EQUIVARIANT DERIVED CATEGORIES FOR TOROIDAL GROUP IMBEDDINGS

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ABSTRACT. Let X denote a projective variety over an algebraically closed field on which a linear algebraic group acts with finitely many orbits. Then, a conjecture of Soergel and Lunts in the setting of Koszul duality and Langlands' philosophy, postulates that the equivariant derived category of bounded complexes with constructible equivariant cohomology sheaves on X is equivalent to a full subcategory of the derived category of modules over a graded ring defined as a suitable graded *Ext*. Only special cases of this conjecture have been proven so far. *The purpose of this paper is to provide a proof of this conjecture for all projective toroidal imbeddings of complex reductive groups.* In fact, we show that the methods used by Lunts for a proof in the case of toric varieties can be extended with suitable modifications to handle the toroidal imbedding case. *Since every equivariant imbedding of a complex reductive group is dominated by a toroidal imbedding, the class of varieties for which our proof applies is quite large.*

We also show that, in general, there exist a countable number of obstructions for this conjecture to be true and that half of these vanish when the odd dimensional equivariant intersection cohomology sheaves on the orbit closures vanish. This last vanishing condition had been proven to be true in many cases of spherical varieties by Michel Brion and the author in prior work.

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1. **Introduction**

This paper concerns a variant of a conjecture attributed to Soergel and Lunts (see [So98], [So01], [Lu95]) for the action of linear algebraic groups on projective varieties with finitely many orbits and over any algebraically closed field k. Let X denote such a projective variety provided with the action of a linear algebraic group G. Recall that a sheaf F on X is equivariant, if it satisfies the following condition: let $\mu, pr_2 : \text{G} \times \text{X} \to \text{X}$ denote the group-action and projection to the second factor, respectively. Then there is given an isomorphism $\phi : \mu^*(F) \to pr_2^*(F)$ satisfying a co-cycle condition on further pull-back to $\text{G} \times \text{G} \times \text{X}$, and which reduces to the identity on pull-back to X by the degeneracy map $x \mapsto (e, x)$. The conjecture asserts that the equivariant derived category $\text{D}_{\text{G},c}^{\text{b}}(\text{X})$ of complexes of sheaves with bounded, equivariant and constructible cohomology sheaves is equivalent to a full subcategory of the derived category of differential graded-modules over a certain differential graded algebra defined as a graded *Ext* ring.

More precisely, the conjecture states the following. Let $\text{C}_{\text{G}.c}^{b}(\text{X})$ denote the category of complexes on a suitable Borel construction $\text{EG} \underset{G}{\times} \text{X}$ associated to X, and with bounded constructible G-equivariant cohomology sheaves. Let L_i denote the the equivariant intersection cohomology complex on the closure of a G-orbit on X, obtained by perverse extension of an irreducible G-equivariant local systems on the corresponding orbit. *We will next replace each such L_i by a complex of injective sheaves in $\text{C}_{\text{G}.c}^{b}(\text{X})$ upto quasi-isomorphism and fixed throughout the paper, but will continue to denote the corresponding complex of injective sheaves by L_i itself.* Let $L = \oplus_i L_i$ and let $\mathcal{B} = \mathcal{B}_{\text{G}}(\text{X}) = Hom(L, L)$ $(= RHom(L, L))$ denote the differential graded algebra where the multiplication is given by composition, and where the *Hom* is computed in $\text{C}_{\text{G},c}^{b}(\text{X})$. (It is straightforward to verify that a different choice of a complex of injective sheaves replacing the L_i up to quasi-isomorphism will provide a quasi-isomorphic differential graded algebra.) Let $Mod(\mathcal{B})$ denote the category of differential graded modules over the differential graded algebra $\mathcal{B}$. Then each $E_i = Hom(L, L_i)$ is an object of $Mod(\mathcal{B})$, and the conjecture says that $\text{D}_{\text{G},c}^{\text{b}}(\text{X})$ is equivalent to the full subcategory of $\text{DMod}(\mathcal{B})$ generated by the E_i and that $\mathcal{B}$ is formal as a differential graded algebra, that is, it is quasi-isomorphic as a differential graded algebra to its cohomology algebra.

When k is the field of complex numbers, one can take the equivariant derived category to be made up of complexes of sheaves of $\mathbb{Q}$-vector spaces and in positive characteristics, one could consider instead the ℓ-adic derived category considered in [Jo93]. Only special cases of this conjecture have been proven so far, all of them for certain classes of complex spherical varieties: the case of toric varieties was considered in [Lu95], the case of smooth complete symmetric G-varieties for the action of a semi-simple adjoint group G was considered in [Gu05] and [Sc11] considered the case of complex flag varieties.

The main goal of this paper is to prove this conjecture for an important sub-class of complex projective spherical varieties, namely, toroidal imbeddings of complex reductive groups.

• Spherical varieties associated to reductive groups, forms a large class of algebraic varieties that includes as special cases, both flag varieties as well as toric varieties, as two extreme cases. Moreover, spherical varieties are constructed using the combinatorial data of colored fans, similar to how toric varieties are constructed from fans. As pointed out above, the conjecture has been already verified for both projective toric varieties and also for flag varieties. Therefore, as the next step, it is important to consider the validity of the conjecture for other important sub-classes of spherical varieties.

• Recall, (see [BK05, Prop. 6.2.5]), that given a complex connected reductive group G and any projective $\text{G} \times \text{G}$-equivariant imbedding X of G, there exists a projective toroidal imbedding $\tilde{\text{X}}$ of G together with a $\text{G} \times \text{G}$-equivariant birational map $\tilde{\text{X}} \to \text{X}$. The above observation shows that the class of projective toroidal imbeddings is, in fact, an important subclass of spherical varieties.

• Moreover, toroidal imbeddings of reductive groups forms a sub-class of spherical varieties that is closest to toric varieties, in the sense that they are also classified by fans and the local structure is that of toric varieties: see [Ti11, 29.1] or [BK05, Sec. 6.2.2]. (More details on toroidal imbeddings, along with some examples may be found in section 2.2.) Thus, our main result, that is, Theorem 1.2, shows that the above mentioned conjecture is indeed true for an important large class of spherical varieties.

Along the way, the same techniques allow us to prove part of the conjecture for horospherical varieties. However, we have decided it may be preferable, for a variety of reasons, to discuss the case of horospherical varieties separately elsewhere. It may be also important to point out that the above conjecture also holds in positive characteristics, though hardly any results are known in this case at present. The reason we restrict to complex spherical varieties in this paper, is largely because we make use of a reduction to the action by a compact group as in Lemma 2.7, though most of our remaining arguments seem to extend to positive characteristics using ℓ-adic étale cohomology in the place of singular cohomology. Moreover, a proof of the conjecture for projective toric varieties in positive characteristics seems to be in place, using certain stack-theoretic machinery in the place of the arguments in [Lu95]. *In view of these, we provide a discussion of equivariant derived categories in section 5 that works in all characteristics, but will restrict to complex algebraic varieties for the rest of the paper.*

Next assume that X is a variety on which a linear algebraic group G acts with only finitely many orbits. For each orbit $\mathcal{O}$, let $\mathcal{L}_\mathcal{O}$ denote an irreducible G-equivariant local system on $\mathcal{O}$. One may identify each $\mathcal{L}_\mathcal{O}$ with its extension by zero to an equivariant sheaf on all of X. Since the cohomology sheaves of any complex $K \in \mathrm{D}^\mathrm{b}_{\mathrm{G,c}}(\mathrm{X})$ are G-equivariant, it is clear that the set $\{\mathcal{L}_\mathcal{O}|\mathcal{O}\}$ of all such G-equivariant local systems as one varies over the G-orbits, forms a set of generators for the derived category $\mathrm{D}^\mathrm{b}_{\mathrm{G,c}}(\mathrm{X})$ in the sense that the smallest triangulated subcategory containing all of $\{\mathcal{L}_\mathcal{O}|\mathcal{O}\}$ and closed under finite sums is $\mathrm{D}^\mathrm{b}_{\mathrm{G,c}}(\mathrm{X})$.[1] Let $\mathrm{IC}^\mathrm{G}(\mathcal{L}_\mathcal{O}[dim\mathcal{O}])$ denote the equivariant intersection cohomology complex on $\bar{\mathcal{O}}$ obtained by starting with the G-equivariant local system $\mathcal{L}_\mathcal{O}$ on $\mathcal{O}$. Since the restriction of each $\mathrm{IC}^\mathrm{G}(\mathcal{L}_\mathcal{O}[dim\mathcal{O}])$ to the orbit $\mathcal{O}$ is the local system $\mathcal{L}_\mathcal{O}$, the set $\{\mathrm{IC}^\mathrm{G}(\mathcal{L}_\mathcal{O}[dim\mathcal{O}])|\mathcal{O}, \mathcal{L}_\mathcal{O}\}$ also generates $\mathrm{D}^\mathrm{b}_{\mathrm{G,c}}(\mathrm{X})$. Let $L_1, \cdots, L_n$ denote the above collection of equivariant intersection cohomology complexes. *We will next replace each such L_i by a complex of injective sheaves in $\mathrm{C}^b_{\mathrm{G.c}}(\mathrm{X})$, but will continue to denote the corresponding complex of injective sheaves by L_i itself. Let $L = \oplus_i L_i$.*

Next let $\mathcal{B}_\mathrm{G}(\mathrm{X}) = Hom(L, L)$. Let $\mathrm{D}(\mathrm{Mod}(\mathcal{B}_\mathrm{G}(\mathrm{X})))$ denote the derived category of differential graded modules over the differential graded algebra $\mathcal{B}_\mathrm{G}(\mathrm{X})$. Then we begin with the following theorem, which may be deduced readily using derived Morita theory. (See also [Lu95, (0.3.1) Proposition] for a variant.) Therefore, we skip its proof.

Theorem 1.1. *Let* X *denote a variety on which a linear algebraic group acts with finitely many orbits. Then, sending $K \in \mathrm{D}^\mathrm{b}_{\mathrm{G,c}}(\mathrm{X}) \mapsto RHom(L, K)$ defines a fully-faithful imbedding of the equivariant derived category $\mathrm{D}^\mathrm{b}_{\mathrm{G,c}}(\mathrm{X})$ into the derived category $\mathrm{D}(\mathrm{Mod}(\mathcal{B}_\mathrm{G}(\mathrm{X})))$. If E_i denotes the image of L_i under this imbedding and if $\mathrm{D}^\mathrm{f}(\mathrm{Mod}(\mathcal{B}_\mathrm{G}(\mathrm{X})))$ denotes the triangulated full sub-category of $\mathrm{D}(\mathrm{Mod}(\mathcal{B}_\mathrm{G}(\mathrm{X})))$ generated by the E_i, $i = 1, \cdots, n$, then $\mathrm{D}^\mathrm{b}_{\mathrm{G,c}}(\mathrm{X})$ is equivalent to $\mathrm{D}^\mathrm{f}(\mathrm{Mod}(\mathcal{B}_\mathrm{G}(\mathrm{X})))$.*

With the above theorem in place, it remains to prove that $\mathcal{B}_\mathrm{G}(\mathrm{X})$ is formal as a differential graded algebra, when X is a projective variety satisfying further assumptions. Sections 2 and 3 are devoted to a detailed discussion of toroidal group imbeddings in characteristic 0, where we prove the formality of the corresponding differential graded algebra thereby settling the conjecture in this case and resulting in the following theorem.

Theorem 1.2. *Let $\bar{\mathrm{G}}$ denote a projective toroidal imbedding of the complex connected reductive group* G *and let $\mathcal{B}_{\mathrm{G\times G}}(\bar{\mathrm{G}})$ denote the differential graded algebra $Hom(L, L)$ $(= RHom(L, L))$ considered above. Then $\mathcal{B}_{\mathrm{G\times G}}(\bar{\mathrm{G}})$ is formal as a differential graded algebra.*

Let $\bar{\mathrm{G}}$ denote such a projective toroidal imbedding of G and let $\bar{\mathrm{T}}$ denote the closure of a maximal torus T in $\bar{\mathrm{G}}$. Our strategy here is to reduce to the case of the toric variety $\bar{\mathrm{T}}$: the picture is however, much more involved than the toric case considered in [Lu95], since one also has an action of the Weyl group on $\bar{\mathrm{T}}$ that needs to be considered, and considerable effort is needed to separate out the Weyl group action from the torus action. We break the entire argument into the following key steps:

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[1]The equivariant derived categories we consider in this paper will always be defined making use of the simplicial Borel construction. The reason for this choice, as well as a comparison with other models of equivariant derived categories, such as those considered in [BL94] is discussed towards the end of this introduction.

1.1. *Step 1.* Let N denote the normalizer of T in G and let diag(T) (resp. diag(N)) denote the image of T (resp. N) in $G \times G$ under the diagonal imbedding of G. Then, the $G \times G$-action on $\bar{G}$ induces an action of $(T \times T)\mathrm{diag}(N)$ on $\bar{T}$. Restricting the $G \times G$-action on $\bar{G}$ to an action by $(T \times T)\mathrm{diag}(N)$ on $\bar{T}$ induces a fully-faithful functor

$$(1.1.1) \qquad\qquad D^b_{G \times G,c}(\bar{G}) \overset{\mathrm{res}}{\to} D^b_{(T \times T)\mathrm{diag}(N),c}(\bar{T}).$$

Moreover, if $D^{b,o}_{(T \times T)\mathrm{diag}(N),c}(\bar{T})$ denotes the full subcategory of $D^b_{(T \times T)\mathrm{diag}(N),c}(\bar{T})$ generated by the $(T \times T)\mathrm{diag}(N)$-equivariant sheaves that are constant along the orbits of $(T \times T)\mathrm{diag}(N)$ on $\bar{T}$, then the same functor induces an equivalence

$$(1.1.2) \qquad\qquad D^b_{G \times G,c}(\bar{G}) \overset{\mathrm{res}}{\to} D^{b,o}_{(T \times T)\mathrm{diag}(N),c}(\bar{T}).$$

(See Theorem 2.5 for further details.)

1.2. *Step 2.* Next we observe that the subgroup $\mathrm{diag}(T) \subseteq (T \times T)\mathrm{diag}(N)$ acts trivially on $\bar{T}$. The quotient $(T \times T)\mathrm{diag}(N)/\mathrm{diag}(T)$ identifies with WT, which is the semi-direct product of W and T, where the W action on T is induced from the action of N(T) on T. For the following discussion, we will abbreviate $(T \times T)\mathrm{diag}(N))$ to $\tilde{N}$. Since $\mathrm{diag}(T)$ acts trivially on $\bar{T}$, one obtains an induced action of WT on $\bar{T}$. Let

$$(1.2.1) \qquad\qquad \psi : E\tilde{N} \underset{\tilde{N}}{\times} \bar{T} \to EWT \underset{WT}{\times} \bar{T}$$

denote the map induced by the identity on $\bar{T}$ and the quotient map $\tilde{N} \to WT$. Since $\mathrm{diag}(T)$ acts trivially on $\bar{T}$, the fibers of the map at every point on $\bar{T}$ can be identified with $B\mathrm{diag}(T)$. Let $G^\bullet$ denote the canonical Godement resolution. Then, clearly the functor $R\psi_* = \psi_* G^\bullet$ sends complexes in $D^{b,o}_{\tilde{N},c}(\bar{T})$ to complexes of dg-modules over the sheaf of dg-algebras $R\psi_*(\mathbb{Q})$, that is, to objects in the derived category $D_{WT}(\bar{T}, R\psi_*(\mathbb{Q}))$. (Here we observe that any complex $K \in D^{b,o}_{\tilde{N},c}(\bar{T})$ comes equipped with a pairing $\mathbb{Q} \otimes K \to K$, which induces the pairing $R\psi_*\mathbb{Q}) \otimes R\psi_*(K) \to R\psi_*(K)$, and hence the structure of a dg-module over $R\psi_*(\mathbb{Q})$ on $R\psi_*(K)$.)

We let $D^{+,o}_{WT,c}(\bar{T}, R\psi_*(\mathbb{Q}))$ denote the full subcategory generated by the objects $R\psi_*(j_!\mathbb{Q})$ where $j : \mathcal{O} \to \bar{T}$ denotes the immersion associated with an $\tilde{N}$-orbit and we vary over such $\tilde{N}$-orbits. Let

$$L\psi^* : D^{+,o}_{WT,c}(\bar{T}, R\psi_*(\mathbb{Q})) \to D^{b,o}_{\tilde{N},c}(\bar{T})$$

denote the functor defined by sending a dg-module M to $\mathbb{Q} \overset{L}{\underset{\psi^{-1}R\psi_*(\mathbb{Q})}{\otimes}} \psi^{-1}(M)$. Then the next key step is that the functor

$$R\psi_* : D^{b,o}_{\tilde{N},c}(\bar{T}) \to D^{+,o}_{WT,c}(\bar{T}, R\psi_*(\mathbb{Q}))$$

is an equivalence of categories with $L\psi^*$ its inverse. (See Proposition 2.9 for further details.)

1.3. *Step 3.* The next step is to separate out the W and T-actions. Since there is no obvious map between the simplicial varieties $EWT \underset{WT}{\times} (\bar{T})$ and $EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T})$, we need to consider the maps $p_1 : EWT \underset{WT}{\times} (ET \times \bar{T}) \to EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T})$ and $p_2 : EWT \underset{WT}{\times} (ET \times \bar{T}) \to EWT \underset{WT}{\times} (\bar{T})$. Let $\pi : ET \times \bar{T} \to \bar{T}/T$ denote the obvious map as well as the induced map $EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}) \to EW \underset{W}{\times} (\bar{T}/T)$. (Here $\bar{T}/T$ denotes the set of all T-orbits on $\bar{T}$ provided with the topology, where the closed sets are the unions of T-orbit closures. Then $\pi : \bar{T} \to \bar{T}/T$ is continuous.) Then we prove the following theorem.

Theorem 1.3. *The restriction functor* $D^b_{G \times G,c}(\bar{G}) \to D^{b,o}_{\tilde{N},c}(\bar{T})$ *and the functors* $R\psi_*, p_2^*, Rp_{1*}$ *and* $R\pi_*$ *induce equivalences of derived categories:*

$$D^b_{G \times G,c}(\bar{G}) \simeq D^{+,o}_{\mathrm{cart},c}(EW \underset{W}{\times} (\bar{T}/T), R\pi_*(\mathcal{A})), \ \text{where } \mathcal{A} = Rp_{1*}p_2^*R\psi_*(\mathbb{Q}).$$

Here $\mathrm{D}^{+,o}_{cart,c}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\bar{\mathrm{T}}/\mathrm{T}),\mathrm{R}\pi_*(\mathcal{A}))$ *denotes the full subcategory of* $\mathrm{D}^+(\mathrm{EW}\underset{\mathrm{W}}{\times}(\bar{\mathrm{T}}/\mathrm{T}),\mathrm{R}\pi_*(\mathcal{A}))$ *generated by applying the functor* $R\pi_*Rp_{1*}p_2^*$ *to the generators of the subcategory* $\mathrm{D}^{+,o}_{\mathrm{WT},c}(\bar{\mathrm{T}},R\psi_*(\mathbb{Q}))$ *(as in Step 2).*

Observe that the t-structure on the derived category on the left is the t-structure whose heart consists of $\mathrm{G}\times\mathrm{G}$-equivariant perverse sheaves on $\bar{\mathrm{G}}$, while the corresponding t-structure on the derived category on the right is obtained by transferring the t-structure from the derived category on the left. (See Proposition 6.4, for example.)

1.4. *Step 4 (Final Step).* Our *next goal* is to show, making use of the equivalences of derived categories provided by the last three steps, that the dg-algebra $\mathcal{B}_{\mathrm{G}\times\mathrm{G}}(\bar{\mathrm{G}})$ is quasi-isomorphic as a dg-algebra to a dg-algebra defined in terms of the toric variety $\bar{\mathrm{T}}$, and provided with a compatible action by W. We will adopt the following convention henceforth: if $\{K_i|i\}$ denotes a finite collection of complexes in an abelian category (with enough injectives), $RHom(\oplus K_i,\oplus_i K_i)=Hom(\oplus_i\hat{K}_i,\oplus\hat{K}_i)$ where $\hat{K}_i$ is a fixed replacement of K_i by a complex of injectives up to quasi-isomorphism. Then we observe the following:

(i) The $\mathrm{G}\times\mathrm{G}$-equivariant local systems on the $\mathrm{G}\times\mathrm{G}$-orbits are constant as observed in [BJ04, Lemma 3.6]. If $\mathcal{O}$ denotes an orbit for the $\mathrm{G}\times\mathrm{G}$-action on $\bar{\mathrm{G}}$, and $\mathrm{IC}^{\mathrm{G}}(\mathcal{O})$ denotes the corresponding equivariant intersection cohomology complex extending the constant sheaf $\mathbb{Q}$ on the orbit $\mathcal{O}$, then it corresponds to $\mathrm{IC}^{\tilde{\mathrm{N}}}(\mathcal{O}_{\tilde{\mathrm{N}}})$ under the equivalence of derived categories provided by Step 1. Here $\mathcal{O}_{\tilde{\mathrm{N}}}$ is the $\tilde{\mathrm{N}}$-orbit on $\bar{\mathrm{T}}$ corresponding to the $\mathrm{G}\times\mathrm{G}$-orbit $\mathcal{O}_{\mathrm{G}}$. Since the equivalence of derived categories in Step 1 is provided by derived functors as in (1.1.2) relating the two derived categories, they preserve the corresponding $RHom$ as well as the composition pairing on the $RHom$s. Therefore, the dga $\mathcal{B}_{\mathrm{G}\times\mathrm{G}}$ is quasi-isomorphic to the dga $\mathcal{B}_{\tilde{\mathrm{N}}}(\bar{\mathrm{T}})=RHom(\oplus_{\mathcal{O}}\mathrm{IC}^{\tilde{\mathrm{N}}}(\mathcal{O}_{\tilde{\mathrm{N}}}),\oplus_{\mathcal{O}}\mathrm{IC}^{\tilde{\mathrm{N}}}(\mathcal{O}_{\tilde{\mathrm{N}}}))$, where the sum varies over all the $\tilde{\mathrm{N}}$-orbits on $\bar{\mathrm{T}}$.

(ii) Under the equivalence of equivariant derived categories provided by Step 2, the equivariant intersection cohomology complex $\mathrm{IC}^{\tilde{\mathrm{N}}}(\mathcal{O}_{\tilde{\mathrm{N}}})$ corresponds to $\mathrm{IC}^{\mathrm{WT}}(\mathcal{O}_{\mathrm{WT}})\otimes R\psi_*(\mathbb{Q})$, where $\mathcal{O}_{\mathrm{WT}}$ denotes the same orbit $\mathcal{O}_{\tilde{\mathrm{N}}}$ of $\tilde{\mathrm{N}}$, but viewed as an orbit for the induced action of WT. Therefore, the equivalence of derived categories in Step 2 similarly provides a quasi-isomorphism between the dgas: $\mathcal{B}_{\tilde{\mathrm{N}}}(\bar{\mathrm{T}})$ and $\mathcal{B}_{\mathrm{WT}}(\bar{\mathrm{T}})=RHom(\oplus_{\mathcal{O}}\mathrm{IC}^{\mathrm{WT}}(\mathcal{O}_{\mathrm{WT}})\otimes R\psi_*(\mathbb{Q}),\oplus_{\mathcal{O}}\mathrm{IC}^{\mathrm{WT}}(\mathcal{O}_{\mathrm{WT}})\otimes R\psi_*(\mathbb{Q}))$.

(iii) Under the equivalence of derived categories provided by Step 3, the complex $\mathrm{IC}^{\mathrm{WT}}(\mathcal{O}_{\mathrm{WT}})\otimes R\psi_*(\mathbb{Q})$ corresponds to $R\pi_*(\mathrm{IC}^{\mathrm{W,T}}(\mathcal{O}_{\mathrm{W,T}})\otimes\mathcal{A})$. Here $\mathrm{IC}^{\mathrm{W,T}}(\mathcal{O}_{\mathrm{W,T}})$ is the same complex as $\mathrm{IC}^{\mathrm{WT}}(\mathcal{O}_{\mathrm{WT}})$, but with the action of W and T separated out: see the proof of Corollary 2.17. This similarly provides a quasi-isomorphism between the dgas: $\mathcal{B}_{\mathrm{WT}}(\bar{\mathrm{T}})\simeq\mathcal{B}_{\mathrm{W,T}}(\bar{\mathrm{T}})=RHom(\oplus_{\mathcal{O}}R\pi_*(\mathrm{IC}^{\mathrm{W,T}}(\mathcal{O}_{\mathrm{W,T}})\otimes\mathcal{A}),\oplus_{\mathcal{O}}R\pi_*(\mathrm{IC}^{\mathrm{W,T}}(\mathcal{O}_{\mathrm{W,T}})\otimes\mathcal{A}))$

The last $RHom$ is taken in the category $\mathrm{C}^{+,o}_{cart,c}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\bar{\mathrm{T}}/\mathrm{T}),\mathrm{R}\pi_*(\mathcal{A}))$, which is the category of complexes whose derived category is $\mathrm{D}^{+,o}_{cart,c}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\bar{\mathrm{T}}/\mathrm{T}),\mathrm{R}\pi_*(\mathcal{A}))$.

Thus, in order to show the dg-algebra $\mathcal{B}_{\mathrm{G}\times\mathrm{G}}(\bar{\mathrm{G}})$ is formal, it suffices to prove that the dg-algebra $\mathcal{B}_{\mathrm{W,T}}(\bar{\mathrm{T}})$ is formal. After having done all of the above work to separate out the W and T-actions, the remaining part of this proof is similar to the proof in the toric case: see [Lu95] and Theorem 3.10. A key step in this proof is that the dg-algebra $R\pi_*(\mathcal{A})=R\pi_*(Rp_{1*}p_2^*R\psi_*(\mathbb{Q}))$ is formal.

Remarks 1.4. 1. The toric variety $\bar{\mathrm{T}}$, with the action of $\mathrm{T}\times\mathrm{T}$, is a non-trivial example of a *toric stack*, since the diagonal torus acts trivially on $\bar{\mathrm{T}}$. Thus, perhaps surprisingly, the theory of toric stacks shows up in the analysis of the equivariant derived categories for toroidal group imbeddings.

