ . Pick some small  $\epsilon$ , and consider for each limit leaf  $\lambda_i$  and each approximating leaf  $\mu_j$  the subspace  $\mu_j(i) \subset \mu_j$  consisting of points which are within  $\epsilon$  of  $\lambda_i$ . Then for sufficiently small  $\epsilon$ , the sets

$\mu_j(i)$  and  $\mu_j(k)$  are disjoint for  $i \neq k$ , since for any lamination with bounded geometry, leaves are uniformly properly embedded in their  $\epsilon$  neighborhoods for some  $\epsilon$ , and therefore pairs of points on incomparable leaves of  $\tilde{\Lambda}$  are a uniform distance apart. On the other hand, by the Leaf Pocket Theorem 5.2, for large  $j$ ,  $\mu_j(i)$  contains a  $\delta$ -net of points in the circle at infinity  $S^1_\infty(\mu_j)$  in the visual metric as seen from  $p_{ij}$ . In particular, the boundary of the convex hull of the limit set of  $\mu_j(i)$  defines a collection of geodesics which separates most of  $\mu_j(i)$  from  $\mu_j(k)$  for  $i \neq k$  in some definite order; the closure of the union of such geodesics is a geodesic lamination  $L_j$  of  $\mu_j$ . This lamination is dual to a planar order tree  $T_j$  whose vertices are the  $i$  for which  $p_{ij}$  is sufficiently close to  $\lambda_i$ . In particular, there are a sequence of inclusions  $T_j \rightarrow T_{j+1} \rightarrow \dots$  and the union  $T_\infty$  is a Hausdorff planar order tree dual to a lamination  $L$  of  $\mathbb{H}^2$ , which is a limit of the  $L_j$  under an appropriate sequence of homeomorphisms (*not* isometries) from  $\mu_j \rightarrow \mathbb{H}^2$ . It is clear that  $L$  does not depend on the choice of approximating leaves, since if we insert some  $\mu_{j+1/2}$  between  $\mu_j$  and  $\mu_{j+1}$  then there are inclusions  $T_j \rightarrow T_{j+1/2} \rightarrow T_{j+1} \rightarrow \dots$  and the limit is unchanged. In particular,  $L$  depends only on  $[C]$ , and therefore the stabilizer of  $[C]$  acts on it by automorphisms.  $\square$

#### 4. THE FILLING LEMMA

This section is devoted to the proof of the following lemma, which is of independent interest. It resolves the disparity between having solid torus guts and solid torus complementary regions, and is used in Section 3 to reduce the construction of actions on circles to the latter case.

**4.1. Filling Lemma.** *Let  $M$  be an orientable 3-manifold and  $\Lambda$  a genuine lamination such that for some decomposition of  $M - \Lambda$  into interstices and guts, the guts are neutered ideal polygon bundles over  $S^1$ . Then  $\Lambda$  can be filled to a genuine lamination  $\bar{\Lambda} \supset \Lambda$ , whose complementary regions are all ideal polygon bundles over  $S^1$ . That is,  $\bar{\Lambda}$  is very full. Moreover, if  $\Lambda$  is tight, so is  $\bar{\Lambda}$ .*

Before giving the proof, let us point out another application. Gabai and Kazez have shown that if an atoroidal manifold has a genuine lamination with some complementary region a solid torus, then homotopic homeomorphisms are isotopic [GK1]. The Filling Lemma extends that result to manifolds with genuine laminations where there is a decomposition of  $M - \Lambda$  with a solid torus gut region. Now we'll prove the Filling Lemma.

*Proof.* First, fill all product complementary regions with foliation. Now, each complementary region  $C$  of  $\Lambda$  is a union of gut pieces  $G_i$  together with a finite collection of interval bundles  $J_k$  (the bases of the  $J_k$  need not be compact). The  $J_k$  are glued to the  $G_i$  along the interstitial annuli. The first step of the proof is to add leaves to reduce to the case where each  $J_k$  is a trivial I-bundle over a surface with a single boundary component. Then, each  $J_k$  is attached to a single  $G_i$  along one interstitial annulus, and each  $C$  contains exactly one gut region. After this basic reduction, we'll add on a product foliation to the boundary leaves of  $C$ , and perturb it so that the new  $C$  is an ideal polygon bundle over  $S^1$ .

4.2. **Basic reduction.** First, we'll insert finitely many leaves so that each I-bundle  $J_k$  has only one interstitial annulus in its boundary. Let  $J_k$  be an I-bundle over a base with at least two boundary components. Pick an interstitial annulus  $A$  bounding  $J_k$ , joining it to a gut region  $G_i$ . Let  $\lambda$  and  $\mu$  be the leaves of  $\Lambda$  that  $A$  runs between.

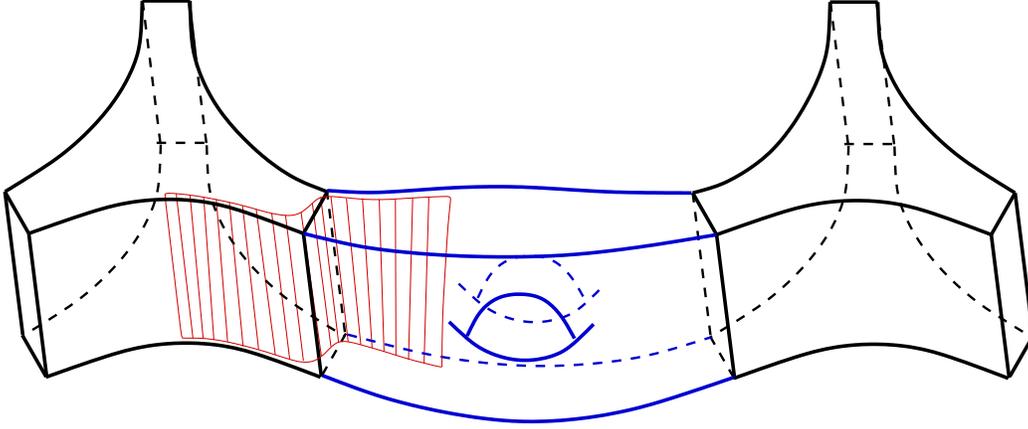


FIGURE 4.3. A  $J_k$  with two boundary components joins two gut regions  $G_i$  and  $G_j$ . The interstitial annulus  $A$  we're considering is one on the left. Also shown is part of the new leaf  $v$  which we're adding. The leaf  $v$  is carried by the branched surface  $B$ .

Build a branched surface  $B$  from  $A$ ,  $\mu$ , and  $\lambda$ , bending  $A$  as shown in Figure 4.3. The leaf we add will be carried by  $B$ , and will consist of one copy of  $A$  glued to stuff parallel to  $\mu$  and  $\lambda$ . Explicitly, take countably many copies of the two leaves  $\lambda$  and  $\mu$  in a product neighborhood, which accumulate only along  $\lambda$  and  $\mu$ . Take the union of this with  $A$  to get a two-complex which is singular along countably many circles accumulating to the two boundary circles of  $A$ . Construct a surface carried by  $B$  by performing a normal sum operation on each singular circle using a consistent orientation. Let  $v$  be the component of this new surface which contains most of  $A$ . Add  $v$  to  $\Lambda$  to get  $\Lambda'$ . To decompose the complement of  $\Lambda'$  into I-bundles and guts, use essentially the same interstitial annuli as for  $\Lambda$ . In the complement of  $\Lambda'$ , the I-bundle  $J_k$  has been split into a  $J'_k$  with one fewer boundary components, and another I-bundle with one boundary component (consult Figure 4.3). Doing this process at most once for each of our original interstitial annuli gives us a new  $\Lambda$  where each  $J_k$  has only one boundary component.

Now, we'll add finitely many more leaves so that each  $J_k$  is a trivial I-bundle. Suppose some I-bundle  $J_1$  is nontrivial, and let  $G$  be the gut region it's attached to via the interstitial annulus  $A_1$ . Let  $\Sigma_1$  be a copy of the base of  $J_1$  sitting in the middle of  $J_1$ . Let  $J_2$  be an I-bundle attached to  $G$  along  $A_2$ . Choose a horizontal surface  $\Sigma_2$  in  $J_2$  which has exactly one boundary component on  $A_2$ . Let  $v$  be the surface created by gluing  $\Sigma_1$  and  $\Sigma_2$  together via an annulus in  $G$ . If we pick  $J_2$  so that  $A_1$  and  $A_2$  are adjacent on  $\partial G$ , the new lamination  $\Lambda' = \Lambda \cup v$  has essentially the same guts as  $\Lambda$ . If  $J_2$  was also twisted, adding  $v$  creates a new  $J'_k$  with two boundary components, so in this case we add a second leaf as in the previous step so that all I-bundles still have

a single boundary component. In any event, there are now fewer twisted I-bundles  $J'_k$  in the complement of  $\Lambda'$ . Thus, after adding finitely many leaves, we can make all of the  $J_k$  trivial I-bundles over surfaces with one boundary component.

**4.4. Main argument.** Consider one of our complementary regions  $C$ . We need to fill  $\Lambda$  so that  $C$  becomes an ideal polygon bundle over  $S^1$ . As a rough outline, we'll begin by foliating each interstitial region  $J_k$  by planes and annuli in a very standard way. Here, by "plane" we really mean a disc with a part of its boundary removed, and similarly for annuli. The induced foliation of each interstitial annulus will consist of a countable collection of circles, and a bunch of lines spiraling out to the circle leaves. Each annulus leaf has a single circle boundary component and many line boundary components. We'll split open the foliation of each  $J_k$  along one of the annulus leaves and glue these foliations together across the gut  $G$  to enlarge our original lamination  $\Lambda$ . If we glue with care, the new leaves are all either annuli or planes. The new interstitial I-bundles will have bases which are annuli of the form  $S^1 \times [0, 1)$ , and so the complementary regions are all ideal polygon bundle over  $S^1$ .

We'll begin by constructing the foliation of each interstitial region. Let  $J$  be an interstitial region  $\Sigma \times I$ , where  $\Sigma$  is has exactly one boundary loop  $\gamma$ . Let  $A = \gamma \times I$ . We will foliate  $J$  by choosing some representation

$$\tilde{\sigma}: \pi_1(\Sigma) \rightarrow \text{Homeo}(I)$$

so that the induced foliation of  $J$  as a foliated bundle has the above properties. We will construct  $\tilde{\sigma}$  from a representation  $\sigma: \pi_1(\Sigma) \rightarrow \widetilde{\text{PSL}}_2\mathbb{R}$  by lifting  $\sigma$  to  $\widetilde{\text{PSL}}_2\mathbb{R}$ . The group  $\widetilde{\text{PSL}}_2\mathbb{R} \subset \widetilde{\text{Homeo}}(S^1)$  acts on  $\mathbb{R}$ , and we can identify  $\text{Homeo}(\mathbb{R})$  with  $\text{Homeo}(I)$ . We'll want the representation  $\sigma$  to be faithful, and also that the non-trivial images are hyperbolic elements. Moreover, we'll need that in the lift

$$\tilde{\sigma}: \pi_1(\Sigma) \rightarrow \widetilde{\text{PSL}}_2\mathbb{R}$$

the rotation number of  $\gamma$  is zero. This way,  $\tilde{\sigma}(\gamma)$  will have countably many fixed points, and will alternate between being increasing and decreasing on the complementary intervals.

Here's why such a  $\sigma$  exists. The group  $\pi_1(\Sigma)$  is a free group, possibly on infinitely many generators. First, consider the case where  $\pi_1(\Sigma)$  is finitely generated. Note that the set of faithful representations is dense in the space of all  $\text{PSL}_2\mathbb{R}$  representations; this is because it is the complement of

$$\bigcup_{g \in \pi_1(\Sigma)} \{\rho: \pi_1(\Sigma) \rightarrow \text{PSL}_2\mathbb{R} \mid \rho(g) = I\}$$

which is a countable union of proper algebraic subvarieties. For the trivial representation  $\tau$ , if we look at the trivial lift  $\tilde{\tau}$  then the rotation number of  $\tilde{\tau}(\gamma)$  is 0. For  $\sigma$  near  $\tau$  for which  $\sigma(\gamma)$  is hyperbolic, the rotation number of  $\tilde{\sigma}(\gamma)$  near  $\tilde{\tau}(\gamma)$  is an integer arbitrarily close to 0. Thus it is 0, and we can construct the needed  $\sigma$ . Moreover, by taking a generic representation  $\tilde{\sigma}$ , we can also require that the stabilizer of all but a countable set of points in  $\mathbb{R}$  will be trivial, and the stabilizer of each element of that countable set will be isomorphic to  $\mathbb{Z}$ .

Now suppose  $\pi_1(\Sigma)$  is infinitely generated. Put a hyperbolic structure on  $\Sigma$  with geodesic boundary. This gives a faithful representation  $\sigma: \pi_1(\Sigma) \rightarrow \mathrm{PSL}_2\mathbb{R}$  where the image of the boundary loop  $\gamma$  is hyperbolic. As  $\pi_1(\Sigma)$  is free, there is a some lift of  $\tilde{\sigma}$  to  $\widetilde{\mathrm{PSL}_2\mathbb{R}}$ . While  $\tilde{\sigma}(\gamma)$  may not have rotation number 0, this time  $\gamma$  is a primitive element of  $H_1(\Sigma)$ ; it is easy to construct a homomorphism  $\tilde{\rho}$  from  $\pi_1(\Sigma)$  to the center of  $\widetilde{\mathrm{PSL}_2\mathbb{R}}$  so that the rotation number of  $\tilde{\rho}(\gamma)$  is minus that of  $\tilde{\sigma}(\gamma)$ . Then  $\tilde{\sigma}': g \mapsto \tilde{\rho}(g)\tilde{\sigma}(g)$  is the required homomorphism. Note that  $\tilde{\sigma}'$  inherits from  $\sigma$  the property that only a countable number of points have non-trivial stabilizer, and that when the stabilizer is non-trivial it is  $\mathbb{Z}$ .

By viewing  $\mathrm{PSL}_2\mathbb{R}$  as a subgroup of  $\mathrm{Homeo}(\mathbb{R}) \cong \mathrm{Homeo}(I)$ , we foliate  $J$  by  $\mathcal{F}$  as a flat foliated bundle with holonomy group  $\tilde{\sigma}(\pi_1(\Sigma))$ . As desired, a countable set of leaves in the interior of  $J$  will be cylinders, and all the rest will be planes. The restriction of the foliation  $\mathcal{F}$  to the annulus  $A$  is a 1-dimensional foliation  $\mathcal{G}$  which is a flat foliated bundle over  $\gamma = \partial\Sigma$  with holonomy generated by  $\tilde{\sigma}(\gamma)$  (see Figure 4.5). The points  $t \in I$  parameterize the leaves  $\lambda^t$  of  $\mathcal{F}$  and the leaves  $\mu^t$  of  $\mathcal{G}$ . Of course,

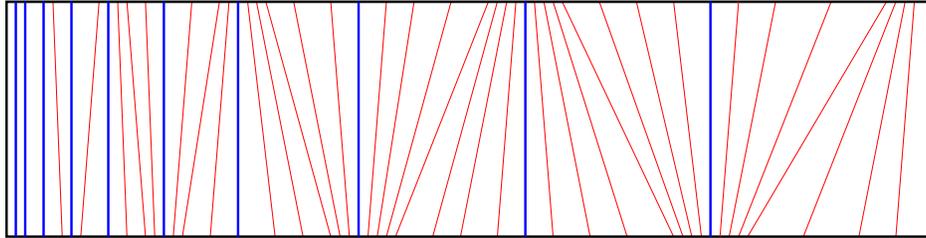


FIGURE 4.5. The element  $\tilde{\sigma}(\gamma) \in \mathrm{Homeo}(I)$  has a countable collection of fixed points, which suspend to (dark) circles, converging to either end. The holonomy alternates between translating in one direction and in the other, which suspends to (light) lines which spiral around the (dark) circles. This figure depicts the restriction of the foliation  $\mathcal{G}$  to one half of  $A$ .

if there is  $\beta \in \pi_1(\Sigma)$  with  $\tilde{\sigma}(\beta)(t) = s$  then  $\lambda^t = \lambda^s$ , so these labels are not unique. Parameterize  $I = [-1, 1]$  in such a way that 0 is an unstable fixed point of  $\tilde{\sigma}(\gamma)$ . So the interior of  $\lambda^0$  is an annulus, and the suspension of the point 0 in  $A$  is a closed loop in  $\mathcal{F}$  which splits  $A$  into two foliated annuli  $A^\pm$ . The interior of each leaf  $\lambda^t$  of  $\mathcal{F}$  is topologically either an annulus or a plane. However, the boundary of each  $\lambda^t$  consists of the leaves

$$\partial\lambda^t = \bigcup_{\beta \in \pi_1(\Sigma)} \mu^{\beta(t)}.$$

For each  $t$ , the set of boundary components consists of at most a single closed loop and countably many lines if  $\lambda^t$  is an annulus, and countably many lines if  $\lambda^t$  is a plane.

