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DEFORMATIONS OF SYMPLECTIC SINGULARITIES AND ORBIT METHOD FOR SEMISIMPLE LIE ALGEBRAS

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Abstract. We classify filtered quantizations of conical symplectic singularities and use this to show that all filtered quantizations of symplectic quotient singularities are spherical symplectic reflection algebras of Etingof and Ginzburg. We further apply our classification and a classification of filtered Poisson deformations obtained by Namikawa to establish a version of the Orbit method for semisimple Lie algebras. Namely, we produce a natural map from the set of coadjoint orbits of a semisimple algebraic group to the set of primitive ideals in the universal enveloping algebra. We show that the map is injective for classical Lie algebras and conjecture that in that case the image consists of the primitive ideals corresponding to one-dimensional representations of W-algebras. Along the way, we get several new results on the Lusztig-Spaltenstein induction for coadjoint orbits.

Dedicated to Sasha Premet, on his 60th birthday, with admiration.

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1. Introduction

1.1. **Filtered deformations of Poisson algebras.** Our general setting is as follows. Let A be a finitely generated Poisson algebra over $\mathbb{C}$ that is equipped with

- an algebra grading $A=\bigoplus_{i=0}^{\infty} A_i$ such that $A_0=\mathbb{C}$ (such a grading will be called *positive*)
- and the Poisson bracket has degree $-d$, where d is a positive integer.

A basic example is as follows. We take a symplectic vector space V and a finite group Γ of its linear symplectomorphisms. Then we can take the algebra $A=\mathbb{C}[V]^{\Gamma}$ of Γ-invariants in $\mathbb{C}[V]$, this is a graded Poisson subalgebra of $\mathbb{C}[V]$ with $d=2$.

We are interested in filtered deformations of A, i.e., in filtered associative algebras $\mathcal{A}$ together with an isomorphism $\operatorname{gr}\mathcal{A}\xrightarrow{\sim} A$ of graded algebras. We only consider two classes of deformations that are compatible with the bracket on A. First, we consider *filtered Poisson deformations*, i.e., commutative algebras $\mathcal{A}^0$ equipped with a Poisson bracket decreasing the filtration degree by d such that $\operatorname{gr}\mathcal{A}^0\xrightarrow{\sim} A$ becomes a Poisson algebra isomorphism. Second, we consider *filtered quantizations* $\mathcal{A}$. Those are non-commutative algebras such that the Lie bracket decreases filtration degree by d and $\operatorname{gr}\mathcal{A}\xrightarrow{\sim} A$ is an isomorphism of Poisson algebras.

We want to describe filtered Poisson deformations and filtered quantizations of A up to an isomorphism (of deformations). This problem doesn't have a nice solution if we do not impose any restrictions on A. We will assume that $X:=\operatorname{Spec} A$ has *symplectic singularities*. In this case, filtered Poisson deformations were classified by Namikawa, [N3], while the classification of quantizations is one of the main results of the present paper (Theorem 3.4).

1.2. **Symplectic singularities.** Recall, following Beauville, [Be], the definition of a variety with symplectic singularities. Let X be a normal algebraic variety such that X^{reg} carries a symplectic form, ω^{reg}. Since X is normal, the form ω^{reg} gives rise to the Poisson bracket on $\mathcal{O}_X$ so X becomes a Poisson algebraic variety.

Let $\widehat{X}$ be a resolution of singularities for X. We say that X *has symplectic singularities* if ω^{reg} extends to a regular form on $\widehat{X}$. In fact, this is independent of the choice of $\widehat{X}$. Below in this section, we will assume that $X:=\operatorname{Spec}(A)$ has symplectic singularities.

Symplectic quotient singularities V/Γ provide an example of symplectic singularities, [Be]. Another classical example arises as follows. We take a semisimple Lie algebra $\mathfrak{g}$ and the nilpotent cone $\mathcal{N}\subset\mathfrak{g}^*$. Then $X:=\mathcal{N}$ has symplectic singularities (for $\widehat{X}$ we can take the Springer resolution $T^*(G/B)$).