2. One may want to compare the above steps with their counterparts in the toric case. There, step 1 is absent and steps 2 and 3 are combined into one step where one proves the result in Theorem 1.3 for projective toric varieties: here none of the complications coming from the maps p_1 and p_2 occur in the toric case, as there is no need to consider p_1 or p_2 in this case. Finally in the analogue of Step 4, one produces a dg-algebra $\mathcal{B}_{\mathrm{T}}(\bar{\mathrm{T}})$ from the complexes $R\pi_*(L_i)$, and proves that $\mathcal{B}_{\mathrm{T}}(\bar{\mathrm{T}})$ is formal as a dga.

The discussion of the proof of Theorem 1.2 in the body of the paper is also broken up, with each major step occupying a clearly marked subsection. See for example, subsection 2.3, which discusses Step 1 of

the proof, subsection 2.4, which discusses Step 2, and subsection 2.5 which discusses Step 3. Finally the last Step (namely Step 4), occupies all of section 3 and completes the proof of Theorem 1.2. Since this is a rather long section, we have further broken this up into Steps 4.1 through 4.3.

In section 4, we discuss the following general result which sheds some light on the validity of the conjecture that the dg-algebra $\mathcal{B}_G(X)$ is formal in general. We recall that a well-known theorem (see [Ka80]) shows that there are a countable number of obstructions m_i, $i = 3, 4, \cdots$, that need to vanish for an A_∞-dg-algebra to be formal. Implicit in the following theorem is the fact that we make use of the cohomology notation for intersection cohomology complexes, where we start with a local system $\mathcal{L}$ or the constant sheaf $\mathbb{Q}$ on the smooth part of a stratified variety: in [BBD82], they start with $\mathcal{L}[d]$ ($\mathbb{Q}[d]$, respectively) where d is the dimension of the open stratum of the stratified variety.

Theorem 1.5. *(i) Suppose the A_∞-dg-algebra $\mathcal{B}$ has the property that $H^i(\mathcal{B}) = 0$ for all odd i. Then all the obstructions m_i, for i odd and $i \geq 3$ vanish.*

(ii) Suppose X is a projective G-spherical variety for a connected reductive group G over a field of characteristic 0. Assume further that X is a simply-connected spherical variety in the sense of [BJ04], that is, the stabilizers in G and B are connected at all points on X, where B is a Borel subgroup of G. Then the conclusions of (i) hold for the dg-algebra $\mathcal{B}_G(X)$ considered in Theorem 1.1.

We begin section 2, with a quick self-contained review of the simplicial model of equivariant derived categories in 2.1, so that the remainder of the paper could be read effortlessly. Some of the more technical details are left to section 5, which is also devoted a detailed comparison of the two well-known models of equivariant derived categories. We provide this comparison for the following two reasons:
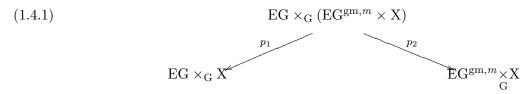
(i) The simplicial Borel construction sending a G-space to $EG\underset{G}{\times}X$ produces a simplicial resolution of X, which is clearly functorial in X. Other geometric models of the Borel construction always involve the choice of such a resolution and can often make the situation quite complicated. For example, in section 3, in the context of the analysis of the toroidal imbedding case, we run into situations where it becomes necessary to relate the equivariant derived categories associated to two groups, G and H, where one is provided with a surjective homomorphism G $\rightarrow$ H. This is quite difficult, and nearly impossible if one uses the geometric approach where one produces approximations of classifying spaces by starting with representations of the groups. In contrast, this is totally effortless in the simplicial setting, so that this forces us to adopt the simplicial methods for defining classifying spaces and associated equivariant derived categories.

(ii) On the other hand, the geometric models of classifying spaces have certain advantages in that their approximations are indeed varieties, and therefore, much of the machinery from the non-equivariant framework adapts easily. In addition, the geometric models seem to be more popular in the literature. Moreover, the conjecture we are considering in this paper is originally stated in terms of the geometric models for classifying spaces: see [Lu95]. [2]

We provide this comparison in as general a context as possible so as to be of use in a wide variety of contexts, that is, for schemes of finite type over perfect fields k of finite ℓ-cohomological dimension for some prime $\ell \neq char(k)$. Here $BG^{gm,m}$ is a degree-m approximation to the classifying space for G and $EG^{gm,m}$ denotes the universal principal G-bundle over $BG^{gm,m}$. Recall this means $U_m = EG^{gm,m}$ is an open G-stable subvariety of a representation W_m of G, so that (i) G acts freely on U_m and a geometric quotient U_m/G exists as a variety and (ii) so that in the family $\{(W_m, U_m)|m \in \mathbb{N}\}$, the codimension of $W_m - U_m$ in W_m goes to ∞ as m approaches ∞. (See Definition 6.2 for more precise details.) BG will denote a simplicial model for the classifying space of G and EG will denote its universal principal G-bundle.

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[2]The reason we have left this discussion still in section 5, is that the results in this section are not used anywhere else in the paper, except to show that the conjecture in question, which was originally stated in the framework of the geometric model of equivariant derived categories carries over to the simplicial model of equivariant derived categories.

Theorem 1.6. *For each fixed $m \geq 0$, we obtain the diagram of simplicial varieties (where p_1 is induced by the projection $EG^{gm,m} \times X \to X$ and p_2 is induced by the projection $EG \times (EG^{gm,m} \times X) \to EG^{gm,m} \times X$):*

(1.4.1)
$$EG \times_G (EG^{gm,m} \times X)$$

$$p_1 \swarrow \qquad \qquad \searrow p_2$$

$$EG \times_G X \qquad \qquad \underset{G}{EG^{gm,m} \times X}$$

(i) For each finite interval $I = [a, b]$ of the integers, with $2m - 2 \geq b - a$,

$$p_1^* : D_G^I(EG \times_G X) \to D_G^I(EG \times_G (EG^{gm,m} \times X)) \ and$$

$$p_2^* : D_G^I(EG^{gm,m} \times_G X) \to D_G^I(EG \times_G (EG^{gm,m} \times X))$$

are equivalences of categories. (Here the superscript I denotes the full subcategory of complexes whose cohomology sheaves vanish outside of the interval I and subscript G denotes the full subcategories of complexes whose cohomology sheaves are G-equivariant.) Moreover, for complexes of $\bar{\mathbb{Q}}_\ell$-adic sheaves (where $\bar{\mathbb{Q}}_\ell$ denotes an algebraic closure of $\mathbb{Q}_\ell$), both the functors p_1^* and p_2^* send complexes that are mixed and pure to complexes that are mixed and pure. [3] *There exists an equivalence of derived categories:*

$$D_G^b(EG^{gm,m} \times_G X) \simeq D_G^b(EG \times_G X)$$

which is natural in X and G. The above equivalences hold in all characteristics with the derived categories of complexes of ℓ-adic sheaves on the étale site and hold in characteristic 0 with the derived categories of complexes of sheaves of $\mathbb{Q}$-vector spaces.

(ii) Moreover, both the maps p_i, $i = 1, 2$, induce isomorphisms on the fundamental groups completed away from the characteristic.

We devote all of section 5 to a detailed discussion of equivariant derived categories leading up to the above theorem. The Appendix provides a supplementary discussion of geometric classifying spaces, concluding with a result that shows how to transfer t-structures under equivalences of triangulated categories.

Conventions.

- In view of the reasons explained above, *the equivariant derived categories we consider will always be defined using the simplicial construction of the classifying spaces of linear algebraic groups.* The equivalence of derived categories stated in the last Theorem, then shows that all our results carry over to the equivariant derived categories defined by means of the procedures discussed in [BL94] or [To99].
- The derived push-forwards we consider in various contexts in the paper, will almost always be defined using a functorial resolution such as that given by the Godement resolution. This has the advantage of preserving any extra structure that is associated to a given sheaf.
- A dga (short for differential graded algebra or a sheaf of such algebras) will mean a chain complex $A^\bullet$, provided with an associative pairing $A^\bullet \otimes A^\bullet \to A^\bullet$ of chain complexes. The dga will be commutative if the above pairing is graded commutative. (A chain complex will always have differentials of degree $+1$.)

The approach we take is similar in principle to that of [Lu95] (and [Gu05]), where the formality of the dg-algebra $\mathcal{B}$ is proven for projective complex toric varieties (for complex complete symmetric varieties, respectively). Both these need to relate the equivariant derived category $D_{G,c}^b(X)$ with the category of sheaves of dg-modules over a sheaf of dg-algebras on a *space* whose points correspond to all the G-orbits on X.

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[3]Recall that a complex of $\bar{\mathbb{Q}}_\ell$-adic sheaves is mixed and pure if it has a finite increasing filtration whose successive quotients have cohomology sheaves that are pure: see [BBD82, 5.1.5].

This intermediate equivalence, strongly needs the existence of *attractive slices* at each point of each G-orbit. Therefore, such a hypothesis is not always satisfied, in general. In the case of toroidal group imbeddings, we show that these hypotheses are met to a large extent.

We define *slices* in a somewhat more general context as follows.

Definition 1.7. Let a linear algebraic group G act on a variety X and let $x \in$ X. A locally closed subvariety $\mathcal{S}$ of X containing x, stable under the action of the isotropy group G_x and satisfying the following two conditions is called a *slice* at x:

(i) There exists a $\mathbb{G}_m$-action on $\mathcal{S}$ commuting with the action of G_x.

(ii) The map $G \underset{G_x}{\times} \mathcal{S} \to$ X sending (g, x) to $g.x$ is an open immersion at (e, x), and the dimension of $\mathcal{S}$ is the codimension of the orbit $G \cdot x$ in X.

Let $\mathbb{G}_m$ act on a variety X with a fixed point x. Let $\lambda : \mathbb{G}_m \times X \to X$ denote this action. We say x is *attractive* if for all y in a Zariski neighborhood of x, we have $lim_{t \to 0} ty = x$. Equivalently, all weights of λ acting on the Zariski tangent space at x are contained in an open half-space.

Assume the situation of Definition 1.7. Let $x \in$ X and let $\mathcal{S}$ denote a slice at x. We say that $\mathcal{S}$ is an *attractive slice*, if x is an attractive fixed point for the given action of $\mathbb{G}_m$ on $\mathcal{S}$. (See [BJ01, appendix] for further details on attractive fixed points.)

2. **Toroidal Group Imbeddings and the Proof of Theorem 1.3**

As pointed out earlier, it is more convenient for us to make use of the simplicial model of the Borel construction and the resulting equivariant derived categories. As this may not be that familiar, we begin this section with a quick review of the simplicial model of the equivariant derived category. A detailed comparison with other geometric models of the equivariant derived category appears in section 5.3.[4]

2.1. **The simplicial model of Equivariant Derived Categories.** This is the simplicial model discussed in detail in [De74], and also in [Jo93, section 6] or [Jo02]. The main advantage of this model comes from the functoriality of the simplicial Borel construction. In view of the various applications, we have decided to make the discussion in this section general enough so that it applies to actions of linear algebraic groups defined over fields k that are perfect and of finite ℓ cohomological dimension for some $\ell \neq char(k)$. Therefore, all objects in the following discussion will be defined and of finite type over such a field k.

Given a linear algebraic group G acting on a variety X, $EG \underset{G}{\times} X$ will now denote the simplicial variety defined by letting $(EG \underset{G}{\times} X)_n = G^{\times n} \times X$ with the face maps $d_i : (EG \underset{G}{\times} X)_n \to (EG \underset{G}{\times} X)_{n-1}$, $i = 0, \cdots n$, induced by the group action $\mu : G \times X \to X$, the group multiplication $G \times G \to G$ and the projection $G \times X \to X$. The i-th degeneracy $s_i : (EG \underset{G}{\times} X)_{n-1} \to (EG \underset{G}{\times} X)_n$, $i = 0, \cdots, n-1$ is induced by inserting the identity element of the group G in the i-th place. This construction is functorial: if $f : X \to Y$ is any G-equivariant map between varieties with G-action, one obtains an induced map $EG \underset{G}{\times} X \to EG \underset{G}{\times} Y$.

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[4]We feel this approach would provide enough information of the constructions used, so that a reader can easily follow the main arguments in the paper, without having to suffer through all the technical fine points. The more technical aspects of the simplicial construction, along with a detailed comparison with other geometric constructions are left to section 5, which can be used as a reference, if the reader so wishes.

Given a Grothendieck topology, Top, on varieties over k, one defines an induced Grothendieck topology $\mathrm{Top}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})$ whose objects are $\mathrm{U_n} \to (\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_n$ in $\mathrm{Top}((\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})_n)$ for some $n \geq 0$. The maps between two such objects and coverings for this topology are defined as in [De74]. When one chooses the étale topology, this site will be denoted $Et(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})$.

2.1.1.   Given a simplicial variety $\mathrm{X}_\bullet$ (for example, $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$), a sheaf F on the site $\mathrm{Top}(\mathrm{X}_\bullet)$ is a collection of sheaves $\{F_m | m \geq 0\}$ with F_m on the transcendental site (in case $k = \mathbb{C}$) or the étale site of $\mathrm{X_m}$, provided with structure maps $\alpha^*(F_m) \to F_n$ for each structure map $\alpha : \mathrm{X_n} \to \mathrm{X_m}$, and satisfying certain compatibility conditions: see [De74, (5.6.6)]. We say that a sheaf F has *descent* (or is *cartesian*) if the above maps $\alpha^*(F_m) \to F_n$ are all isomorphisms. $\mathrm{D}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})$ will denote the derived category of complexes of sheaves of $\mathbb{Q}$-vector spaces on the simplicial variety $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ when everything is defined over the complex numbers, and will denote the derived category of ℓ-adic sheaves on the étale site $Et(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})$, in general. In this framework, $\mathrm{D_G}(\mathrm{X})$ will denote the full subcategory of $\mathrm{D}(\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X})$ consisting of complexes of sheaves so that the cohomology sheaves have descent. Moreover, for each finite interval $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$ with $-\infty < a \leq b < \infty$, $\mathrm{D_G^I}(\mathrm{X})$ will denote the full subcategory of $\mathrm{D_G}(\mathrm{X})$ consisting of complexes K for which $\mathcal{H}^i(K) = 0$ for all $i \notin I$.

Remark 2.1. It may be important to point out that, in order to construct the equivariant derived category, one needs to begin with the category of all sheaves on the simplicial variety $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$: that is the only way to ensure that the category of sheaves have important properties like having enough injectives. Then one restricts to the *full subcategory* of all complexes of sheaves whose cohomology sheaves are equivariant (that is, cartesian) to obtain the equivariant derived category: the fact that the full subcategory of equivariant sheaves is closed under extensions is needed to ensure the equivariant derived category, so defined, is a triangulated category as shown in [Hart66, p. 38]. One needs to adopt such a construction in the geometric models of equivariant derived categories as well: see [BL94, Chapter 1.8], for example. The only exception to this is when the group G is discrete: in this case, the category of equivariant sheaves have enough injectives as shown by Grothendieck (see [Groth57], [BL94, Chapter 1.8]): therefore, in this case, we may work with the category of complexes which are equivariant in each degree. We make use of this observation in *Step 4.1* in the proof of Theorem 1.2.

2.1.2.   Observe that if $f_\bullet : \mathrm{X}_\bullet \to \mathrm{Y}_\bullet$ is a map of simplicial varieties, then the induced push-forward $f_{\bullet,*} = \{f_{n,*} | n \geq\}$, which is not a single functor, but a collection of functors, indexed by $n \geq 0$. This issue is rather technical, and becomes relevant only where it is important to compute the cohomology of the fibers of the simplicial map $f_\bullet$ as a simplicial scheme. The solution is to use the simplicial topology as in [Jo02], and we invoke that in section 5.4, as well as in the proof of Theorem 1.6 in section 5.

Terminology 2.2. *We will adopt the following terminology throughout the rest of the paper. If* G *is a linear algebraic group acting on a variety* X, $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ *will always denote the simplicial variety defined above. In particular,* BG *will denote the corresponding simplicial variety when* $\mathrm{X} = \mathrm{Spec}\, k$. *The geometric model for* $\mathrm{EG}\underset{\mathrm{G}}{\times}\mathrm{X}$ *considered in (5.1.2), which is an ind-scheme, will always be denoted* $\{\mathrm{EG}^{\mathrm{gm},m}\underset{\mathrm{G}}{\times}\mathrm{X} | m\}$.

2.2. **Toroidal imbeddings of connected reductive groups: basic definitions and examples.** Throughout the rest of this section we will assume the base field is the field of complex numbers. Let G denote a connected reductive group. Viewing G as a $\mathrm{G} \times \mathrm{G}$-homogeneous space for the action of $\mathrm{G} \times \mathrm{G}$ by left and right multiplication, $\mathrm{G} \simeq (\mathrm{G} \times \mathrm{G})/\mathrm{diag}(\mathrm{G})$. (Here $\mathrm{diag}(\mathrm{G})$ denotes the group G imbedded diagonally in $\mathrm{G} \times \mathrm{G}$.)

G-*spherical varieties* may be defined as normal varieties X, equipped with a G-action so that a Borel subgroup B has an open dense B-orbit. It is known (see [Kn91, Remark 2.2]) that then X contains only finitely many B-orbits as well as G-orbits. Moreover, such a spherical variety X may be viewed as a partial compactification of the homogeneous space G/H, which denotes the open G-orbit on X. When

the group G is replaced by G × G and H by the diagonal imbedding of G in G × G (with the diagonal G acting on G × G by both left and right-multiplication), we obtain *spherical imbeddings of the group* G.

Choose a maximal torus T of G, and denote by N its normalizer in G; the quotient N/T is the Weyl group W. Let B denote a fixed Borel subgroup containing T. G-spherical varieties are classified by *colored* fans (that is, fans with the extra structure of colors) in the valuation cone associated to G/H (see [Kn91]). The *colors* of X correspond to B-stable prime divisors in X that are not G-stable, and contain a G-orbit. Equivalently, the colors correspond to the closures in X of the B-stable prime divisors in the open G-orbit, so that the closure in X contains a G-orbit. Spherical varieties that are G × G-*equivariant imbeddings of* G are classified by colored fans in $X_*(T) \otimes \mathbb{R}$ with support in the negative Weyl chamber: see [BK05, 6.2.4 Proposition]. (Here $X_*(T)$ denotes the weight-lattice.)

Toroidal imbeddings form an important special class of G-spherical varieties, defined as follows.

Definition 2.3. (Toroidal imbeddings) An equivariant imbedding of the connected reductive group G is *toroidal*, precisely when there are no colors, that is, every B-stable prime divisor in X is either G-stable or does not contain a G-orbit. Toroidal imbeddings of the group G are classified by fans in the negative Weyl chamber: see [BK05, 6.2.4 Proposition].

Therefore, given a G × G equivariant imbedding X of G, one may find a G × G equivariant toroidal imbedding X̃ provided with a birational G × G-equivariant map X̃ → X, by replacing the given colored fan with the fan obtained by removing all the colors.

Moreover, such a toroidal imbedding is *complete*, if and only if the corresponding fan has support the whole Weyl chamber. By considering W-translates of the above fans, one obtains a W-invariant fan in $X_*(T) \otimes \mathbb{R}$. Then a toroidal imbedding will be *smooth* if and only if the corresponding W-invariant fan has the property that every cone is generated by part of a basis of the free abelian group $X_*(T)$.

Before proceeding further, we will next discuss a few examples of toroidal group imbeddings.

Examples 2.4.   (i) *All toric varieties are toroidal imbeddings of the open dense torus.*
 (ii) *The simplest example of group imbeddings, other than toric varieties, are that of* GL_2. *In this case there are at least 4 distinct* $GL_2 \times GL_2$-*equivariant imbeddings of* GL_2, *namely,* $\mathbb{A}^4$, $\mathbb{P}^4$, *and* $\widetilde{\mathbb{A}}^4$, $\widetilde{\mathbb{P}}^4$, *where the last two are blow-ups of the first two at the origin in* $\mathbb{A}^4$. *Only the latter two are toroidal imbeddings, and clearly* $\widetilde{\mathbb{P}}^4$ *is the only toroidal imbedding that is projective. (See* [AKP, 8.2].*)*
(iii) *Let* G *be semi-simple of adjoint type. Then* $X_*(T)$ *has a basis consisting of the fundamental weights. Therefore, the fan consisting of the Weyl chambers and their faces is smooth. The corresponding (smooth) toroidal imbedding is the wonderful compactification of* G. *(See* [BK05, section 6.1].*) In fact, an alternate definition of a toroidal group imbedding of the reductive group* G *is as a group imbedding* X *of* G, *so that the quotient map* $G \to G_{ad}$ *(which denotes the adjoint group) extends to a morphism of schemes* X *to the wonderful compactification of* G_{ad}: *see* [BK05, Definition 6.2.2].
(iv) *If* G *is semi-simple but no longer of adjoint type, then we may consider the same fan, but now it is almost never smooth. For example, if* $G = SL_3$ *then we get a singular toroidal compactification of* G. *It can be constructed geometrically as the normalization of the wonderful compactification of* PGL_3 *in the function field* $k(SL_3)$. *This construction works more generally for any semi-simple* G: *the normalization in* $k(G)$ *of the wonderful compactification of the adjoint group yields a canonical imbedding of* G, *which is toroidal and projective but (again) almost never smooth. (See* [BK05, section 6.2.A] *for additional details.)*

For the rest of the paper, we will adopt the terminology introduced in the outline of the proof of Theorem 1.2 in the introduction.

2.3. **Step 1 of the proof of Theorem 1.2.** Let Ḡ denote a normal G × G-variety which contains G as an open orbit. We will assume that Ḡ is a toroidal imbedding of G. Note that the normalizer (or stabilizer) of diag(T) in G × G equals (T × T)diag(N); moreover, the centralizer of diag(T) in G × G equals T × T, since T is its own centralizer in G. We also have the exact sequence

$$1 \longrightarrow \mathrm{diag}(T) \longrightarrow (T \times T)\mathrm{diag}(N) \longrightarrow WT \longrightarrow 1,$$

where WT denotes the semi-direct product of T with W (acting naturally on T).

Let $\bar{T}$ denote the closure of T in $\bar{G}$; then $(T \times T)\text{diag}(N)$ acts on $\bar{T}$ via its quotient WT, where W acts by conjugation and T by left multiplication. This yields a (restriction) functor

$$(2.3.1) \qquad\qquad res : D^b_{G \times G, c}(\bar{G}) \longrightarrow D^b_{(T \times T)\text{diag}(N), c}(\bar{T}).$$

The first main result of this section is the following.

Theorem 2.5. *(i) The above functor is fully-faithful.*

(ii) Moreover, if $D^{b,o}_{(T \times T)\text{diag}(N), c}(\bar{T})$ *denotes the full subcategory of* $D^b_{(T \times T)\text{diag}(N), c}(\bar{T})$ *generated by the* $(T \times T)\text{diag}(N)$-*equivariant sheaves that are constant along the orbits of* $(T \times T)\text{diag}(N)$ *on* $\bar{T}$, *then the above functor induces an equivalence*

$$D^b_{G \times G, c}(\bar{G}) \overset{res}{\to} D^{b,o}_{(T \times T)\text{diag}(N), c}(\bar{T}).$$

(iii) It sends the $G \times G$-*equivariant intersection cohomology complex* $IC^{G \times G}(\mathbb{Q}_{\mathcal{O}})$ *on a* $G \times G$-*orbit* $\mathcal{O}$ *to the corresponding* $(T \times T)\text{diag}(N)$-*equivariant intersection cohomology complex* $IC^{(T \times T)\text{diag}(N)}(\mathbb{Q}_{\mathcal{O}'})$ *where* $\mathcal{O}'$ *is the* $(T \times T)\text{diag}(N)$-*orbit corresponding to the* $G \times G$-*orbit* $\mathcal{O}$.

Proof. Observe that

$$D^b_{G \times G, c}(\bar{G}) = D^b_{G \times G, c}(EG \times EG \underset{G \times G}{\times} \bar{G}) \text{ while,}$$

$$D^b_{(T \times T)\text{diag}(N), c}(\bar{T}) = D_{G \times G, c}(EG \times EG \underset{G \times G}{\times} (G \times G \underset{(T \times T)\text{diag}(N)}{\times} (\bar{T}))).$$

Therefore, the map denoted *res* is the pull-back map ϕ^* induced by the map

$$\phi : EG \times EG \underset{G \times G}{\times} (G \times G \underset{(T \times T)\text{diag}(N)}{\times} (\bar{T})) \to EG \times EG \underset{G \times G}{\times} \bar{G},$$

where ϕ itself is induced by the natural map $G \times G \underset{(T \times T)\text{diag}(N)}{\times} \bar{T} \to \bar{G}$.

In order to prove ϕ^* is fully-faithful, it is enough to show that the natural map $M \to R\phi_* \phi^*(M)$ is a quasi-isomorphism for all $M \in D^b_{G \times G, c}(\bar{G})$. Unfortunately, the map ϕ is neither proper nor smooth, which makes it difficult to prove this directly. Therefore, we adopt a rather indirect technique. We first replace all the linear algebraic groups by their maximal compact subgroups. We may choose a maximal compact subgroup $K \subset G$ such that $N_K := K \cap N$ is a maximal compact subgroup of N; then $T_K := T \cap K$ is the largest compact sub-torus of T. Lemma 2.6 below shows then that it suffices to prove that the induced restriction

$$res_K : D^b_{K \times K, c}(\bar{G}) \longrightarrow D^b_{(T_K \times T_K)\text{diag}(N_K), c}(\bar{T})$$

is fully-faithful. This functor identifies with ϕ_K^* where ϕ_K is the natural map

$$(2.3.2) \qquad EK \times EK \underset{K \times K}{\times} (K \times K \underset{(T_K \times T_K)\text{diag}(N_K)}{\times} (\bar{T})) \to EK \times EK \underset{K \times K}{\times} \bar{G}.$$

Observe that ϕ_K is proper. Therefore, one has a projection formula which provides the (natural) identification

$$R\phi_{K*}\phi_K^*(M) \simeq M \otimes R\phi_{K*}\phi_K^*(\underline{\mathbb{Q}}), M \in D^b_{(T_K \times T_K)\text{diag}(N_K), c}(\bar{T}).$$

Therefore, it suffices to prove that

$$(2.3.3) \qquad\qquad R\phi_{K*}(\underline{\mathbb{Q}}) = R\phi_{K*}(\phi_K^*\underline{\mathbb{Q}}) \overset{\simeq}{\to} \underline{\mathbb{Q}}.$$

Secondly, the properness of ϕ_K shows that one has proper-base-change, so that it suffices to prove that ϕ_K is surjective and its fibers are $\mathbb{Q}$-acyclic and connected: observe that this will prove (2.3.3) and, making use of the adjunction between ϕ_K^* and $R\phi_{K*}$, that the functor ϕ_K^* is fully-faithful. This is worked out in Lemma 2.7 below, which will complete the proof that the functor ϕ^* is fully-faithful.