Now we'll extend  $\mathcal{F}$  slightly into  $G$  in the following way. Let  $\lambda = \lambda^0$ , which is topologically a cylinder, and let  $\mu^{0\alpha}$  denote the non-circle boundary components of  $\lambda$ . Recall  $\mu^0$  denotes the boundary circle of  $\lambda^0$ . We split  $A$  into two foliated annuli  $A^\pm$

along the circle  $\mu^0$ , and we denote the foliations by  $\mathcal{G}^\pm$ . See Figure 4.6. We extend the

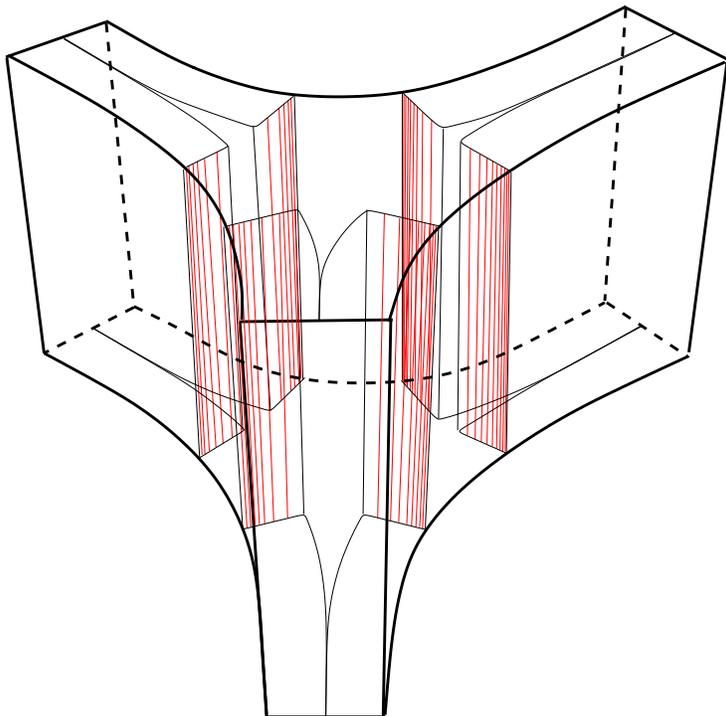


FIGURE 4.6. The foliated annuli  $A_j$  are split open to annuli foliated by  $\mathcal{G}_j^\pm$ , which are glued together in  $G$  by matching their holonomy.

foliation to a pair of solid tori  $A^+ \times I \subset G$  and  $A^- \times I \subset G$  which are interval bundles over annuli neighborhoods of  $\partial A$  in  $\partial G - A$ . We foliate these solid tori  $A^\pm \times I$  by the product of the foliations  $\mathcal{G}^\pm$  with  $I$ . This completes our construction of the foliation of the interstitial region  $J$ .

We do this procedure simultaneously for all the interstitial regions  $J_k$  which bound  $G$ . For each object  $X$  constructed in  $J$ , let  $X_k$  denote the corresponding object constructed in  $J_k$ .

**4.7. Gluing the interstitial foliations.** We would like to glue the foliated annuli ends of the  $A_k^\pm \times I$  together to foliate a product neighborhood of  $\partial G - \bigcup_k A_k$  (as suggested by Figure 4.6). Consider a pair of interstitial annuli  $A_1$  and  $A_2$  which are separated by an annulus of  $\partial G - \bigcup_k A_k$ . The task is to glue the foliated annulus  $A_1^-$  to the foliated annulus  $A_2^+$ , being careful about the topology of the resulting leaves. Doing the gluing amounts to finding a conjugacy between the dynamics of  $\tilde{\sigma}_1(\gamma_1)$  on  $[-1, 0]$  and  $\tilde{\sigma}_2(\gamma_2)$  on  $[0, 1]$  with the orientation reversed. There are many choices of such conjugacies; the set of fixed orbits of  $\tilde{\sigma}_1(\gamma_1)$  on  $[-1, 0]$  is a countably infinite discrete sequence accumulating only at  $-1$ , and the set of fixed orbits of  $\tilde{\sigma}_2(\gamma_2)$  on  $[0, 1]$  is a countably infinite discrete sequence accumulating only at  $1$ . These suspend to closed circles in  $\mathcal{G}_1^-$  and  $\mathcal{G}_2^+$  respectively, which can be glued together. On a complementary interval in  $A_1^-$  and  $A_2^+$ , the dynamics of  $\tilde{\sigma}_1(\gamma_1)$  and  $\tilde{\sigma}_2(\gamma_2)$  have no fixed points, so the set of conjugacies is parameterized by  $\text{Homeo}(I)$ . We need to

choose these conjugacies, simultaneously for all adjacent pairs  $(A_k^-, A_{k+1}^+)$ , so that in the resulting foliation the leaves containing the middle leaves  $\lambda_k^0$  are annuli.

Consider one of the leaves  $\lambda_k^0$ , whose boundary lines are denoted  $\mu_k^{0\alpha}$ . Since all but countably many leaves of each  $\mathcal{F}_j$  are planes, it is easy to glue so that boundary lines  $\mu_k^{0\alpha}$  are glued to lines of some  $\mathcal{G}_j$  which are boundaries of planar leaves  $\lambda_j^{0\alpha}$  of  $\mathcal{F}_j$ . However, this is not enough to ensure that the new leaf  $(\lambda_k^0)'$  which contains  $\lambda_k^0$  is an annulus. The leaf  $(\lambda_k^0)'$  is built up of pieces which are leaves from the  $\mathcal{F}_j$ , and so far we've only ensured that those pieces that touch  $\lambda_k^0$  are planar. Moreover, we have to worry about whether the planar pieces of  $(\lambda_k^0)'$  could be glued together in a cycle, adding to the fundamental group of  $(\lambda_k^0)'$ . We'll solve these problems by building up the gluings inductively. More precisely, start by gluing one boundary line of  $\lambda_k^0$  to a planar leaf. Now glue another one, making sure we don't add any fundamental group. Now glue a boundary component of one of the planar leaves we added to a new planar leaf. Now go work on some other  $\lambda_j^0$  for a bit, etc. At each stage, we can avoid creating any fundamental group since we've only made finitely many gluings. We can formalize the order of induction as follows. Note that when we're done, each leaf  $(\lambda_k^0)'$  will be built up from an annulus and infinitely many planes glued together along lines in their boundaries. The dual graph to the pattern of gluings is a rooted tree  $T_k$ , where the root corresponds to the single annulus piece, and every vertex has valence the cardinality of the natural numbers. So start with a finite set of such trees  $T_k$  and choose a bijection from the union of their edges to the natural numbers, so that each embedded path from a root vertex outward is labelled by an increasing sequence of numbers. Then do the gluings in the order of the labeling. Since there are only countably many bad choices for any given gluing, we can find a good choice arbitrarily close to any initial guess, and choose them so that the size of the biggest unglued gap goes to zero; thus this process defines the full gluing map of all the annuli pairs  $(A_k^-, A_{k+1}^+)$ .

Next, we'll "blow air" into  $\mathcal{F}$  along the leaves  $(\lambda_k^0)'$  to get the final lamination. We can extend  $\mathcal{F}$  to a singular foliation  $\mathcal{F}^s$  by adding a finite set of annuli  $B_j$ , parallel to the annuli components of  $\partial G - \bigcup_k A_k$ , which end in pairs on the circles  $\mu_k^0$ . The only singular leaf of  $\mathcal{F}^s$  is the single branched surface  $B$  which is the union of the  $(\lambda_k^0)'$  and the  $B_j$ . Then  $B$  is topologically a train-track bundle over  $S^1$  with fiber the train-track  $T$  obtained from an ideal polygon  $P$  by collapsing the cusps of  $P$  to single branches. It is clear that  $B$  can be split open to a  $\partial P$  bundle over  $S^1$ , by Denjoying each leaf  $(\lambda_k^0)'$  before gluing on the annuli  $B_j$ . Thus  $\mathcal{F}^s$  can be split open to a nonsingular lamination  $\Lambda \subset C$  with a single complementary region which is an ideal polygon bundle over  $S^1$ . Repeating the construction for every complementary region  $C$  of the original  $\Lambda$ , we obtain a lamination of  $M$  all of whose complementary regions are ideal polygon bundles over  $S^1$ .

**4.8. Tightness is preserved.** To finish the proof of the Filling Lemma, it remains to check that the construction preserves tightness. We'll make this clear by getting an explicit description of the quasi-isometry type of the original complementary regions, and understanding how the added leaves are embedded in these regions.

Let  $\Lambda$  be a genuine lamination with solid torus guts, and take a metric on  $M$  which makes the leaves of  $\Lambda$  hyperbolic. Let  $\tilde{G}$  be the universal cover of a complementary

region. We want to describe its quasi-isometry type (with respect to the induced path metric). Each leaf of  $\partial\tilde{G}$  is isometric to  $\mathbb{H}^2$ . The boundaries of interstitial annuli lift to quasigeodesics in leaves of  $\partial G$  bounding quasiconvex regions which are the lifts of the base surfaces of the interstitial regions. There are only finitely many interstitial annuli, and so the modulus of quasi-isometry of these geodesics is uniformly bounded. The gut regions lift to solid tori of bounded thickness which are contained in bounded neighborhoods of these quasigeodesics in the various boundary leaves of  $\partial\tilde{G}$  which they border. Thus, up to coarse quasi-isometry,  $\tilde{G}$  is a *tree of hyperbolic planes*. That is, it's obtained from countably many copies of  $\mathbb{H}^2$ , glued together in pairs along convex subsets bounded by complete geodesics. A finite number of copies of  $\mathbb{H}^2$  come together along a boundary geodesic, and the pattern of gluings is treelike—every loop in the union is contained in some copy of  $\mathbb{H}^2$ .

More precisely, each interstitial region is covered by a product  $\tilde{\Sigma}_i \times I$  which is uniformly quasi-isometric to a region  $\tilde{\Sigma}_i \subset \mathbb{H}^2$  bounded by a collection of complete geodesics. Each leaf  $\lambda$  of  $\partial\tilde{G}$  is tiled by isometric copies of the  $\tilde{\Sigma}_i$ . Pairs of leaves of  $\partial\tilde{G}$  are glued together along such tiles, and  $n$ -tuples are glued together around each boundary geodesic of  $\partial\tilde{\Sigma}_i$ . We can make a graph with two kinds of vertices: the first kind correspond to the boundary geodesics of tiles  $\partial\tilde{\Sigma}_i$ . The second kind correspond to the tiles themselves  $\tilde{\Sigma}_i$ . The edges correspond to a choice of a tile and a boundary leaf in that tile. This gives a graph which is a simplicial tree. (Similar trees of hyperbolic planes arise as quasi-isometric models for Cayley graphs of Baumslag-Solitar groups [FM], but there the tiles are glued along regions bounded by horocycles, not geodesics.)

Looking back at our construction, the final lamination  $\bar{\Lambda}$  was obtained from  $\Lambda$  by inserting new leaves into complementary regions to  $\Lambda$  so that the added leaves  $\lambda$  are everywhere transverse to the  $I$ -bundle regions and parallel to the boundary in the gut regions. Such leaves are covered in  $\tilde{G}$  by hyperbolic planes which project homeomorphically to the  $\mathbb{H}^2$  slices in our quasi-isometric model. It is clear that a sequence of such slices cannot converge to a pair of distinct slices, and so the leaf space of the leaves inside  $\tilde{G}$  is Hausdorff. Thus  $\bar{\Lambda}$  is tight if  $\Lambda$  is. This completes the proof of the Filling Lemma.  $\square$

## 5. LEAF POCKET THEOREM

Let  $\mathcal{F}$  be a taut foliation with hyperbolic leaves. Consider a leaf  $\lambda$  of  $\mathcal{F}$ , and a point  $p$  in  $\lambda$ . Given a geodesic ray  $r$  starting at  $p$ , we can ask: Can we choose a transversal  $\tau$  at  $p$  so that the holonomy of  $\tau$  is defined along all of the ray  $r$ ? A yes answer is equivalent to the other leaves not pulling away from  $\lambda$  along  $r$  too fast. Thurston showed the surprising fact that there are always many directions where the corresponding ray has this property. In fact, the set of directions where this is true is dense in the tangent space to  $\lambda$  at  $p$ . This section is devoted to the proof of this theorem, generalized from foliations to essential laminations. The existence of so many directions of “pinching” of the leaves of  $\mathcal{F}$  allows one to piece together the circles at infinity of the individual leaves into a universal circle (see Section 6).

We'll begin with the definition of a *marker*, which is a way of keeping track of how the leaves of an essential lamination come together.

**5.1. Definition.** *Let  $\Lambda$  be an essential lamination of  $M$  with hyperbolic leaves. A marker for  $\Lambda$  is a map*

$$m : I \times \mathbb{R}^+ \rightarrow \widetilde{M}$$

with the following properties:

1. *There is a closed set  $K \subset I$  such that for each  $k \in K$ , the image of  $k \times \mathbb{R}^+$  in  $\widetilde{M}$  is a geodesic ray in a leaf of  $\widetilde{\Lambda}$ . Further, for  $k \in I - K$ ,*

$$m(k \times \mathbb{R}^+) \subset \widetilde{M} - \widetilde{\Lambda}.$$

*We call these rays the horizontal rays of the marker.*

2. *For each  $t \in \mathbb{R}^+$ , the interval  $m(I \times t)$  is a tight transversal. Further, there is a separation constant  $\epsilon$  for  $\Lambda$ , such that*

$$\text{length}(m(I \times t)) < \epsilon/3.$$

*We call these intervals the vertical intervals of the marker.*

The main theorem of this section is:

**5.2. Leaf Pocket Theorem for Laminations.** *Let  $\Lambda$  be an essential lamination of an atoroidal 3-manifold  $M$ . Then for every leaf  $\lambda$  of  $\widetilde{\Lambda}$ , the set of endpoints of markers is dense in  $S_\infty^1(\lambda)$ .*

This theorem generalizes Thurston's corresponding theorem for foliations given in [Thu]. The proof here is completely topological, in contrast to Thurston's proof which relies on the existence of harmonic measures for foliations. See Remark 5.11 for more on harmonic measures and their relationship to the Leaf Pocket Theorem. As such, our conclusion is slightly different from that of [Thu]. Precisely, Thurston showed the following. For any  $\epsilon > 0$  there is a  $t > 0$  such that if  $\lambda$  and  $\mu$  are a pair of leaves which are joined by a transversal  $\tau$  of length  $\leq t$ , then for a random walk  $\gamma$  in  $\lambda$  starting at  $\tau \cap \lambda$ , the holonomy transport of  $\tau$  along  $\gamma$  has uniformly bounded length with probability at least  $1 - \epsilon$ .

The rest of the section is devoted to the proof of the Leaf Pocket Theorem.

*Proof.* In outline, the proof goes like this. A *minimal set* of  $\Lambda$  is a nonempty closed union of leaves which is minimal with respect to that property. A closed union of leaves  $\Sigma$  is minimal if and only if every leaf in  $\Sigma$  is dense in  $\Sigma$ . The main case of the theorem is for leaves of  $\widetilde{\Lambda}$  which cover leaves in a minimal set. While not every leaf of  $\Lambda$  is contained in a minimal set, the closure of every leaf in  $\Lambda$  contains some minimal set. Once we know the theorem for leaves in minimal sets, it is not hard to prove the theorem for all leaves.

To continue the outline, let  $\Sigma$  be a minimal set of  $\Lambda$ . First we show that some leaf of  $\Sigma$  has nontrivial fundamental group. Then there is a closed geodesic  $\gamma$  in a leaf of  $\Sigma$ . Using the holonomy around  $\gamma$ , we construct an "immersed sawblade" in  $\Lambda$ , from which we build a collection of markers in  $\widetilde{\Lambda}$ . We show that any leaf in  $\widetilde{\Sigma}$  intersects at least one of these markers. We then show that in any given leaf  $\lambda$  of  $\widetilde{\Lambda}$ , the endpoints of these markers is dense in  $S_\infty^1(\lambda)$ .

Let's begin the proof in the case of a minimal set  $\Sigma$  of  $\Lambda$ , that is, for leaves  $\lambda$  of  $\tilde{\Lambda}$  which lie in the inverse image  $\tilde{\Sigma} \subset \tilde{\Lambda}$  of  $\Sigma$ .

**5.3. Each leaf of  $\tilde{\Sigma}$  has a marker.** First, we claim that  $\Sigma$  contains a leaf with a non-trivial closed geodesic. Suppose to the contrary that every leaf of  $\Sigma$  is simply connected. Since  $M$  is atoroidal, the lamination  $\Lambda$  has hyperbolic leaves, and so each leaf of  $\Sigma$  is a hyperbolic plane. Since the holonomy on  $\Sigma$  is trivial, it's easy to construct a nontrivial invariant transverse measure supported on  $\Sigma$ . In codimension 1, Plante showed that leaves in the support of an invariant transverse measure have polynomial area growth [Pla]. This is a contradiction as the hyperbolic plane has exponential growth. So there is a leaf  $\lambda$  of  $\Sigma$  which contains a closed geodesic  $\gamma$ .