Return to the general situation: $X := \operatorname{Spec}(A)$ with symplectic singularities, where A a positively graded algebra. Namikawa has constructed a "Cartan space" $\mathfrak{P}(=\mathfrak{P}_X)$ and a "Weyl group" $W(=W_X)$ for X. The latter acts on the former as a crystallographic reflection group. In the case of the nilpotent cone, we recover the Cartan subalgebra and the Weyl group of $\mathfrak{g}$.

The simplest form of the main result of [N3] can be stated as follows: the filtered Poisson deformations are canonically indexed by the points of the quotient $\mathfrak{P}/W$. In this paper, we will show that the filtered quantizations of A are indexed by the points of the same quotient $\mathfrak{P}/W$.

Let us explain how the quantizations are constructed in the general case. The variety X admits distinguished partial resolutions called *$\mathbb{Q}$-factorial terminalizations*. The filtered Poisson deformations/quantizations of X are essentially produced by taking global sections of Poisson deformations/quantizations of $\mathcal{O}_{\tilde{X}}$, where $\tilde{X}$ is a $\mathbb{Q}$-factorial terminalization of X. The filtered Poisson deformations/quantizations of $\mathcal{O}_{\tilde{X}}$ are parameterized by the points of $\mathfrak{P}$. Moreover, W-conjugate parameters give rise to isomorphic algebras of global sections, see e.g. [BPW, Proposition 3.10].

In the case when $X = \mathcal{N}$ our result says that all filtered quantizations of $\mathbb{C}[\mathcal{N}]$ are obtained as the central reductions of the universal enveloping algebra $U(\mathfrak{g})$.

In the case of $A = \mathbb{C}[V]^\Gamma$ filtered deformations of A were constructed algebraically by Etingof and Ginzburg, [EG], as spherical subalgebras $eH_{t,c}e$ in symplectic reflection algebras $H_{t,c}$, where $t \in \mathbb{C}$ and c is a vector in a space of dimension $\dim \mathfrak{P}$. The algebra $eH_{t,c}e$ is a filtered Poisson deformation of $\mathbb{C}[V]^\Gamma$ when $t = 0$, and is a filtered quantization when $t = 1$. Results of Bellamy, [Bel], show that any filtered Poisson deformation of $\mathbb{C}[V]^\Gamma$ is isomorphic to $eH_{0,c}e$ for some c. The results of the present paper show that every filtered quantization of $\mathbb{C}[V]^\Gamma$ has the form $eH_{1,c}e$ for some c. The connection to deformations of $\mathbb{Q}$-terminalizations turns out to be an important tool to study the symplectic reflection algebras, [Lo10].

1.3. **Orbit method.** One of the general principles of Lie representation theory is that interesting irreducible representations of Lie groups or Lie algebras should have to do with the orbits of the coadjoint representation of the corresponding group. The most famous manifestation of this principle is Kirillov's orbit method, [Ki], that describes irreducible unitary representations of nilpotent Lie groups. Namely, let G be a nilpotent Lie group. Kirillov has constructed a natural bijection between $\mathfrak{g}^*/G$, the set of coadjoint G-orbits, and the set of isomorphism classes of irreducible unitary representations of G.

An algebraic version of this result was found by Dixmier, [D1], see also [D2, Section 6]. Namely, let $\mathfrak{g}$ be a nilpotent Lie algebra over $\mathbb{C}$. Consider the universal enveloping algebra $U(\mathfrak{g})$. Recall that by a *primitive ideal* in an associative algebra one means the annihilator of a simple module. One of the points of considering primitive ideals is that, while the set of isomorphism classes of irreducible representations is huge and wild for almost all $\mathfrak{g}$, the set of primitive ideals has reasonable size and it is often possible to describe it. In [D1], Dixmier has proved that the set $\operatorname{Prim}(\mathfrak{g})$ of primitive ideals in $U(\mathfrak{g})$ is in a natural bijection with $\mathfrak{g}^*/G$.

For a long time, there was, and still is, a question of how to adapt the Orbit method to semisimple Lie groups or algebras that are far more interesting than nilpotent ones from the representation theoretic perspective. In this paper, we study the algebraic version and seek to find an analog of Dixmier's result. The classification of primitive ideals in

$U(\mathfrak{g})$ is known, thanks to the work of Barbasch, Joseph, Lusztig, Vogan and others, see [CM, Section 10] for a review, but a connection with (co)adjoint orbits is very subtle and indirect. The algebraic version of the Orbit method was studied previously, for example, in [V1, McG].