To prove the second statement, it suffices to show that ϕ^* induces an equivalence on the corresponding hearts, that is, at the level of the equivariant sheaves. First observe that $\bar{G}$ is an scs variety (that is, a spherical simply connected variety) in the sense of [BJ04, section 1]. Thus the isotropy subgroup of any $G \times G$-orbit $\mathcal{O} \subset \bar{G}$ is connected, by [loc. cit., Lemma 3.6], which implies that all the $G \times G$-equivariant

local systems on $\mathcal{O}$ are constant. Any constructible $G \times G$-equivariant sheaf on $\bar{G}$ has a finite filtration by $G \times G$-equivariant sheaves that are extension by zero of the constant sheaves on the $G \times G$-orbits. Also, by [BK05, Proposition 6.2.3], every such orbit $\mathcal{O}$ intersects $\bar{T}$ along a unique orbit of $(T \times T)\mathrm{diag}(N)$, and this sets up a bijection between $G \times G$-orbits in $\bar{G}$ and $(T \times T)\mathrm{diag}(N)$-orbits in $\bar{T}$, which preserves the closure relations. This completes the proof of the second statement. (Observe that the stabilizers of the $(T \times T)\mathrm{diag}(N)$-orbits in $\bar{T}$ are not connected in general: for example, the stabilizer of the open orbit is $\mathrm{diag}(N)$. Thus, one obtains $(T \times T)\mathrm{diag}(N)$-equivariant sheaves on these orbits that are not constant. This is the need to restrict to the full subcategory $D^{b,o}_{(T \times T)\mathrm{diag}(N),c}(\bar{T})$ in order for the restriction functor to be an equivalence.)

The third statement follows from the local structure of toroidal group imbeddings: see [Ti11, Theorem 29.1]. Here are the details to prove it. Let $\mathcal{O}'$ denote a $(T \times T)\mathrm{diag}(N)$-orbit on $\bar{T}$ and let $\mathcal{O}' \to \bar{\mathcal{O}}'$ denote the corresponding open immersion of $\mathcal{O}'$ into its closure. We stratify $\bar{\mathcal{O}}'$ so that one obtains a sequence of open immersions $U_0 = \mathcal{O}' \overset{j'_0}{\to} U'_1 \overset{j'_1}{\to} U'_2 \cdots U'_{n-1} \overset{j'_{n-1}}{\to} U'_n = \bar{O}'$, so that each of the strata $U_i - U_{i-1}$ is a (disjoint) union of $(T \times T)\mathrm{diag}(N)$-orbits. Let $j' : U_i \to U_{i+1}$ denote an open immersion of $(T \times T)\mathrm{diag}(N)$-stable subvarieties appearing in the above factorization of $\mathcal{O}' \to \bar{\mathcal{O}}'$. Let $j : V_i \to V_{i+1}$ denote the open immersion of the corresponding $G \times G$-stable subvarieties in the factorization of $\mathcal{O} \to \bar{\mathcal{O}}$, where $\mathcal{O}$ is the $G \times G$-orbit corresponding to $\mathcal{O}'$. Recalling that the equivariant intersection cohomology complexes are suitable perverse extensions, now it suffices to show that $\phi^*(\sigma_{\leq n} Rj_*(L)) \simeq \sigma_{\leq n} Rj'_*(\phi^*(L))$ for any constant sheaf L on V_i, where $\sigma_{\leq n}$ denotes the cohomology truncation that kills cohomology sheaves in degrees greater than n.

For a given point $x \in V_{i+1}$, the local structure in [Ti11, Theorem 29.1] shows that, after possibly replacing x by a translate of x by an element $(g_1, g_2) \in G \times G$, one may take the intersection with an open neighborhood V_x so that $j_{|V_x \cap V_i} : V_x \cap V_i \to V_x \cap V_{i+1}$ identifies with $\mathrm{id}_{R_u(B) \times R_u(B^-)} \times j' : R_u(B) \times R_u(B^-) \times U_i \to R_u(B) \times R_u(B^-) \times U_{i+1}$. (Here B is a Borel subgroup of G containing T and B^- is its opposite Borel subgroup also containing T. Then $R_u(B) \times R_u(B^-)$ denotes the product of the unipotent radicals of B and B^-.) Now ϕ^* corresponds to pull-back by the inclusion of U_i in $R_u(B) \times R_u(B^-) \times U_i$ so that $\phi^*(L)$ is the pull-back of the sheaf L on $V_x \cap V_i = R_u(B) \times R_u(B^-) \times U_i$ to U_i. Therefore, on this open neighborhood V_x, $\sigma_{\leq n}(Rj_*(L)) = id \boxtimes \sigma_{\leq n} Rj'_*(\phi^*(L))$ and therefore, $\phi^*(\sigma_{\leq n} Rj_*(L)) = \sigma_{\leq n} Rj'_*(\phi^*(L))$. $\qquad\square$

Lemma 2.6. *Let* G *denote a connected reductive group acting on a variety* X. *Let* K *denote a maximal compact subgroup of* G. *Then the restriction functor:*

$$D^b_{G,c}(X) \to D^b_{K,c}(X)$$

is fully-faithful.

Proof. We consider the map $\psi : EG \underset{G}{\times} (G \underset{K}{\times} X) \to EG \underset{G}{\times} X$ induced by the left-action of G on X. Observe that the functor ψ^* identifies with the restriction functor considered in the lemma. Using the simplicial models for the Borel construction, one sees that the above map in simplicial degree n is the map $\psi_n : G^n \times G \underset{K}{\times} X \to G^n \times X$. This is induced by the map $\psi_0 : G \underset{K}{\times} X \to X$, which being locally trivial in the complex topology, is cohomologically proper. One may verify therefore that each ψ_n is cohomologically proper, so that proper-base-change and the projection formula hold and that therefore, it suffices to verify that the fibers of ψ_0 are acyclic with $\mathbb{Q}$-coefficients. But the fibers of ψ_0 identify with G/K, which is acyclic with rational coefficients. Now an argument exactly as in the case of the functor ϕ^*_K in the last Theorem, proves that the functor ψ^* is fully-faithful. $\qquad\square$

Lemma 2.7. *The fibers of* ϕ_K *are acyclic with* $\mathbb{Q}$-*coefficients, where* ϕ_K *is the map in* (2.3.2). *Moreover,* ϕ_K *restricts to an isomorphism over an open dense subset of* G ($\subseteq \bar{G}$) *for the complex topology.*

Proof. Recall the Cartan decomposition $G = KTK$; also, $T = T_K T_r$, where $T_r := \exp(i\mathrm{Lie}(T_K))$ denotes the real part of T. Thus, $G = KT_r K$. We will first consider fibers of ϕ_K on the open orbit, that is, for

the map

$$\phi_K^o : (K \times K) \underset{(T_K \times T_K).\text{diag}(N_K)}{\times} T \to G.$$

Since this map is $K \times K$-equivariant, it suffices to consider the fiber at a point $t \in T_r$. We claim that this fiber is isomorphic to K^t/N_K^t, where K^t (resp. N_K^t) denotes the centralizer of t in K (resp. N_K).

Let $x, y \in K$ and $z \in T$ such that $xzy^{-1} = t$. Write $z = z_K z_r \in T_K T_r = T$. Using the action of $1 \times T_K \subset (T_K \times T_K)\text{diag}(N_K)$, we may assume that $z_K = 1$, that is, $z = z_r$. Then $t = (xy^{-1})(yzy^{-1})$ and $xy^{-1} \in K$, $yzy^{-1} \in \exp(i\text{Lie}(K))$. By the uniqueness in the decomposition $G = K\exp(i\text{Lie}(K))$, we obtain $xy^{-1} = 1$ and $t = yzy^{-1}$. So $y = x$, and z is conjugate to t in K. As $z, t \in T_r$, they are conjugate in N_K, that is, there exists $n \in N_K$ such that $t = nzn^{-1}$. Then $t = nx^{-1}txn^{-1}$, that is, $x \in K^t n$. We may replace (x, y, z) with $(xn^{-1}, yn^{-1}, nzn^{-1})$ by using the action of $\text{diag}(N_K)$, to get (xn^{-1}, xn^{-1}, t), where $xn^{-1} \in K^t$ is unique up to multiplication by N_K^t. As $(T_K \times T_K)\text{diag}(N_K) = (1 \times T_K)\text{diag}(N_K)$, this yields the claim.

Since K^t is a compact connected Lie group with maximal compact torus T_K, and N_K^t is its normalizer in K^t, the claim implies the fibers over G are $\mathbb{Q}$-acyclic. Moreover, the fiber at any point $t \in T$ with regular real part t_r just consists of this point.

Next we consider the fibers of ϕ_K over an arbitrary $G \times G$-orbit $\mathcal{O}$ in $\bar{G}$. It suffices to show that any such fiber is again isomorphic to K^t/N_K^t for some $t \in T_r$. For this, we recall the structure of $\mathcal{O}$ (see [BJ04, (5.1.2), (5.1.3)] which discusses projective reductive varieties, which includes group imbeddings): there exists a Levi subgroup L of G containing T such that $L_K := L \cap K$ is a maximal compact subgroup of L, and

$$\mathcal{O} = (K \times K) \underset{L_K \times L_K}{\times} L/L_{\mathcal{O}},$$

where $L_{\mathcal{O}} \subseteq L$ is a central subgroup. Moreover $\bar{T} \cap \mathcal{O} = \text{diag}(N_K) \underset{(\text{diag}(N_K) \cap L)}{\times} T/L_{\mathcal{O}}$ (note that $L_{\mathcal{O}} \subset T$ as $L_{\mathcal{O}}$ is central in L). So we obtain an isomorphism

$$(K \times K) \underset{(T_K \times T_K)\text{diag}(N_K)}{\times} (\bar{T} \cap \mathcal{O}) \cong (K \times K) \underset{(T_K \times T_K)\text{diag}(N_K \cap L)}{\times} T/L_{\mathcal{O}}$$

and a commutative triangle

$$(K \times K) \underset{(T_K \times T_K)\text{diag}(N_K))}{\times} (\bar{T} \cap \mathcal{O}) \longrightarrow \mathcal{O} = (K \times K) \underset{L_K \times L_K}{\times} L/L_{\mathcal{O}}$$

$$(K \times K)/(L_K \times L_K)$$

where the horizontal arrow is the pull-back of φ_K over $\mathcal{O}$. This identifies the fibers of ϕ_K over the orbit $\mathcal{O}$ with $K \times K$-translates of the fibers of the map

$$(L_K \times L_K) \underset{(T_K \times T_K)\text{diag}(N_K))}{\times} (T/L_{\mathcal{O}}) \to L/L_{\mathcal{O}}.$$

Therefore, this reduces to the case of the fibers over the open orbit with the group G replaced by L. $\square$

In the remainder of this section, we will need to consider the derived direct images of maps between simplicial varieties of the form $EG \underset{G}{\times} X \to EG \underset{G}{\times} Y$ where X and Y are varieties provided with the action of a linear algebraic group G and $f : X \to Y$ is a G-equivariant map. Most of the time, this can be handled as in section 2.1.2, but occasionally it is helpful to make use of the methods of [Jo02], which enables one to consider a single derived functor Rf_*. Since this is technical issue, we prefer to discuss it later in 5.4, rather than here.

2.4. **Step 2 of the proof of Theorem 1.2.** Throughout the rest of the paper, we will let

(2.4.1) $$\tilde{\mathrm{N}} := (\mathrm{T} \times \mathrm{T})\mathrm{diag}(\mathrm{N}).$$

Recall that $\tilde{\mathrm{N}}$ acts on $\bar{\mathrm{T}}$ via its quotient $\tilde{\mathrm{N}}/\mathrm{diag}(\mathrm{T}) \simeq \mathrm{WT}$. Let

(2.4.2) $$\psi : \mathrm{E}\tilde{\mathrm{N}} \underset{\tilde{\mathrm{N}}}{\times} \bar{\mathrm{T}} \to \mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}}$$

denote the map induced by the identity on $\bar{\mathrm{T}}$ and the quotient map $\tilde{\mathrm{N}} \to \mathrm{WT}$. Since $\mathrm{diag}(\mathrm{T})$ acts trivially on $\bar{\mathrm{T}}$, the fibers of the simplicial map ψ at every point can be identified with $\mathrm{Bdiag}(\mathrm{T})$. Recall $\mathrm{D}^{b,o}_{\tilde{\mathrm{N}},c}(\bar{\mathrm{T}})$ denotes the full subcategory of the derived category $\mathrm{D}^{b}_{\tilde{\mathrm{N}},c}(\bar{\mathrm{T}})$ generated by the constant sheaves on each orbit of $\tilde{\mathrm{N}}$ on $\bar{\mathrm{T}}$. Let $\mathrm{G}^{\bullet}$ denote the canonical Godement resolution. Then, clearly the functor $R\psi_{*} = \psi_{*}\mathrm{G}^{\bullet}$ sends complexes in $\mathrm{D}^{b,o}_{\tilde{\mathrm{N}},c}(\bar{\mathrm{T}})$ to complexes of dg-modules over the sheaf of dg-algebras $R\psi_{*}(\mathbb{Q})$, that is, to objects in the derived category $\mathrm{D}_{\mathrm{WT}}(\bar{\mathrm{T}}, R\psi_{*}(\mathbb{Q}))$. We let $\mathrm{D}^{+,o}_{\mathrm{WT},c}(\bar{\mathrm{T}}, R\psi_{*}(Q))$ denote the full subcategory generated by the objects $R\psi_{*}(j_{!}\mathbb{Q})$ where $j : \mathcal{O} \to \bar{\mathrm{T}}$ denotes the immersion associated with an $\tilde{\mathrm{N}}$-orbit and we vary over such $\tilde{\mathrm{N}}$-orbits. Let

$$L\psi^{*} : \mathrm{D}^{+,o}_{\mathrm{WT},c}(\bar{\mathrm{T}}, R\psi_{*}(\mathbb{Q})) \to \mathrm{D}^{b,o}_{\tilde{\mathrm{N}},c}(\bar{\mathrm{T}})$$

denote the functor defined by sending a dg-module M to $\mathbb{Q} \underset{\psi^{-1}R\psi_{*}(\mathbb{Q})}{\overset{L}{\otimes}} \psi^{-1}(M)$.

Remark 2.8. Observe that the derived category $\mathrm{D}^{b,o}_{\tilde{\mathrm{N}},c}(\bar{\mathrm{T}})$ has the following t-structures: either the standard one, where the heart is the full subcategory of complexes whose cohomology sheaves are $\tilde{\mathrm{N}}$-equivariant and vanish in all degrees except 0 (see, for example, [BBD82, 1.3.2]), or the t-structure obtained by gluing as in [BBD82, 2.2], where the heart is the full subcategory of $\tilde{\mathrm{N}}$-equivariant perverse sheaves.

Proposition 2.9. *(i) The functor $R\psi_{*} : \mathrm{D}^{b,o}_{\tilde{\mathrm{N}},c}(\bar{\mathrm{T}}) \to \mathrm{D}^{+,o}_{\mathrm{WT},c}(\bar{\mathrm{T}}, R\psi_{*}(\mathbb{Q}))$ is an equivalence of categories with inverse $L\psi^{*}$. (The derived category on the right is generated by the dg-modules over the dga $R\psi_{*}(\mathbb{Q})$ of the form $j_{!}j^{*}R\psi_{*}(\mathbb{Q}) = R\psi_{*}j_{!}j^{*}(\mathbb{Q})$, as $j : \mathcal{O} \to \bar{\mathrm{T}}$ varies over the $\tilde{\mathrm{N}}$-orbits on $\bar{\mathrm{T}}$.)*

(ii) As a consequence, the derived category $\mathrm{D}^{+,o}_{\mathrm{WT},c}(\bar{\mathrm{T}}, R\psi_{*}(\mathbb{Q}))$ obtains induced t-structures, induced from the t-structures on $\mathrm{D}^{b,o}_{\tilde{\mathrm{N}},c}(\bar{\mathrm{T}})$ by the functor $R\psi_{*}$.

Proof. A key observation is that if $j : \mathcal{O} \to \bar{\mathrm{T}}$ is the locally-closed immersion associated to an $\tilde{\mathrm{N}}$-orbit on $\bar{\mathrm{T}}$, then $R\psi_{*}(j_{!}j^{*}(\mathbb{Q})) = j_{!}j^{*}(R\psi_{*}(\mathbb{Q}))$. This follows readily since the map ψ is a fibration with fibers $\mathrm{Bdiag}(\mathrm{T})$ and one makes use of the simplicial model of classifying spaces. In the simplicial setting, the map $\psi_{n} : (\mathrm{E}\tilde{\mathrm{N}} \underset{\tilde{\mathrm{N}}}{\times} \bar{\mathrm{T}})_{n} \to (\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}})_{n}$ is of the form $p_{n} \times id$, where $p_{n} : (\tilde{\mathrm{N}})^{n-1} \to (\mathrm{WT})^{n-1}$ is the surjection induced by the surjection $\tilde{\mathrm{N}} \to \mathrm{WT}$. Moreover, if $j_{n} : (\mathrm{E}\tilde{\mathrm{N}} \underset{\tilde{\mathrm{N}}}{\times} \mathcal{O})_{n} = (\tilde{\mathrm{N}})^{n-1} \times \mathcal{O} \to (\mathrm{E}\tilde{\mathrm{N}} \underset{\tilde{\mathrm{N}}}{\times} \bar{\mathrm{T}})_{n} = (\tilde{\mathrm{N}})^{n-1} \times \bar{\mathrm{T}}$ is the map induced by j, then $j_{n} = id^{n-1} \times j$, with a similar assertion holding for the map induced by j, $(\mathrm{EWT} \underset{\mathrm{WT}}{\times} \mathcal{O})_{n} \to (\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}})_{n}$, which will be also denoted j_{n}. Therefore (denoting by j also the map $\{j_{n}|n\}$ of simplicial varieties),

$$R\psi_{*}(j_{!}j^{*}(\mathbb{Q})) = \{R\psi_{n*}(j_{n!}j_{n}^{*}(\mathbb{Q})|n\} = \{Rp_{n*}(\mathbb{Q}) \boxtimes j_{!}(j^{*}(\mathbb{Q}))|n\} = \{j_{n!}j_{n}^{*}(Rp_{n*}(\mathbb{Q}) \boxtimes \mathbb{Q})|n\} = j_{!}j^{*}R\psi_{*}(\mathbb{Q}).$$

Therefore, one obtains the identifications

$$R\psi_{*} \circ L\psi^{*}(j_{!}R\psi_{*}(j^{*}\mathbb{Q})) = j_{!}j^{*}(R\psi_{*}(\mathbb{Q})) \text{ and } L\psi^{*} \circ R\psi_{*}(j_{!}j^{*}\mathbb{Q}) = j_{!}j^{*}(\mathbb{Q}).$$

Here we have denoted the restriction of the map ψ to any of the orbits also by ψ. Since the derived category $\mathrm{D}^{b,o}_{\tilde{\mathrm{N}},c}(\bar{\mathrm{T}})$ ($\mathrm{D}^{+,o}_{\mathrm{WT},c}(\bar{\mathrm{T}}, R\psi_{*}(\mathbb{Q}))$) is generated by the $j_{!}j^{*}(\mathbb{Q})$ ($j_{!}(R\psi_{*}j^{*}(\mathbb{Q}))$, respectively) as one varies over the $\tilde{\mathrm{N}}$-orbits, and $L\psi^{*}$ is left-adjoint to $R\psi_{*}$, the statement (i) in the proposition follows.

Proposition 6.4 shows that when one has an equivalence of triangulated categories, one may transfer the t-structure on one of the triangulated categories to the other. This proves the statement (ii) and completes the proof of the Proposition.   □

Definition 2.10. Henceforth we call the t-structure obtained by gluing the standard t-structures on a stratified variety, shifted by the codimensions of the strata (as in [BBD82, 2.2]), *the t-structure obtained by gluing.*

One starts with either the standard t-structures or the t-structures obtained by gluing on the derived categories $\mathrm{D}^{\mathrm{b}}_{\mathrm{G}\times\mathrm{G},\mathrm{c}}(\bar{\mathrm{G}})$ and $\mathrm{D}^{b,o}_{(\mathrm{T}\times\mathrm{T})\mathrm{diag}(\mathrm{N}(\mathrm{T})),\mathrm{c}}(\bar{\mathrm{T}})$.

Corollary 2.11. *We obtain the equivalences of derived categories:*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{G}\times\mathrm{G},\mathrm{c}}(\bar{\mathrm{G}})\overset{\mathrm{res}}{\to}\mathrm{D}^{b,o}_{(\mathrm{T}\times\mathrm{T})\mathrm{diag}(\mathrm{N}(\mathrm{T})),\mathrm{c}}(\bar{\mathrm{T}})\overset{\mathrm{R}\psi_*}{\to}\mathrm{D}^{+,o}_{\mathrm{WT},c}(\bar{\mathrm{T}},\mathrm{R}\psi_*(\mathbb{Q})).$$

The derived category $\mathrm{D}^{+;o}_{\mathrm{WT},c}(\bar{\mathrm{T}},R\psi_*(\mathbb{Q}))$ *obtains induced t-structures, induced from the t-structures on* $\mathrm{D}^{b;o}_{\tilde{\mathrm{N}},c}(\bar{\mathrm{T}})$ *by the functor* $R\psi_*$.

Moreover, if $\mathrm{IC}^{\tilde{\mathrm{N}}}(\mathcal{O}_{\tilde{\mathrm{N}}})$ *denotes the* $\tilde{\mathrm{N}}$ *-equivariant intersection cohomology complex on the closure of the* $\tilde{\mathrm{N}}$*-orbit* $\mathcal{O}_{\tilde{\mathrm{N}}}$ *obtained from the constant local system on the corresponding orbit, then* $R\psi_*(\mathrm{IC}^{\tilde{\mathrm{N}}}(\mathcal{O}_{\tilde{\mathrm{N}}}) = \mathrm{IC}^{\mathrm{WT}}(\mathcal{O}_{\mathrm{WT}}) \otimes R\psi_*(\mathbb{Q})$. *Here* $\mathcal{O}_{\mathrm{WT}}$ *is the same* $\tilde{\mathrm{N}}$*-orbit viewed as an orbit for the action of* WT *and* $\mathrm{IC}^{\mathrm{WT}}(\mathcal{O}_{\mathrm{WT}})$ *denotes the* WT*-equivariant intersection cohomology complex on the closure of the orbit* $\mathcal{O}_{\mathrm{WT}}$ *obtained from the constant local system on* $\mathcal{O}_{\mathrm{WT}}$.

Proof. All conclusions but the last are clear by combining Theorem 2.5 and Proposition 2.9. To see the last conclusion, first observe that $\mathrm{IC}^{\tilde{\mathrm{N}}}(\mathcal{O}_{\tilde{\mathrm{N}}}) \simeq \psi^*(\mathrm{IC}^{\mathrm{WT}}(\mathcal{O}_{\mathrm{WT}})$ and the map ψ is a locally trivial fibration. Therefore, the last conclusion follows from the projection formula.   □

2.5. **Step 3 of the proof of Theorem 1.2: proof of Theorem 1.3.** It is important for us to be able to separate the W and T actions on $\bar{\mathrm{T}}$. We proceed to do this.

First we define actions of finite groups on simplicial varieties.

Definition 2.12. (*Finite group actions on simplicial varieties*) If $\mathrm{X}_\bullet$ is a simplicial variety, an action of the finite group W on $\mathrm{X}_\bullet$ corresponds to a W-action on each X_n so that the structure maps of the simplicial variety $\mathrm{X}_\bullet$ are all W-equivariant. The two results below, follow readily from the fact that WT is the semi-direct product of W and T.

Proposition 2.13. *(i) If* X *is a variety, then giving a* WT-*action on* X *is equivalent to providing* T *and* W *actions on* X *so that the* T-*action* $\mu : \mathrm{T} \times \mathrm{X} \to \mathrm{X}$ *is* W-*equivariant. Equivalently, the diagram*

(2.5.1)
$$
\begin{array}{ccc}
\mathrm{W} \times \mathrm{T} \times \mathrm{X} & \xrightarrow{id \times \mu_{\mathrm{T}}} & \mathrm{W} \times \mathrm{X} \\
\downarrow{\scriptstyle \mu'_{\mathrm{W}}} & \xrightarrow{\mu_{\mathrm{T}}} & \downarrow{\scriptstyle \mu_{\mathrm{W}}} \\
\mathrm{T} \times \mathrm{X} & \longrightarrow & \mathrm{X}
\end{array}
$$

commutes, where μ_W (μ_{T}) *denote the* W *(T) action and* μ'_{W} *denotes the diagonal action of* W *on* $\mathrm{T} \times \mathrm{X}$. *Equivalently, the following relations hold (where $\circ$ denotes generically any of the actions):*

(2.5.2) $w \circ (t \circ x) = (w \circ t) \circ (w \circ x), w \in \mathrm{W}, t \in \mathrm{T}, \mathrm{x} \in \mathrm{X}.$

(ii) Moreover, in this case the simplicial variety $\mathrm{ET}\underset{\mathrm{T}}{\times}\mathrm{X}$ *has an induced action by* W.