(Note: In [Pla], Plante was concerned only with foliations, but his proof applies to laminations as well. The only new issue is the step which involves passing to a *co-orientable* finite cover, since not all laminations are virtually co-orientable. However, one can find an open neighborhood  $N(\Lambda)$  of  $\Lambda$  where  $\Lambda$  can be extended to a foliation, and pass if necessary to a finite cover of  $N(\Lambda)$  where the foliation becomes co-orientable. That is, one can find a 3-manifold  $\hat{N}$  and a compact co-oriented lamination  $\hat{\Lambda}$  in  $\hat{N}$  which maps by either an embedding or a double cover to a submanifold of  $M$ , taking  $\hat{\Lambda}$  to  $\Lambda$ . Plante's theorem applies to  $\hat{\Lambda}$ , and therefore also to  $\Lambda$ .)

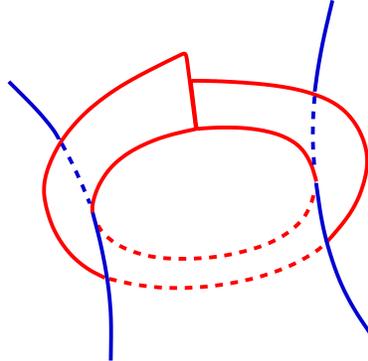


FIGURE 5.4. Holonomy transport of some small transversal around  $\gamma$  sweeps out an immersed sawblade.

Now we'll use  $\gamma$  to show that each leaf of  $\tilde{\Sigma}$  intersects a marker. We think of  $\gamma$  as an interval, whose endpoints are the same point in  $\lambda$ . If  $\Sigma$  is a closed isolated leaf then constructing markers is trivial, so we assume this is not the case. As  $\lambda$  is in a minimal set, the geodesic  $\gamma$  is a limit of leaves of  $\Lambda$  on at least one side. Let  $\tau$  be a tight transversal with lower endpoint in  $\lambda$  to a non-isolated side of  $\lambda$ , sufficiently small so that the holonomy transport of  $\tau$  around  $\gamma$  exists. Moreover, if  $\lambda$  is not isolated in  $\Sigma$ , choose  $\tau$  on a non-isolated side in  $\Sigma$ . Then, by minimality of  $\Sigma$ , every leaf of  $\Sigma$  intersects  $\tau$ .

Consider the holonomy transport  $T: I \times \gamma \rightarrow M$  of  $\tau$  around  $\gamma$ . Even though  $\gamma(0) = \gamma(1)$ , we do not usually have  $T(I, \gamma(0)) = T(I, \gamma(1))$ . However, either  $T(I, \gamma(1)) \subset$

$T(I, \gamma(0))$  or vice versa. See Figure 5.4. By reversing the orientation on  $\gamma$  if necessary, we can ensure  $T(I, \gamma(1)) \subset T(I, \gamma(0))$ . We will call the image of  $T$  an *immersed sawblade*, and denote it  $G$ . The universal cover  $\tilde{G}$  of  $G$  lifts to  $\tilde{M}$  where it runs along a lift of  $\lambda$ . See Figure 5.5 for one possible configuration which makes clear why  $G$  is

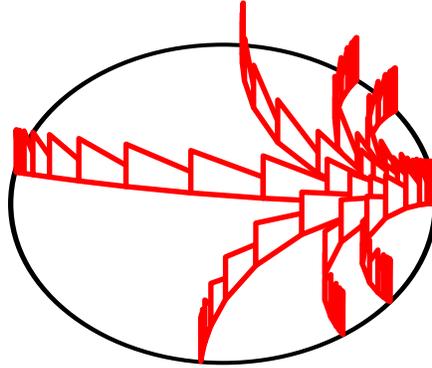


FIGURE 5.5. Lifts of a sawblade  $\tilde{G}$  running along  $\lambda$ .

called an immersed sawblade. If we take  $\tau$  short enough so that  $G$  has height less than  $1/3$  a separation constant for  $\Lambda$ , a lift of the immersed sawblade  $G$  must contain the image of a marker  $m$ . More precisely, consider any lift of  $\tau$  to  $\tilde{M}$  and look at the lift of  $\tilde{G}$  which contains it. The sawblade structure of  $\tilde{G}$  forces holonomy of  $\tau$  to be defined for all time in the positive direction along the lift of  $\gamma$ . Sweeping the lift of  $\tau$  along the lift of  $\gamma$  gives a marker  $m$  which is contained in  $\tilde{G}$ . Actually,  $m$  isn't quite a marker because the definition requires that each horizontal ray be a geodesic. Shrinking  $\tau$  and thus  $G$  if necessary, we can ensure that each horizontal ray has uniformly small geodesic curvature in its leaf. Then each horizontal ray of  $m$  is a  $K$ -quasigeodesic for a uniform  $K$  which we can make arbitrarily close to 1. We can then straighten these to genuine geodesics to construct a real marker. Making  $K$  small enough, the final marker lies in a small  $\delta$ -neighborhood of  $\tilde{G}$ .

Taking the possible lifts of  $\tau$  to  $\tilde{M}$  gives us an equivariant family  $\mathcal{M}$  of markers in  $\tilde{M}$ . Now consider any  $\lambda'$  in  $\Sigma$  and some leaf  $\tilde{\lambda}'$  in  $\tilde{\Sigma}$  which covers it. As noted above,  $\lambda'$  must intersect the transversal  $\tau$ . Thus one of the markers in  $\mathcal{M}$  intersects  $\tilde{\lambda}'$ . So every leaf of  $\tilde{\Sigma}$  intersects a marker.

**5.6. Endpoints of markers.** Consider the family  $\mathcal{M}$  of markers coming from the lifts of  $\tau$  above. Let  $\lambda$  be any leaf of  $\tilde{\Sigma}$ . Consider the collection of geodesic rays in  $\lambda$  which are horizontal rays of markers of  $\mathcal{M}$ . These rays originate from the intersections of lifts of  $\tau$  with  $\lambda$ . We want to show that the endpoints of these rays are dense in  $S^1_\infty(\lambda)$ . Equivalently, let  $\mathcal{G}$  denote the set of all lifts of  $\tilde{G}$  to  $\tilde{M}$ , and look at the intersections of  $\mathcal{G}$  with  $\lambda$ . We need to show that the endpoints of these quasigeodesic rays are dense in  $S^1_\infty(\lambda)$ .

First, we'll show there are infinitely many intersections of  $\mathcal{G}$  with  $\lambda$ . Since  $\Sigma$  is minimal, there is a uniform  $R$  such that for any point  $p$  in any leaf of  $\Sigma$ , the disk in that leaf about  $p$  of radius  $R$  intersects  $\tau$ . This is because the function of leafwise distance to  $\tau$  is upper semi-continuous on  $\Sigma$ . Applying this to  $\lambda$ , we have that any point  $p \in \lambda$  is within  $R$  of a lift of  $\tau$ , and therefore within  $R$  of an intersection of  $\mathcal{G}$  and  $\lambda$ . Thus, there are infinitely many quasigeodesic rays  $\gamma_i$  contained in  $\lambda \cap \mathcal{G}$ .

Next, we'll show that any two distinct  $\gamma_i$  and  $\gamma_j$  have disjoint endpoints in  $S_\infty^1(\lambda)$  (in particular, Figure 5.5 depicts an impossible configuration). Downstairs in  $M$ , choose a  $\delta$  so that the leafwise  $\delta$ -neighborhood of  $G$  deformation retracts to  $G$ . Note that  $\gamma_i$  and  $\gamma_j$  come from distinct lifts of  $\tilde{G}$  in  $\mathcal{G}$ , as otherwise some lift of  $\tau$  intersects  $\lambda$  twice. Lifting the  $\delta$ -neighborhood of  $G$  upstairs, we see that in  $\lambda$  the  $\delta$ -neighborhoods of  $\gamma_i$  and  $\gamma_j$  are disjoint. By shrinking  $\tau$  if necessary, we can make the uniform quasigeodesic constant for all the  $\gamma_k$  so small, that the  $\delta/3$  neighborhood of any  $\gamma_k$  contains a geodesic ray. If  $\gamma_i$  and  $\gamma_j$  had a common endpoint, consider the corresponding geodesic rays that they are  $\delta/3$ -close to. There are points on these geodesic rays that are less than  $\delta/3$  apart, and so the distance between  $\gamma_i$  and  $\gamma_j$  is less than  $\delta$ . But this contradicts that  $\gamma_i$  and  $\gamma_j$  have disjoint  $\delta$ -neighborhoods in  $\lambda$ . So  $\gamma_i$  and  $\gamma_j$  have disjoint endpoints.

From now on, we'll replace the original  $\gamma_i$  by the corresponding geodesics. Note that the  $\gamma_i$  have disjoint  $\delta/3$ -neighborhoods, and that the  $\gamma_i$  are an  $R + \delta$  net for  $\lambda$ . Now suppose that the endpoints of the  $\gamma_i$  are not dense. Let  $I$  be an open interval in  $S_\infty^1(\lambda)$  containing no endpoints. Let  $J$  be a closed interval contained in  $I$ . Let  $H$  be the half-space associated to the smaller interval  $J$ . As the  $\gamma_i$  are a net, there must be an infinite sequence of disjoint  $\gamma_i$ , call them  $\alpha_k$ , which intersect  $H$ . Passing to a subsequence, we can assume that the terminal endpoints of the  $\alpha_k$  converge to a point  $p$  in  $S_\infty^1(\lambda)$ , and that the initial endpoints converge to a point  $q$  in  $\lambda \cup S_\infty^1(\lambda)$ . The point  $q$  is in  $H \cup J$ , and, by assumption,  $p$  can't be in  $I$ . So  $p \neq q$ . But then the  $\alpha_k$  converge to the geodesic ray joining  $q$  to  $p$ , which contradicts that the  $\alpha_k$  have disjoint  $\delta/3$ -neighborhoods. Thus the endpoints of markers are dense in  $S_\infty^1(\lambda)$ . This proves the theorem in the case of leaves in minimal sets.

**5.7. Properties of markers.** Before moving on, we note that we actually know more than just that the endpoints of the markers are dense in  $S_\infty^1(\lambda)$ . If  $\Sigma$  is not a compact leaf, then every leaf of  $\Sigma$  intersects the *interior* of  $\tau$  many times. Thus there must be a marker  $m$  for each leaf  $\lambda$  in  $\tilde{\Sigma}$  so that  $\lambda$  is in the interior of marker (in the sense that the vertical intervals of the marker intersect  $\lambda$  in their interior). If, on the other hand,  $\Sigma$  is a compact leaf then all the markers we construct have endpoints in  $\Sigma$ , and extend only to one side of  $\Sigma$ . In this case, we repeat the construction on the other side of  $\gamma$  to get dense sets of markers extending in both directions.

**5.8. Leaves not in minimal sets.** The basic idea here is that leaves that are not in minimal sets get very close to leaves which are in minimal sets, and pick up markers that way. Fix a leaf  $\lambda$  of  $\Lambda$ , and a point  $p$  in  $\lambda$ . For each unit vector  $v$  tangent to  $\lambda$  at  $p$ , we need to show that there is a marker intersecting  $\lambda$  whose endpoint in  $S_\infty^1(\lambda)$  is arbitrarily close (in the visual sense) to  $v$ . Fix an  $\epsilon > 0$ . Let  $S_\epsilon$  be the sector of  $\lambda$  bounded by two geodesic rays starting at  $p$ , so that the angle between the two rays

is  $\epsilon$ , and so that  $v$  points down the center of  $S_\epsilon$ . We claim the closure of  $S_\epsilon$  in  $\Lambda$  must contain a minimal set. Pick a sequence of points  $m_i$  in  $S_\epsilon$  marching out towards infinity so that the disk about  $m_i$  in  $\lambda$  of radius  $i$  is contained in  $S_\epsilon$ . If  $m$  is any limit point in  $M$  of the  $m_i$ , the closure of  $S_\epsilon$  contains the entire leaf containing  $m$ , and thus contains any minimal set in the closure of that leaf. Let  $\Sigma$  be such a minimal set.

Fix a point  $q$  in a leaf  $\sigma$  of  $\Sigma$ . As  $q$  is in the closure of  $S_\epsilon$ , there exist points  $p_i$  in  $S_\epsilon$  which converge to  $q$  in  $M$ . We can assume that locally near  $q$ , the  $p_i$  all lie to a fixed side of  $\sigma$ . Let  $v_i$  be the unit tangent vector to  $\lambda$  at  $p_i$  which points directly away from the base point  $p$ . By passing to a subsequence, we can assume that the  $v_i$  converge to a vector  $w$  tangent to  $\sigma$  at  $q$ . As  $\sigma$  is in a minimal set, there exists a marker whose endpoint in  $S_\infty^1(\sigma)$  makes an angle of  $< \epsilon/2$  with  $w$ . Initially, that marker might not come near the point  $q$ , but at the cost of shortening the vertical intervals we can drag it over so that one of the vertical intervals contains  $q$ . Moreover, we can choose this marker so that it lies to the same side of  $\sigma$  as the  $p_i$ 's. This marker intersects  $\lambda$  in horizontal rays which nearly contain the geodesic rays given by  $(p_i, v_i)$ . For  $i$  large enough, these horizontal rays must be contained in the enlarged sector  $S_{2\epsilon}$ . Thus the endpoints of these markers make a visual angle of at most  $2\epsilon$  with our initial vector  $v$ . It follows that the endpoints of markers are dense in  $S_\infty^1(\lambda)$ .  $\square$

5.9. *Remark.* Note that the proof actually shows a little more about the set of markers than just that their endpoints are dense in  $S_\infty^1(\lambda)$ . Namely, if  $\lambda$  is non-compact, then the endpoints of markers which intersect  $\lambda$  in their interior are dense. If  $\lambda$  is compact, then the endpoints of markers which extend to a fixed side of  $\lambda$  are dense.

5.10. *Remark.* Notice that the ‘‘essential’’ hypothesis in this theorem is excessive. All that is necessary is that  $\Lambda$  is compact, codimension one, and has hyperbolic leaves. Also, while Theorem 5.2 generalizes Thurston’s original theorem from foliations to all essential laminations, this does not allow one to generalize his theorem on the existence of universal circles to the case of genuine laminations.

5.11. *Remark.* A more analytic proof of this theorem might run along the following lines: The set of harmonic probability measures supported on a compact lamination  $\Lambda$  is a compact convex set; the extremal points in this set (those which can’t be written as a nontrivial convex sum of harmonic probability measures) are the ergodic harmonic measures [Gar, Can2]. The holonomy transport along a random walk in a leaf preserves the (infinitesimal) harmonic measure of a transversal, on average. It follows by an analytic argument that the holonomy of some transversal does not blow up for a.e. random walk on a leaf in the support of an ergodic harmonic measure. If  $\lambda$  is a leaf not in the support of any ergodic harmonic measure, a random walk on  $\lambda$  is dispersive, and must eventually wander arbitrarily close to the support of an ergodic measure.

## 6. UNIVERSAL CIRCLES

Let  $M$  be an orientable 3-manifold with a taut foliation  $\mathcal{F}$ . In the last section, we saw that the leaves of  $\mathcal{F}$  come together in many directions. In this section, we’ll explain how this allows the construction of a master *universal circle* by assembling the

circles at infinity of the leaves of  $\tilde{\mathcal{F}}$ . This universal circle comes equipped with an action of  $\pi_1(M)$ . A simple example to keep in mind is that of a surface bundle over  $S^1$ , where one can identify the circles at infinity of any two leaves using the product structure of  $\tilde{M}$ . In this case, the universal circle can be identified with any particular circle at infinity. In general, the branching behavior of the leaf space makes the relation between the universal circle and the circle at infinity of a leaf more complicated.

We'll start with the precise definition of a *universal circle*. Recall that two leaves are in  $\tilde{\mathcal{F}}$  are *comparable* if they can be joined by an arc transverse to  $\tilde{\mathcal{F}}$ . Then:

**6.1. Definition.** A universal circle,  $S_{univ}^1$ , for  $\mathcal{F}$  is the following data:

1. A representation

$$\rho_{univ}: \pi_1(M) \rightarrow \text{Homeo}(S_{univ}^1).$$

2. For every leaf  $\lambda$  of  $\tilde{\mathcal{F}}$ , a monotone map

$$\phi_\lambda: S_{univ}^1 \rightarrow S_\infty^1(\lambda).$$

3. For every  $\alpha \in \pi_1(M)$ , the following diagram commutes:

$$\begin{array}{ccc} S_{univ}^1 & \xrightarrow{\rho_{univ}(\alpha)} & S_{univ}^1 \\ \phi_\lambda \downarrow & & \downarrow \phi_{\alpha(\lambda)} \\ S_\infty^1(\lambda) & \xrightarrow{\alpha} & S_\infty^1(\alpha(\lambda)) \end{array}$$

4. For any leaf  $\mu$ , the associated gaps are the maximal connected intervals in  $S_{univ}^1$  mapped to points by  $\phi_\mu$ . The complement of the gaps in  $S_{univ}^1$  is the core associated to  $\mu$ . We require that for any pair  $(\mu, \lambda)$  of incomparable leaves, the core associated to  $\lambda$  is contained in a single gap associated to  $\mu$ , and vice versa.

The purpose of this section is to prove:

**6.2. Theorem (Thurston).** Let  $\mathcal{F}$  be a taut foliation of an orientable 3-manifold with hyperbolic leaves. Then there exists a universal circle for  $\mathcal{F}$ .