To explain our result, we need the notion of a *Dixmier algebra* (due to Vogan, [V1]). Let G be a semisimple algebraic group, $\mathfrak{g}$ its Lie algebra.

Definition 1.1. By a Dixmier algebra we mean an algebra $\mathcal{A}$ equipped with a rational action of G together with a quantum comoment map $U(\mathfrak{g}) \to \mathcal{A}$ such that $\mathcal{A}$ is finitely generated as a $U(\mathfrak{g})$-module.

In Theorem 5.3, we will produce a natural embedding, $\mathbb{O}_1 \mapsto \mathcal{A}(\mathbb{O}_1)$, of $\mathfrak{g}^*/G$ into the set of isomorphism classes of Dixmier algebras. All algebras occurring in the image are *completely prime*, i.e., have no zero divisors.

For the classical Lie algebras, we can get a stronger result. Let $\mathcal{J}(\mathbb{O}_1)$ denote the kernel of $U(\mathfrak{g}) \to \mathcal{A}(\mathbb{O}_1)$. In Theorem 5.3, we will see that the map $\mathbb{O}_1 \mapsto \mathcal{J}(\mathbb{O}_1)$ is injective provided $\mathfrak{g}$ is classical (we also expect this to be the case when $\mathfrak{g}$ is exceptional). The ideals in the image correspond to 1-dimensional representations of W-algebras and, for $\mathfrak{g}$ classical, we expect that all such ideals occur in the image[1].

1.4. **Birational induction.** Let us elaborate on how the embedding of $\mathfrak{g}^*/G$ into the set of completely prime Dixmier algebras is constructed. First, we will identify $\mathfrak{g}^*/G$ with the set of equivalence classes of *birationally minimal induction data* defined as follows.

Take a Levi subalgebra $\mathfrak{l} \subset \mathfrak{g}$, a nilpotent orbit $\mathbb{O}' \subset \mathfrak{l}^*$, and $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. Following Lusztig and Spaltenstein, [LS], we include $\mathfrak{l}$ into a parabolic subalgebra $\mathfrak{p} = \mathfrak{l} \ltimes \mathfrak{n}$. We write $\mathfrak{p}^\perp$ for the annihilator of $\mathfrak{p}$ in $\mathfrak{g}^*$. Consider the generalized Springer map $G \times^P (\xi + \overline{\mathbb{O}'} + \mathfrak{p}^\perp) \to \mathfrak{g}^*$. Here we view $\xi + \overline{\mathbb{O}'} + \mathfrak{p}^\perp$ as a closed subvariety in $\mathfrak{n}^\perp$, this subvariety is P-stable. The image of the generalized Springer map is the closure of a single G-orbit and it is generically finite onto its image. We denote this map considered as a map onto its image by π.

The following definition introduces an important terminology to be used in the paper.

Definition 1.2. Recall that $\mathfrak{l}$ stands for a Levi subalgebra of $\mathfrak{g}$, $\mathbb{O}'$ for a nilpotent orbit in $\mathfrak{l}^*$ and ξ for an element of $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$.

- A triple $(\mathfrak{l}, \mathbb{O}', \xi)$ is called an *induction datum*.
- The orbit $\mathbb{O}_\xi$ is said to be *induced* from $(\mathfrak{l}, \mathbb{O}', \xi)$.
- When π is generically injective, i.e., birational, we will say that $\mathbb{O}_\xi$ is *birationally induced* from $(\mathfrak{l}, \mathbb{O}', \xi)$.
- We say that $\mathbb{O}'$ is *birationally rigid* if it cannot be birationally induced from a proper Levi.
- If π is birational and $\mathbb{O}'$ is birationally rigid, then we say that $(\mathfrak{l}, \mathbb{O}', \xi)$ is a *birationally minimal induction datum*.

The group G acts naturally on the set of induction data and one can show that the map $(\mathfrak{l}, \mathbb{O}', \xi) \mapsto \mathbb{O}_\xi$ is a bijection between the set of G-equivalence classes of the birationally minimal induction data and the set $\mathfrak{g}^*/G$ of coadjoint orbits, see Theorem 4.4.