Proof. We skip the proofs as these may be deduced in a straightforward manner from the fact that WT is the semi-direct product of W and T.   □

Lemma 2.14. *Assume the situation of Proposition 2.13. Given $t_i \in \mathrm{T}$ and $w_i \in \mathrm{W}$, $i = 1, 2$ and $x \in \mathrm{X}$, one obtains the following identifications:*

$$t_1 w_1 \circ (w \circ x) = t_1 \circ (w_1 \circ (w \circ x)), t_2 w_2 t_1 w_1 \circ (w \circ x) = t_2 w_2 t_1 \circ (w_1 \circ (w \circ x)) = t_2 \circ (w_2 \circ (t_1 \circ (w_1 \circ w \circ x)))$$

Proof. Again the proof is skipped since this follows from a straight-forward calculation. □

One may now consider the action of W on the simplicial variety $\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}$, where we let W act diagonally on the two factors ET and $\bar{\mathrm{T}}$. We let $\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}})$ denote the simplicial variety which is the diagonal of the resulting bisimplicial variety.

Lemma 2.15. *Now we obtain an equivalence of categories:*

$$\mathrm{D}^{\mathrm{b}}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}), \mathbb{Q}) \simeq \mathrm{D}^{\mathrm{b}}_{\mathrm{WT,c}}(\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}}, \mathbb{Q}) = \mathrm{D}^{\mathrm{b}}_{\mathrm{WT,c}}(\bar{\mathrm{T}}, \mathbb{Q})$$

where the subscript cart (c) denotes the complexes of sheaves having cartesian (constructible, respectively) cohomology sheaves. (Recall that a sheaf $F = \{F_n | n\}$ on a simplicial variety $\mathrm{X}_\bullet$ is cartesian, if all the maps $\phi : \alpha^*(F_m) \to F_n$ are isomorphisms for each structure map $\alpha : \mathrm{X}_n \to \mathrm{X}_m$ of the simplicial space $\mathrm{X}_\bullet$. See 2.1.1 for further details.)

The above equivalence extends to an equivalence of the corresponding bounded below derived categories:

$$\mathrm{D}^{+}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}), \mathbb{Q}) \simeq \mathrm{D}^{+}_{\mathrm{WT,c}}(\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}}, \mathbb{Q}) = \mathrm{D}^{+}_{\mathrm{WT,c}}(\bar{\mathrm{T}}, \mathbb{Q}).$$

The above equivalence of derived categories preserves both the standard t-structures and the t-structures obtained by gluing.

Proof. Since there is no direct map between the simplicial varieties $\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}})$ and $\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}}$, we make use of the intermediate simplicial variety: $\mathrm{EWT} \underset{\mathrm{WT}}{\times} (\mathrm{ET} \times \bar{\mathrm{T}})$ which maps to both the above simplicial varieties. Let

(2.5.3)
$$p_1 : \mathrm{EWT} \underset{\mathrm{WT}}{\times} (\mathrm{ET} \times \bar{\mathrm{T}}) \to \mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}) \text{ and}$$
$$p_2 : \mathrm{EWT} \underset{\mathrm{WT}}{\times} (\mathrm{ET} \times \bar{\mathrm{T}}) \to \mathrm{EWT} \underset{\mathrm{WT}}{\times} (\bar{\mathrm{T}})$$

denote the maps defined as follows. p_1 is induced by the maps sending EWT to EW by taking the quotient by T and $\mathrm{ET} \times \bar{\mathrm{T}}$ to $\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}$. (Recall $(\mathrm{EWT} \underset{\mathrm{WT}}{\times} (\mathrm{ET} \times \bar{\mathrm{T}}))_\mathrm{n} = (\mathrm{WT})^\mathrm{n} \times \mathrm{T}^{\mathrm{n}+1} \times \bar{\mathrm{T}}$, while $(\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}))_\mathrm{n} = \mathrm{W}^\mathrm{n} \times \mathrm{T}^\mathrm{n} \times \bar{\mathrm{T}}$, so it should be clear what $p_{1,\mathrm{n}}$ is.) The second map p_2 is induced by the map that drops the factor ET.

One may observe the need to work with the simplicial model of classifying spaces in being able to define the first map p_1. The fibers of both these simplicial maps are ET which is acyclic. Moreover, these maps are cohomologically proper and the projection formula holds for both of the maps, so that one concludes readily that the natural map $K \to Rp_{1*}p_1^*(K)$ is a quasi-isomorphism for any complex $K \in \mathrm{D}^{+}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}), \mathbb{Q})$ and that the natural map $K \to Rp_{2*}p_2^*(K)$ is a quasi-isomorphism for any complex $K \in \mathrm{D}^{+}_{\mathrm{WT,c}}(\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}}, \mathbb{Q})$. Therefore, it follows that both the functors p_i^* are fully-faithful at the level of the bounded below derived categories. The heart of both the above bounded derived categories is the category of WT-equivariant sheaves on $\bar{\mathrm{T}}$. (This is clear for the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{WT,c}}(\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}}, \mathbb{Q})$. To see the same for the derived category $\mathrm{D}^{\mathrm{b}}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}), \mathbb{Q})$, one makes use of Proposition 2.13.) Therefore, both the functors p_i^* are equivalences of the bounded derived categories, thereby proving the first statement in the lemma. Since both the functors p_i^* are exact, it is clear that they preserve the standard *t*-structures. It is clear that the maps p_i preserve the stratification on $\bar{\mathrm{T}}$ by WT-orbits. Since the map $(p_1)_\mathrm{n} = \mathrm{WT}^\mathrm{n} \times \mathrm{T}^{\mathrm{n}+1} \times \bar{\mathrm{T}} \to \mathrm{W}^\mathrm{n} \times \mathrm{T}^\mathrm{n} \times \bar{\mathrm{T}}$ is given by the identity map on the factor $\bar{\mathrm{T}}$, one may readily see that the functor p_1^* commutes with the functors $j_!$, j^*, Rj_* and $Rj^!$ associated to the locally closed immersion $j : \mathrm{U}_\mathrm{i} \to \mathrm{U}_{\mathrm{i}+1}$ of strata appearing in the stratification of $\bar{\mathrm{T}}$ by

WT-orbits. The same holds for the functor p_2^*. These observations then show that the above equivalences preserve the t-structures obtained by gluing.

To extend this equivalence for the corresponding bounded below derived categories, one proceeds as follows. Let $D = D_{cart,c}^+(EWT \underset{WT}{\times} (ET \times \bar{T}), \mathbb{Q})$. Observe that the inclusion functor $D^{\leq n} \to D$ has a right adjoint given by the cohomology truncation functor $\sigma_{\leq n}$: here $D^{\leq n}$ denotes the full subcategory of D consisting of complexes that have trivial cohomology above degree n. Next observe that the functors $Rp_{i*} = p_{i*}$, $i = 1, 2$, and hence are exact functors. Similarly p_i^*, $i = 1, 2$, are exact functors. Therefore all of the above functors commute with the functors $\sigma_{\leq n}$ and if $K \in D$, then $p_{i*}(\sigma_{\leq n}K)$ are bounded complexes for each n. Since p_i^* is an equivalence at the level of the above bounded derived categories with inverse p_{i*}, it follows that $K = \lim_{n \to \infty} \sigma_{\leq n}(K) \simeq \lim_{n \to \infty} p_i^*(p_{i*}\sigma_{\leq n}K) \simeq p_i^*(\lim_{n \to \infty} p_{i*}(\sigma_{\leq n}K))$ and that $K \simeq \lim_{n \to \infty} \sigma_{\leq n}K \overset{\cong}{\to} \lim_{n \to \infty} Rp_{i*}p_i^*(\sigma_{\leq n}K) \simeq Rp_{i*}p_i^*(\lim_{n \to \infty} \sigma_{\leq n}K) = Rp_{i*}p_i^*(K)$, thereby proving that the functors p_i^* also induce equivalences of the corresponding bounded below derived categories. □

Next we consider the subcategory $D_{WT,c}^+(\bar{T}, R\psi_*(\mathbb{Q}))$. Now $Rp_{1*}(p_2^*(R\psi_*(\mathbb{Q}))$ defines a sheaf of dgas on $EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T})$.

Definition 2.16. Henceforth, we will denote the sheaf of dgas $Rp_{1*}p_2^*(R\psi_*(\mathbb{Q}))$ on $EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T})$ by $\mathcal{A}$.

Recall that $R\psi_*(\mathbb{Q})$ is a complex of sheaves on $EWT \underset{WT}{\times} (\bar{T})$ and that $\mathcal{A} = Rp_{1*}p_2^*(R\psi_*(\mathbb{Q}))$ is a complex of sheaves on $EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T})$.

Corollary 2.17. *The equivalence of categories in Lemma 2.15 extends to an equivalence of categories:*

$$D_{cart,c}^+(EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}), \mathcal{A}) \simeq D_{WT,c}^+(EWT \underset{WT}{\times} \bar{T}, R\psi_*(\mathbb{Q}) = D_{WT,c}^+(\bar{T}, R\psi_*(\mathbb{Q})) \text{ and}$$

$$D_{cart,c}^{+,o}(EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}), \mathcal{A}) \simeq D_{WT,c}^{+,o}(EWT \underset{WT}{\times} \bar{T}, R\psi_*(\mathbb{Q}) = D_{WT,c}^{+,o}(\bar{T}, R\psi_*(\mathbb{Q}))$$

where $D_{cart,c}^{+,o}(EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}), \mathcal{A})$ *is the full subcategory of* $D(EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}), \mathcal{A})$ *generated by the functor* $Rp_{1*}p_2^*$ *applied to the generators* $\{R\psi_*(j_!j^*(\mathbb{Q}_{|\mathcal{O}}))|\mathcal{O} \text{ a } WT - \text{orbit on } \bar{T}\}$ *of the subcategory* $D_{WT,c}^{+,o}(\bar{T}, R\psi_*(\mathbb{Q}))$.

The t-structures on the derived categories on the left are obtained by transferring the t-structures on the corresponding derived categories on the right. Moreover,

$$Rp_{1*}p_2^*(R\psi_*(IC^{\tilde{N}}(\mathcal{O}_{\tilde{N}}))) = IC^{W,T}(\mathcal{O}_{WT}) \otimes Rp_{1*}p_2^*(R\psi_*(\mathbb{Q})).$$

where $IC^{W,T}(\mathcal{O}_{WT})$ *denotes the same equivariant intersection cohomology complex* $IC^{WT}(\mathcal{O}_{WT})$ *for the action* WT, *but viewed as an object on* $EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T})$.

Proof. All but the last conclusions are clear in view of the observations above, and also from the observation that the composite functors $Rp_{i*} \circ p_i^*$ and $p_i^* \circ Rp_{i*}$ are still the identity at the level of the above derived categories of dg-modules. To see the last conclusion, first observe that

(2.5.4) $$IC^{WT}(\mathcal{O}_{WT}) = \oplus_{w \in W/W_{\mathcal{O}_T}} IC^T(w\mathcal{O}_T),$$

where $W_{\mathcal{O}_T}$ denotes the stabilizer in W of the T-orbit $\mathcal{O}_T$, and $IC^T(w\mathcal{O}_T)$ denotes the intersection cohomology complex obtained by extending the constant local system on $w\mathcal{O}_T$. (This may be deduced from the fact that the equivariant intersection cohomology complexes $IC^{WT}(\mathcal{O}_{WT})$ are perverse extensions of the constant sheaf $\mathbb{Q}_{\mathcal{O}_{WT}} = \oplus_{w \in W/W_{\mathcal{O}_T}} \mathbb{Q}_{\mathcal{O}_T}$.) Therefore, $p_2^*(IC^{WT}(\mathcal{O}_{WT})) = p_1^*(IC^{W,T}(\mathcal{O}_{WT}))$: recall $IC^{W,T}(\mathcal{O}_{WT})$ denotes the same equivariant intersection cohomology complex $IC^{WT}(\mathcal{O}_{WT})$ for the action WT, but viewed as an object on $EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T})$. This is possible, in view of (2.5.4).

Next observe from Corollary 2.11 that $R\psi_*(IC^{\tilde{N}}(\mathcal{O}_{\tilde{N}})) = IC^{WT}(\mathcal{O}_{WT}) \otimes R\psi_*(\mathbb{Q})$. Therefore, the last identification in the corollary follows from a projection formula. □

Let $\bar{T}/T$ denote the space whose points correspond to the T-orbits on $\bar{T}$ equipped with the topology as in the introduction. Let

(2.5.5) $\pi : ET \underset{T}{\times} \bar{T} \to \bar{T}/T$

denote the obvious map. The fact that the action of T on $\bar{T}$ is equivariant with respect to the W-action shows that one has an induced action of W on the set of T-orbits on $\bar{T}$. Denoting this W-action on $\bar{T}/T$ also by $\circ$, observe that $w \circ [x] = [w \circ x]$, for any $w \in W$ and $x \in \bar{T}$, where $[x]$ denotes the T-orbit of x. Now W acts on both $ET \underset{T}{\times} \bar{T}$ and $\bar{T}/T$.

Lemma 2.18. *Assume the above situation. Then, with respect to the above actions, the map π is W-equivariant.*

Proof. Observe that the map π sends $(t_1, t_2, \cdots, t_{n-1}, x) \in (ET \underset{T}{\times} \bar{T})_n$ to $[x]$. Now one verifies that

$$\pi(w, (t_1, \cdots, t_{n-1}, x)) = \pi(w \circ t_1, \cdots, w \circ t_{n-1}, w \circ x) = [w \circ x] = w \circ [x] = w \circ \pi(t_1, \cdots, t_{n-1}, x).$$

This proves the map π is W-equivariant. $\square$

Therefore, we will also let the map $EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}) \to EW \underset{W}{\times} (\bar{T}/T)$ induced by π be denoted π.

Definition 2.19. We define $D^{+,o}_{cart,c}(EW \underset{W}{\times} (\bar{T}/T), R\pi_*(\mathcal{A}))$ to be the full subcategory of $D(EW \underset{W}{\times} (\bar{T}/T), R\pi_*(\mathcal{A}))$ generated by $R\pi_*(G_\alpha)$, as G_α varies over the generators of the derived category $D^{+,o}_{cart,c}(EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}), \mathcal{A})$ as in Corollary 2.17.

One may now define

$$R\pi_* : D^{+,o}_{cart,c}(EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}), \mathcal{A}) \to D^{+,o}_{cart,c}(EW \underset{W}{\times} (\bar{T}/T), R\pi_*(\mathcal{A}))$$

by once again making use of the canonical Godement resolutions. A left derived functor

$$L\pi^* : D^{+,o}_{cart,c}(EW \underset{W}{\times} (\bar{T}/T), R\pi_*(\mathcal{A})) \to D^{+,o}_{cart,c}(EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}), \mathcal{A})$$

may be defined by taking flat-resolutions of $M \in D^+_{cart,c}(EW \underset{W}{\times} (\bar{T}/T), R\pi_*(\mathcal{A}))$ and letting

$$L\pi^*(M) = \mathcal{A} \overset{L}{\underset{\pi^{-1} R\pi_*(\mathcal{A})}{\otimes}} \pi^{-1}(M).$$

So defined, $L\pi^*$ will send objects in $D_{cart,c}(EW \underset{W}{\times} (\bar{T}/T), R\pi_*(\mathcal{A}))$ to objects in $D_{cart,c}(EW \underset{W}{\times} (ET \underset{T}{\times} \bar{T}), \mathcal{A})$.

Remark 2.20. Observe that we define $R\pi_*$ making use of the canonical Godement resolution. Therefore $R\pi_*$ will be functorial at the level of complexes. Similarly, by making use of functorial flat resolutions, one makes the functor $L\pi^*$ also functorial at the level of complexes.

Since $L\pi^*$ will be left adjoint to a functor $R\pi_*$ (defined at the level of the corresponding unbounded derived categories), one obtains natural transformations $L\pi^* \circ R\pi_* \to id$ and $id \to R\pi_* \circ L\pi^*$. To show these are quasi-isomorphisms, it suffices to restrict to the hearts of the corresponding derived categories. For this, we will resort to a variant of the arguments used for toric (and also horospherical) varieties.

In fact the W-symmetric toric variety $\bar{T}$ is a toric variety for the torus T provided with an extra action by the finite group W that permutes the various T-orbits of the same type. Therefore let Y denote a T-orbit in $\bar{T}$ and let U_Y denote the (unique) affine open T-stable neighborhood in $\bar{T}$ so that Y is the only closed T-orbit in U_Y. Similarly, let Y' denote a T-orbit on $\bar{T}$ and let $U_{Y'}$ denote the affine open T-stable neighborhood of Y' in $\bar{T}$ so that Y' is the only closed T-orbit in $U_{Y'}$. Assume that either $\overline{Y}' - Y' \supseteq Y$ (in which case $U_{Y'} \subseteq U_Y$), or that $Y \nsubseteq U_{Y'}$. Observe that under the W-action on $\bar{T}$, the T-orbit Y (Y')

is sent to another T-orbit $w\mathrm{Y}$ ($w\mathrm{Y}'$, respectively) depending on $w \in \mathrm{W}$. Therefore, as shown in [Lu95, Proof of Theorem 2.6] we observe that the restriction

$$(2.5.6) \qquad \mathrm{H}^*_{\mathrm{T}}(\mathrm{U}_{\mathrm{Y}}, \mathcal{A}) \to \mathrm{H}^*_{\mathrm{T}}(\mathrm{U}_{\mathrm{Y}} - \mathrm{U}_{\mathrm{Y}'}, \mathcal{A})$$

is an isomorphism: to see this one may observe that Y is a deformation retract of both U_{Y} and $\mathrm{U}_{\mathrm{Y}} - \mathrm{U}'_{\mathrm{Y}}$ and $R\psi_*(\mathbb{Q})$, and therefore $\mathcal{A}$ have constant cohomology sheaves. (See (3.0.3) for the latter.) Then each $w \in \mathrm{W}$ sends this to the restriction-isomorphism:

$$(2.5.7) \qquad \mathrm{H}^*_{\mathrm{T}}(w\mathrm{U}_{\mathrm{Y}}, \mathcal{A}) \to \mathrm{H}^*_{\mathrm{T}}(w\mathrm{U}_{\mathrm{Y}} - w\mathrm{U}'_{\mathrm{Y}}, \mathcal{A}).$$

Next let Y' be as before, but choose Y to be another T-orbit on $\bar{\mathrm{T}}$ so that $\mathrm{Y} \not\subseteq \mathrm{U}_{\mathrm{Y}'}$. Let $j : \mathrm{U}_{\mathrm{Y}'} \to \bar{\mathrm{T}}$ denote the open immersion and let $i : \bar{\mathrm{T}} - \mathrm{U}_{\mathrm{Y}'} \to \mathrm{X}$ denote the corresponding closed immersion. Let $\pi : \mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}} \to \bar{\mathrm{T}}/\mathrm{T}$ denote the map considered in (2.5.5). We will consider the following commutative diagram

$$(2.5.8)$$

$$
\begin{array}{ccc}
\mathrm{ET} \underset{\mathrm{T}}{\times} \mathrm{U}_{w\mathrm{Y}'} & \xrightarrow{\; j_{w\mathrm{Y}'} \;} & \mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}} \\
\downarrow{\scriptstyle \pi_{\mathrm{U}_{w\mathrm{Y}'}}} & & \downarrow{\scriptstyle \pi} \\
\mathrm{U}_{w\mathrm{Y}'}/\mathrm{T} & \xrightarrow{\; j_{w\mathrm{Y}'}/\mathrm{T} \;} & \bar{\mathrm{T}}/\mathrm{T}
\end{array}
$$

where the maps $j_{w\mathrm{Y}'}$, $j_{w\mathrm{Y}'}/\mathrm{T}$ and $\pi_{\mathrm{U}_{w\mathrm{Y}'}}$ are defined by the above diagram.

Lemma 2.21. *Assume the above situation. Then, (i) $R\pi_* j_{w\mathrm{Y}'!} j^*_{w\mathrm{Y}'}(\mathcal{A}) \simeq j_{w\mathrm{Y}'/\mathrm{T}!} j^*_{w\mathrm{Y}'/\mathrm{T}} R\pi_*(\mathcal{A})$.*

(ii) The equivariant derived category $\mathrm{D}^{+,\mathrm{o}}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}), \mathcal{A})$ *is generated by objects of the form* $\oplus_{w \in \mathrm{W}/\mathrm{W}_{\mathrm{Y}'}} j_{w\mathrm{Y}'!} j^*_{w\mathrm{Y}'}(\mathcal{A})$, *where $\mathrm{W}_{\mathrm{Y}'}$ denotes the stabilizer in* W *of the T-orbit Y'.*

Proof. (i) We obtain this from the isomorphism (2.5.7), by observing that, therefore, $\mathrm{H}^*_{\mathrm{T}, w\mathrm{U}_{\mathrm{Y}} - w\mathrm{U}'_{\mathrm{Y}}}(w\mathrm{U}_{\mathrm{Y}}, \mathcal{A}) = 0$.

(ii) Recall that the generators of the derived category $\mathrm{D}^{+,\mathrm{o}}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}), \mathcal{A})$ are obtained by applying the functor $Rp_{1*} p_2^*$ to the generators of the derived category $\mathrm{D}^{+,\mathrm{o}}_{\mathrm{WT,c}}(\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}}, R\psi_*(\mathbb{Q}))$. The generators of the latter are $R\psi_*(j_{\mathcal{O}_{\mathrm{WT}}!}(\mathbb{Q}_{|\mathcal{O}_{\mathrm{WT}}}))$ as one varies over the WT-orbits $\mathcal{O}_{\mathrm{WT}}$ on $\bar{\mathrm{T}}$. But each WT-orbit $\mathcal{O}_{\mathrm{WT}} = \sqcup_{w \in \mathrm{W}/\mathrm{W}_{\mathrm{Y}'}} w\mathrm{Y}'$, where Y' is a T-orbit on $\bar{\mathrm{T}}$, with $\mathrm{W}_{\mathrm{Y}'}$ denoting the stabilizer of the T-orbit Y' for the action of W on $\bar{\mathrm{T}}/\mathrm{T}$. $\qquad \square$

Proposition 2.22. *The natural transformations $K \to R\pi_* L\pi^*(K)$, $K \in \mathrm{D}^{+,\mathrm{o}}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}), \mathcal{A})$ and $L\pi^* R\pi_*(L) \to L$ for any $L \in \mathrm{D}^{+,\mathrm{o}}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\bar{\mathrm{T}}/\mathrm{T}), R\pi_*(\mathcal{A}))$ are quasi-isomorphisms. Therefore, the functors $R\pi_*$ and $L\pi^*$ induce an equivalence of categories.*

Proof. The key observation (see Lemma (2.21) (ii)) is that the derived category $\mathrm{D}^{+,\mathrm{o}}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}}), \mathcal{A})$ is generated by the objects $\oplus_{w \in \mathrm{W}/\mathrm{W}_{\mathrm{Y}'}} j_{w\mathrm{Y}'!} j^*_{w\mathrm{Y}'}(\mathcal{A})$ while the derived category $\mathrm{D}^{+,\mathrm{o}}_{\mathrm{cart,c}}(\mathrm{EW} \underset{\mathrm{W}}{\times} (\bar{\mathrm{T}}/\mathrm{T}), R\pi_*(\mathcal{A}))$ is generated by the objects $\oplus_{w \in \mathrm{W}/\mathrm{W}_{\mathrm{Y}'}} j_{w\mathrm{Y}'/\mathrm{T}!} j^*_{w\mathrm{Y}'/\mathrm{T}} R\pi_*(\mathcal{A})$ as Y' varies among the T-orbits on $\bar{\mathrm{T}}$. Lemma (2.21) shows that the functor $R\pi_*$ sends $\oplus_{w \in \mathrm{W}/\mathrm{W}_{\mathrm{Y}'}} j_{w\mathrm{Y}'!} j^*_{w\mathrm{Y}'}(\mathcal{A})$ to $\oplus_{w \in \mathrm{W}/\mathrm{W}_{\mathrm{Y}'}} j_{w\mathrm{Y}'/\mathrm{T}!} j^*_{w\mathrm{Y}'/\mathrm{T}} R\pi_*(\mathcal{A})$. One may readily see that $L\pi^*(\oplus_{w \in \mathrm{W}/\mathrm{W}_{\mathrm{Y}'}} j_{w\mathrm{Y}'/\mathrm{T}!} j^*_{w\mathrm{Y}'/\mathrm{T}} R\pi_*(\mathcal{A})) = \oplus_{w \in \mathrm{W}/\mathrm{W}_{\mathrm{Y}'}} j_{w\mathrm{Y}'!} j^*_{w\mathrm{Y}'} L\pi^* R\pi_*(\mathcal{A}) = \oplus_{w \in \mathrm{W}/\mathrm{W}_{\mathrm{Y}'}} j_{w\mathrm{Y}'!} j^*_{w\mathrm{Y}'}(\mathcal{A})$. Therefore, one readily sees that the natural transformations $L\pi^* \circ R\pi_* \to id$ and $id \to R\pi_* \circ L\pi^*$ are isomorphisms proving the proposition. $\qquad \square$

Proof of Theorem 1.3. Combining Proposition 2.22, Corollaries 2.11 and 2.17, we obtain Theorem 1.3.

3. **Proof of Theorem 1.2: Step 4.**

The remainder of this section is devoted to a proof of the remaining aspects of Theorem 1.2. We begin with the following result, which applies to any group action.