Moreover, one has:

**6.3. Theorem.** If  $M$  is atoroidal, the action of  $\pi_1(M)$  on the universal circle is faithful.

Theorem 6.2 is announced in [Thu], but unfortunately this paper is mostly unwritten. Thurston has outlined a construction of  $S_{univ}^1$  in several lectures, and given many details. An alternate construction of  $S_{univ}^1$  when  $\mathcal{F}$  is co-orientable is given in [Call]. As a public service, we give a complete proof here. The basic idea is to look at certain sections of the circle bundle  $E_\infty$  over the leaf space  $L$ , namely the ‘‘special sections’’ which are canonically defined. The set of all special sections,  $\mathcal{S}$ , has a natural circular order, and can be completed to form  $S_{univ}^1$ . The map  $S_{univ}^1 \rightarrow S_\infty^1(\lambda)$  comes from restricting sections in  $\mathcal{S}$  to  $\lambda$ . Since special sections are canonically defined, they are invariant under the action of  $\pi_1(M)$  on  $E_\infty$ , and this gives the action of  $\pi_1(M)$  on  $S_{univ}^1$ . The hard work in proving Theorem 6.2 is all in the Leaf Pocket

Theorem 5.2, which is used to define the correct sections. The arguments in this section are mostly formal.

**6.4. Monotone maps of circles.** We'll warm up to our study of special sections by investigating some generalities about monotone maps between circles. A *monotone map*  $\phi: S^1 \rightarrow S^1$  is a degree one map which does not reverse (but might degenerate) the cyclic ordering on triples of points. Note that it is implicit in this definition that we are considering maps between *oriented* circles. A map is monotone if for every  $p \in S^1$ , the preimage  $\phi^{-1}(p)$  is contractible. This use of the word monotone agrees with the standard use from decomposition theory, in the sense of R. L. Moore.

**6.5. Definition.** A monotone relation between an ordered pair of circles  $S_1^1$  and  $S_2^1$  is a third circle  $S_{12}^1$  and two monotone maps  $\phi_i: S_{12}^1 \rightarrow S_i^1$  for  $i = 1, 2$ .

For ease of notation, we denote a monotone relation between two circles by the name of the source of the two monotone maps to these circles. The monotone maps are part of the data, of course.

The following lemma gives a pushout construction for monotone maps.

**6.6. Lemma.** Consider three circles  $S_1^1, S_2^1$ , and  $S_3^1$ . Let  $S_{12}^1$  be a monotone relation between  $S_1^1$  and  $S_2^1$ , and  $S_{23}^1$  a relation between  $S_2^1$  and  $S_3^1$ . Then there is a canonical (leftmost) monotone relation  $S_{13}^1$  between  $S_1^1$  and  $S_3^1$ .

*Proof.* The mapping cylinder of a monotone relation is literally a cylinder whose interior is foliated as a product, but for which distinct intervals of leaves converge to a single point in the target circle (see Figure 6.7). The mapping cylinders of our two

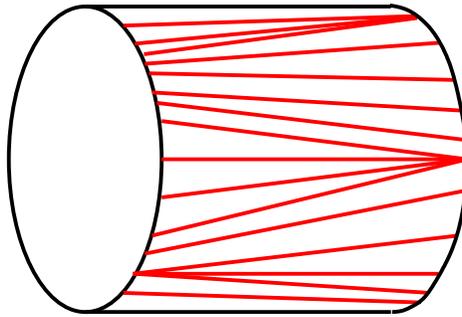


FIGURE 6.7. The mapping cylinder of a monotone relation.

monotone relations can be glued along the intermediate  $S_2^1$  to give a cylinder  $C$  with a foliation which is singular along the middle and two boundary circles. Using a leftmost rule, we will resolve these singularities to construct the mapping cylinder of the composite. If  $p \in S_2^1$  is singular, the leaf through  $p$  consists of the union of a cone and a line, or of two cones. (Throughout, consult Figure 6.8.) The boundary of this singular leaf is a train track with 3 or 4 branches and a single switch at  $p$ . If the train track has 3 branches, we can split it open until the switch is on a boundary circle of  $C$ . If the track has 4 branches, we resolve the switch by pushing the upper left branch over the right branch, and then split open as before. The regions

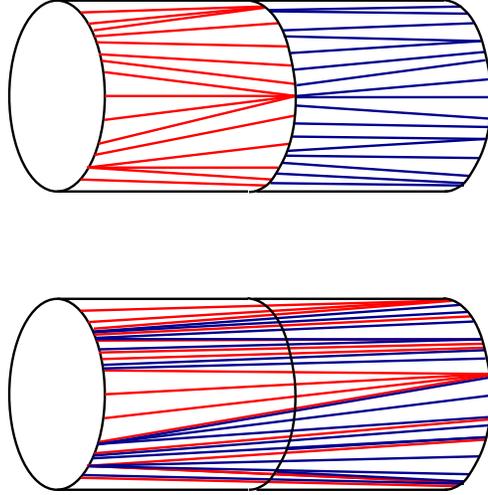


FIGURE 6.8. Monotone relations can be composed by splitting the associated singular foliations according to the *leftmost rule*.

bounded by split open tracks can be foliated by a product foliation. The result is a foliation of the interior of  $C$  which is singular along the two boundary circles, which are canonically identified with  $S^1_1$  and  $S^1_3$ . This gives the needed relation.  $\square$

If we think of monotone relations between *ordered pairs*, then Lemma 6.6 shows us how to compose monotone relations. The topological realization of the pushout as a canonical splitting of a singular foliation of a cylinder implies that the composition of monotone relations is *associative*.

Lemma 6.6 generalizes to continuous families of monotone maps, as follows. The reader may wish to skip ahead to Section 6.10 to see how this material is needed in the sequel before proceeding. Consider a continuous family of monotone maps

$$\phi_t: S^1 \times t \rightarrow S^1 \times t$$

parameterized by  $t$  in an interval  $I$ . Let  $C = S^1 \times I$  be a parameterized cylinder, and let  $\phi: C \rightarrow C$  be the map associated to the  $\phi_t$ , which preserves the  $I$ -coordinate.

The cylinder  $C$  is foliated by  $\{\text{point}\} \times I$ . The image of this foliation under  $\phi$  is denoted by  $\mathcal{F}_\phi$ , and  $\mathcal{F}_\phi$  is said to be *monotonely parameterized by  $\phi$* . The foliation is singular, since distinct leaves are not disjoint. While leaves may overlap, they can't cross. It is clear from the definition that a monotone parameterization of  $C$  gives a monotone relation between the circles of  $\partial C$ .

**6.9. Lemma.** *Let  $C = S^1 \times I$  be a parameterized cylinder, and let  $\mathcal{F}$  be a singular foliation of  $C$  transverse to every  $S^1 \times \{\text{point}\}$ . Then  $\mathcal{F}$  is monotonely parameterized by a canonical leftmost  $\phi$ .*

*Proof.* We can think of the singular foliation  $\mathcal{F}$  as a foliation by train tracks, in the generalized sense that continuous families of branches may coalesce at a switch. A dense subset of  $\mathcal{F}$  can be exhausted by finite train-tracks  $F_i$ . These can be desingularized, pushing left branches over right branches to the right boundary where they can be split open, and right branches under left branches to the left boundary

where they can be split open. This gives a family of trivial (product) laminations in  $C$  of the form  $S_i \times I$  where  $S_i$  is a finite set, which maps by a monotone family of maps to the train track  $F_i$ . As the  $F_i$  accumulate in  $C$ , so do the  $S_i$ , and in the limit we construct a monotone family of maps from  $C$  foliated as a product to  $C$  foliated by  $\mathcal{F}$ . This limiting map gives our canonical monotone parameterization of  $\mathcal{F}$ . It is easy to check that the map does not depend up to homeomorphism on the choice of exhaustion by  $F_i$ , and is natural.  $\square$

**6.10. Leftmost sections for comparable leaves.** Fix a co-orientation of  $\tilde{\mathcal{F}}$ . For a pair  $(\lambda, \mu)$  of comparable leaves in  $\tilde{\mathcal{F}}$ , we will write  $\lambda > \mu$  to mean that  $\lambda$  lies above  $\mu$  with respect to our co-orientation. Consider such a pair  $\lambda > \mu$ . Let  $I \subset L$  be the interval joining  $\lambda$  to  $\mu$ . Then  $C = E_\infty|_I$  is topologically a cylinder foliated by circles  $S_\infty^1(\nu)$  for  $\lambda \geq \nu \geq \mu$ . In this subsection, we will define certain sections of  $C$ , the *leftmost sections*. The leftmost sections of  $C$  are defined in terms of the endpoints of markers, so we'll begin with a discussion of markers. Let  $m$  be a marker for  $\tilde{\mathcal{F}}$  which intersects only leaves in  $I$ . Then the endpoints of the horizontal rays of  $m$  form an interval in  $C \subset E_\infty$  which is transverse to the circle fibers. From now on, we will abuse notation and refer to the interval of endpoints in  $E_\infty$  as a marker. By the Leaf Pocket Theorem 5.2, the markers are dense in each circle fiber  $S_\infty^1(\nu)$  of  $C$ . Moreover, even if we only look at markers which extend to a fixed side of  $S_\infty^1(\nu)$ , then these markers are dense in  $S_\infty^1(\nu)$  (see Remark 5.9). The key to understanding the whole set of markers is the following:

**6.11. Lemma.** *Let  $m_1$  and  $m_2$  be two markers in  $C$ . If  $m_1$  and  $m_2$  are not disjoint, their union  $m_1 \cup m_2$  is also an interval transverse to the circle fibers (in particular,  $m_1 \cap m_2$  is a subinterval of each  $m_i$ ).*

*Proof.* Suppose the markers intersect. Let  $\lambda$  be a leaf where the endpoints of  $m_1$  and  $m_2$  are the same, and  $\mu$  be a leaf where the endpoints of the  $m_i$  differ. Remember that we required that the vertical intervals in our markers are very thin, and have length  $< \epsilon/3$  for some separation constant  $\epsilon$ . In particular, the vertical intervals of the  $m_i$  from  $\lambda$  to  $\mu$  have length  $< \epsilon/3$ . Let  $\gamma_i^\lambda$  denote the horizontal ray of  $m_i$  in  $\lambda$  and similarly with  $\gamma_i^\mu$ . Shortening the markers horizontally if necessary, we can make  $\gamma_1^\lambda$  lie in an  $\epsilon/3$  neighborhood of  $\gamma_2^\lambda$ . If we start at some point  $p$  on  $\gamma_1^\lambda$ , we can join it to a point  $q$  in  $\gamma_2^\mu$  by a path of length  $< \epsilon$  by going up along  $m_1$  to  $\gamma_1^\lambda$ , over to  $\gamma_2^\lambda$  in  $\lambda$ , and down  $m_2$  to  $\gamma_2^\mu$ . But the  $\gamma_i^\mu$  diverge in  $\mu$ , so the distance in  $\mu$  between  $p$  and  $q$  can be made arbitrarily large. But the distance in  $\tilde{M}$  between  $p$  and  $q$  is at most  $\epsilon$ . This contradicts that  $\epsilon$  is a separation constant for  $\mathcal{F}$ , as  $\mu$  is not quasi-isometrically embedded in its  $\epsilon$ -neighborhood.  $\square$

By the above lemma, we can amalgamate intersecting markers into one larger interval. If we take a maximal such union of markers, we get an interval  $m$  which is quite possibly open at either end. We'll abuse notation and call such an  $m$  a marker. Because of the density of markers, the ends  $m$  can't wiggle violently, and the closure of  $m$  is a closed interval transverse to the circle fibers of  $C$ . The closures of two maximal markers can intersect only in at most their endpoints. Thus, the set of all markers gives us something which approximates a singular foliation of  $C$ , not unlike

Figure 6.8. Later, we'll describe how to “integrate” the set of markers into a proper singular foliation.

Now we're in a position to define the *leftmost sections* of  $C$ . First, an *admissible section* is a section  $\tau: I \rightarrow C$  whose image does not cross, but might run into, any marker. The *leftmost section* starting at  $p \in S_\infty^1(\mu)$  is an admissible section  $\tau: I \rightarrow C$  which is anticlockwisemost among all such sections in the following sense: for any leaf  $\nu \in I$ , the value  $\tau(\nu)$  is anticlockwisemost among all endpoints of admissible sections starting at  $p$  and ending on  $S_\infty^1(\nu)$ .

**6.12. Lemma.** *The leftmost section exists and is continuous.*

*Proof.* For a fixed  $p$  in  $S_\infty^1(\mu)$ , we will construct a series of approximations to the leftmost section. Consider a finite set of markers  $\mathcal{M}$  which intersects every leaf  $S_\infty^1(\nu)$ . Consider paths  $\gamma: [0, 1] \rightarrow C$  which are almost sections in the sense that the composition of  $\gamma$  and the projection  $[0, 1] \rightarrow I$  is a monotone surjection. It makes sense to say such a path is admissible. Define  $\tau_{\mathcal{M}}$  to be the leftmost admissible path in this sense. Explicitly,  $\tau_{\mathcal{M}}$  is constructed by starting at  $p$ , heading left until we hit a marker  $m \in \mathcal{M}$ , going up along  $m$  until it ends, then left again until we hit a marker, etc. See Figure 6.13

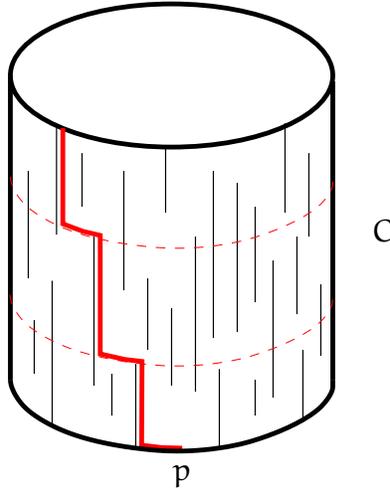


FIGURE 6.13. An approximate leftmost section  $\tau_{\mathcal{M}}$ .

Note that if we add additional elements to  $\mathcal{M}$ , the intersection of  $\tau_{\mathcal{M}}$  with any fixed  $S_\infty^1(\nu)$  moves to the right. The leftmost section  $\tau$  to the full set of markers is essentially the righthanded limit of all the  $\tau_{\mathcal{M}}$ . To be precise, let's work in the universal cover of  $C$  which is  $\mathbb{R} \times I$ , and consider the  $\tau_{\mathcal{M}}$  to be based at some fixed lift  $\tilde{p}$  of  $p$ . Then we define a section  $\tau: I \rightarrow C$  by

$$\tau(\nu) = \sup_{\mathcal{M}} (\min(\tau_{\mathcal{M}} \cap S_\infty^1(\nu))).$$

Here's why this supremum exists: First note that we can restrict the supremum to only markers  $\mathcal{M}$  which contain some fixed set of markers  $\mathcal{M}_0$ . Then, for any  $\mathcal{M} \supset \mathcal{M}_0$ , the path  $\tau_{\mathcal{M}}$  lies to the left of the the rightward analogue of  $\tau_{\mathcal{M}_0}$ .

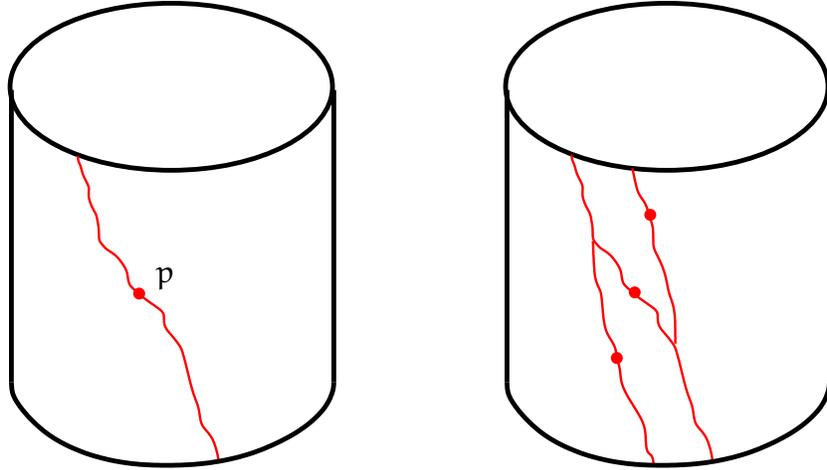


FIGURE 6.15. Some special sections, showing how they can coalesce.

Because of the density of markers in *every* circle  $S_\infty^1(\lambda)$ , it is not hard to see that  $\tau$  is continuous. Since the supremum was taken over all finite subsets  $\mathcal{M}$ , the section  $\tau$  is admissible with respect to the full set of markers. Finally, if  $\gamma$  is any admissible section, it lies to the right of any  $\tau_{\mathcal{M}}$ , and hence to the right of  $\tau$ . So  $\tau$  is the promised leftmost section starting at  $p$ .  $\square$

While a given leftmost section is continuous, the leftmost sections may not vary continuously as a function of  $p$ . However, any two leftmost sections do not cross, though they can coalesce. Symmetrically, given a point  $q$  in the top boundary component  $S_\infty^1(\lambda)$ , we can talk about the rightmost (downward) section of  $C$  starting at  $q$ . Note that rightmost sections starting at  $S_\infty^1(\lambda)$  and leftmost sections starting at  $S_\infty^1(\mu)$  also do not cross.