Lemma 3.1. *Let* G *denote a linear algebraic group acting on a variety* X. *Then for the action of* G *on* $(EG \underset{G}{\times} X)_n$ *given by* $g \circ (g_0, \cdots, g_{n-1}, x) = (gg_0g^{-1}, \cdots, gg_{n_1}g^{-1}, g \circ x)$, *with* $(g_0, \cdots, g_{n-1}, x) \in G^n \times X$, $g \in G$, *the structure maps of the simplicial variety* $EG \underset{G}{\times} X$ *are all* G-*equivariant.*

Proof. We first verify this for $n = 1$. In this case, the face map $d_0 = pr_2$, (which is the projection to the second factor $G \times X \to X$), and $d_1 = \mu : G \times X \to X$ (which is the group action). Then $g \circ d_0(g_0, x) = g \circ x = d_0(gg_0g^{-1}, g \circ x) = d_0(g \circ (g_0, x))$ and $g \circ d_1(g_0, x) = g \circ (g_0 \circ x) = (gg_0) \circ x = d_1(gg_0g^{-1}, g \circ x) = d_1(g \circ (g_0, x))$. The face map s_0 sends $x \in X$ to $(e, x) \in G \times X = (EG \underset{G}{\times} X)_1$. Now, $g \circ s_0(x) = g \circ (e, x) = (geg^{-1}, g \circ x) = (e, g \circ x) = s_0(g \circ (e, x))$. The proof that the remaining structure maps are all G-equivariant is similar, and is therefore skipped. $\qquad\square$

Next we show that the category of equivariant sheaves admits a simpler formulation when the group is a discrete group. This seems to be rather well-known: see [BL94, Chapter I, section 8] for related results. Let W denote a discrete group acting on a variety X and let R denote a commutative Noetherian ring. We will assume X is provided with a (Grothendieck) topology $\mathrm{Top}(X)$. Let $\mathrm{Sh}^W(X, R) = \mathrm{Sh}^W(X)$ denote the following category. Its objects are sheaves of F of R-modules on $\mathrm{Top}(X)$ so that one is provided with a homomorphism $W \to \mathrm{Aut}(F)$, where $\mathrm{Aut}(F)$ denotes the automorphism group of F as a sheaf of R-modules. Morphisms between two such sheaves F' and F are morphisms of sheaves compatible with the given extra structure.

Lemma 3.2. *Assume the above situation.*

(i) *Then the categories* $\mathrm{Sh}^W(X)$ *and* $\mathrm{Sh}^W(EW \underset{W}{\times} X)$ *are equivalent, where the category* $\mathrm{Sh}^W(EW \underset{W}{\times} X)$ *denotes the full subcategory of cartesian sheaves of* R-*modules on the simplicial variety* $EW \underset{W}{\times} X$.

(ii) *If* $F = \{F_n | n\} \in \mathrm{Sh}^W(EW \underset{W}{\times} X)$, *then each* $F_n \in \mathrm{Sh}^W((EW \underset{W}{\times} X)_n)$, *when* $(EW \underset{W}{\times} X)_n$ *is provided with the* W-*action as in Lemma 3.1.*

(iii) *If* $X = \{x_o\}$ *denotes a point, then we obtain the equivalence:* $\mathrm{Sh}^W(BW) \simeq \mathrm{Sh}^W(\{x_o\})$, *and the latter category is the category of* R-*modules provided with* W-*actions.*

Proof. For each $w \in W$, we let $w : X \to X$ denote the automorphism induced by w. Now, one may observe that giving a homomorphism $W \to \mathrm{Aut}(F)$ corresponds to giving for each $w \in W$, an isomorphism $w^{-1}(F) \to F$ (or equivalently an isomorphism $F \to w_*(F)$) of sheaves of R-modules, and which are compatible as w varies in W. In view of the fact that the group W is discrete, one may now readily show that the category $\mathrm{Sh}^W(X)$ is equivalent to the category of sheaves F_0 on $\mathrm{Top}(X)$ provided with an isomorphism $\phi : \mu^*(F_0) \to pr_2^*(F_0)$, satisfying a cocycle condition on further pull-back to $(EW \underset{W}{\times} X)_2$, and so that $s_0^*(\phi)$ is the identity. (Here $\mu, pr_2 : G \times X \to X$ are the group action and projection to the second factor, respectively, while $s_0 : X \to G \times X$ is the map $x \mapsto (x, e)$, where e denotes the identity of G and $x \in X$.)

It is well-known that this latter category is equivalent to $\mathrm{Sh}^W(EW \underset{W}{\times} X)$. In fact, given an $F_0 \in \mathrm{Sh}^W(X)$, one defines $F = \{F_n | n\} \in \mathrm{Sh}^W(EW \underset{W}{\times} X)$, by letting $F_n = ((d_0)^n)^*(F_0)$. The inverse of this functor sends any such $F = \{F_n | n\}$ to F_0. Finally, one may see from the above description that if $F = \{F_n | n\}$ belongs to $\mathrm{Sh}^W(EW \underset{W}{\times} X)$, then $F_0 \in \mathrm{Sh}^W(X)$. Since all the structure maps of the simplicial variety $EW \underset{W}{\times} X$ have been shown to be W-equivariant (see Lemma 3.1), it follows that $F_n = ((d_0)^n)^*(F_0) \in \mathrm{Sh}^W((EW \underset{W}{\times} X)_n)$. The last statement is clear since $EW \underset{W}{\times} \{x_o\} = BW$. $\qquad\square$

Step 4.1. The *first result* we proceed to establish in this section is that the dga $\mathcal{A}$, considered in Definition 2.16 and the dga $R\pi_*(\mathcal{A})$, (considered in Definition 2.19) are formal as dgas.

Definition 3.3. A complex of sheaves K of $\mathbb{Q}$-vector spaces on $\mathrm{EW} \underset{\mathrm{W}}{\times} \mathrm{X}$ is W-equivariant, if each term K^n in the complex is a W-equivariant sheaf and the differentials of $\overset{\bullet}{K}$ are also W-equivariant, that is, $K^\bullet$ is a complex in the abelian category $\mathrm{Sh}^{\mathrm{W}}(\mathrm{EW} \underset{\mathrm{W}}{\times} \mathrm{X}, \mathbb{Q})$.

We next consider the commutative diagram:

(3.0.1)
$$
\begin{array}{ccccc}
\mathrm{E}\tilde{\mathrm{N}} \underset{\tilde{\mathrm{N}}}{\times} \bar{\mathrm{T}} & \xrightarrow{\ \alpha\ } & \mathrm{B}\tilde{\mathrm{N}} & \xrightarrow{\ \beta\ } & \mathrm{BWdiag(T)} \\
\downarrow{\scriptstyle\psi} & & \downarrow{\scriptstyle\bar{\psi}} & & \downarrow{\scriptstyle\tilde{\psi}} \\
\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}} & \xrightarrow{\ \bar{\alpha}\ } & \mathrm{BWT} & \xrightarrow{\ \bar{\beta}\ } & \mathrm{BW}.
\end{array}
$$

Here $\tilde{\mathrm{N}} = (\mathrm{T} \times \mathrm{T})\mathrm{diag(N)}$ and the torus T in WT denotes the quotient torus $(\mathrm{T} \times \mathrm{T})/\mathrm{diag(T)}$. One may identify this torus with the sub-torus $(1 \times \mathrm{T}) \subseteq \mathrm{T} \times \mathrm{T} \subseteq (\mathrm{T} \times \mathrm{T})\mathrm{diagN(T)}$. Wdiag(T) denotes the quotient $((\mathrm{T} \times \mathrm{T})\mathrm{diag(N)}/(1 \times \mathrm{T}))$. The maps α and $\bar{\alpha}$ are the obvious maps. The map β corresponds to taking the quotient of $\tilde{\mathrm{N}}$ by $(1 \times \mathrm{T})$ and the map $\bar{\beta}$ corresponds to taking the quotient of WT by the corresponding torus which also identifies with $(1 \times \mathrm{T})$.

Next, one may observe that the composite map $\bar{\beta} \circ \bar{\alpha} \circ p_2 : \mathrm{EWT} \underset{\mathrm{WT}}{\times} (\mathrm{ET} \times \bar{\mathrm{T}}) \to \mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}} \to \mathrm{BW}$ factors also as the composite map $\phi \circ \pi \circ p_1$, where $\phi : \mathrm{EW} \underset{\mathrm{W}}{\times} (\bar{\mathrm{T}}/\mathrm{T}) \to \mathrm{BW}$ is the map induced sending all of $\bar{\mathrm{T}}/\mathrm{T}$ to a point, p_i, $i = 1, 2$ are the maps defined in (2.5.3) and π is the map in (2.5.5). Therefore,

(3.0.2)
$$
\begin{aligned}
\mathcal{A} &= Rp_{1*}p_2^*(\bar{\alpha}^*\bar{\beta}^*(R\tilde{\psi}_*(\mathbb{Q}))) \\
&= Rp_{1*}p_1^*\pi^*\phi^*(R\tilde{\psi}_*(\mathbb{Q})) \\
&= \pi^*\phi^*(R\tilde{\psi}_*(\mathbb{Q})) \otimes_{\mathbb{Q}} Rp_{1*}(\mathbb{Q}) \\
&= \pi^*\phi^*(R\tilde{\psi}_*(\mathbb{Q})).
\end{aligned}
$$

The last equality follows from the observation that $Rp_{1*}(\mathbb{Q}) \simeq \mathbb{Q}$.

Proposition 3.4. [5] *(i) The sheaf of commutative dgas, $R\tilde{\psi}_*(\mathbb{Q}) = \tilde{\psi}_*G^\bullet(\mathbb{Q})$, (where $G^\bullet$ is the canonical Godement resolution), is W-equivariant and formal as a sheaf of commutative dgas and whose dga-structure is compatible with the W-action.*

(ii) The corresponding statements also hold for $R\psi_*(\mathbb{Q})$ on $\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}}$, $\mathcal{A}$ on $\mathrm{EW} \underset{\mathrm{W}}{\times} (\mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}})$ and $R\pi_*(\mathcal{A})$ on $\mathrm{EW} \underset{\mathrm{W}}{\times} (\bar{\mathrm{T}}/\mathrm{T})$.

Proof. Step 4.1.1. The two squares in the diagram (3.0.1) are clearly cartesian. Observe that the map

$$
\psi_n : (\mathrm{E}\tilde{\mathrm{N}} \underset{\tilde{\mathrm{N}}}{\times} \bar{\mathrm{T}})_n = \tilde{\mathrm{N}}^{n-1} \times \bar{\mathrm{T}} \to (\mathrm{WT})^{n-1} \times \bar{\mathrm{T}} = (\mathrm{EWT} \underset{\mathrm{WT}}{\times} \bar{\mathrm{T}})_n
$$

is a principal bundle with fiber $(\mathrm{diag(T)})^{n-1}$. The same holds for each of the maps $\bar{\psi}_n$ and $\tilde{\psi}_n$. Therefore, one has base-change, and one obtains the quasi-isomorphisms:

(3.0.3)
$$
\bar{\alpha}^*\bar{\beta}^*R\tilde{\psi}_*(\mathbb{Q}) \xrightarrow{\simeq} \bar{\alpha}^*R\bar{\psi}_*(\beta^*\mathbb{Q}) = R\psi_*(\alpha^*\beta^*(\mathbb{Q})) = R\psi_*(\mathbb{Q}).
$$

Moreover, the above isomorphisms are compatible with the structure of sheaves of dg-algebras. Therefore, the statements in (ii) for $R\psi_*(\mathbb{Q})$ follow from those in (i). Proof of (i) will occupy the remaining part of Step 4.1.1 and Step 4.1.2. Steps 4.1.3 and 4.1.4 below provide a detailed proof of the statements in (ii) for $\mathcal{A}$ and $R\pi_*(\mathcal{A})$.

---

[5]As pointed out by one of the referees, it is not essential in this Proposition that W be the Weyl group $\mathrm{N(T)/T}$. The same arguments should work if W is replaced by a finite group (which we will continue to denote by W for want of a better notation), so that Wdiag(T) is the semi-direct product of W and diag(T), with diag(T) normal in Wdiag(T).

Next observe from Lemma 3.1 that each $(\mathrm{BWdiag}(T))_n$ has a W-action (induced from the action of the bigger group $\mathrm{Wdiag}(T)$) and that the structure maps of the simplicial variety $\mathrm{BWdiag}(T)$ are compatible with these W-actions. Therefore, W acts on the simplicial variety $\mathrm{BWdiag}(T)$. In fact, the above action identifies with the action of W induced by the conjugation action of W on $\mathrm{Wdiag}(T)$. [6] Moreover, the simplicial map $\tilde{\psi} : \mathrm{BWdiag}(T) \to \mathrm{BW}$ is a W-equivariant map in each simplicial degree. Therefore, $R\tilde{\psi}_*(\mathbb{Q}) = \tilde{\psi}_* \mathrm{G}^\bullet(\psi^*(\mathbb{Q}))$ is a complex of W-equivariant sheaves on BW.

One may also observe that that the pairing $\tilde{\psi}^*(\mathbb{Q}) \boxtimes \tilde{\psi}^*(\mathbb{Q}) \to \Delta_*(\tilde{\psi}^*(\mathbb{Q}))$ is compatible with the W-actions, when W-acts diagonally on $\mathrm{BWdiag}(T) \times \mathrm{BWdiag}(T)$.

Next the commutative diagram

$$
\begin{array}{ccc}
\mathrm{BWdiag}(T) & \xrightarrow{\ \Delta\ } & \mathrm{BWdiag}(T) \times \mathrm{BWdiag}(T) \\
\downarrow & & \downarrow \\
\mathrm{BW} & \xrightarrow{\ \Delta\ } & \mathrm{BW} \times \mathrm{BW},
\end{array}
$$

and the pairing $\tilde{\psi}^*(\mathbb{Q}) \boxtimes \tilde{\psi}^*(\mathbb{Q}) \to \Delta_*(\tilde{\psi}_*(\mathbb{Q}))$, (along with the fact that the Godement resolution $\mathrm{G}^\bullet$ is functorial in its argument and preserves multiplicative pairings), shows that one obtains the pairings

(3.0.4) $$R\tilde{\psi}_*(\mathbb{Q}) \boxtimes R\tilde{\psi}_*(\mathbb{Q}) \to \Delta_*(R\tilde{\psi}_*(\mathbb{Q})),$$

which are also compatible with the W-actions when W acts diagonally on the left. Therefore, it follows that the dga-structure on $R\tilde{\psi}_*(\mathbb{Q})$ is compatible with the W-action. Since the maps $\bar{\alpha}$, $\bar{\beta}$, ϕ and π are all W-equivariant maps, it also follows that the dga-structures on $R\psi_*(\mathbb{Q})$, $\mathcal{A}$ and on $R\pi_*(\mathcal{A})$ are also compatible with the W-action.

Step 4.1.2. Next we show the sheaf of commutative dg-algebras $R\psi_*(\mathbb{Q})$ is formal. In view of the fact that the quasi-isomorphisms in (3.0.3) are compatible with the dga-structures, it suffices to show that $R\tilde{\psi}_*(\mathbb{Q})$ is formal as a sheaf of dgas.

Let $\mathbb{H}(\mathrm{Bdiag}(T)) = R\tilde{\psi}_*(\mathbb{Q})_{x_o}$ where the right-hand-side denotes the stalk at the base point x_o in BW. (Observe that this identification of the stalks is possible because we are using the simplicial topology as in 2.1.2 and because $(\mathrm{BW})_0 = \{x_o\}$.) This is a commutative dga provided with an action by W and whose cohomology is $\mathrm{H}^*(\mathrm{Bdiag}(T))$. *A key point to observe in view of the equivalence of categories in Lemma 3.2(iii) is that, in order to show the sheaf of dgas $R\tilde{\psi}_*(\mathbb{Q})$ on BW is formal as a sheaf of dgas, and compatible with the given W-action, it suffices to show that dga at the stalk $\mathbb{H}(\mathrm{Bdiag}(T)) = R\tilde{\psi}_*(\mathbb{Q})_{x_o}$ is formal, and compatible with the given W-action.*

We adopt a rather standard argument to do this: see [Lu95, (3.5) Lemma]. Key use is made of the fact that the cohomology of this dga is a polynomial ring generated by elements in degree 2. Let Z^2 denote the cycles in degree 2 of the dga $\mathbb{H}(\mathrm{Bdiag}(T))$. Let $K = d^{-1}(Z^2) \subseteq \mathbb{H}(\mathrm{Bdiag}(T))^1$ and $N = ker(d : K \to Z^2)$. Since $\mathbb{H}(\mathrm{Bdiag}(T))$ has no cohomology in degree 1, one can find a W-stable subring S of $\mathbb{H}(\mathrm{Bdiag}(T))^0$ so that $d(S) = N$. (In fact, let $S = d^{-1}(N)$.) Since the dga $\mathbb{H}(\mathrm{Bdiag}(T))$ has an action by W, the differentials of the dga $\mathbb{H}(\mathrm{Bdiag}(T))$ are compatible with the W-action and therefore, K, N and S all are stable with respect to the given action of W.

Now one lets $\mathcal{B}$ denote the free graded commutative (or super-commutative) algebra on $S \oplus K \oplus Z^2$, with the differential $d : S \to N \subseteq K$ and $d : K \to Z^2$ defined to be induced by the differential of $\mathbb{H}(\mathrm{Bdiag}(T))$. Then the map sending S (K, Z^2) to S (K, Z^2) defines a map of dg-algebras $\mathcal{B} \to \mathbb{H}(\mathrm{Bdiag}(T))$ which is a quasi-isomorphism. The quotient of $\mathcal{B}$ by the ideal generated by S, K and $d(K)$ will then map isomorphically to the cohomology algebra $\mathrm{H}^*(\mathrm{Bdiag}(T))$. Since all the objects above are stable with respect to the action of W, we observe that that the quasi-isomorphisms $\mathbb{H}(\mathrm{Bdiag}(T)) \leftarrow \mathcal{B} \to \mathrm{H}^*(\mathrm{Bdiag}(T))$ are compatible with the W-actions. This completes the proof that $R\tilde{\psi}_*(\mathbb{Q})$ is formal as a

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[6]In particular, this also shows that one gets an induced action of W on $\mathrm{Bdiag}(T)$, (viewed as a sub-simplicial scheme of $\mathrm{BWdiag}(T)$), which is in fact induced by the conjugation action of W on $\mathrm{diag}(T)$.

dga on BW, and that therefore $R\psi_*(\mathbb{Q})$ is formal as a sheaf of dgas on $\text{EWT} \underset{\text{WT}}{\times} \bar{\text{T}}$, thereby completing the proof of (i).

Step 4.1.3. Recall that $\mathcal{A}$ is defined as $Rp_{1*}p_2^*(R\psi_*(\mathbb{Q}))$ (using the terminology as in the proof of Lemma 2.15). Next we consider the sheaf of commutative dg-algebras $p_2^*(R\psi_*(\mathbb{Q}))$ on $\text{EWT} \underset{\text{WT}}{\times} (\text{ET} \times \bar{\text{T}})$ which is formal. Recall this means, there exists a sheaf of commutative dg-algebras $K \in \text{D}(\text{EWT} \underset{\text{WT}}{\times} (\text{ET} \times \bar{\text{T}}), \mathbb{Q})$ so that there exists a diagram $p_2^*(R\psi_*(\mathbb{Q}) \leftarrow K \to \mathcal{H}^*(p_2^*(R\psi_*(\mathbb{Q})))$ of sheaves of commutative dgas, where the maps are all quasi-isomorphisms of sheaves of dgas and $\mathcal{H}^*(p_2^*(R\psi_*(\mathbb{Q})))$ is the cohomology algebra of $p_2^*(R\psi_*(\mathbb{Q}))$. Observe that $Rp_{1*}(K)$ is a sheaf of commutative dg-algebras on $\text{EW} \underset{\text{T}}{\times} (\text{ET} \times \bar{\text{T}})$. Next we make use of the observation that the fibers of p_1 are all ET and hence acyclic, so that $R^n p_{1*}(\mathbb{Q}) = 0$ for $n \neq 0$ and $= \mathbb{Q}$ for $n = 0$. Since $R^n p_{1*}(\mathbb{Q}) = 0$ for all $n > 0$, it follows that the natural map $p_{1*}(\mathcal{H}^*(p_2^*(R\psi_*(\mathbb{Q})))) \to Rp_{1*}(\mathcal{H}^*(p_2^*(R\psi_*(\mathbb{Q}))))$ is a quasi-isomorphism. It follows that we obtain the diagram of sheaves of commutative dgas on $\text{EW} \underset{\text{W}}{\times} (\text{ET} \underset{\text{T}}{\times} \bar{\text{T}})$:

$$Rp_{1*}(p_2^*(R\psi_*(\mathbb{Q}))) \overset{\sim}{\leftarrow} Rp_{1*}(K) \overset{\sim}{\rightarrow} Rp_{1*}(\mathcal{H}^*(p_2^*(R\psi_*(\mathbb{Q})))) \overset{\sim}{\leftarrow} p_{1*}(\mathcal{H}^*(p_2^*(R\psi_*(\mathbb{Q}))))$$

Clearly the last sheaf of dgas is formal. This completes the proof of the statement that the sheaf of commutative dgas $\mathcal{A} = Rp_{1*}(p_2^*(R\psi_*(\mathbb{Q})))$ on $\text{EW} \underset{\text{W}}{\times} (\text{ET} \underset{\text{T}}{\times} \bar{\text{T}})$ is formal. Since $\mathcal{A}$ is formal it is quasi-isomorphic as a sheaf of commutative dgas to its cohomology algebra, which is $\mathcal{H}^*(\mathcal{A})$, which is the constant sheaf associated to $H^*(\text{Bdiag}(\text{T}), \mathbb{Q})$.

Step 4.1.4. Since $\bar{\text{T}}$ is a projective toric variety for the torus T, the arguments in [Lu95, Theorem 3.1] or [Gu05, section 5] apply verbatim to prove that $R\pi_*(\mathcal{H}^*(\mathcal{A}))$ is formal as a sheaf of commutative dgas. In fact $R\pi_*(\mathcal{H}^*(\mathcal{A})) = R\pi_*(\mathbb{Q}) \otimes_{\mathbb{Q}} \mathcal{H}^*(\bar{\mathcal{A}})$, where $\bar{\mathcal{A}}$ is defined in the line following (3.0.5). Therefore, it suffices to observe that the arguments in [Lu95, Theorem 3.1] or [Gu05, section 5] are compatible with the action of W. Recall these arguments are essentially a sheafified variant of the arguments discussed above in Step 2, sheafified on the space $\bar{\text{T}}/\text{T}$. As discussed in (2.5.6), each T-orbit Y on $\bar{\text{T}}$ has an open T-stable neighborhood U_Y of which Y is a deformation retract and the W-action sends the T-orbit Y to another T-orbit $w\text{Y}$ and the neighborhood U_Y to a neighborhood U_wY of $w\text{Y}$. The sheafified variant of the argument in Step 2 is carried out by applying the constructions in Step 2 to the sheaf of dg-algebras $R\pi_*(\mathbb{Q}_{|\text{U}_\text{Y}})$, which produces a free graded commutative dga $\mathcal{B}_{\text{U}_\text{Y}}$ on the image of U_Y in $\bar{\text{T}}$ and then by showing that for a smaller open neighborhood $\text{U}_{\text{Y}'} \subseteq \text{U}_\text{Y}$ associated to another T-orbit Y', the dg-algebra $\mathcal{B}_{\text{U}_\text{Y}}$ restricts to $\mathcal{B}_{\text{U}_{\text{Y}'}}$. Since the W-action preserves the type of the T-orbits, it preserves the closure relations among these T-orbits, sending the neighborhood $\text{U}_{\text{Y}'}$ to $\text{U}_{w\text{Y}'}$ and U_Y to $\text{U}_{w\text{Y}}$. Therefore it is clear that the above restrictions are compatible with the W-action. This completes the proof of the proposition. □

Step 4.2. As observed in Proposition 3.4, $R\psi_*(\mathbb{Q})$ is obtained as the pull-back $\bar{\alpha}^*(\bar{\beta}^*(R\tilde{\psi}_*(\mathbb{Q}))$. Since $R\tilde{\psi}_*(\mathbb{Q}))$ is formal, we may in fact replace this by its cohomology. Therefore, by applying the projection formula to (3.0.2), one may also observe that

$$(3.0.5) \qquad\qquad R\pi_*(\mathcal{A}) = \phi^*(R\tilde{\psi}_*(\mathbb{Q}))) \otimes_{\mathbb{Q}} R\pi_*(\mathbb{Q}).$$

We will denote

$$(3.0.6) \qquad\qquad \phi^*(R\tilde{\psi}_*(\mathbb{Q}))) \text{ by } \bar{\mathcal{A}}.$$

Lemma 3.5. $R\pi_*(\text{IC}^{\text{W},\text{T}}(\mathcal{O}_{\text{W},\text{T}}) \otimes \mathcal{A}) = R\pi_*(\text{IC}^{\text{W},\text{T}})(\mathcal{O}_{\text{W},\text{T}}) \otimes \bar{\mathcal{A}}.$

Proof. Observe that $\mathcal{A}$ being formal, we may replace $\mathcal{A}$ by its cohomology sheaves. The arguments in (3.0.2) show that $\mathcal{A}$ is also constant, that is, we may replace $\mathcal{A}$ by a graded $\mathbb{Q}$-vector space. Then the conclusion follows readily. □

3.0.7.   Recall that we showed in Step 4 (see 1.4), that the dgas $\mathcal{B}_{\mathrm{G}\times\mathrm{G}}(\bar{\mathrm{G}})$ and $\mathcal{B}_{\mathrm{W},\mathrm{T}}(\bar{\mathrm{T}})$ are quasi-isomorphic, where the dga $\mathcal{B}_{\mathrm{W},\mathrm{T}}(\bar{\mathrm{T}})$ is defined in Step 4 (iii) as

$$RHom(\oplus_{\mathcal{O}}R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}})(\mathcal{O}_{\mathrm{W},\mathrm{T}}\otimes\mathcal{A}), \oplus_{\mathcal{O}}R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}})(\mathcal{O}_{\mathrm{W},\mathrm{T}}\otimes\mathcal{A}).$$

In view of Lemma 3.5, this identifies with

$$RHom(\oplus_{\mathcal{O}}R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}})(\mathcal{O}_{\mathrm{W},\mathrm{T}})\otimes\bar{\mathcal{A}}, \oplus_{\mathcal{O}}R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}})(\mathcal{O}_{\mathrm{W},\mathrm{T}})\otimes\bar{\mathcal{A}}).$$

Step 4.3. *Formality of the dg-algebra $\mathcal{B}_{\mathrm{W},\mathrm{T}}(\bar{\mathrm{T}})$. In order to establish this formality, we consider projective resolutions in Lemma 3.6 and Proposition 3.7, discussed below.*

Lemma 3.6. *Let $\mathrm{C}_{\mathrm{cart,c}}^{+,\mathrm{o}}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\mathrm{ET}\underset{\mathrm{T}}{\times}\bar{\mathrm{T}}),\mathcal{A})$ $(\mathrm{C}_{\mathrm{cart,c}}^{+,\mathrm{o}}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\bar{\mathrm{T}}/\mathrm{T}),R\pi_*(\mathcal{A}))$ denote the category of complexes whose associated derived category is $\mathrm{D}_{\mathrm{cart,c}}^{+,\mathrm{o}}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\mathrm{ET}\underset{\mathrm{T}}{\times}\bar{\mathrm{T}}),\mathcal{A})$ $(\mathrm{D}_{\mathrm{cart,c}}^{+,\mathrm{o}}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\bar{\mathrm{T}}/\mathrm{T}),R\pi_*(\mathcal{A}))$, respectively).*

(i) Let Y' denote a T-orbit on $\bar{\mathrm{T}}$. Let $\mathrm{W}_{\mathrm{Y}'}$ denote the stabilizer of Y' in W. Then $P = \oplus_{w\in\mathrm{W}/\mathrm{W}_{\mathrm{Y}'}}j_{w\mathrm{Y}'!}j_{w\mathrm{Y}'}^*(\mathcal{A})$ is a projective object in $\mathrm{C}_{\mathrm{cart,c}}^{+,\mathrm{o}}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\mathrm{ET}\underset{\mathrm{T}}{\times}\bar{\mathrm{T}}),\mathcal{A})$ in the sense that $Hom_{\mathcal{A}}(P,\ )$ preserves quasi-isomorphisms in the second argument, where $Hom_{\mathcal{A}}$ denotes the Hom in the category $\mathrm{C}_{\mathrm{cart,c}}^{+,\mathrm{o}}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\mathrm{ET}\underset{\mathrm{T}}{\times}\bar{\mathrm{T}}),\mathcal{A})$ of sheaves of dg-modules over $\mathcal{A}$. Similarly,

$$Q = \oplus_{w\in\mathrm{W}/\mathrm{W}_{\mathrm{Y}'}}j_{w\mathrm{Y}'/\mathrm{T}!}j_{w\mathrm{Y}'/\mathrm{T}}^*(R\pi_*(\mathcal{A})) \text{ is a projective object in } \mathrm{C}_{\mathrm{cart,c}}^{+,\mathrm{o}}(\mathrm{EW}\underset{\mathrm{W}}{\times}(\bar{\mathrm{T}}/\mathrm{T}),R\pi_*(\mathcal{A})).$$

(ii) Every sheaf of dg-modules over $\mathcal{H}^*(\mathcal{A})$ has a bounded above resolution by projectives as in (i). The same holds for sheaves of dg-modules over $\mathcal{H}^*(R\pi_*(\mathcal{A}))$.