**6.14. Monotone relations between comparable leaves.** We will give two different points of view on how the leftmost sections of  $C$  give a monotone relation from  $S_\infty^1(\mu)$  to  $S_\infty^1(\lambda)$ .

Given a  $\nu \in I$  and a  $p$  in  $S_\infty^1(\nu)$ , we define the *special section* of  $C$ , denoted  $\tau_p$ , to be the section which is the leftmost section going up from  $S_\infty^1(\nu)$  to  $S_\infty^1(\lambda)$ , and the rightmost section going downward from  $S_\infty^1(\nu)$  to  $S_\infty^1(\mu)$  (see Figure 6.15). By the above, two special sections do not cross, though they may coalesce. Let  $\mathcal{S}$  denote the set of all special sections, that is, all sections of  $C$  which are of the form  $\tau_p$  for at least one  $p$ . The paths in  $\mathcal{S}$  give us a singular foliation of  $C$  of the type discussed in Lemma 6.9. By that lemma, the singular foliation can be split open to give a monotone relation from  $S_\infty^1(\mu)$  to  $S_\infty^1(\lambda)$ .

Another point of view on this monotone relation is the following, which is a warm up for the general construction of the universal circle. Because  $\mathcal{S}$  consists of non-crossing paths, it has a natural circular order as follows (see the proof of Theorem 3.8 for the definition of a circular order). Consider three distinct sections

$$(\tau_{p_1}, \tau_{p_2}, \tau_{p_3}).$$

The non-crossing property guarantees that if  $(\tau_{p_1}(\nu), \tau_{p_2}(\nu), \tau_{p_3}(\nu))$  are clockwise ordered for some  $\nu \in I$ , then for any other  $\nu'$ , the triple  $(\tau_{p_1}(\nu'), \tau_{p_2}(\nu'), \tau_{p_3}(\nu'))$  is either clockwise ordered or degenerate. Thus, we say that  $(\tau_{p_1}, \tau_{p_2}, \tau_{p_3})$  are clockwise ordered if for some  $\nu$  the points  $(\tau_{p_1}(\nu), \tau_{p_2}(\nu), \tau_{p_3}(\nu))$  are clockwise ordered. (Actually, one can have three distinct  $\tau_{p_i}$  which intersect each circle in only two points, but such paths still have a natural circular order by perturbing the  $\tau_{p_i}$  slightly to disjoint paths).

Let  $\bar{\mathcal{S}}$  be the closure of  $\mathcal{S}$  in the space of sections  $I \rightarrow C$ . It is easy to check that every  $\tau \in \bar{\mathcal{S}}$  is admissible, and that any two sections in  $\bar{\mathcal{S}}$  do not cross. We claim that  $\bar{\mathcal{S}}$  is isomorphic, as a circularly ordered set, to a circle. For this, we just need to check that  $\bar{\mathcal{S}}$  is compact under the topology induced by the circular order. But this is the case because the topology induced by the order is the same as the one induced as a subspace of the space of all sections. Henceforth, we will denote the circle  $\bar{\mathcal{S}}$  by  $S_{\mu\lambda}^1$ .

For any  $\nu \in I$ , we have a monotone map from  $S_{\mu\lambda}^1$  to  $S_\infty^1(\nu)$  by evaluation of sections. In particular, we have our promised monotone relation by looking at the restrictions to  $\mu$  and  $\lambda$ . The circle  $S_{\mu\lambda}^1$  should be thought of as the universal circle of the foliation restricted to  $I$ .

**6.16. The universal circle for  $\mathbb{R}$ -covered foliations.** Before doing the general case, let's construct the universal circle in the case where the leaf space  $L = \mathbb{R}$ . For  $p \in S_\infty^1(\nu)$  we can define the *special section*,  $\tau_p$ , of  $E_\infty$  as above (that is, it's the leftmost section upward from  $p$ , and the rightmost section downward). Let  $\mathcal{S}$  denote the set of all such special sections, which has a natural circular order. Moreover, we have surjective monotone maps  $\mathcal{S} \rightarrow S^1(\nu)$  for each  $\nu \in L$  via evaluation.

We claim that  $\mathcal{S}$  is invariant under the action of  $\pi_1(M)$  on  $E_\infty$ . The only thing to worry about is that if  $\mathcal{F}$  is not co-orientable, then the action of  $\pi_1(M)$  on  $\tilde{\mathcal{F}}$  can reverse the co-orientation we used to define leftmost. However, since  $M$  is orientable, if  $\gamma \in \pi_1(M)$  flips the co-orientation then  $\gamma$  also reverses the orientation of all the circle fibers of  $E_\infty$ . This has the affect of exchanging leftmost upwards paths with rightmost downwards paths and vice versa. Thus  $\mathcal{S}$  is preserved by  $\pi_1(M)$ .

To construct the universal circle, we just need to complete  $\mathcal{S}$  into a circle. Up to homeomorphism, there is a unique way to embed  $\mathcal{S}$  into a circle  $S^1$  so that the embedding respects the circular order and is continuous with respect to the topology on  $\mathcal{S}$  induced by its order. The closure of the image of  $\mathcal{S}$  might omit some gaps, which we can collapse to get a monotone map from  $\mathcal{S}$  to a circle  $S_{\text{univ}}^1$ , which is the promised *universal circle*. Because no  $\tau_p$  is isolated to either side in  $\mathcal{S}$ , it follows that  $\mathcal{S}$  actually embeds in  $S_{\text{univ}}^1$ .

The action of  $\pi_1(M)$  on  $\mathcal{S}$  extends naturally to an action on  $S_{\text{univ}}^1$ . Likewise, we have natural monotone maps from  $S_{\text{univ}}^1$  to each  $S_\infty^1(\nu)$ . It easy to check that  $S_{\text{univ}}^1$  has all the properties of Definition 6.1, and so we've proved Theorem 6.2 in the case where  $L = \mathbb{R}$ .

**6.17. Turning corners.** To build a universal circle in general, we need to figure out how to relate the circles at infinity of incomparable leaves. Recall that two incomparable leaves  $\lambda_1$  and  $\lambda_2$  are part of the same cataclysm if they are both limits of a

sequence  $\nu_i$  of comparable leaves (see Section 3.4). The following lemma is the key to constructing special sections in the non- $\mathbb{R}$ -covered case.

**6.18. Lemma (Turning corners).** *Let  $\lambda_1$  and  $\lambda_2$  be a pair of incomparable leaves of the same cataclysm. Then there is a canonical monotone relation between  $S_\infty^1(\lambda)$  and  $S_\infty^1(\mu)$ .*

*Proof.* Roughly, the monotone relation is the inverse limit of the circles at infinity in the approximating leaves of the cataclysm. We'll assume that  $\lambda_1$  and  $\lambda_2$  are a limit from below. That is, there are intervals in the leaf space  $I_i = [\mu, \lambda_i]$ , whose intersection is  $I = [\mu, \lambda_1) = [\mu, \lambda_2)$  (this is the situation shown inside the circle in Figure 6.24). Let  $C_i$  denote the cylinder  $E_\infty|_{I_i}$ , and  $C$  the half-open cylinder  $E_\infty|_I$ . Let  $\mathcal{M}_i$  be the set of markers heading downward from  $S_\infty^1(\lambda_i)$  into  $C_i$ . Essentially, the needed monotone relation is obtained by completing the set of markers  $\mathcal{M} = \mathcal{M}_1 \cup \mathcal{M}_2$  into a circle.

First we claim that if  $m_1 \in \mathcal{M}_1$  and  $m_2 \in \mathcal{M}_2$  then  $m_1$  and  $m_2$  are disjoint in  $C$ . If not, as in the proof of Lemma 6.11, we can join points  $x_i \in \lambda_i$  by a path  $\gamma$  of length less than a separation constant  $\epsilon$ . A leaf  $\mu \in I$  close to the  $\lambda_i$  intersects  $\gamma$  in a pair of points  $y_i$ , one close to each  $x_i$ . The distance between the  $y_i$  in  $\mu$  can be made arbitrary large by choosing  $\mu$  close enough to the  $\lambda_i$ . As the  $y_i$  are distance less than  $\epsilon$  in  $\widetilde{M}$ , this violates that  $\epsilon$  is a separation constant.

We can define a map  $f_i: \mathcal{M} \rightarrow S_\infty^1(\lambda_i)$  as follows. For a marker  $m \in \mathcal{M}$ , look at the (possibly half-open) marker  $m \cap C_i$ . Denote by  $\overline{m}$  the closure of  $m \cap C_i$  in  $C_i$ . If  $m$  is in  $\mathcal{M}_i$ , then of course  $\overline{m}$  is just  $m$ . In general, the density of markers implies that  $\overline{m}$  is an interval transverse to the circle fibers of  $C_i$ . We define  $f_i(m)$  to be the point of intersection of  $\overline{m}$  with  $S_\infty^1(\lambda_i)$ . Consider pairs of markers in  $\mathcal{M}_1$  and  $\mathcal{M}_2$ . Because the  $\lambda_i$  are incomparable, the intersection of these pairs of markers with any  $S_\infty^1(\nu)$  in  $C$  are unlinked. This implies that for  $i \neq j$ ,  $f_i(\mathcal{M}_j)$  consists of a single point  $p_i \in S_\infty^1(\lambda_i)$ . The set  $\mathcal{M}$  has a circular order coming from intersecting markers with  $C$ . The completion/blow-down of  $\mathcal{M}$  is a circle  $S_{\lambda_1, \lambda_2}^1$  which is the needed relation.

Explicitly, we can construct  $S_{\lambda_1, \lambda_2}^1$  by cutting each  $S_\infty^1(\lambda_i)$  at  $p_i$  and gluing the resulting intervals together to form a circle. The map of the relation  $f_i: S_{\lambda_1, \lambda_2}^1 \rightarrow S_\infty^1(\lambda_i)$  is just the identity on the interval coming from  $S_\infty^1(\lambda_i)$ , and maps the interval coming from  $S_\infty^1(\lambda_j)$  to the point  $p_i$ .  $\square$

**6.19. Remark.** The construction in Lemma 6.18 has the following important property. Take two points  $q_i \in \lambda_i$  on a pair of incomparable leaves situated as above. Consider the rightmost (downward) section  $\tau_i$  on  $C_i$ . We claim that these sections  $\tau_i$  differ on  $C$ . Pick a sequence of leaves  $\mu_j$  converging to the  $\lambda_i$ . Denote by  $\mathcal{M}_i(j)$  the intersection of  $\mathcal{M}_i$  with  $\mu_j$ . The subsets  $\mathcal{M}_i(j)$  are nonempty for large  $j$ , and they are *disjoint*. Moreover, no pair of points in  $\mathcal{M}_1(j)$  links a pair of points in  $\mathcal{M}_2(j)$ . Because the  $\tau_i$  are rightmost sections, it follows that either  $\tau_1(j) = \tau_1(\mu_j)$  is separated from  $\tau_2(j)$  for some  $j$  by pairs of points in  $\mathcal{M}_1(j)$ , or else  $(m_1(j), \tau_1(j), m_2(j))$  is anticlockwise ordered for each pair of markers  $m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2$ . Conversely, either  $\tau_2(j)$  is separated from  $\tau_1(j)$  for some  $j$  by pairs of points in  $\mathcal{M}_2(j)$ , or else  $(m_2(j), \tau_2(j), m_1(j))$  is anticlockwise ordered for each pair of markers  $m_1 \in \mathcal{M}_1, m_2 \in \mathcal{M}_2$ . It follows that either the  $\tau_i(j)$  are separated from each other by  $\mathcal{M}_i(j)$ , or else for large  $j$ , there are



*Proof.* Suppose that  $\nu$  does not lie on a line between  $\lambda$  and  $\mu$ . Then the paths  $\gamma_{\lambda\nu}$  and  $\gamma_{\mu\nu}$  intersect in a nontrivial subpath  $\gamma$  ending at  $\nu$ . Note that the orientations of  $\gamma_{\lambda\nu}$  and  $\gamma_{\mu\nu}$  agree on  $\gamma$ , and that  $\gamma$  contains at least one turn. The sections  $\tau_p$  and  $\tau_q$  coalesce as they pass through that turn, and so  $\tau_p(\nu) = \tau_q(\nu)$ .  $\square$

On the other hand, we have:

**6.23. Lemma.** *Let  $\tau_p$  and  $\tau_q$  be two distinct sections, and  $A$  a line in  $L$  which lies between  $\tau_p$ . Then  $\tau_p$  and  $\tau_q$  differ on  $A$ .*

*Proof.* If  $\lambda$  and  $\mu$  are comparable, then as  $\tau_p$  and  $\tau_q$  are distinct they must differ on the interval  $I = [\lambda, \mu]$  in  $L$ . As the line  $A$  contains  $I$ , the two sections differ on  $A$ . So now assume that  $\lambda$  and  $\mu$  are incomparable. As in Figure 6.24,  $A$  must pass through

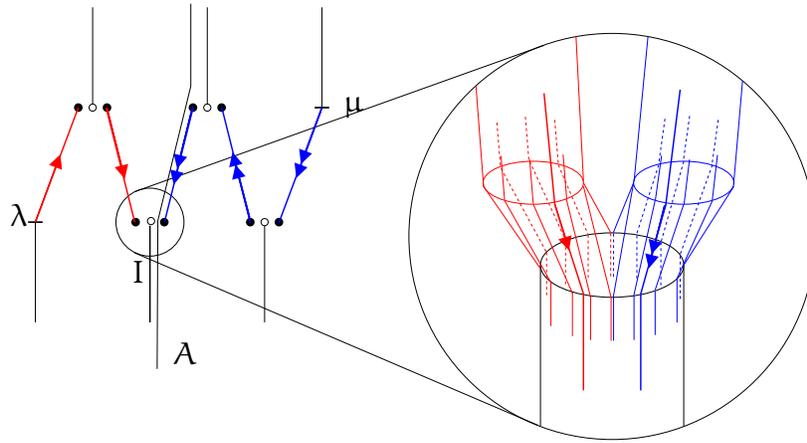


FIGURE 6.24. Where the sections can differ.

a turn of  $\gamma_{\lambda\mu}$ . Thus we have an interval  $I$  of  $A$  where the two sections come in via different branches of the cataclysm. So the situation looks like the magnified picture in Figure 6.24. As noted in Remark 6.19, the way the monotone relation works for incomparable leaves implies that the two sections differ on  $I$ .  $\square$

Now let  $\mathcal{S}$  denote the set of all special sections  $\tau_p$ . We want to put a natural circular order on  $\mathcal{S}$  by declaring that a triple  $(\tau_{p_1}, \tau_{p_2}, \tau_{p_3})$  is clockwise ordered if the restrictions of the  $\tau_{p_i}$  to some line  $A \subset L$  are clockwise ordered. To know that this makes sense, we need to check:

**6.25. Lemma.** *Given three distinct sections  $(\tau_{p_1}, \tau_{p_2}, \tau_{p_3})$ , there exists a line  $A$  so that the restriction of the  $\tau_i$  to  $A$  has a non-degenerate circular order. Moreover, this circular order is independent of the choice of  $A$ .*

*Proof.* Let  $\lambda_i$  be the leaf where  $p_i \in S_\infty^1(\lambda_i)$ . The *midpoint* of the  $\lambda_i$  is defined analogously to the midpoint of three points in an  $\mathbb{R}$ -tree. More precisely, we let  $\Gamma$  be the union of the three paths  $\gamma_{\lambda_i\lambda_j}$ . The midpoint is constructed by making  $\Gamma$  Hausdorff by amalgamating the cataclysms, taking the midpoint in the resulting tree, and then pulling back that midpoint to  $L$ . The midpoint consists of either a single point in  $L$  or several points of the same cataclysm. Given Lemma 6.22, if  $\nu$  is a leaf where all

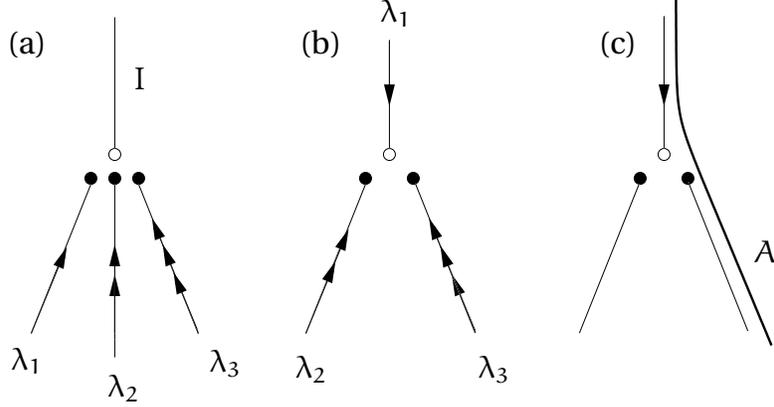


FIGURE 6.26. Some possible cases.

the  $\tau_{p_i}$  differ, then for each pair  $(i, j)$  there is a line  $A_{ij}$  containing  $v$  which intersects  $\bar{\gamma}_{\gamma_i \gamma_j}$ . So a natural place to look for the needed line is near the midpoint of the  $\lambda_i$ . There are several cases. First, consider the case where the midpoint is several points of the same cataclysm. There are two subcases corresponding to Figure 6.26(a) and (b). In case (a), the proof of Lemma 6.23 shows that the three sections differ on the interval  $I$  shown. In case (b), first notice that  $\tau_{p_2}$  and  $\tau_{p_3}$  differ on the interval  $I$ . Therefore,  $\tau_{p_1}$  differs from at least one of  $\tau_{p_2}$  and  $\tau_{p_3}$  on  $I$ . Without loss of generality, assume that  $\tau_{p_1} \neq \tau_{p_2}$ . Take  $A$  to be the line shown in Figure 6.26(c). As  $A$  is between  $\lambda_1$  and  $\lambda_3$ , the sections  $\tau_{p_1}$  and  $\tau_{p_3}$  differ on  $A$ . As  $I \subset A$ , we know that the  $\tau_{p_i}$  are distinct on  $A$ , and so have a non-degenerate circular order.