Proof. In view of the equivalence of categories provided by Proposition 2.22 and Remark 2.20, the first assertion in (i) follows from the second assertion in (i). For the second assertion in (i), we simply observe that

$$Hom_{R\pi_*(\mathcal{A})}(\oplus_{w\in\mathrm{W}/\mathrm{W}_{\mathrm{Y}'}}j_{w\mathrm{Y}'/\mathrm{T}!}j_{w\mathrm{Y}'/\mathrm{T}}^*(R\pi_*(\mathcal{A})),K) = \oplus_{w\in\mathrm{W}/\mathrm{W}_{\mathrm{Y}'/\mathrm{T}}}Hom_{R\pi_*(\mathcal{A})}(j_{w\mathrm{Y}'/\mathrm{T}}^*(R\pi_*(\mathcal{A})),j_{w\mathrm{Y}'/\mathrm{T}}^*(K)).$$

But $j_{w\mathrm{Y}'/\mathrm{T}}^*(R\pi_*(\mathcal{A}))$ $(j_{w\mathrm{Y}'/\mathrm{T}}^*(K))$ is the stalk of $R\pi_*(\mathcal{A})$ $(K$, respectively) at the orbit $w\mathrm{Y}'$. This proves the second assertion in (i). We skip the proof of the assertions in (ii).   □

Let $\mathcal{O}_{\mathrm{WT}}$ denote a WT-orbit on $\bar{\mathrm{T}}$. Then $\mathcal{O}_{\mathrm{WT}}$ is a disjoint union of T-orbits on $\bar{\mathrm{T}}$ permuted under the action of W: therefore, we denote $\mathcal{O}_{\mathrm{WT}} = \mathrm{W}.\mathcal{O}_{\mathrm{T}}$. Observe also that the dg-algebra $R\pi_*(\mathcal{A})$ being formal can be viewed as a dg-algebra over $\mathrm{H}^*(B\mathrm{T})$. Recall that $\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})$ denotes the equivariant intersection cohomology complex on the WT-orbit $\mathcal{O}_{\mathrm{WT}}$ for the action of the group WT (and extending the constant sheaf $\mathbb{Q}$ on $\mathcal{O}_{\mathrm{WT}}$), but viewed as a complex on $\mathrm{EW}\underset{\mathrm{W}}{\times}(\mathrm{ET}\underset{\mathrm{T}}{\times}\bar{\mathrm{T}})$

Proposition 3.7. *(i) Let $x\in\bar{\mathrm{T}}^{\mathrm{T}}$ denote a fixed point. Let W_x denote the stabilizer of x in W and $\mathrm{W}x = \mathrm{W}/\mathrm{W}_x$ the corresponding W-orbit. Then denoting the sum $\oplus_{w\in\mathrm{W}/\mathrm{W}_x}R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})\otimes\mathcal{A})_{wx}$ by $R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})\otimes\mathcal{A})_{\mathrm{W}x}$, we obtain:*

$$(3.0.8)\qquad R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})\otimes\mathcal{A})_{\mathrm{W}x} = \oplus_{w\in\mathrm{W}/\mathrm{W}_x}R\pi_*(\mathrm{WIC}^{\mathrm{T}}(\mathcal{O}_{\mathrm{T}}))_{wx}\otimes_{\mathbb{Q}}\bar{\mathcal{A}}$$

where $\mathrm{WIC}^{\mathrm{T}}(\mathcal{O}_{\mathrm{T}})$ denotes a sum of intersection cohomology complexes $\oplus\mathrm{IC}^{\mathrm{T}}(\mathcal{O}_{\mathrm{T}})$ where the sum varies over all the disjoint W-translates of a given T-orbit $\mathcal{O}_{\mathrm{T}}$. Therefore, the cohomology $\mathrm{H}^*(R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}}))\otimes\bar{\mathcal{A}})_{\mathrm{W}x}$ forms a projective module over $\mathrm{H}^*(B\mathrm{T})\otimes\mathrm{H}^*(\bar{\mathcal{A}})$.

(ii) Every object $R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}}))\otimes\bar{\mathcal{A}}$ has a projective resolution $\{\cdots\to P^{-n}\to P^{-n+1}\to\cdots P^0\}$ in $\mathrm{D}_{\mathrm{W}}(\bar{\mathrm{T}}/\mathrm{T},\mathrm{H}^*(R\pi_*(\mathbb{Q})\otimes\bar{\mathcal{A}}))$ so that the given augmentation $P^0\to R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}}))\otimes\bar{\mathcal{A}}$ is a quasi-isomorphism at each stalk of the form $\mathrm{W}x$, $x\in\bar{\mathrm{T}}^{\mathrm{T}}$, it induces a surjection at each stalk $\mathrm{W}x$, $x\in\bar{\mathrm{T}}$ and each P^{-i} is of the form

$$\oplus_{w\in\mathrm{W}/\mathrm{W}_x}j_{\mathrm{U}_{wx}!}j_{\mathrm{U}_{wx}}^*(\mathrm{H}^*(R\pi_*(\mathbb{Q})\otimes\bar{\mathcal{A}}))[n_{\mathrm{U}_{wx}}]$$

as U_{wx} varies over neighborhoods of points $wx\in\bar{\mathrm{T}}/\mathrm{T}$ and where $n_{\mathrm{U}_{wx}}$ are integers.

(iii) The complexes P^i, for $i < 0$, are supported at points in $\bar{\mathrm{T}} - \bar{\mathrm{T}}^{\mathrm{T}}$ and hence viewed as modules over $\mathbb{H}^*(\mathrm{BT}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{H}^*(\bar{\mathcal{A}})$ *are torsion.*

Proof. (i) is a straight-forward calculation making use Proposition 3.5(iii) and the following observation. If $\mathcal{O}_{\mathrm{T}}$ denotes an orbit of T on $\bar{\mathrm{T}}$, and $\mathrm{W}.\mathcal{O}_{\mathrm{T}} = \mathcal{O}_{\mathrm{WT}}$ denotes the orbit for the corresponding WT-action, then as observed in (2.5.4),

$$(3.0.9) \qquad \mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}}) = \oplus_{w \in \mathrm{W}/\mathrm{W}_{\mathcal{O}_{\mathrm{T}}}} \mathrm{IC}^{\mathrm{T}}(\mathcal{O}_{\mathrm{T}}),$$

where $\mathrm{W}_{\mathcal{O}_{\mathrm{T}}}$ denotes the stabilizer of the $\mathcal{O}_T$-orbit in W. Applying $R\pi_*$, and making use of (3.0.2), we therefore obtain:

$$R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}}) \otimes \mathcal{A}) = R\pi_*(\mathrm{WIC}^{\mathrm{T}}(\mathcal{O}_{\mathbf{T}})) \otimes_{\mathbb{Q}} \bar{\mathcal{A}}.$$

Then take the stalks at wx and the sum $\oplus_{w \in \mathrm{W}/\mathrm{W}_x}$ to obtain (3.0.8).

Next use the fact that the global equivariant intersection cohomology of a projective toric variety is a free module over the cohomology ring of the classifying space of the torus, and the stalk cohomology of the intersection cohomology complex at a T-fixed point on $\bar{\mathrm{T}}$ is isomorphic to the global intersection cohomology of the link at that point: see [Lu95, (4.0.4) Theorem and (4.2.2)] and also [BJ04, Theorem 1.1]. This completes the proof of (i).

Then (ii) is an immediate consequence of (i), Lemma 3.6 and the formality of the dgas $R\pi_*(\mathcal{A}) = R\pi_*(\mathbb{Q}) \otimes \bar{\mathcal{A}}$. (ii) shows that the augmentation $P^0 \to R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})) \otimes \bar{\mathcal{A}}$ induces a quasi-isomorphism at every $\mathrm{W}x$, for every $x \in \bar{\mathrm{T}}^{\mathrm{T}}$. Therefore, it follows that each $\mathcal{H}^i(P^\bullet)$, for $i < 0$, are torsion modules over $\mathrm{H}^*(\mathrm{BT}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{H}^*(\bar{\mathcal{A}})$. $\qquad\square$

Recall that $\bar{\mathrm{T}}$ is the closure in the projective variety $\bar{\mathrm{G}}$ of the torus T: therefore, $\bar{\mathrm{T}}$ is projective. Next recall from (3.0.2) that ϕ denotes the map $\phi : \mathrm{EW} \underset{\mathrm{W}}{\times} (\bar{\mathrm{T}}/\mathrm{T}) \to \mathrm{BW}$ sending $\bar{\mathrm{T}}/\mathrm{T}$ to the base point in BW.

Lemma 3.8. *The stalk of*

$$R\phi_*(RHom_{R\pi_*(\mathbb{Q})}(\oplus R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}}), \oplus R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})))))$$

at the base point in BW*, where the sum varies over all the* WT*-orbits in* $\bar{\mathrm{T}}$*, identifies with the dg-algebra* $\mathcal{B}_{\mathrm{T}}(\bar{\mathrm{T}})$*, which is the dg-algebra considered in Theorem 1.2 for the projective toric variety* $\bar{\mathrm{T}}$*, viewed as an imbedding of the torus* T*. In particular, its cohomology is a torsion-free module over* $H^*(\mathrm{BT}, \mathbb{Q})$*.*

Proof. The decomposition of the WT-orbits into the corresponding T-orbits and therefore, the induced decomposition as in (3.0.9) shows that the sum $\oplus R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})$ in fact, runs over all the T-equivariant intersection cohomology complexes on the T-orbits on $\bar{\mathrm{T}}$. Then the stalk considered above (using the simplicial topology as in 2.1.2) identifies with the dg-algebra $RHom_{R\pi_{o*}(\mathbb{Q})}(\oplus R\pi_{o*}(\mathrm{IC}^{\mathrm{T}}(\mathcal{O}_{\mathrm{T}})), \oplus R\pi_{o*}(\mathrm{IC}^{\mathrm{T}}(\mathcal{O}_{\mathrm{T}})))$, where $\pi_o : \mathrm{ET} \underset{\mathrm{T}}{\times} \bar{\mathrm{T}} \to \bar{\mathrm{T}}/\mathrm{T}$ is the quotient map. By [Lu95, the proof of (0.1.1) Theorem] this identifies with the dg-algebra $\mathcal{B}_{\mathrm{T}}(\bar{\mathrm{T}})$. The last statement is then deduced as in [Lu95, (4.0.3) Theorem] from the observation that the global equivariant intersection cohomology complex of a projective toric variety is a free module over the cohomology ring of the classifying space of the torus as observed earlier. One applies this observation to the projective toric varieties forming the closures of the T-orbits on $\bar{\mathrm{T}}$. $\qquad\square$

Remark 3.9. In keeping with the terminology used in Theorem 1.2, the dg -algebra $\mathcal{B}_{\mathrm{T}}(\bar{\mathrm{T}})$ should be denoted $\mathcal{B}_{\mathrm{T} \times \mathrm{T}}(\bar{\mathrm{T}})$. However, since in this case T is commutative, we adopt the simpler notation as above.

Theorem 3.10. *Assume next that the toroidal imbedding $\bar{\mathrm{G}}$ and therefore, the toric variety $\bar{\mathrm{T}}$ is projective. Then (i) the dg algebra $\mathcal{B}_{\mathrm{W},\mathrm{T}}(\bar{\mathrm{T}})$ is formal and (ii) $\mathcal{B}_{\mathrm{W},\mathrm{T}}(\bar{\mathrm{T}}) = (\mathcal{B}_{\mathrm{T}}(\bar{\mathrm{T}}) \otimes R\tilde{\psi}_*(\mathbb{Q}))^{\mathrm{W}}$.*

Proof. Let $P^\bullet \to \oplus R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})) \otimes \bar{\mathcal{A}}$ denote a projective resolution, where each P^{-i} is a sum of terms of the form as in Proposition 3.7(ii) and where the sum varies over all the WT-orbits in $\bar{\mathrm{T}}$. Therefore $P = \oplus_i P^{-i}[i] \to \oplus R\pi_*(\mathrm{IC}^{\mathrm{W},\mathrm{T}}(\mathcal{O}_{\mathrm{WT}})) \otimes \bar{\mathcal{A}}$ is a quasi-isomorphism with P a projective object in $\mathrm{C}_{\mathrm{W}}(\bar{\mathrm{T}}/\mathrm{T}, \mathrm{H}^*(\mathrm{R}\pi_*(\mathbb{Q}) \otimes \bar{\mathcal{A}}))$.

Observe next that the dga $\bar{\mathcal{A}}$ is formal, and since each P^{-i} is of the form given in Proposition 3.7(ii), the differentials of each P^{-i} are in fact trivial. Therefore, the spectral sequence for the total complex for $\mathcal{H}om(P^\bullet, P^\bullet)$ degenerates. Therefore, now $\mathcal{B}_{W,T}(\bar{T})$ identifies with the total complex of $Hom(P^\bullet, P^\bullet)$, where the differentials of $P^\bullet$ (that is, $\{d : P^n \to P^{n+1}|n\}$) provide the structure of a chain-complex on $Hom(P^\bullet, P^\bullet)$.

We will presently provide two somewhat different proofs to show that the dg algebra $\mathcal{B}_{W,T}(\bar{T})$ is formal. The first starts with the observation that the dg-algebra $\mathcal{B}_{W,T}(\bar{T}) = R\Gamma(\mathrm{BW}, \widetilde{\mathcal{B}}_{W,T}(\bar{T})) = (\widetilde{\mathcal{B}}_{W,T}(\bar{T}))^W$ for the sheaf of dgas $\widetilde{\mathcal{B}}_{W,T}(\bar{T}) = \mathcal{H}om(P^\bullet, P^\bullet)$ on BW.

A *key observation* we make is that the stalk of $\mathcal{H}^*(\widetilde{\mathcal{B}}_{W,T}(\bar{T}))$ (at the base point in BW) is a torsion-free module over $\mathrm{H}^*(\mathrm{BT}, \mathbb{Q}) \otimes_{\mathbb{Q}} \mathrm{H}^*(\mathrm{R}\tilde{\psi}_*(\mathbb{Q}))$. When $\bar{\mathrm{G}}$ and hence $\bar{\mathrm{T}}$ are projective, this is clear in view of the identification of the stalks of

$$R\phi_*(RHom_{R\pi_*(\mathbb{Q})}(\oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT})), \oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT}))))$$

at the base point in BW with the dg-algebra $\mathcal{B}_T(\bar{T})$ as shown in Lemma 3.8, along with the observation that $\bar{\mathcal{A}} = \phi^*(R\tilde{\psi}_*(\mathbb{Q}))$. (In fact, see (3.0.10) below.) Therefore, Proposition 3.7 shows that the cohomology of the sheaf of dg-algebras $\widetilde{\mathcal{B}}_{W,T}(\bar{T})$, which identifies with the cohomology of the complex $\mathcal{H}om(P^\bullet, P^\bullet)$, vanishes in all degrees except 0. (Here, we again make use of the observation that the dga $\bar{\mathcal{A}}$ is formal, and since each P^{-i} is of the form given in Proposition 3.7(ii), the differentials of each P^{-i} are in fact trivial.)

Now the diagram of dgas

$$\mathcal{H}^0(\widetilde{\mathcal{B}}_{W,T}(\bar{T})) \leftarrow \sigma_{\leq 0}(\widetilde{\mathcal{B}}_{W,T}(\bar{T})) \to \widetilde{\mathcal{B}}_{W,T}(\bar{T})$$

(where $\sigma_{\leq 0}$ is the functor that kills the above cohomology in negative degrees) shows that the sheaf of dg-algebras $\widetilde{\mathcal{B}}_{W,T}(\bar{T})$ is formal on BW. Finally one simply takes W-invariants, observing that taking W-invariants is an exact functor since we are working with rational coefficients. This proves (i).

We will next provide a proof (ii). Now one may observe that with $\bar{\mathcal{A}} = \phi^*(R\tilde{\psi}_*(\mathbb{Q}))$,

(3.0.10)
$$\mathcal{B}_{W,T}(\bar{T}) = R\Gamma(\mathrm{BW}, \mathrm{R}\phi_*(RHom_{R\pi_*(\mathbb{Q}) \otimes \bar{\mathcal{A}}}(\oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT})) \otimes_{\mathbb{Q}} \bar{\mathcal{A}}, \oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT})) \otimes_{\mathbb{Q}} \bar{\mathcal{A}})))$$
$$= R\Gamma(\mathrm{BW}, \mathrm{R}\phi_*(RHom_{R\pi_*(\mathbb{Q})}(\oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT})), \oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT})) \otimes_{\mathbb{Q}} \bar{\mathcal{A}})))$$
$$= R\Gamma(\mathrm{BW}, \mathrm{R}\phi_*(RHom_{R\pi_*(\mathbb{Q})}(\oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT})), \oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT}))) \otimes_{\mathbb{Q}} R\tilde{\psi}_*(\mathbb{Q}))).$$

The last equality follows from the projection formula as well as the observation that $\bar{\mathcal{A}}$ is formal and constant (see (3.0.5)), while the one above follows from standard properties of $RHom$. Here the sum $\oplus$ varies over all WT-orbits in $\bar{T}$. Now observe that

(3.0.11) $$R\phi_*(RHom_{R\pi_*(\mathbb{Q})}(\oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT})), \oplus R\pi_*(\mathrm{IC}^{W,T}(\mathcal{O}_{WT}))))$$

is a complex of W-equivariant sheaves on BW. Lemma 3.8 shows that its stalk at the base point identifies with the dga $\mathcal{B}_T(\bar{T})$ associated to $\bar{T}$, viewed as a toric variety for T. Therefore, we obtain (ii):

$$\text{that is, the dga } \mathcal{B}_{W,T}(\bar{T}) = (\mathcal{B}_T(\bar{T}) \otimes R\tilde{\psi}_*(\mathbb{Q}))^W.$$

Finally we will provide a second proof of (i). Since $R\tilde{\psi}_*(\mathbb{Q})$ was already observed to be formal as a dga on BW, (that is, as a dga with W-action), it suffices to show that $\mathcal{B}_T(\bar{T})$ is formal as a dga with W-action. The arguments for the formality of $\mathcal{B}_T(\bar{T})$ (see for example, the proof of [Lu95, Proposition 4.1.2]) show that $H^i(\mathcal{B}_T(\bar{T})) = 0$ for all $i \neq 0$, where H^i denotes the cohomology of the corresponding complex $Hom(P^\bullet, P^\bullet)$.

Next let $\sigma_{\leq 0}\mathcal{B}_T(\bar{T})$ denote the truncation functor that kills the cohomology above degree 0. Then it is clear that we obtain maps of dgas

$$\mathrm{H}^0(\mathcal{B}_T(\bar{T})) \leftarrow \sigma_{\leq 0}(\mathcal{B}_T(\bar{T})) \to \mathcal{B}_T(\bar{T})$$

which are both quasi-isomorphisms and compatible with the given W-actions. (The compatibility with the W-action should be clear from (3.0.9), which shows that the T-orbits on $\bar{\mathrm{T}}$ may be grouped into WT-orbits.) Therefore, the induced maps of dgas

$$\mathrm{H}^0(\mathcal{B}_\mathrm{T}(\bar{\mathrm{T}})) \otimes_\mathbb{Q} \mathrm{H}^*(\mathrm{R}\tilde{\psi}_*(\mathbb{Q})) \leftarrow \sigma_{\leq 0}(\mathcal{B}_\mathrm{T}(\bar{\mathrm{T}})) \otimes_\mathbb{Q} \mathrm{H}^*(\mathrm{R}\tilde{\psi}_*(\mathbb{Q})) \rightarrow \mathcal{B}_\mathrm{T}(\bar{\mathrm{T}}) \otimes_\mathbb{Q} \mathrm{H}^*(\mathrm{R}\tilde{\psi}_*(\mathbb{Q}))$$

are also quasi-isomorphisms and compatible with the W-actions. Since $R\tilde{\psi}_*(\mathbb{Q})$ was observed to be formal as a dga on BW, one may now replace $\mathrm{H}^*(\mathrm{R}\tilde{\psi}_*(\mathbb{Q}))$ by $R\tilde{\psi}_*(\mathbb{Q})$ and obtain the same conclusions. Therefore, we see that $\mathcal{B}_{\mathrm{W},\mathrm{T}}(\bar{\mathrm{T}}) = (\mathcal{B}_\mathrm{T}(\bar{\mathrm{T}}) \otimes_\mathbb{Q} R\tilde{\psi}_*(\mathbb{Q}))^\mathrm{W}$ is formal as a dga. $\qquad\square$

In view of the arguments in 3.0.7, this completes the proof of Theorem 1.2.

4. A General Obstruction Theory for Formality of the Dg-algebra $\mathcal{B}$ and Conclusions for scs varieties

We start with the observation that the dg-algebra $\mathcal{B}_\mathrm{G}(\mathrm{X})$ in Theorem 1.1 is only an A_∞-dg-algebra (or only an associative dg-algebra), because the multiplicative structure is given by composition and hence not commutative in general. (Recall that an A_∞-dg algebra means a dg-algebra where the multiplication is coherently homotopy associative.) A well-known result of Kadeishvili (see [Ka80]) shows that, nevertheless for any A_∞ dg-algebra $\mathcal{B}$ (over $\mathbb{Q}$) there exists an A_∞-structure on the cohomology algebra $H^*(\mathcal{B})$, so that $\mathcal{B}$ is quasi-isomorphic as an A_∞-dg-algebra to $H^*(\mathcal{B})$ with the above A^∞-structure. We follow the exposition of [LPWZ09, section 2] for this. Since we are working over $\mathbb{Q}$, we obtain a (non-canonical) decomposition of each $\mathbb{Q}$-vector space $\mathcal{B}^n$ as

(4.0.1) $$\mathcal{B}^n = Z^n \oplus L^n = C^n \oplus H^n \oplus L^n.$$

Here C^n denotes the co-boundaries, Z^n the co-cycles and $H^n = Z^n/C^n$. We will identify $\oplus_n H^n(\mathcal{B})$ as imbedded in $\mathcal{B}$ by co-cycle-sections $H^n \subset \mathcal{B}^n$. *Clearly there are many different choices of H^n and L^n.* Let $p : \mathcal{B} \to \mathcal{B}$ be a projection to $H := \oplus_n H^n$, and let $G : \mathcal{B} \to \mathcal{B}$ be a homotopy from $id_\mathcal{B}$ to p. Hence we have $id_\mathcal{B} - p = \partial G + G\partial$. We will define the map G as follows: for every n, $G^n : \mathcal{B}^n \to \mathcal{B}^{n-1}$ is the map which satisfies

- $G^n = 0$ when restricted to L^n and H^n, and
- $G^n = (\partial^{n-1}|_{L^{n-1}})^{-1}$ when restricted to C^n.

(Observe that $\partial^{n-1}_{|L^{n-1}} : L^{n-1} \to C^n$ is a bijection.) Therefore, the image of G^n is L^{n-1}. It follows that $G^{n+1}\partial^n = Pr_{L^n}$ and $\partial^{n-1}G^n = Pr_{C^n}$, where $Pr_{L^n} : \mathcal{B}^n \to L^n$ and $Pr_{C^n} : \mathcal{B}^n \to C^n$ are the projections.

Next we define a sequence of linear maps $\lambda_n : \mathcal{B}^{\otimes n} \to \mathcal{B}$ of degree $2 - n$ as follows. There is no map λ_1, but we formally define the "composite" $G\lambda_1$ by $G\lambda_1 = -id_\mathcal{B}$. λ_2 is the multiplication of $\mathcal{B}$, namely, $\lambda_2(a_1 \otimes a_2) = a_1 \cdot a_2$. For $n \geq 3$, λ_n is defined by the recursive formula

(4.0.2) $$\lambda_n = \sum_{\substack{s+t=n, \\ s,t \geq 1}} (-1)^{s+1} \lambda_2 \circ (G\lambda_s \otimes G\lambda_t).$$

Using p to denote both the map $\mathcal{B} \to \mathcal{B}$ and also (since the image of p is $H^*(\mathcal{B})$) the map $\mathcal{B} \to H^*(\mathcal{B})$; we also use λ_i both for the map $\mathcal{B}^{\otimes i} \to \mathcal{B}$ and for its restriction $(H^*(\mathcal{B}))^{\otimes i} \to H^*(\mathcal{B}) \to \mathcal{B}$. Then the above mentioned result of Kadeishvili can be stated as follows:

Theorem 4.1. *(See* [LPWZ09, Theorem 2.2, Proposition 2.3].*) (i) Let $m_i = p\lambda_i$. Then $(H^*(\mathcal{B}), m_2, m_3, \cdots)$ is an A_∞-algebra.*

(ii) Let $\{\lambda_n\}$ be defined as above. For $i \geq 1$ let $f_i = -G\lambda_i : (H^*(\mathcal{B}))^{\otimes i} \to \mathcal{B}$. Then $f := \{f_i\}$ is a map of A_∞ dg-algebras so that f_1 is a quasi-isomorphism.

We define the A_∞-dg-algebra $\mathcal{B}$ to be *formal*, if all the $m_i = 0$ for $i > 2$. Now we conclude this discussion with the

Proof of Theorem 1.5.

(i) follows readily from the fact that the $m_i = p\lambda_i$ and that the map $\lambda_i : \mathcal{B}^{\otimes i} \to \mathcal{B}$ has degree $2 - i$ which is odd if i is odd. Now the map m_i is the composition $H^*(\mathcal{B})^{\otimes i} \to \mathcal{B}^{\otimes i} \overset{\lambda_i}{\to} \mathcal{B} \overset{p}{\to} \mathbb{H}^*(\mathcal{B})$. Therefore, when $H^i(\mathcal{B}) = 0$ for all odd i, one may readily see that the map m_i is trivial for all odd i.

(ii) Let X denote a projective G-spherical variety as in (ii). Then it was shown in [BJ04, Lemma 3.6] and [BJ01, Theorem 4] that the only G-equivariant local systems on the G-orbits $\mathcal{O}$ on X are constant, that the odd dimensional intersection cohomology sheaves $\mathcal{H}^i(\mathrm{IC}(\mathbb{Q}_{|\mathcal{O}}))$, as well as the odd dimensional intersection cohomology groups $\mathrm{IH}^i(\bar{\mathcal{O}})$ vanish. (Here $\mathbb{Q}_{|\mathcal{O}}$ denotes the constant sheaf $\mathbb{Q}$ on the G-orbit $\mathcal{O}$ in X.) A spectral sequence argument now shows that $H^i(\mathcal{B}_G(X)) = 0$ for all odd i, for the corresponding dg-algebra $\mathcal{B}_G(X)$. Therefore, the result in (i) applies. $\qquad\square$

Remark 4.2. One may interpret the conclusion of the last theorem as saying that *half the obstructions for the formality of the dg-algebra $\mathcal{B}_G$ vanish* for the spherical varieties considered there. Clearly, this includes a large class of spherical varieties.

5. Comparison of Equivariant Derived Categories

The equivariant derived categories associated to the action of a group G on a space are usually defined as certain full subcategories of the derived category on the Borel construction associated to the group action. Different models for the Borel construction, therefore provide different models for the equivariant derived categories. The geometric model which had been introduced in [BL94] in the topological framework and in the scheme-theoretic framework in [To99] and [MV99] complements the simplicial model which was discussed in [De74], [Fr83] and [Jo93], each with its own advantages and dis-advantages. For example, the discussion of the Weyl-group action in section 3 (see Lemma 2.15) as well as the need to handle modules over sheaves of dgas that are only bounded below, seems to require the simplicial model. In fact the situation considered in Proposition 2.9 and Lemma 2.15 is a special case of a more general situation where one needs to consider actions of two groups G and H provided with a surjective map from G to H acting on spaces compatibly and then relate the corresponding equivariant derived categories. In this case, the simplicial model seems to be able to handle the situation easily, while the approach making use of geometric classifying spaces simply does not work, since one has to find representations of both G and H and relate them suitably. We encounter several instances of such situations in section 2, which is the reason we have chosen to work with the simplicial model.

However, since the geometric model is perhaps more suited for handling properties like the weight filtration (used in [Jo17]), and also more commonly used in the literature dealing with equivariant derived categories, we felt it important to provide a comparison between the two models, which is what is done in the rest of this section. In view of the various applications, we have decided to make the discussion in this section general enough so that it applies to actions of linear algebraic groups defined over fields that are perfect and of finite ℓ cohomological dimension for some $\ell \neq char(k)$.

5.1. **Equivariant Derived Categories: Version I.** Presently we proceed to define a model for the equivariant derived category that is valid in all characteristics, making use of an algebraic model for the classifying space BG for a linear algebraic group as in [MV99] and [To99]. (The construction in [BL94] is similar in spirit, and may have served as a motivation for the constructions in [MV99] and [To99], but applies mostly to actions of topological groups on topological spaces.)

Definition 5.1. We will often use $\mathrm{EG}^{\mathrm{gm},m}$ to denote the m-th term of an admissible gadget $\{\mathrm{U_m}|m\}$: the superscript *gm* stands for *geometric*. This is discussed in more detail in Definition 6.2. (Recall this means $\mathrm{U}_m = \mathrm{EG}^{\mathrm{gm},m}$ is an open G-stable subvariety of a representation W_m of G, so that (i) G acts freely on U_m and a geometric quotient U_m/G exists as a variety and (ii) so that in the family $\{(\mathrm{W}_m, \mathrm{U}_m)|m \in \mathbb{N}\}$, the codimension of $\mathrm{W}_m - \mathrm{U}_m$ in W_m goes to ∞ as m approaches ∞.)

Making use of $\mathrm{EG}^{\mathrm{gm},m}$ we may now define a characteristic free algebraic model for the equivariant derived category.

5.1.1. **Convention.** Henceforth, we will adopt the following conventions. If X is a variety defined over k, we will consider ℓ-adic sheaves on X_{et}, which denotes the étale site of X. (One may also consider sheaves of $\mathbb{Z}/\ell^\nu$-modules, or sheaves of R-modules for a commutative Noetherian ring that is torsion with torsion prime to $char(k)$. This way, it becomes possible to consider ℓ-adic sheaves where $\mathbb{Q}_\ell$ is replaced by $\bar{\mathbb{Q}}_\ell$, which is the algebraic closure of $\mathbb{Q}_\ell$: see [BBD82, 2.2.18].) If X is a variety defined over the complex numbers, we may consider sheaves of $\mathbb{Q}$-vector spaces on the transcendental site of $X(\mathbb{C})$ or ℓ-adic sheaves on X_{et}. We will denote by $D(X)$ ($D^+(X)$, $D^b(X)$) the unbounded derived category (the bounded below derived category, the bounded derived category, respectively) of complexes of sheaves of $\mathbb{Q}$-vector spaces on $X(\mathbb{C})$ or ℓ-adic sheaves on X_{et} depending on the context.

Observe that, if k is algebraically closed,

$$H^i_{et}(EG^{gm,m}, \mathbb{Q}_\ell) = 0 \text{ for all } 0 < i \leq 2m-2 \text{ and } H^0_{et}(EG^{gm,m}, \mathbb{Q}_\ell) = \mathbb{Q}_\ell.$$

This follows from the fact that $EG^{gm,m} = U_m$ which is an open G-stable subvariety of codimension at least $m > 1$. (It may be deduced from the hypothesis in the definition of the admissible gadgets in Definition (6.2) that $codim_{W_m}(W_m \setminus U_m) = m(codim_W(Z))$.) The corresponding results also hold with $\mathbb{Z}$ or $\mathbb{Q}$-coefficients over an algebraically closed field of characteristic 0. (Here we apply Lemma 6.1 with $c = m$.) Therefore, for each fixed finite interval $I = [a, b]$ of integers, $a \leq b$, and each integer $m \geq 0$, we now define

$$D^I(EG^{gm,m}\underset{G}{\times}X) = \{K \in D(EG^{gm,m}\underset{G}{\times}X))|\mathcal{H}^i(K) = 0, i \notin I\}.$$

Now we let, for each I, with $2m - 2 \geq |I| = b - a$,

(5.1.2)
$$D^{I,gm}_{G,m}(X) = \text{ the full subcategory of } D^I(EG^{gm,m}\underset{G}{\times}X) \text{ consisting of those K}$$
$$\text{such that there exists an } L \in D(X) \text{ so that } \pi_m^*(K)\overset{\cong}{\leftarrow}p_{2,m}^*(L).$$

Here $\pi_m : EG^{gm,m} \times X \to EG^{gm,m}\underset{G}{\times}X$ is the quotient map and $p_{2,m} : EG^{gm,m} \times X \to X$ is the projection. In case we need to clarify the choice of the geometric classifying spaces, we will denote $D^{I,gm}_{G,m}(X)$ by $D^I_G(EG^{gm,m}\underset{G}{\times}X)$. One observes that if $I \subseteq J$, then one obtains a fully faithful imbedding $D^{I,gm}_{G,m}(X) \to D^{J,gm}_{G,m}(X)$, so that varying I, one obtains a filtration of $D^{b,gm}_{G,m}(X)$, which is defined as above except the vanishing of the cohomology sheaves $\mathcal{H}^i(K)$ is for all i outside of some finite interval I depending on K.

Next let $i_m^G = i_m\underset{G}{\times}id_X : EG^{gm,m}\underset{G}{\times}X \to EG^{gm,m+1}\underset{G}{\times}X$ denote the map induced by the regular immersion $i_m : EG^{gm,m} \to EG^{gm,m+1}$, which is the map defined in Definition 6.2(5). Pullback along this map defines the inverse system $\{D^{I,gm}_{G,m}(X)|m\}$. We proceed to take the limit of this inverse system of categories: we discuss this in a more general setting as follows. Given an inverse system of categories $\{\mathbf{C}_i|i \in \mathbb{N}\}$ and functors $\mathbf{F}_{i-1,i} : \mathbf{C}_i \to \mathbf{C}_{i-1}$ for each $i \in \mathbb{N}$, we define the inverse limit category $\underset{i}{\lim}\mathbf{C}_i$ to be the following category:

(1) The objects are pairs $(\{C_i|i\}, \{\phi_{i-1,i}|i\})$ where $C_i \in \mathbf{C}_i$ for each $i \in \mathbb{N}$ and $\phi_{i-1,i} : \mathbf{F}_{i-1,i}(C_i)\overset{\cong}{\to}C_{i-1}$ for any $i \geq 2$.
(2) A morphism f between two objects $(\{C_i|i\}, \{\phi_{i-1,i}|i\})$, $(\{D_i|i\}, \{\psi_{i-1,i}|i\})$ is a set of arrows $\{f_i : C_i \to D_i|i\}$ so that the diagram

$$
\begin{array}{ccc}
\mathbf{F}_{i-1,i}(C_i) & \xrightarrow{\phi_{i-1,i}} & C_{i-1} \\
\downarrow{\scriptstyle \mathbb{F}_{i-1,i}(f_i)} & & \downarrow{\scriptstyle f_{i-1}} \\
\mathbf{F}_{i-1,i}(D_i) & \xrightarrow[\psi_{i-1,i}]{} & D_{i-1}
\end{array}
$$

commutes. Composition of morphisms is component-wise.

Similarly if $\{\mathbf{F}_{\alpha,\beta} : \mathbf{C}_\alpha \to \mathbf{C}_\beta | \alpha, \beta \in \mathrm{J}\}$ is a sequence of functors indexed by the filtered category J, the category $\mathrm{colim}_J \mathbf{C}_\alpha$ may be described as follows: objects have the form (C, α) for some $\alpha \in \mathrm{J}$ and $C \in Ob(\mathbf{C}_\alpha)$. The set of morphisms from (C, α) to (D, β) is given by $\mathrm{colim}_{\gamma \geq \alpha, \beta} Hom_{\mathbf{C}_\gamma}(\mathbf{F}_{\alpha,\gamma}(C), \mathbf{F}_{\beta,\gamma}(D))$.

Let $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$, $a < b$ denote a finite interval of integers. Now one defines

$$(5.1.3) \qquad \mathrm{D}_\mathrm{G}^{\mathrm{I,gm}}(\mathrm{X}) = \lim_{\infty \leftarrow \mathrm{m}} \{\mathrm{D}_{\mathrm{G},m}^{\mathrm{I,gm}}(\mathrm{X})|\mathrm{m}\} \text{ and } \mathrm{D}_\mathrm{G}^{\mathrm{b,gm}}(\mathrm{X}) = \mathrm{colim}_\mathrm{I} \mathrm{D}_\mathrm{G}^{\mathrm{I,gm}}(\mathrm{X})$$

where the last colimit is over all finite intervals $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$. Making use of the above discussion, we define for each prime ℓ different from the residue characteristics, and each finite interval $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$ of integers

$$(5.1.4) \qquad \mathrm{D}_\mathrm{G}^{\mathrm{I,gm}}(\mathrm{X}, \mathbb{Z}_\ell) = \lim_{\infty \leftarrow \nu} \{\mathrm{D}_\mathrm{G}^{\mathrm{I,gm}}(\mathrm{X}, \mathbb{Z}/\ell^\nu)|\nu\} \text{ and } \mathrm{D}_\mathrm{G}^{\mathrm{I,gm}}(\mathrm{X}, \mathbb{Q}_\ell) = \mathrm{D}_\mathrm{G}^{\mathrm{I,gm}}(\mathrm{X}, \mathbb{Z}_\ell) \otimes \mathbb{Q}_\ell.$$

One also defines the ℓ-adic derived categories

$$(5.1.5) \qquad \mathrm{D}_{\mathrm{G},m}^{\mathrm{b,gm}}(\mathrm{X}, \mathbb{Q}_\ell) = \mathrm{colim}_\mathrm{I}\{\mathrm{D}_{\mathrm{G},m}^{\mathrm{I,gm}}(\mathrm{X}, \mathbb{Q}_\ell)|\mathrm{I}\}.$$

When everything is defined over the complex numbers, one may define $\mathrm{D}_\mathrm{G}^{\mathrm{I,gm}}(\mathrm{X}, \mathbb{Q})$ and $\mathrm{D}_{\mathrm{G,m}}^{\mathrm{b,gm}}(\mathrm{X}, \mathbb{Q})$ similarly.

Proposition 5.2. *Let R denote a commutative Noetherian ring that is torsion, with torsion prime to $char(k)$.*

(i) Let $\{\widetilde{\mathrm{EG}}^{\mathrm{gm},m}|m\}$ denote the geometric classifying space defined with respect to the choice of another admissible gadget. Then, for each fixed $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$ with $2m - 2 \geq |\mathrm{I}| = \mathrm{b} - \mathrm{a}$, the projections

$$pr_1 : (\mathrm{EG}^{\mathrm{gm},m} \times \widetilde{\mathrm{EG}}^{\mathrm{gm},m}) \underset{\mathrm{G}}{\times} \mathrm{X} \to \mathrm{EG}^{\mathrm{gm,m}} \underset{\mathrm{G}}{\times} \mathrm{X} \text{ and } pr_2 : (\mathrm{EG}^{\mathrm{gm},m} \times \widetilde{\mathrm{EG}}^{\mathrm{gm},m}) \underset{\mathrm{G}}{\times} \mathrm{X} \to \widetilde{\mathrm{EG}}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{X}$$

induce equivalences

$$pr_1^* : \mathrm{D}_\mathrm{G}^\mathrm{I}(\mathrm{EG}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{X}, \mathrm{R}) \to \mathrm{D}_\mathrm{G}^\mathrm{I}((\mathrm{EG}^{\mathrm{gm,m}} \times \widetilde{\mathrm{EG}}^{\mathrm{gm},m}) \underset{\mathrm{G}}{\times} \mathrm{X}, \mathrm{R}) \text{ and}$$

$$pr_2^* : \mathrm{D}_\mathrm{G}^\mathrm{I}(\widetilde{\mathrm{EG}}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{X}, \mathrm{R}) \to \mathrm{D}_\mathrm{G}^\mathrm{I}((\mathrm{EG}^{\mathrm{gm,m}} \times \widetilde{\mathrm{EG}}^{\mathrm{gm},m}) \underset{\mathrm{G}}{\times} \mathrm{X}, \mathrm{R}).$$

(ii) For a fixed $\mathrm{I} = [\mathrm{a}, \mathrm{b}]$, with $2m - 2 \geq |\mathrm{I}| = \mathrm{b} - \mathrm{a}$, the induced map $i_m^{G,*} : \mathrm{D}_{\mathrm{G},m+1}^{\mathrm{I,gm}}(\mathrm{X}, \mathrm{R}) \to \mathrm{D}_{\mathrm{G},m}^{\mathrm{I,gm}}(\mathrm{X}, \mathrm{R})$ is an equivalence of categories.

Proof. (i) follows readily since both $\mathrm{EG}^{\mathrm{gm},m}$ and $\widetilde{\mathrm{EG}}^{\mathrm{gm},m}$ are $2m - 2$-connected. In view of this, the functor pr_i^* (for $i = 1, 2$) has an inverse given by $\tau_{\leq b} Rpr_{i*}$, where $\tau_{\leq b}$ denotes the cohomology truncation to degrees $\leq b$. Therefore, the functors pr_i^* are fully-faithful. Now to see these functors are equivalences of derived categories, it suffices to observe that all complexes in the above equivariant derived categories are quasi-isomorphic to complexes obtained by pull-back from X.

Next we consider (ii). One may take $\widetilde{\mathrm{EG}}^{\mathrm{gm},m} = \mathrm{EG}^{\mathrm{gm},m+1}$ which show the projections $pr_1 : (\mathrm{EG}^{\mathrm{gm},m} \times (\mathrm{EG})^{\mathrm{gm},m+1}) \underset{\mathrm{G}}{\times} \mathrm{X} \to \mathrm{EG}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{X}$ and $pr_2 : (\mathrm{EG}^{\mathrm{gm},m} \times (\mathrm{EG})^{\mathrm{gm},m+1}) \underset{\mathrm{G}}{\times} \mathrm{X} \to \mathrm{EG}^{\mathrm{gm},m+1} \underset{\mathrm{G}}{\times} \mathrm{X}$ both induce equivalences:

$$pr_1^* : \mathrm{D}_\mathrm{G}^\mathrm{I}(\mathrm{EG}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{X}, \mathrm{R}) \to \mathrm{D}_\mathrm{G}^\mathrm{I}((\mathrm{EG}^{\mathrm{gm},m} \times \mathrm{EG}^{\mathrm{gm},m+1}) \underset{\mathrm{G}}{\times} \mathrm{X}, \mathrm{R}) \text{ and}$$

$$pr_2^* : \mathrm{D}_\mathrm{G}^\mathrm{I}(\mathrm{EG}^{\mathrm{gm},m+1} \underset{\mathrm{G}}{\times} \mathrm{X}, \mathrm{R}) \to \mathrm{D}_\mathrm{G}^\mathrm{I}((\mathrm{EG}^{\mathrm{gm},m} \times \mathrm{EG}^{\mathrm{gm},m+1}) \underset{\mathrm{G}}{\times} \mathrm{X}, \mathrm{R}).$$

If $\Delta : \mathrm{EG}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{X} \to (\mathrm{EG}^{\mathrm{gm},m} \times (\mathrm{EG})^{\mathrm{gm},m+1}) \underset{\mathrm{G}}{\times} \mathrm{X}$ is the diagonal imbedding, then Δ is a section to pr_1 and $i_m^\mathrm{G} = pr_2 \circ \Delta$. Since pr_1^* is an equivalence, so is Δ^*; since Δ^* and pr_2^* are equivalences, it follows so is i_m^{G*}. (Here, it may be worthwhile to observe that for every complex $K \in \mathrm{D}_\mathrm{G}^\mathrm{I}(\mathrm{EG}^{\mathrm{gm},m+1} \underset{\mathrm{G}}{\times} \mathrm{X}, \mathrm{R})$ there exists a complex $L \in \mathrm{D}^\mathrm{I}(\mathrm{X}, \mathrm{R})$ so that $\pi_{m+1}^*(K) \overset{\simeq}{\leftarrow} p_{2,m+1}^*(L)$ following the notations as in (5.1.2).) $\qquad \square$

Remark 5.3. Observe that, for each fixed m, and each fixed finite interval I, one obtains the equivalences:

$$D_G^{I,gm}(X, \mathbb{Q}_\ell) = \lim_{\infty \leftarrow m} D_{G,m}^{I,gm}(X, \mathbb{Q}_\ell) = \lim_{\infty \leftarrow m} (\lim_{\infty \leftarrow \nu} D_{G,m}^{I,gm}(X, \mathbb{Z}/\ell^\nu)) \otimes \mathbb{Q}_\ell)$$

$$\cong (\lim_{\infty \leftarrow m} \lim_{\infty \leftarrow \nu} D_{G,m}^{I,gm}(X, \mathbb{Z}/\ell^\nu)) \otimes \mathbb{Q}_\ell \cong (\lim_{\infty \leftarrow \nu} \lim_{\infty \leftarrow m} D_{G,m}^{I,gm}(X, \mathbb{Z}/\ell^\nu)) \otimes \mathbb{Q}_\ell.$$

All but the next-to-last equivalence are clear from the definition, and this equivalence follows from Proposition 5.2(ii).

5.2. **The role of stratifications.** Next we will consider the role of stratifications. A G-stratification of the variety X is a decomposition of X into finitely many disjoint locally closed smooth equi-dimensional and G-stable subvarieties called *strata*. Let the stratification be denoted $\mathcal{S} = \{S_\alpha|\alpha\}$. Since the Borel construction is functorial, and $EG^{gm,m}/G$ is smooth, such a stratification of X defines a stratification $\{EG^{gm,m} \underset{G}{\times} S_\alpha|\alpha\}$ of the Borel construction $EG^{gm,m} \underset{G}{\times} X$ for each m. This stratification of $EG^{gm,m} \underset{G}{\times} X$ will be denoted $\mathcal{S}_{G,m}$. These are evidently compatible as m varies.

Given an interval I of the integers, we let $D_{G,m}^{I,gm}(X, \mathcal{S}_{G,m})$ denote the full subcategory of $D^{I,gm}(EG^{gm,m} \underset{G}{\times} X, \mathcal{S}_{G,m})$ consisting of complexes K whose cohomology sheaves are G-equivariant. $D_{G,m}^{b,gm}(X, \mathcal{S}_{G,m})$ will denote the full subcategory of complexes K that belong to $D^{I,gm}(EG^{gm,m} \underset{G}{\times} X, \mathcal{S}_{G,m})$ for some I. One takes the 2-limit of the categories $\{D_{G,m}^{I,gm}(EG^{gm,m} \underset{G}{\times} X, \mathcal{S}_{G,m})|m\}$ as $m \to \infty$ to define $D_G^{I,gm}(X, \mathcal{S})$.

Next we discuss how the t-structures on the equivariant derived category $D_{G,m}^{b,gm}(X, \mathcal{S}_{G,m})$ behave as one varies m and also the stratifications.

First observe that the map $i_m : EG^{gm,m} \to EG^{gm,m+1}$ is a regular closed immersion, for each m. Therefore so is the induced map $i_m^G : EG^{gm,m} \underset{G}{\times} X \to EG^{gm,m+1} \underset{G}{\times} X$. Given a G-stratification $\mathcal{S} = \{S_\alpha|\alpha\}$ of X, let $i_{S_\alpha}^{(m)} = id \underset{G}{\times} i : EG^{gm,m} \underset{G}{\times} S_\alpha \to EG^{gm,m} \underset{G}{\times} X$ denote the induced closed immersion.

A perversity function p defined on a stratified variety Y will be defined as a non-decreasing function on codimension of the strata, so that the value on the open stratum will be 0. We will view p as defined on the strata themselves. (For the most part, we will only consider the middle perversity, which is defined by $m(S) =$ the codimension of S in Y.) Recall that the standard t-structure on a derived category of complexes with bounded cohomology is one whose heart consists of complexes that have non-trivial cohomology only in degree 0. Then one may start with standard t-structures defined on each of the strata S shifted by the perversity $p(S)$ and obtain a non-standard t-structure on the bounded derived category, $D^b(Y)$ by *gluing* as in [BBD82, Definition 2.1.2].

Proposition 5.4. *Let $\mathcal{S}_G$ denote a fixed G-stable stratification of the G-variety X and let $\mathcal{S}_{G,m}$ denote the induced stratification on $EG^{gm,m} \underset{G}{\times} X$ for the action of G on X.*

(i) *Let the derived categories $D_{G,m+1}^{b,gm}(X, \mathcal{S}_{G,m+1})$ and $D_{G,m}^{b,gm}(X, \mathcal{S}_{G,m})$ be provided with the t-structures obtained by gluing with respect to a fixed perversity function and with respect to a G-stable stratification of X. Then the functor $i_m^{G,*}$ preserves the above t-structures.*

(ii) *Let $\mathcal{T}_G = \{\mathcal{T}_\beta|\beta\}$ denote a G-stable stratification that is a refinement of the stratification $\mathcal{S}_G$. Then any complex in $D_{G,m}^{b,gm}(X)$ whose cohomology sheaves are local systems on each stratum of $\mathcal{S}_{G,m}$ clearly belongs $D^{b,gm}(EG^{gm,m} \underset{G}{\times} X, \mathcal{T}_{G,m})$. This induces the inclusion functor*

$$D_{G,m}^{b,gm}(X, \mathcal{S}_{G,m}) \to D_{G,m}^{b,gm}(X, \mathcal{T}_{G,m}).$$

This functor preserves the t-structures on either side obtained by gluing.