In the case that the midpoint is a single point  $\mu$ , take any line  $A$  containing  $\mu$ . The midpoint  $\mu$  lies in every  $\gamma_{\lambda_i \lambda_j}$ , and so  $A$  lies between each pair  $(\lambda_i, \lambda_j)$ . By Lemma 6.23, the  $\tau_i$  restricted to  $A$  are distinct, and so have a non-degenerate circular order.

It remains to prove that the circular order is independent of the choice of the line  $A$ . Because of Lemma 6.22, this is simple to check in each of the above cases, and we leave this to the reader.  $\square$

Now that we've defined special sections in general and shown they have a circular order, the proof of Theorem 6.2 follows as in the  $\mathbb{R}$ -covered case by completing  $S$  into a circle. The only difference is that now we need to check Property (4) of Definition 6.1, but this is clear from the construction in Lemma 6.18, and from Remark 6.19 immediately following it.

**6.27. Remark.** Note that there is no claim of *uniqueness* for the universal circles constructed by this or other methods. Even if one asks for minimal universal circles, it is not *a priori* clear that they should be unique. On the other hand, for  $\mathbb{R}$ -covered foliations or those with one-sided branching, there is a unique minimal universal circle [Cal2, Cal3].

**6.28. Faithfulness of the action.** Finally, we conclude this section by proving Theorem 6.3, namely that the action of  $\pi_1(M)$  on  $S^1_{\text{univ}}$  is faithful provided  $M$  is atoroidal. If  $\mathcal{F}$  is  $\mathbb{R}$ -covered, then since  $M$  is atoroidal there is a transverse pseudo-Anosov flow

which can be split open to a pair of very full genuine laminations  $\Lambda^\pm$  transverse to each other and to  $\mathcal{F}$ . These laminations are covered by genuine laminations  $\tilde{\Lambda}^\pm$  which arise from a pair of invariant laminations  $\Lambda_{\text{univ}}^\pm$  of  $S_{\text{univ}}^1$ . If  $\gamma$  acts as the identity on  $S_{\text{univ}}^1$ , it acts as the identity on the leaf space of either genuine lamination, and therefore it fixes the singular flowlines of the pseudo-Anosov flow (see [Cal2] Corollary 5.3.16); in particular, it acts as a nontrivial translation along these flowlines. But the dynamics of the flow expands the transverse measure on one singular lamination and contracts the transverse measure on the other, and therefore the action on the leaf spaces of the singular foliations is nontrivial, giving a contradiction. So we can assume that the leaf space  $L$  has branching (in this case, we will not need the assumption that  $M$  is atoroidal).

Let  $K$  be the kernel of the map  $\pi_1(M) \rightarrow \text{Homeo}(S_{\text{univ}}^1)$ . Let  $\lambda$  be any leaf of  $\tilde{\mathcal{F}}$ , and  $k$  a non-identity element in  $K$ . We claim that  $k\lambda$  and  $\lambda$  are comparable and distinct. Note that  $k$  can't fix  $\lambda$  setwise, because if it did it would act identically on  $S_\infty^1(\lambda)$  and hence identically on  $\lambda$ . If  $\lambda$  and  $k\lambda$  are incomparable, we know that the maps  $S_{\text{univ}}^1 \rightarrow S_\infty^1(\lambda)$  and  $S_{\text{univ}}^1 \rightarrow S_\infty^1(k\lambda)$  blow down different gaps. As  $k$  acts identically on  $S_{\text{univ}}^1$ , this is impossible.

Fix a branch point  $\lambda$  in  $L$ . By our above observations, the set  $K\lambda$  is infinite and contained in a line  $A \subset L$ . Now consider a leaf  $\mu$  which is in the same cataclysm as  $\lambda$ . Again,  $K\mu$  is contained in a line  $B$ . For each  $k$ , the leaf  $k\mu$  is in the same cataclysm as  $k\lambda$ . So there are infinitely many pairs of non-separable points  $(k\lambda, k\mu)$  where one point is in  $A$  and other in  $B$ . This is not possible for two lines in  $L$ , and so we have a contradiction. Thus the action on  $S_{\text{univ}}^1$  is faithful in the branching case. This completes the proof of Theorem 6.3.

## 7. GROUP ACTIONS ON 1-MANIFOLDS AND LEFT ORDERINGS

In this section, we discuss the algebraic interdependence between the existence of actions of a group  $G$  on various 1-manifolds. The main reason for this is to improve the usefulness of the (non)existence of such a group action as a criterion for the existence of foliations and laminations. In particular, we want to promote an action of  $\pi_1(M)$  on  $S^1$  to an action on  $\mathbb{R}$ . This is because it is much easier to decide whether a group  $G$  admits a faithful action on  $\mathbb{R}$  than on  $S^1$ .

We'll begin with a more algebraic criterion for a group  $G$  to act faithfully on  $\mathbb{R}$ . A group  $G$  is *left-orderable* if there is a total order on  $G$  which is invariant under left multiplication. If  $G$  has a faithful representation into  $\text{Homeo}^+(\mathbb{R})$ , then  $G$  is left-orderable for the following reason. Pick an infinite sequence of points  $p_1, p_2, \dots \in \mathbb{R}$  such that the intersection of the stabilizers of these points in  $G$  is trivial. Then say  $\alpha > \beta$  if for the smallest  $i$  where one of  $\alpha$  and  $\beta$  does not fix  $p_i$ , we have  $\alpha(p_i) > \beta(p_i)$ . Conversely, for a finitely generated group  $G$ , it is not hard to show that the existence of a left-invariant order gives rise to a faithful representation in  $\text{Homeo}^+(\mathbb{R})$ , so these conditions are actually equivalent (see e.g. [Ghys]). For actions on circles, one can use the analogous notion of a circular order to the same effect.

**7.1. Lifting representations.** Given a group  $G$  and a representation

$$\rho: G \rightarrow \text{Homeo}(S^1),$$

when does  $\rho$  lift to a representation  $\tilde{\rho}: G \rightarrow \text{Homeo}(\mathbb{R})$ ? Here we want the lift to respect the universal covering map  $\pi: \mathbb{R} \rightarrow S^1$  in the sense that

$$\pi \circ \tilde{\rho}(g) = \rho \circ \pi(g) \quad \text{for all } g \text{ in } G.$$

The topological group  $\text{Homeo}(S^1)$  has two connected components; the component of the identity is the subgroup  $\text{Homeo}^+(S^1)$  of orientation preserving homeomorphisms. If  $H^1(G; \mathbb{Z}/2\mathbb{Z}) = 0$ , every homomorphism from  $G$  to  $\text{Homeo}(S^1)$  has image in  $\text{Homeo}^+(S^1)$ .

There is a universal central extension of  $\text{Homeo}^+(S^1)$

$$1 \rightarrow \mathbb{Z} \rightarrow \widetilde{\text{Homeo}}^+(S^1) \rightarrow \text{Homeo}^+(S^1) \rightarrow 1$$

where the group  $\widetilde{\text{Homeo}}(S^1)$  is the group of periodic homeomorphisms of  $\mathbb{R}$  with period 1; i.e. those that commute with the translation  $Z: t \rightarrow t + 1$ .

Since  $\widetilde{\text{Homeo}}^+(S^1)$  is the universal central extension of  $\text{Homeo}^+(S^1)$ , the obstruction to lifting a homomorphism  $\rho: G \rightarrow \text{Homeo}^+(S^1)$  is an element of  $H^2(G; \mathbb{Z})$  called the *Euler class*  $e(\rho)$  of  $\rho$  (for more, see [Ghys]). We will give a geometric interpretation of  $e(\rho)$  later, but for the moment we study the implications for  $G = \pi_1(M)$  where  $M$  is a rational homology sphere.

**7.2. Theorem.** *Let  $M$  be a rational homology sphere and  $\rho: \pi_1(M) \rightarrow \text{Homeo}^+(S^1)$  a faithful homomorphism. Then the commutator subgroup  $[\pi_1(M), \pi_1(M)]$  is left-orderable.*

*Proof.* Let  $G = \pi_1(M)$ . By assumption  $H_1(M; \mathbb{Z})$  is finite, so that  $e(\rho) \in H^2(G; \mathbb{Z}) \cong H_1(M; \mathbb{Z})$  is torsion. We will show that  $\rho$  restricted to  $[G, G]$  lifts to  $\widetilde{\text{Homeo}}^+(S^1)$ .

For each  $g \in G$ , there is a lift of  $\rho(g)$  to  $\tilde{\rho}(g) \in \widetilde{\text{Homeo}}^+(S^1)$ . This lift is not unique, and distinct lifts differ by powers of the generator  $Z$  of the center of  $\widetilde{\text{Homeo}}^+(S^1)$ . If  $c \in [G, G]$ , we can express  $c$  as a product of commutators

$$c = \prod_i [g_i, h_i].$$

Consider the element in  $\widetilde{\text{Homeo}}^+(S^1)$  given by

$$\tilde{\rho}(c) = \prod_i [\tilde{\rho}(g_i), \tilde{\rho}(h_i)].$$

Since  $Z$  is central, this depends only the description of  $c$  as a product of commutators, not on the choices of the  $\{\tilde{\rho}(g_i), \tilde{\rho}(h_i)\}$ . We will show that  $\tilde{\rho}(c)$  is independent of the choice of expression as a product of commutators. Then  $\tilde{\rho}: [G, G] \rightarrow \widetilde{\text{Homeo}}^+(S^1)$  will be a homomorphism which is a lift of  $\rho$  as required.

If we have another description of  $c$  as a product of commutators

$$c = \prod_i [g'_i, h'_i]$$

then the relation

$$\text{id} = \prod_i [g_i, h_i] \left( \prod_j [g'_j, h'_j] \right)^{-1}$$

determines a map from a surface  $\Sigma \rightarrow M'$ , and therefore an element  $[\Sigma] \in H_2(G; \mathbb{Z})$ . To see that  $\tilde{\rho}$  is well-defined on  $[G, G]$  it suffices to show that

$$Z^n = \prod_i [\tilde{\rho}(g_i), \tilde{\rho}(h_i)] \left( \prod_j [\tilde{\rho}(g'_j), \tilde{\rho}(h'_j)] \right)^{-1}$$

is trivial in  $\widetilde{\text{Homeo}}^+(S^1)$ . But by the definition of the Euler class, we have that

$$n = e([\Sigma])$$

is zero, because  $e$  is torsion. So the representation lifts to  $\widetilde{\text{Homeo}}^+(S^1)$  on  $[G, G]$ , and we are done.  $\square$

In case of a general action on  $S^1$  we have:

**7.3. Theorem.** *Let  $M$  be an irreducible rational homology sphere, and*

$$\rho: \pi_1(M) \rightarrow \text{Homeo}(S^1)$$

*a faithful homomorphism. Let  $K \subset \pi_1(M)$  be the kernel of the homomorphism from  $\pi_1(M)$  to  $\mathbb{Z}/2\mathbb{Z}$  defined by the orientation of  $\rho$ . Then the subgroup  $[K, K]$  is left-orderable.*

*Proof.* Consider the manifold  $M'$  corresponding to  $K$ . If  $M'$  is also a rational homology sphere, then we are done by the preceding theorem. If instead  $H^1(M) \neq 0$ , then  $K = \pi_1(M')$  itself is left-orderable for reasons that have nothing to do with the existence of  $\rho$ , namely the following theorem of from [BRW, Cor. 3.4]:

**7.4. Theorem (Boyer, Rolfsen, Wiest).** *A compact, orientable, irreducible 3-manifold with  $H^1(M) \neq 0$  has left-orderable fundamental group.*

In either case, we're done.  $\square$

**7.5. Remark.** The necessity of passing to a dihedral cover in general in Theorem 7.3 reflects the fact that finite dihedral groups act faithfully on  $S^1$ . For instance, let  $M$  be a 3-manifold with a dihedral cover  $N \rightarrow M$ , where  $\pi_1(M)/\pi_1(N) = D_n$ , such that  $\pi_1(N)$  is left-orderable. Let  $\rho: \pi_1(M) \rightarrow \text{Homeo}(S^1)$  have image the standard action of the dihedral group  $D_n$  on  $S^1$ . The kernel is exactly  $\pi_1(N)$ , and we can modify  $\rho$  to a monotonely equivalent *faithful* action  $\rho'$  by blowing up the (finite) orbit of some point in  $S^1$ , inserting a faithful action of  $\pi_1(N)$  at some blown up interval, and transporting this action around the orbit by  $D_n$ .

**7.6. Corollary.** *Let  $M$  be an orientable atoroidal 3-manifold which contains a taut foliation, a tight essential lamination with solid torus guts, or a pseudo-Anosov flow. Then there is a finite cover of  $M$  with left-orderable fundamental group. The covering group is either an abelian group, or a  $\mathbb{Z}/2\mathbb{Z}$  extension of an abelian group.*

*Proof.* By the results of previous sections, in every case of the hypothesis of the theorem, there is a faithful representation of  $\pi_1(M)$  into  $\text{Homeo}(S^1)$ . If  $H^1(M; \mathbb{Z}) \neq 0$ , then  $\pi_1(M)$  is left-orderable by Theorem 7.4. Otherwise, the result follows from Theorem 7.3.  $\square$

**7.7. The Euler class and multisections.** A more geometric way to think about Euler classes is via foliated bundles and multisections. A representation  $\rho: \pi_1(M) \rightarrow \text{Homeo}^+(S^1)$  determines a foliated  $S^1$  bundle  $E$  over  $M$ , by taking the quotient of the trivial bundle  $\widetilde{M} \times S^1$  by  $\pi_1(M)$  acting via

$$\gamma(m, \theta) = (\gamma(m), \rho(\gamma)(\theta)).$$

Another interpretation of the Euler class  $e$  of  $\rho$  is the obstruction to finding a section of this foliated bundle. For any integer  $n$ , the class  $ne$  is the obstruction to finding an *order  $n$  multisection* of this bundle. The existence of such a multisection gives a representation of  $\pi_1(M)$  on the group of homeomorphisms of  $n$  copies of  $\mathbb{R}$ , as follows. An order  $n$  multisection can be parameterized on a small open set  $B \subset M$  by  $n$  sections  $s_i: B \rightarrow E$ . The “labels” on these sections can be analytically continued along a path in  $M$ , but after traveling around a loop  $\gamma \subset M$ , the labels are permuted by some element in the symmetric group on  $n$  letters:

$$\sigma(\gamma) \in S_n.$$

Pick a point  $p$  in  $B$ , and consider the cover of the fiber  $S_p^1$  by  $n$  copies of  $\mathbb{R}$ , where the basepoint of the  $i$ th copy of  $\mathbb{R}$  maps to  $s_i(p)$ . As we wind around a loop  $\gamma$  based at  $p$ , we can see how much holonomy transport of some leaf twists in  $S_p^1$  *relative* to the section  $s_i$ . But after traveling around the loop, the section  $s_i$  gets relabelled as  $s_{\sigma(\gamma)(i)}$ . This holonomy transport gives an identification of the  $i$ th copy of  $\mathbb{R}$  with the  $\sigma(\gamma)(i)$ th copy of  $\mathbb{R}$ , and one gets a representation of  $\pi_1(M)$  in the group  $\text{Homeo}^+(\mathbb{R} \times \{1, \dots, n\})$ .

**7.8. Flips and canceling Euler classes.** Let  $M$  be a 3-manifold, and suppose that  $M$  contains a taut foliation, a tight essential lamination with solid torus guts, or a pseudo-Anosov flow. Then there is a faithful representation  $\pi_1(M) \rightarrow \text{Homeo}(S^1)$ . We would like to find a subgroup  $K$  of  $\pi_1(M)$ , of as large index as possible, where the representation lifts to  $\text{Homeo}^+(\mathbb{R})$ . From what we have said above, it might seem that the worst case is when  $H^1(M; \mathbb{Z}/2\mathbb{Z}) \neq 0$ . But there are some instances where the nature of these representations helps out, using the flips discussed in Section 3.3.

Suppose  $\Lambda$  is a tight essential lamination of  $M$  with solid torus guts, and suppose further that there is a homomorphism  $\sigma: \pi_1(M) \rightarrow \mathbb{Z}/2\mathbb{Z}$  such that the core of each gut region maps to the identity. Let  $\widehat{M}$  be the 2-fold cover corresponding to the kernel of  $\sigma$ . Then  $\Lambda$  lifts to a tight essential lamination of  $\widehat{M}$  with two solid torus gut complementary regions for each solid torus gut region of  $\Lambda$ . By the Filling Lemma 4.1, we can assume the complementary regions to  $\Lambda$  were actually ideal polygon bundles over  $S^1$ , and the same is therefore true for  $\widehat{\Lambda}$ . The complementary regions to  $\widehat{\Lambda}$  come in pairs  $C_1^i, C_2^i$  corresponding to the complementary

regions  $C^i$  to  $\Lambda$ . The representation  $\rho: \pi_1(M) \rightarrow \text{Homeo}(S^1)$  restricts to a representation  $\rho: \pi_1(\widehat{M}) \rightarrow \text{Homeo}(S^1)$ . If, for each complementary region  $C^i$ , we perform a flip of *exactly one* of the  $C_j^i$ , we get a *new* representation  $\rho'$  which is evidently both orientation-preserving and has Euler class 0. In particular,  $\rho'$  lifts to a faithful representation in  $\text{Homeo}^+(\mathbb{R})$ , and  $\pi_1(\widehat{M})$  is left-orderable. Unfortunately, we see no means of exploiting this trick for algorithmic purposes, since the condition that  $\sigma$  be trivial on the core of every gut region seems hard to verify in practice.

**7.9. Actions on leaf spaces of taut foliations.** For completeness, we include a proof of the following fact mentioned in the introduction:

**7.10. Theorem.** *Let  $\mathcal{F}$  be a taut foliation of  $M$ . Then  $\pi_1(M)$  admits a faithful action on  $L$  without a global fixed point.*

*Proof.* Let  $K$  be the kernel of the holonomy representation. Then  $K$  fixes every leaf  $\lambda$  of  $\mathcal{F}$ . In particular,  $K$  is a surface group, and is therefore left-orderable, and acts faithfully on  $\mathbb{R}$ . We can insert this action at some point  $\lambda \in L$  and transport it around by the action of  $\pi_1(M)/K$ . Geometrically, we can realize such an action by changing the foliation  $\mathcal{F}$  by a monotone equivalence.  $\square$

**7.11. Corollary.** *If  $M$  admits a co-orientable  $\mathbb{R}$ -covered foliation  $\mathcal{F}$ , then  $\pi_1(M)$  is left-orderable.*

## 8. ALGORITHMIC ISSUES

Let  $G$  be a finitely presented group which has a short-lex automatic structure. In this section, we describe computational techniques for proving that  $G$  is not left-orderable. The existence of a short-lex automatic structure (see [ECH<sup>+</sup>] for definitions) implies that the word problem for  $G$  is efficiently solvable. In fact, there is a set of generators  $S$  of  $G$  for which there is a fast algorithm for reducing a word  $w$  in  $S$  to the canonical word  $w'$  which is lexicographically first among all shortest words equal to  $w$  in  $G$ .

If  $G$  is orderable, consider the positive cone  $P = \{g \in G \mid g > 1\}$ . Then  $P \cdot P \subset P$  and  $G$  is the disjoint union  $P \cup P^{-1} \cup \{1\}$ . Conversely, any  $P$  with these two properties gives rise to a left invariant order via  $a > b$  if and only if  $b^{-1}a \in P$ . Let  $B(r)$  be the ball in  $G$  of radius  $r$  about 1, that is, all words in  $S$  of length at most  $r$ . For fixed  $r$ , we can consider the following

**8.1. Question.** *Does there exist a  $P \subset B(r)$  such that  $(P \cdot P) \cap B(r) \subset P$  and  $B(r)$  is the disjoint union  $P \cup P^{-1} \cup 1$ ?*

If  $G$  is orderable, the answer to (8.1) is yes. If the answer to (8.1) is no, then  $G$  is not orderable. It is not hard to show that if  $G$  is non-orderable then the answer to (8.1) is no for large enough  $r$ . The idea is that if the answer to (8.1) is always yes, then one can construct an global positive cone by taking an inverse limit of the partial positive cones  $P(r) \subset B(r)$ , picking at each stage a  $P(r) \subset B(r)$  which has infinitely many extensions to larger  $B(R)$ .

For any particular  $r$ , the automatic structure on  $G$  combined with the fact that  $B(r)$  is finite means that (8.1) is algorithmically decidable. We'll now give a simple

recursive algorithm to do this. The algorithm follows a similar line to the proof of Theorem 9.1, and the reader is advised to read that proof before proceeding. Fix  $r$  and set  $B = B(r)$ . The following recursive procedure,  $\text{constructP}$ , has the property that  $\text{constructP}(\{ \})$  returns *true* if and only if the answer to (8.1) is yes.

```

constructP( P such that  $P \subset B$ ):
  while  $(P \cdot P) \cap B \not\subset P$ :
     $P := (P \cdot P) \cap B$ 
  if  $1 \in P$ :
    return false
  if  $B = P \cup P^{-1} \cup \{1\}$ :
    return true
   $g :=$  a shortest word in  $B - (P \cup P^{-1} \cup \{1\})$ 
  return  $\text{constructP}(P \cup \{g\})$  or  $\text{constructP}(P \cup \{g^{-1}\})$ 

```

Let's get a rough grip on how long this algorithm takes in practice. A bad case is when  $P$  exists as then we have to construct it. Note that even if we're handed  $P$  by an oracle, it still takes about  $(\#P)^2 = (1/4)(\#B)^2$  multiplications to *check* that  $P$  satisfies the conditions in (8.1). If we were to compute a multiplication table for  $B(r)$  in advance (using  $(1/2)(\#B)^2$  multiplications), we could do all further multiplications by table lookup at essentially no cost. So in cases where we end up constructing  $P$ , we can make the running time  $C(\#B)^2$  and this is roughly the best possible.

However, if the answer to (8.1) is no, in practice one needs far fewer multiplications and only a few ( $\leq 4$ ) levels of recursion for the algorithm to finish. So in practice, a good strategy seems to be to stop the algorithm if the recursion depth is greater than 5, and just assume the final answer would have been yes.

The groups we're interested in are the fundamental groups of hyperbolic 3-manifolds. These do have short-lex automatic structures. The problem is that  $B(r)$  has exponential growth with  $\#B(r) \approx AC^r$ . For the 2-generator groups we looked at in Section 10,  $C$  is usually a little less than 3. In practice, the size of  $B(r)$  makes it very difficult to decide (8.1) if  $r$  is bigger than, say, 7. Given this, it is remarkable that among the small volume closed hyperbolic 3-manifolds in the Hodgson-Weeks census there are quite a few whose fundamental groups can be shown to be non-orderable by this method (see Section 10).

The algorithm can be easily modified to keep track of the origin of an element of  $P$  as a product of the elements  $g$  that have been added to  $P$ . This lets one generate a proof that the group is non-orderable. In fact, this was how we found the proof of Theorem 9.2.

## 9. THE WEEKS MANIFOLD

The Weeks manifold is the smallest known hyperbolic 3-manifold. In this section, we will show that its fundamental group cannot act faithfully on  $\mathbb{R}$  or  $S^1$ . The Weeks manifold  $W$  is the  $(5/2, 5/1)$  Dehn surgery on the Whitehead link (our convention is that  $+1$  surgery on a component of this link yields the trefoil). It is arithmetic, and its volume is about 0.942707362776. Its fundamental group is

$$G = \langle a, b \mid bababAb^2A, ababaBa^2B \rangle$$

where  $A = a^{-1}$  and  $B = b^{-1}$ .

We'll begin with the easy case of actions on  $\mathbb{R}$ .

**9.1. Theorem.** *The fundamental group  $G$  of the Weeks manifold is non-orderable.*

*Proof.* Suppose that  $G$  is orderable. Consider the positive cone

$$P = \{g \in G \mid g > 1\}.$$

Then  $P \cdot P \subset P$  and  $G$  is the disjoint union  $P \cup P^{-1} \cup \{1\}$ . Switching  $>$  around if necessary, we can assume that  $a > 1$ , that is  $a \in P$ . We now consider the various possibilities.

**Case  $b \in P$ .** As  $P \cdot P \subset P$ , we have  $ab, bab, abab$ , etc. contained in  $P$ .

**Subcase  $aB \in P$ .** Then  $(abab) \cdot (aB) \cdot a \cdot (aB)$  is in  $P$ . But  $ababaBa^2B = 1$  in  $G$ , and so  $1 \in P$ , a contradiction.

**Subcase  $bA \in P$ .** Then  $(babA) \cdot (bA) \cdot b \cdot (bA)$  is in  $P$ . But  $bababAb^2A = 1$  in  $G$ , and so  $1 \in P$ , a contradiction.

**Case  $B \in P$ .** Using the relations  $R_1$  and  $R_2$  we have

$$BaB^2a^2Ba^2B = b^{-1}R_1^{-1}b \cdot R_2 = 1$$

in  $G$ . But then  $BaB^2a^2Ba^2B \in P$ , a contradiction.

This shows that such a  $P$  does not exist, and so  $G$  is non-orderable.  $\square$

Next we'll show:

**9.2. Theorem.** *The fundamental group of the Weeks manifold has no faithful action on  $S^1$ .*

*Proof.* Suppose that  $G$  acts faithfully on a circle. Since  $H_1(M, \mathbb{Z}/2) = 0$ , the action is orientation preserving. The Euler class  $e$  of the action must be nontrivial as the previous theorem shows that  $G$  can't action faithfully on  $\mathbb{R}$ . Now  $H^2(M) \cong H_1(M) = \mathbb{Z}/5 \oplus \mathbb{Z}/5$ , and so  $e$  has order 5. By Section 7.7, this means that  $G$  acts faithfully on the union of 5 lines. Let  $N$  be the stabilizer of one of the lines. The subgroup  $N$  acts faithfully on that line and so is left-orderable. The action on the set of lines is transitive, since 5 is the smallest multiple of  $e$  which vanishes; thus  $N$  has index 5 in  $G$ . To prove the theorem, we'll show that no subgroup of  $G$  of index 5 is left-orderable.

One can check that the only subgroups of  $G$  of index 5 are normal. So  $N$  is normal and  $G/N \cong \mathbb{Z}/5$ . Thus  $N$  is the kernel of some homomorphism  $G \rightarrow \mathbb{Z}/5$ , and there are 6 possibilities for  $N$ . Consider the automorphisms of  $G$  given by

$$\phi: a \mapsto b, b \mapsto a \quad \text{and} \quad \psi: a \mapsto aB, b \mapsto a.$$

(To check that  $\psi$  induces an automorphism, use the relation  $BaB^2a^2Ba^2B = 1$  mentioned above.) Together,  $\phi$  and  $\psi$  generate a subgroup of the outer automorphism group isomorphic to  $D_6$  (in fact, this is the whole isometry group of  $W$ ).

In any event, it is easy to check that under the action of  $\langle \phi, \psi \rangle$  there are two orbits of subgroups of index 5 in  $G$ . Representatives of these orbits are  $N_1$ , the kernel of the homomorphism sending  $a \mapsto 0$  and  $b \mapsto 1$ , and  $N_2$ , the kernel of  $a \mapsto 1$  and  $b \mapsto -1$ .

First we'll show that  $N_1$  is non-orderable. Let  $P$  be the positive cone, and assume that  $a \in P$ . The proof is completed by considering the following cases. The needed identities in  $G$  can be checked in number of ways, e.g. by using automatic groups, multiplying matrices, or for the determined, the relations and a chalkboard.

**Case**  $baB \in P$ . Then  $a(baB)^3(a^3(baB))^2a^2(baB)(a^3(baB))^2(baB)^2 = 1$  in  $P$ .

**Case**  $bAB \in P$ .

**Subcase**  $abaB \in P$ . Then

$$a^2(abaB)a^2(bAB)^2a(abaB)a(a(abaB))^2(a^2(abaB))^2a(abaB) = 1 \text{ in } P.$$

**Subcase**  $bABA \in P$ . Set  $W = ((bAB)(bABA)(bAB))^2$ . Then

$$a^2W(bABA)^2a(bAB)(bABA)(bAB)^2(bABA)W(bAB) = 1 \text{ in } P.$$

This completes the proof that  $N_1$  is non-orderable.

Now consider  $N_2$ . We can assume that  $ab \in P$ . The cases are:

**Case**  $ba \in P$ . Then

**Subcase**  $a^2bA \in P$ . Then

$$(ab)(a^2bA)^2(ab)^2(a^2bA)^2((ab)^2(ba)^2)^2 = 1 \text{ in } P.$$

**Subcase**  $aBA^2 \in P$ . Then

$$(aBA^2)^2(ba)^2(ab)^2(ba)^2(ab)(aBA^2)(ba)^2((ab)^2(ba)^2(ab)^2(ba))^2 = 1 \text{ in } P.$$

**Case**  $AB \in P$ .

**Subcase**  $a^2bA \in P$ . Then

$$((a^2bA)(ab)^2(a^2bA)^2(ab)^2)^2(a^2bA)^2(AB)(ab)(a^2bA)^2(ab)^2(a^2bA)^2(AB)^2 = 1 \text{ in } P.$$

**Subcase**  $aBA^2 \in P$ . Then  $(ab)(AB)^3(aBA^2)^3 = 1$  in  $P$ .

This proves that  $N_2$  is non-orderable. This completes the proof that  $G$  does not act faithfully on  $S^1$ .  $\square$

In fact, one can show more:

**9.3. Theorem.** *Let  $G$  be the fundamental group of the Weeks manifold. Then any homomorphism  $G \rightarrow \text{Homeo}(S^1)$  has image which is either trivial or  $\mathbb{Z}/5\mathbb{Z}$ .*

*Proof.* Consider a nontrivial action of  $G$  on the circle. Let  $K$  be the kernel of  $G \rightarrow \text{Homeo}(S^1)$ . We can apply the proofs of Theorems 9.1 and 9.2 to  $G/K$  *unless* one of the elements we added to  $P$  is in  $K$ . In other words, we have a contradiction unless one of

$$a, b, aB, bA, B, baB, bAB, abaB, bABA, ba, a^2bA, aBA^2, AB, a^2bA, aBA^2$$

is in  $K$ . But it is easy to check that the quotient of  $G$  by the normal closure of any of the above words is  $\mathbb{Z}/5\mathbb{Z}$ . So  $G/K = \mathbb{Z}/5\mathbb{Z}$  and we're done.  $\square$

As a corollary of these non-existence results, we get the following:

**9.4. Corollary.** *The Weeks manifold does not admit a taut foliation, a tight essential lamination, or a pseudo-Anosov flow.*

*Proof.* As the fundamental group of the Weeks manifold can't act on  $S^1$ , the first and last assertions follow immediately from Theorem 6.3 and Corollary 3.9. For tight essential laminations, though, to apply Theorem 3.2 we need the additional hypothesis that the lamination has solid torus guts. So it remains to show:

**9.5. Claim.** *Let  $\Lambda$  be a tight genuine lamination of the Weeks manifold  $W$ . Then the complementary regions of  $\Lambda$  have a decomposition where all the gut regions are solid tori.*

This will follow from Agol's work on volumes of 3-manifolds with tight laminations. A decomposition of  $W - \Lambda$  into interstices and guts has *minimal guts* if in every complementary region  $C$ , the interstitial regions are the whole characteristic I-bundle of  $C$ . Such a decomposition is unique up to isotopy. In [Agol], Agol shows:

**9.6. Theorem (Agol).** *Let  $M$  be a hyperbolic 3-manifold, and  $\Lambda$  a tight genuine lamination. Let  $G$  be the guts of the minimal decomposition of  $M - \Lambda$ . Then*

$$\text{vol}(M) \geq -2v_3\chi(G),$$

where  $v_3$  is the volume of the regular ideal tetrahedron in  $\mathbb{H}^3$ .

So now let  $G_i$  be a gut region of the minimal decomposition of  $W - \Lambda$ . By Agol's theorem, we must have  $\chi(G_i) = 0$  as  $\text{vol}(W) \approx 0.942 < 2v_3 \approx 2.02988$ . Thus the boundary of  $G_i$  consists of tori. The boundary of  $G_i$  is incompressible outward into  $W$ , and so as  $W$  is atoroidal, the boundary of  $G_i$  is compressible into  $G_i$ . Therefore  $G_i$  is a solid torus. This proves the claim, and thus the corollary.  $\square$

## 10. FURTHER EXAMPLES

For the smallest manifolds in the Hodgson-Weeks census [W] of closed hyperbolic 3-manifolds we tried to determine which act faithfully on  $\mathbb{R}$ . We looked at the census manifolds of volume  $< 3$  which are  $\mathbb{Z}/2$ -homology spheres (so any action would be orientation preserving). There are 128 such manifolds. Using the algorithm of Section 8, we showed that at least 44 of them can't act faithfully on  $\mathbb{R}$ . Conversely, we showed that at least 3 of them have such actions. We also found that at least 60 have essential laminations. See the table below for details. We would have liked to give more examples where the fundamental group can't act faithfully on  $S^1$ , but the only manifold where we were able to succeed at this was the Weeks manifold.