Proof. We will first consider (i). Let i denote generically the closed immersions $\mathrm{EG}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{S} \to \mathrm{EG}^{\mathrm{gm},m+1} \underset{\mathrm{G}}{\times} \mathrm{S}$
and $\mathrm{EG}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{X} \to \mathrm{EG}^{\mathrm{gm},m+1} \underset{\mathrm{G}}{\times} \mathrm{X}$. Recall the t-structure obtained by gluing on $\mathrm{D}_{\mathrm{G},m}^{\mathrm{b,gm}}(\mathrm{X})$ is such that

(5.2.1) $\mathrm{D}_{\mathrm{G},m}^{\mathrm{b,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G},m})^{\leq 0} = \{K \in \mathrm{D}_{\mathrm{G},m}^{\mathrm{b,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G},m}) | \mathcal{H}^{\mathrm{i}}(\mathrm{i}_{\mathrm{S}}^{(m)})^{*})(\mathrm{K})) = 0, \mathrm{i} > \mathrm{p(S)}\}$ and

(5.2.2) $\mathrm{D}_{\mathrm{G},m}^{\mathrm{b,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G},m})^{\geq 0} = \{K \in \mathrm{D}_{\mathrm{G},m}^{\mathrm{b,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G},m}) | \mathcal{H}^{\mathrm{i}}(\mathrm{Ri}_{\mathrm{S}}^{(m)})^{!})(\mathrm{K})) = 0, \mathrm{i} < \mathrm{p(S)}\}$

Here $i_S^{(m)} : \mathrm{EG}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{S} \to \mathrm{EG}^{\mathrm{gm},m} \underset{\mathrm{G}}{\times} \mathrm{X}$ is the closed immersion corresponding to a stratum S. Since
$i_{\mathrm{S}}^{(m)*} i^* = i^* i_{\mathrm{S}}^{(m+1)*}$, it follows readily that i^* sends $\mathrm{D}_{\mathrm{G},m+1}^{\mathrm{I,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G},m})^{\leq 0}$ to $\mathrm{D}_{\mathrm{G},m}^{\mathrm{I,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G},m})^{\leq 0}$. To see
that i^* sends $\mathrm{D}_{\mathrm{G},m+1}^{\mathrm{I,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G,m}})^{\geq 0}$ to $\mathrm{D}_{\mathrm{G},m}^{\mathrm{I,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G},m})^{\geq 0}$, one needs to observe that $i_S^{(m)} = id_{\mathrm{EG}^{\mathrm{gm},m}} \times_{\mathrm{G}} i_{\mathrm{S}}$,
$i_S^{(m+1)} = id_{EG^{gm,m+1}} \times_{\mathrm{G}} i_S$ and that $i = i' \times_{\mathrm{G}} id$, where $i' : \mathrm{EG}^{\mathrm{gm},m} \to \mathrm{EG}^{\mathrm{gm},m+1}$ is the closed immersion.
One also needs the fact that for every complex $K \in \mathrm{D}_{\mathrm{G},m}^{\mathrm{I,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G},m})$ there exists a complex $L \in \mathrm{D}^{\mathrm{I}}(\mathrm{X})$ so
that $p_2^*(L) \simeq \pi_m^*(K)$.

  (ii) follows from [BBD82, Proposition 2.1.14]. $\square$

Proposition 5.5. *Assume in addition to the above hypotheses that* G *acts with finitely many orbits on
the variety* X. *Let* $\mathcal{S}$ *denote the stratification of* X *by the* G-*orbits. Then* $\mathrm{D}_{\mathrm{G},m}^{\mathrm{b,gm}}(\mathrm{X}, \mathcal{S}_{\mathrm{G},m}) = \mathrm{D}_{\mathrm{G},m}^{\mathrm{b,gm}}(\mathrm{X})$,
for every m.

Proof. This is clear since any G equivariant sheaf is a local system on each orbit. $\square$

5.3. Equivariant Derived Categories: Version II.
This is based on the simplicial model of the Borel
construction and we discussed this already in section 2.1. Therefore, we proceed to discuss the proof of
Theorem 1.6, next.

Proof of Theorem 1.6. (i) The fact that p_1^* is fully faithful follows from the observation that the
geometric fibers of the map p_1 are $\mathrm{EG}^{\mathrm{gm},m}$ (base extended to the algebraic closure of the base field)
which is acyclic in degrees 1 through $2m - 2$. (See Lemma 6.1.) Its inverse is the functor $\tau_{\leq b} Rp_{1*}$, where
$\tau_{\leq b}$ is the functor killing cohomology above degree b. Now the fact that p_1^* is an equivalence follows from
observing that it induces an equivalence on the hearts of the corresponding derived categories provided
with the usual t-structures, where the heart consists of complexes with trivial cohomology in all degrees
except 0. Both p_1 and p_2 are simplicial maps which are smooth in each degree: therefore, both p_1^* and
p_2^* send complexes that are mixed and pure to complexes that are mixed and pure.

  The geometric fibers of the map p_2 identify with the simplicial variety EG (base extended to the
algebraic closure of the base field) which is acyclic. The equivalence in (5.4.1) and [Jo02, Theorem 4.2],
show readily that the functor p_2^* is fully-faithful. That it is also an equivalence now follows by observing
that p_2^* induces an equivalence on the hearts of the corresponding derived categories provided with the
usual t-structures, where the heart consists of complexes with trivial cohomology in all degrees except 0.
These observations complete the proof of the statements in (i).

  (ii) We make use of the construction of the geometric classifying spaces through the admissible gadgets
discussed in (6.1.1), but first over the algebraic closure of the base field. In the terminology there, if we
are working over the complex numbers, then with the complex topology, we now observe that U_{m+1} is
in fact the *join* of U_m with U, where the join $\mathrm{U}_m * \mathrm{U}$ denotes the homotopy pushout

(5.3.1)
$$\begin{array}{ccc} \mathrm{U}_m \times \mathrm{U} & \longrightarrow & \mathrm{U} \\ \downarrow & & \downarrow \\ \mathrm{U}_m & \longrightarrow & \mathrm{U}_m * \mathrm{U} \end{array}$$

In the étale framework, the corresponding statement holds when U_m (U) is replaced by the completed
étale topological types, with the completion at a prime ℓ different from the characteristic. Thus, in case

the base field is the complex numbers, U_m is the iterated join of U m-times and in positive characteristics, $(U_m)_{et}\widehat{\ell} \simeq (U_{et}\widehat{\ell}) * \cdots * (U_{et}\widehat{\ell})$ (the m-fold join). Therefore $EG^{\mathrm{gm},m}$ ($EG^{\mathrm{gm},m}_{et}\widehat{\ell}$) is highly connected in the first case (the second case respectively) in the sense that the homotopy groups are all trivial through a sufficiently high degree. (A point to observe here is that the join of any two connected simplicial sets is simply connected. Completion preserves a simplicial set being connected. Therefore, since the join is already simply connected, one may check if the join $(U_m)_{et}\widehat{\ell} * (U_{et})\widehat{\ell}$ is ℓ-complete on homology with Z/l^ν-coefficients. Therefore, the join of two ℓ-complete simplicial sets is ℓ-complete.)

Next observe that the map p_1 is a map of simplicial varieties, which in degree n is given by the map

$$p_{1,n} : G^{n-1} \times (EG^{\mathrm{gm.m}} \times X) \to G^{n-1} \times X.$$

Therefore, taking the completed étale topological types provides an inverse system of maps of simplicial sets for each fixed n:

(5.3.2) $\qquad\qquad p_{1,n,et} : (G)_{et}\widehat{\ell}^{\,n-1} \times (EG^{\mathrm{gm},m}_{et}\widehat{\ell}) \times X_{et}\widehat{\ell} \to (G)_{et}\widehat{\ell}^{\,n-1} \times X_{et}\widehat{\ell}$

This is simply the projection to all factors except $EG^{\mathrm{gm},m}_{et}\widehat{\ell}$, so that the fibers identify with $EG^{\mathrm{gm},m}_{et}\widehat{\ell}$. Varying n, this defines an inverse system of maps of bisimplicial sets

(5.3.3) $\qquad\qquad\qquad\qquad\qquad \{E^\alpha_{\bullet,\bullet} \to B^\alpha_{\bullet,\bullet} | \alpha\}$

with the second index corresponding to the n in (5.3.2). Since all the varieties are connected, it follows that for each fixed n and α, the simplicial set $B^\alpha_{\bullet,n}$ is connected. Therefore, one may invoke [Wa78, Lemma 5.2] to conclude that on taking the diagonals, we obtain an inverse system of fibrations $\{\Delta(E_{\bullet,\bullet})^\alpha \to \Delta(B_{\bullet,\bullet})^\alpha | \alpha\}$. The fibers are clearly $EG^{\mathrm{gm},m}_{et}\widehat{\ell}$, which we have observed above are highly connected if m is sufficiently large. Therefore it follows that the map p_1 induces an isomorphism on the fundamental groups completed at ℓ, when everything is defined over the algebraic closure of k.

Observe that the map $EG^{\mathrm{gm},m} \times X \to EG^{\mathrm{gm},m} \underset{G}{\times} X$ is a locally trivial principal fibration in the étale topology. Therefore, a similar analysis as for the case of the map p_1, will prove that the map p_2 will induce an isomorphism on the fundamental groups completed at ℓ, again when everything is defined over the algebraic closure of k.

Next we consider the general case, where the base field k is no longer required to be algebraically closed. In this case we will let $\bar{k}$ denote the algebraic closure of k. For this discussion we will denote all objects defined over the base field k with a subscript o, while objects over $\bar{k}$ will be denoted as before. It suffices to show that local systems on $EG_o \underset{G_o}{\times} X_o$ and $EG^{\mathrm{gm},m}_o \underset{G_o}{\times} X_o$ correspond $1-1$ if $m >> 0$. Observe a local system on either of these defines, by pull-back a local system on $EG_o \underset{G_o}{\times} (EG^{\mathrm{gm},m}_o \times X_o)$ and by further pull-back a local system on $EG \underset{G}{\times} (EG^{\mathrm{gm},m} \times X)$. Observe that the group of automorphisms of the latter over $EG_o \underset{G_o}{\times} (EG^{\mathrm{gm},m}_o \times X_o)$ is the Galois group $Gal_k(\bar{k})$. By what we just proved in the paragraphs above, local systems on $EG \underset{G}{\times} (EG^{\mathrm{gm},m} \times X)$ correspond $1-1$ with local systems on $EG \underset{G}{\times} X$ as well as local systems on $EG^{\mathrm{gm},m} \underset{G}{\times} X$ which are also equivariant for the action of $Gal_k(\bar{k})$. Therefore, by $Gal_k(\bar{k})$-equivariance, these local systems descend to local systems on $EG_o \underset{G_o}{\times} X_o$ and $EG^{\mathrm{gm},m}_o \underset{G_o}{\times} X_o$. This completes the proof of the second statement and hence of the theorem.

5.4. Derived functors for maps of simplicial varieties. It is shown in [Jo02] and also ([Jo93]), that associated to any given Grothendieck topology on varieties, there is another Grothendieck topology that one can put on any simplicial variety $X_\bullet$. As we show below, this topology plays a key role in being able to define derived functors of the direct image functor for maps between simplicial varieties, and therefore often comes in handy. Therefore, we proceed to summarize the main features of this construction. Let Top denote a Grothendieck topology defined on algebraic varieties over the given field k. Let $X_\bullet$ denote a simplicial variety over k. Then we define the objects of the topology, $STop(X_\bullet)$, to consist of all maps $u_\bullet : U_\bullet \to X_\bullet$ of simplicial varieties so that each $u_n : U_n \to X_n$ belongs to the topology $Top(X_n)$.

Morphisms between two such objects will be defined to be commutative triangles. One defines a family of maps $\{v_{\bullet\alpha} : V_{\bullet\alpha} \to U_{\bullet}|\alpha\}$ to be a covering if each $\{v_{n_\alpha} : V_{n_\alpha} \to U_n|\alpha\}$ is a *covering* in $\mathrm{Top}(X_n)$. We will often call this *the simplicial topology associated to the given topology* Top. It is shown in [Jo02, section 1] that the topology STop and the associated topos of sheaves on it inherits all the good properties from the given topology Top: for example, the category underlying $\mathrm{STop}(X_\bullet)$ is closed under finite inverse limits and there are enough points on the site $\mathrm{STop}(X_\bullet)$ if each $\mathrm{Top}(X_n)$ has the corresponding property.

Observe that for each fixed integer $n \geq 0$, there is a map of sites $\eta_n : \mathrm{Top}(X_n) \to \mathrm{STop}(X_\bullet)$ defined by sending the (simplicial) object $U_\bullet$ in $\mathrm{STop}(X_\bullet)$ to U_n. Let R denote a commutative Noetherian ring and let $\mathrm{C}^+(\mathrm{Sh}(\mathrm{Top}(X_\bullet), R))$ $(\mathrm{C}^+(\mathrm{Sh}(\mathrm{STop}(X_\bullet), R)))$ denote the category of all complexes of sheaves of R-modules on $\mathrm{Top}(X_\bullet)$ $(\mathrm{STop}(X_\bullet)$, respectively), that are bounded below. Then one defines a functor $\eta : \mathrm{C}^+(\mathrm{Sh}(\mathrm{Top}(X_\bullet), R)) \to \mathrm{C}^+(\mathrm{Sh}(\mathrm{STop}(X_\bullet), R))$ by sending a complex of sheaves $\{K_n|n\}$ on $\mathrm{Top}(X_\bullet)$ to the total complex of the double complex $\{\eta_{n*}(K_n)|n\}$. One verifies readily the following properties of this functor (see [Jo02, sections 1 and 3]):

(i) If $f : X_\bullet \to Y_\bullet$ is a map of simplicial varieties, then $\eta(f_*(K)) = {}_sf_*(\eta(K))$, $K \in \mathrm{C}^+(\mathrm{Sh}(\mathrm{STop}(X_\bullet), R))$ and $\eta(f^*(L)) = {}_sf^*(\eta(L))$, $L \in \mathrm{C}^+(\mathrm{Sh}(\mathrm{STop}(Y_\bullet), R))$. Here ${}_sf_*$ $({}_sf^*)$ denotes the push-forward functor (the pull-back functor) associated to f on the simplicial topology. Both of these statements may be checked at the stalks and may be deduced from the basic properties of stalks of sheaves computed on the simplicial topology as in [Jo02, section 3].

(ii) If $j : U_\bullet \to X_\bullet$ is an open immersion in each degree, then $\eta(j_!(K)) = {}_sj_!(\eta(K))$, $K \in \mathrm{C}^+(\mathrm{Sh}(\mathrm{STop}(U_\bullet), R))$ and ${}_sj_!$ is the corresponding functor on the simplicial topology.

(iii) If $f : X_\bullet \to Y_\bullet$ is a map of simplicial varieties and $K \in \mathrm{C}^+(\mathrm{Sh}(\mathrm{STop}(X_\bullet), R))$ so that (K_n, f_n) is cohomologically proper, that is, satisfies the conclusions of the proper base-change theorem, then the cohomology of the stalks of $Rf_*(K)$ computed on the simplicial topology $\mathrm{STop}(Y_\bullet)$ identify with the cohomology of the fibers of the simplicial map f with respect to K. (See [Jo02, (4.2) Theorem].)

Next let G denote a linear algebraic group acting on a variety X. Then it is shown in [Jo02, (3.8.2) Definition, (3.10) Corollary] that one may define the notion of sheaves with descent (or cartesian sheaves) on the site $\mathrm{STop}(EG \underset{G}{\times} X)$ and that if $\mathrm{D}_G(\mathrm{STop}(EG \underset{G}{\times} X))$ denotes the full subcategory of $\mathrm{D}(\mathrm{STop}(EG \underset{G}{\times} X))$ consisting of complexes whose cohomology sheaves have descent, then the functor η induces a *fully-faithful* functor (where the superscript $+$ denotes the bounded below derived categories):

$$(5.4.1) \qquad\qquad \mathrm{D}_G^+(X) \to \mathrm{D}_G^+(\mathrm{STop}(EG \underset{G}{\times} X)).$$

Remark 5.6. Therefore, if X is a variety for which the derived category $\mathrm{D}_G^+(X)$ is generated by a collection of complexes $\{K_\alpha|\alpha\}$, by taking the full subcategory of $\mathrm{D}_G^+(\mathrm{STop}(EG \underset{G}{\times} X))$ generated by the images of these complexes, one obtains an equivalence of categories. Therefore, we will henceforth denote this full subcategory by $\mathrm{D}_G^+(\mathrm{STop}(EG \underset{G}{\times} X))$ which enables us to work with the simplicial topology throughout.

6. Appendix: Background material

Next we consider the following background material needed for the definition of the geometric classifying spaces in section 2. We start with the following lemma.

Lemma 6.1. *Let* V *denote a representation of the linear algebraic group* G, *all defined over a perfect field* k *of finite ℓ-cohomological dimension for some prime* $\ell \neq char(k)$. *Let* $U \subseteq V$ *denote an open* G-*stable subvariety so that the complement* $V - U$ *has codimension* $c > 1$ *in* V.

(i) *Then denoting by* $\bar{k}$ *the algebraic closure of* k,

$$\mathrm{H}^n_{\mathrm{et}}(U \underset{\mathrm{Spec}\,k}{\times} \mathrm{Spec}\,\bar{k}, \mathbb{Z}/\ell^\nu) = 0 \text{ for all } 0 < n < 2c-1 \text{ and } \mathrm{H}^0_{\mathrm{et}}(U \underset{\mathrm{Spec}\,k}{\times} \mathrm{Spec}\,\bar{k}, \mathbb{Z}/\ell^\nu) = \mathbb{Z}/\ell^\nu.$$

(ii) *For any variety* X,

$$R^n f_*(\mathbb{Z}/\ell^\nu) = 0 \text{ for all } 0 < n < 2c-1, \text{ and } R^0 f_*(\mathbb{Z}/\ell^\nu) = \mathbb{Z}/\ell^\nu$$

where $f : U \times X \to X$ denotes the projection.

(iii) In case the field $k = \mathbb{C}$, the corresponding results also hold for $\mathbb{Z}$ and $\mathbb{Q}$ in the place of $\mathbb{Z}/\ell^{\nu}$.

Proof. (i) It suffices to consider the case k is algebraically closed. Then (i) follows from the long-exact sequence

$$\cdots \to H^n_{et,V-U}(V, \mathbb{Z}/\ell^{\nu}) \to H^n_{et}(V, \mathbb{Z}/\ell^{\nu}) \to H^n_{et}(U, \mathbb{Z}/\ell^{\nu}) \to H^{n+1}_{et,V-U}(V, \mathbb{Z}/\ell^{\nu}) \cdots \to$$

and the fact that $H^i_{et,V-U}(V, \mathbb{Z}/\ell^{\nu}) = 0$ for all $i < 2c$ while $H^i_{et}(V, \mathbb{Z}/\ell^{\nu}) = 0$ for all $i > 0$, $H^0_{et}(V, \mathbb{Z}/\ell^{\nu}) = \mathbb{Z}/\ell^{\nu}$. These complete the proof of (i).

The assertion $H^i_{et,V-U}(V, \mathbb{Z}/\ell^{\nu}) = 0$ for all $i < 2c$ is a cohomological semi-purity statement. We provide a short proof of this statement due to the lack of an adequate reference. Since the base field is assumed to be perfect, one may find an open subvariety V_0 of V so that $Y_0 = (V - U) \cap V_0$ is smooth and nonempty. Now one has a long-exact sequence in cohomology:

$$\cdots \to H^i_{et,Y_1}(V, \mathbb{Z}/\ell^{\nu}) \to H^i_{et,Y}(V, \mathbb{Z}/\ell^{\nu}) \to H^i_{et,Y_0}(V_0, \mathbb{Z}/\ell^{\nu}) \to H^{i+1}_{et,Y_1}(V, \mathbb{Z}/\ell^{\nu}) \to \cdots$$

where $Y = V - U$, $Y_1 = Y - Y_0$. We may also assume without loss of generality that Y is irreducible. By cohomological purity, $H^i_{et,Y_0}(V_0, \mathbb{Z}/\ell^{\nu}) = 0$ for all $i < 2c$ (in fact, for all $i \neq 2c$). Since Y_1 is of dimension strictly less than the dimension of Y, an ascending induction on the dimension of Y enables one to assume $H^i_{et,Y_1}(V, \mathbb{Z}/\ell^{\nu}) = 0$ for all $i < 2\mathrm{codim}_{Y_1}(V)$. (One may start the induction when $\dim(Y) = 0$, since in that case Y is smooth.) Since $\mathrm{codim}_Y(V) < \mathrm{codim}_{Y_1}(V)$, the long exact sequence above now proves $H^i_{et,Y}(V, \mathbb{Z}/\ell^{\nu}) = 0$ for all $i < 2\mathrm{codim}_Y(V)$.

(ii) follows readily from (i). We skip the proof of (iii) which follows along the same lines as the proofs of (i) and (ii). $\qquad\square$

Since different choices are possible for such geometric classifying spaces, we proceed to consider this in the more general framework of *admissible gadgets* as defined in [MV99, section 4.2]. The following definition is a variation of the above definition in [MV99].

6.1. **Admissible gadgets associated to a given G-variety.** We shall say that a pair (W, U) of smooth varieties over k is a *good pair* for G if W is a k-rational representation of G and $U \subsetneq W$ is a G-invariant non-empty open subset on which G acts freely and so that U/G is a variety. It is known (*cf.* [To99, Remark 1.4]) that a good pair for G always exists.

Definition 6.2. A sequence of pairs $\{(W_m, U_m)|m \geq 1\}$ of smooth varieties over k is called an *admissible gadget* for G, if there exists a good pair (W, U) for G such that $W_m = W^{\times m}$ and $U_m \subsetneq W_m$ is a G-invariant open subset such that the following hold for each $m \geq 1$.

(1) $(U_m \times W) \cup (W \times U_m) \subseteq U_{m+1}$ as G-invariant open subvarieties.

(2) $\{\mathrm{codim}_{U_{m+1}}(U_{m+1} \setminus (U_m \times W))|m\}$ is a strictly increasing sequence, that is, $\mathrm{codim}_{U_{m+2}}(U_{m+2} \setminus (U_{m+1} \times W)) > \mathrm{codim}_{U_{m+1}}(U_{m+1} \setminus (U_m \times W))$.

(3) $\{\mathrm{codim}_{W_m}(W_m \setminus U_m)|m\}$ is a strictly increasing sequence, that is, $\mathrm{codim}_{W_{m+1}}(W_{m+1} \setminus U_{m+1}) > \mathrm{codim}_{W_m}(W_m \setminus U_m)$.

(4) U_m has a free G-action, the quotient U_m/G is a smooth quasi-projective variety over k and $U_m \to U_m/G$ is a principal G-bundle.

Lemma 6.3. *Let* U *denote a smooth quasi-projective variety over a field* K *with a free action by the linear algebraic group* G *so that the quotient* U/G *exists as a smooth quasi-projective variety over* K. *Then if* X *is any smooth quasi-projective variety over* K, *the quotient* $U \underset{G}{\times} X \cong (U \underset{\mathrm{Spec}\,K}{\times} X)/G$ *(for the diagonal action of* G*) exists as a scheme over* K.

Proof. This follows, for example, from [MFK94, Proposition 7.1]. $\qquad\square$

An *example* of an admissible gadget for G can be constructed as follows: start with a good pair (W, U) for G. The choice of such a good pair will vary depending on G. Choose a faithful k-rational representation R of G of dimension n, that is, G admits a closed immersion into GL(R). Then G acts

freely on an open subset U of $W = R^{\oplus n} = \text{End}(R)$ so that U/G is a variety. (For e.g. $U = GL(R)$.) Let $Z = W \setminus U$.

Given a good pair (W, U), we now let

(6.1.1) $$W_m = W^{\times m}, U_1 = U \text{ and } U_{m+1} = (U_m \times W) \cup (W \times U_m) \text{ for m} \geq 1.$$

Setting $Z_1 = Z$ and $Z_{m+1} = U_{m+1} \setminus (U_m \times W)$ for $m \geq 1$, one checks that $W_m \setminus U_m = Z^m$ and $Z_{m+1} = Z^m \times U$. In particular, $\text{codim}_{W_m}(W_m \setminus U_m) = m(\text{codim}_W(Z))$ and $\text{codim}_{U_{m+1}}(Z_{m+1}) = (m+1)d - m(\dim(Z)) - d = m(\text{codim}_W(Z))$, where $d = \dim(W)$. Moreover, $U_m \to U_m/G$ is a principal G-bundle and the quotient $V_m = U_m/G$ exists as a smooth quasi-projective scheme.

We conclude with the following Proposition that shows how t-structures may be transferred under an equivalence of derived categories. Though this result is fairly well-known, we could not find a suitable reference: that is the reason for including it here. But we skip its proof.

Proposition 6.4. *Let* $\mathbf{T}$ *and* $\mathbf{T}'$ *denote two triangulated categories, with* $F : \mathbf{T}' \to \mathbf{T}$ *an equivalence of categories together with a left-adjoint* $G : \mathbf{T} \to \mathbf{T}'$*, so that the natural transformations* $id_{\mathbf{T}} \to F \circ G$ *and* $G \circ F \to id_{\mathbf{T}'}$ *are natural isomorphisms. Let* $(\mathbf{T}'^{\leq 0}, \mathbf{T}'^{\geq 0})$ *denote a t-structure on* $\mathbf{T}'$*.*

Then $(\mathbf{T}^{\leq 0}, \mathbf{T}^{\geq 0})$ *is a t-structure on* $\mathbf{T}$ *where* $\mathbf{T}^{\leq 0}$ *(*$\mathbf{T}^{\geq 0}$*) is defined as the strict full subcategory of* $\mathbf{T}$ *whose objects are isomorphic to objects of the form* $F(A)$*, A an object of* $\mathbf{T}'^{\leq 0}$ *(A an object of* $\mathbf{T}'^{\geq 0}$*, respectively).*

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