The algorithm was implemented starting from group presentations generated by SnapPea and using the automatic groups program KBMAG [Holt] to solve the word problem.

To show manifolds contain essential laminations, we used two techniques. The first is that for many simple knots, such as two-bridge knots or most knots under 11 crossings, every nontrivial Dehn surgery contains an essential lamination (see [Gab, pg. 8] and the references therein). In particular, this is true for the knots that appear in the table below. In some cases, such as for two-bridge knots, one can take the lamination to be a taut foliation [Del].

The second technique also uses Dehn filling, but is more complicated to explain. Let  $M$  be a 3-manifold with torus boundary whose interior is hyperbolic. Gabai and Mosher [Mos] proved that  $M$  contains a pair of laminations  $\Lambda^\pm$  coming from

a pseudo-Anosov flow associated to a finite depth foliation. When  $M$  fibers over  $S^1$ , these are just the suspensions of the stable and unstable laminations of the surface homeomorphism. There is a *degeneracy slope* in  $\partial M$  associated to  $\Lambda^\pm$  with the following property: for any slope  $\alpha$  with  $\Delta(\delta, \alpha) > 1$ , the laminations  $\Lambda^\pm$  remain essential in the Dehn filling  $M(\alpha)$  (for more see [Mos, Thm. B] or the summary of this theory in [Bri2]).

In general, it is quite difficult to determine the degeneracy slope. Here, we used the following trick. Suppose  $M$  has Dehn fillings  $M(\alpha_1), \dots, M(\alpha_n)$  which have finite fundamental group. The  $M(\alpha_i)$  don't contain any essential laminations. Now consider some other filling  $M(\beta)$ . If  $M(\beta)$  also has no essential lamination, we must have  $\Delta(\alpha_i, \delta) \leq 1$  and  $\Delta(\beta, \delta) \leq 1$  where  $\delta$  is the degeneracy slope. So if there does not exist such a  $\delta$ , we can conclude that  $M(\beta)$  has an essential lamination. What's more, that essential lamination in  $M(\beta)$  is very full, because the laminations  $\Lambda^\pm$  are themselves very full.

Below is the table summarizing our findings. In the Ord column, N means that the fundamental group is non-orderable, O means that it is orderable, and blank means unknown. The Lam column lists an L if the manifold is known to contain an essential lamination. If the manifold is known to contain an essential lamination, the final column gives the reason. In that column,  $K(p/q)$  means the complement of the  $p/q$  two-bridge knot, and numbers of the form  $\delta_{20}$  refer to the complements of the corresponding knots in the standard table [Rol]. For the trick with the degeneracy slope, we give the particular expression as a Dehn filling that was used.

Finally, the reason that the indicated manifolds have orderable fundamental group is that they have taut foliations and are integral homology spheres. In such cases, the action on the universal circle lifts to a faithful action on  $\mathbb{R}$ .

Name	Volume	Hom	Ord	Lam	Reason for knowing laminar
m003(-3, 1)	0.9427073628	$\mathbb{Z}/5 + \mathbb{Z}/5$	N		
m003(-2, 3)	0.9813688289	$\mathbb{Z}/5$		L	Is m004(5, 1) and m004 is $K(2/5)$ .
m003(-4, 3)	1.2637092387	$\mathbb{Z}/5 + \mathbb{Z}/5$	N	L	Degeneracy test as m003(-4, 3).
m004(1, 2)	1.3985088842	0	O	L	Is m004(1, 2) and m004 is $K(2/5)$ .
m003(-4, 1)	1.4236119003	$\mathbb{Z}/35$	N		
m004(3, 2)	1.4406990067	$\mathbb{Z}/3$		L	Is m004(3, 2) and m004 is $K(2/5)$ .
m004(7, 1)	1.4637766449	$\mathbb{Z}/7$		L	Is m004(7, 1) and m004 is $K(2/5)$ .
m004(5, 2)	1.5294773294	$\mathbb{Z}/5$		L	Is m004(5, 2) and m004 is $K(2/5)$ .
m003(-5, 3)	1.5435689115	$\mathbb{Z}/35$	N	L	Degeneracy test as m003(-5, 3).
m007(1, 2)	1.5435689115	$\mathbb{Z}/21$	N	L	Degeneracy test as m011(3, 2).
m007(4, 1)	1.5831666606	$\mathbb{Z}/21$	N		
m007(3, 2)	1.5831666606	$\mathbb{Z}/3 + \mathbb{Z}/9$	N		
m006(-3, 2)	1.6496097158	$\mathbb{Z}/15$	N	L	Degeneracy test as m006(-3, 2).
m015(5, 1)	1.7571260292	$\mathbb{Z}/7$		L	Is m015(5, 1) and m015 is $K(-2/7)$ .
m007(-3, 2)	1.8243443222	$\mathbb{Z}/3 + \mathbb{Z}/3$	N	L	Degeneracy test as m007(-3, 2).
m016(-3, 2)	1.8854147256	$\mathbb{Z}/39$	N	L	Degeneracy test as m016(-3, 2).
m017(-3, 2)	1.8854147256	$\mathbb{Z}/7 + \mathbb{Z}/7$	N	L	Degeneracy test as m017(-3, 2).
m006(3, 2)	1.8859142560	$\mathbb{Z}/45$	N	L	Degeneracy test as m006(3, 2).
m011(2, 3)	1.9122102501	0		L	Is m222(-2, 1) and m222 is $\delta_{20}$ .
m006(4, 1)	1.9222971095	$\mathbb{Z}/35$	N		
m006(-2, 3)	1.9537083154	$\mathbb{Z}/35$	N	L	Degeneracy test as m006(-2, 3).
m006(2, 3)	1.9627376578	$\mathbb{Z}/55$	N	L	Degeneracy test as m006(2, 3).
m017(-1, 3)	1.9627376578	$\mathbb{Z}/7 + \mathbb{Z}/7$	N		

Name	Volume	Hom	Ord	Lam	Reason for knowing laminar
m023(-4, 1)	2.0143365838	$\mathbb{Z}/3 + \mathbb{Z}/3$			
m007(5, 2)	2.0259452819	$\mathbb{Z}/33$	N		
m006(-5, 2)	2.0288530915	$\mathbb{Z}/5$		L	Is m015(1, 2) and m015 is $K(-2/7)$ .
m036(-3, 2)	2.0298832128	$\mathbb{Z}/3 + \mathbb{Z}/15$			
m007(-6, 1)	2.0555467489	$\mathbb{Z}/3 + \mathbb{Z}/3$			
m007(-5, 2)	2.0656708385	$\mathbb{Z}/3$		L	Is m015(-1, 2) and m015 is $K(-2/7)$ .
m015(-5, 1)	2.1030952907	$\mathbb{Z}/3$		L	Is m015(-5, 1) and m015 is $K(-2/7)$ .
m016(3, 2)	2.1145676931	$\mathbb{Z}/33$	N	L	Degeneracy test as m016(3, 2).
m015(3, 2)	2.1145676931	$\mathbb{Z}/7$		L	Is m015(3, 2) and m015 is $K(-2/7)$ .
m011(4, 1)	2.1243017573	$\mathbb{Z}/43$			
m017(1, 3)	2.1557385676	$\mathbb{Z}/35$	N		
m011(-2, 3)	2.1557385676	$\mathbb{Z}/53$	N	L	Degeneracy test as m011(-2, 3).
m034(4, 1)	2.1847555751	$\mathbb{Z}/7$		L	Is s385(-2, 1) and s385 is $10_{125}$ .
m034(-4, 1)	2.1959641187	$\mathbb{Z}/25$	N		
m011(-3, 2)	2.2082823597	$\mathbb{Z}/57$	N	L	Degeneracy test as m011(-3, 2).
m011(4, 3)	2.2102443409	$\mathbb{Z}/25$		L	Degeneracy test as m011(4, 3).
m011(1, 4)	2.2109517391	$\mathbb{Z}/23$		L	Degeneracy test as m011(1, 4).
m015(-3, 2)	2.2267179039	0	O	L	Is m015(-3, 2) and m015 is $K(-2/7)$ .
m015(7, 1)	2.2267179039	$\mathbb{Z}/9$		L	Is m015(7, 1) and m015 is $K(-2/7)$ .
m038(1, 2)	2.2597671326	0		L	Is m372(-2, 1) and m372 is $9_{46}$ .
m015(5, 2)	2.2662435733	$\mathbb{Z}/9$		L	Is m015(5, 2) and m015 is $K(-2/7)$ .
m026(-4, 1)	2.2726318636	$\mathbb{Z}/13$			
m011(-1, 4)	2.2757758101	$\mathbb{Z}/49$	N	L	Degeneracy test as m011(-1, 4).
m023(-3, 2)	2.2944383001	$\mathbb{Z}/3$		L	Is m032(5, 1) and m032 is $K(-2/9)$ .
m038(-5, 1)	2.3126354033	$\mathbb{Z}/17$			
m017(-5, 1)	2.3188118677	$\mathbb{Z}/7 + \mathbb{Z}/7$	N	L	Degeneracy test as m022(-3, 2).
m016(-5, 1)	2.3188118677	$\mathbb{Z}/23$			
m019(4, 1)	2.3207602675	$\mathbb{Z}/7$		L	Is m199(3, 1) and m199 is $9_{42}$ .
m022(1, 3)	2.3380401178	$\mathbb{Z}/35$	N		
m016(-1, 4)	2.3522069054	$\mathbb{Z}/73$	N	L	Degeneracy test as m026(2, 3).
m017(-1, 4)	2.3522069054	$\mathbb{Z}/63$	N		
m019(-2, 3)	2.3641969332	$\mathbb{Z}/63$	N	L	Degeneracy test as m019(-2, 3).
m022(5, 1)	2.3705924006	$\mathbb{Z}/3 + \mathbb{Z}/7$			
m019(-4, 1)	2.3803358221	$\mathbb{Z}/41$	N	L	Degeneracy test as m026(-2, 3).
m022(5, 2)	2.4224625169	$\mathbb{Z}/7$		L	Is m032(-5, 1) and m032 is $K(-2/9)$ .
m019(4, 3)	2.4444077795	$\mathbb{Z}/27$		L	Degeneracy test as m019(4, 3).
m022(-1, 3)	2.4540294422	$\mathbb{Z}/7 + \mathbb{Z}/7$			
m026(4, 1)	2.4631393944	$\mathbb{Z}/51$			
m029(-3, 2)	2.4682321967	$\mathbb{Z}/5 + \mathbb{Z}/9$	N		
m036(3, 2)	2.4682321967	$\mathbb{Z}/3 + \mathbb{Z}/9$	N	L	Degeneracy test as m036(3, 2).
m022(-5, 1)	2.4878225918	$\mathbb{Z}/7 + \mathbb{Z}/7$	N		
m023(-6, 1)	2.4903791858	$\mathbb{Z}/15$			
m038(3, 2)	2.5026593054	$\mathbb{Z}/5$		L	Is m289(2, 1) and m289 is $K(-3/11)$ .
m034(-5, 1)	2.5065758445	$\mathbb{Z}/29$	N		
m034(5, 1)	2.5144043349	$\mathbb{Z}/11$			
m070(-3, 1)	2.5274184773	$\mathbb{Z}/11$		L	Degeneracy test as m117(-3, 2).
m038(-5, 2)	2.5274184773	$\mathbb{Z}/19$			
m036(-5, 1)	2.5274184773	$\mathbb{Z}/33$			
m030(5, 2)	2.5303032876	$\mathbb{Z}/63$	N		
m023(-5, 2)	2.5415850101	$\mathbb{Z}/3 + \mathbb{Z}/3$			
m038(5, 1)	2.5495466001	$\mathbb{Z}/13$			
m026(-5, 1)	2.5667347900	$\mathbb{Z}/21$			

Name	Volume	Hom	Ord	Lam	Reason for knowing laminar
m160(1, 2)	2.5689706009	$\mathbb{Z}/3 + \mathbb{Z}/5$			
m036(-1, 3)	2.5751620736	$\mathbb{Z}/57$			
m030(1, 3)	2.5854830480	$\mathbb{Z}/7 + \mathbb{Z}/7$		L	Degeneracy test as m030(1, 3).
m160(-3, 2)	2.5953875937	$\mathbb{Z}/3 + \mathbb{Z}/9$	N		
m036(-5, 2)	2.6095439552	$\mathbb{Z}/51$			
m027(-4, 1)	2.6122234482	$\mathbb{Z}/77$			
m027(4, 3)	2.6172815707	$\mathbb{Z}/25$		L	Degeneracy test as m027(4, 3).
m081(1, 3)	2.6244624283	$\mathbb{Z}/37$	N		
m036(5, 1)	2.6285738915	$\mathbb{Z}/3$		L	Is s580(-2, 1) and s580 is $10_{145}$ .
m032(5, 2)	2.6294053953	0	O	L	Is m032(5, 2) and m032 is $K(-2/9)$ .
m034(-1, 3)	2.6414714456	$\mathbb{Z}/31$		L	Degeneracy test as m034(-1, 3).
m036(1, 3)	2.6536080625	$\mathbb{Z}/51$		L	Degeneracy test as m082(-3, 2).
m034(-2, 3)	2.6555425236	$\mathbb{Z}/35$			
m034(1, 3)	2.6646126469	$\mathbb{Z}/23$		L	Degeneracy test as m034(1, 3).
m160(2, 1)	2.6735274161	$\mathbb{Z}/3$		L	Is m372(2, 1) and m372 is $9_{46}$ .
m032(7, 1)	2.6822267321	$\mathbb{Z}/5$		L	Is m032(7, 1) and m032 is $K(-2/9)$ .
m069(4, 1)	2.6954841673	$\mathbb{Z}/65$	N	L	Degeneracy test as m081(-3, 2).
m069(-1, 3)	2.6954841673	$\mathbb{Z}/39$			
m030(5, 3)	2.7067833105	$\mathbb{Z}/77$	N	L	Is Haken. See [Dun].
m120(-3, 2)	2.7124588084	0		L	Is m199(-3, 1) and m199 is $9_{42}$ .
m116(-1, 3)	2.7589634387	$\mathbb{Z}/7$		L	Is s580(2, 1) and s580 is $10_{145}$ .
m081(-1, 3)	2.7725163132	$\mathbb{Z}/59$	N		
m160(-2, 3)	2.8022537823	$\mathbb{Z}/3 + \mathbb{Z}/11$			
m221(3, 1)	2.8281220883	$\mathbb{Z}/21$			
m142(3, 2)	2.8281220883	$\mathbb{Z}/19$			
m206(1, 2)	2.8281220883	$\mathbb{Z}/5$			
m082(2, 3)	2.8458961160	$\mathbb{Z}/83$	N		
m070(4, 3)	2.8472238006	$\mathbb{Z}/85$	N		
m069(4, 3)	2.8472238006	$\mathbb{Z}/99$			
m137(-5, 1)	2.8656302333	0			
m070(-2, 3)	2.8669017766	$\mathbb{Z}/61$	N	L	Degeneracy test as m070(-2, 3).
m069(-2, 3)	2.8669017766	$\mathbb{Z}/27$		L	Degeneracy test as m069(-2, 3).
m069(-4, 1)	2.8733431176	$\mathbb{Z}/31$			
m070(-4, 1)	2.8733431176	$\mathbb{Z}/7$			
m100(2, 3)	2.8824943873	$\mathbb{Z}/85$			Is Haken. See [Dun].
m082(-2, 3)	2.9027039980	$\mathbb{Z}/79$		L	Degeneracy test as m082(-2, 3).
m221(-1, 2)	2.9133321143	$\mathbb{Z}/7$			
m116(1, 3)	2.9169341134	$\mathbb{Z}/41$			
m120(-5, 1)	2.9356518985	$\mathbb{Z}/17$			
m078(2, 3)	2.9398104423	$\mathbb{Z}/37$			
m145(2, 3)	2.9400386172	$\mathbb{Z}/47$	N		
m078(5, 2)	2.9438596478	$\mathbb{Z}/43$			
m249(3, 1)	2.9545326040	$\mathbb{Z}/3 + \mathbb{Z}/5$			
m145(3, 2)	2.9582502906	$\mathbb{Z}/13$			
m117(3, 2)	2.9605565159	$\mathbb{Z}/53$	N		
m117(-5, 1)	2.9607151670	$\mathbb{Z}/19$			
m154(2, 3)	2.9670703390	$\mathbb{Z}/77$			
m078(-2, 3)	2.9696321386	$\mathbb{Z}/17$			
m100(-2, 3)	2.9709840073	$\mathbb{Z}/77$		L	Degeneracy test as m100(-2, 3).
m117(1, 3)	2.9760925194	$\mathbb{Z}/55$			
m078(-5, 2)	2.9769925267	$\mathbb{Z}/7$		L	Is m199(-1, 2) and m199 is $9_{42}$ .
m159(3, 2)	2.9781624873	$\mathbb{Z}/35$			

Name	Volume	Hom	Ord	Lam	Reason for knowing laminar
m137(5, 1)	2.9868370451	0			

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