Now pick a coadjoint orbit $\mathbb{O}_1$ and let $(\mathfrak{l}, \mathbb{O}', \xi)$ be the corresponding birationally minimal induction datum. Let G/H denote the open G-orbit in $G \times^P (\overline{\mathbb{O}'} + \mathfrak{n})$. This is a finite cover

[1]This conjecture has been recently proved in [T].

of some nilpotent orbit $\mathbb{O}$ and hence a symplectic variety. Moreover, $X := \operatorname{Spec}(\mathbb{C}[G/H])$ has symplectic singularities and $\mathfrak{P} = (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$, see Proposition 4.7. We let $\mathcal{A}(\mathbb{O}_1)$ to be the quantization of $\mathbb{C}[X]$ corresponding to the parameter ξ. This is the Dixmier algebra that we assign to $\mathbb{O}_1$.

Remark 1.3. In this paper, we only need to consider induction from coadjoint orbits. More generally, one can consider induction from their covers. Many geometric results of this paper generalize to that setting, see, for example, [Ma2],[N5].

1.5. **Content of the paper.** Let us describe the content of this paper.

In Section 2 we discuss symplectic singularities, their $\mathbb{Q}$-factorial terminalizations and their deformations. In Section 2.1 we introduce the notion of a symplectic singularity following Beauville, give some examples and discuss the $\mathbb{Q}$-factorial terminalizations. In Section 2.2 we recall the notion of a filtered Poisson deformation of a not necessarily affine Poisson variety and the classification of such deformations of $\mathbb{Q}$-factorial terminalizations following [N1]. In Section 2.3 we apply this to studying Poisson deformations of conical symplectic singularities. In Section 2.4 we recall results of Namikawa, [N4, N3], on the universal conical Poisson deformation of a conical symplectic singularity. The only new results in Section 2 are contained in Section 2.5, where we study the negative degree Poisson derivations of the algebra of functions on a conical symplectic singularity.

In Section 3 we study quantizations of conical symplectic singularities. We start, Section 3.1, by recalling the general definition of a filtered quantization and classification results in the symplectic case obtained in [BeKa, Lo5]. In Section 3.2 we produce quantizations of a conical symplectic singularity starting from those of its $\mathbb{Q}$-terminalization following [BPW]. Also in that section we state the main classification result, Theorem 3.4. This theorem is then proved in the three subsequent sections. Then, in Section 3.6 we treat the case of symplectic quotient singularities and show that in this case all quantizations are spherical symplectic reflection algebras of Etingof and Ginzburg, [EG]. Finally, in Section 3.7, we study the question of when filtered Poisson deformations (resp., quantizations) are isomorphic as filtered Poisson (resp., associative) algebras.

In Section 4 we study various questions related to the geometry of coadjoint orbits. In Section 4.1 we recall some basic results on the Lusztig-Spaltenstein induction. In Section 4.2 we recall sheets in semisimple Lie algebras, introduce the related notion of *birational sheets* and state a result, Theorem 4.4, describing their structure. In Section 4.3 we study $\mathbb{Q}$-terminalizations of normalizations of nilpotent orbit closures giving conceptual proofs of results previously obtained by Namikawa, [N2], and Fu, [F]. Then we compute Weyl groups of these normalizations and their suitable covers, Section 4.4, generalizing results of Namikawa, [N3, Section 2]. We use results of Sections 4.3 and 4.4 to prove Theorem 4.4 in Section 4.5.

In Section 5 we establish our version of the orbit method. We start by recalling results and constructions related to W-algebras. In Section 5.1 we present their construction following [Lo1, Lo2]. Then in Section 5.2 we recall the restriction functor for Harish-Chandra bimodules from [Lo2] and its basic properties. Section 5.3 is the main part, there we state and prove Theorem 5.3 that provides a map $\mathfrak{g}^*/G \to \operatorname{Prim}(\mathfrak{g})$. We discuss the image of this map in Section 5.4.

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2. Symplectic singularities and their deformations

2.1. **Symplectic singularities and $\mathbb{Q}$-terminalizations.** Let X be a normal Poisson algebraic variety over $\mathbb{C}$ such that the smooth locus X^{reg} is a symplectic variety. Let ω^{reg} denote the symplectic form on X^{reg}. Following [Be, Definition 1.1], we say that X has *symplectic singularities* if there is a projective resolution of singularities $\rho:\widehat{X}\to X$ such $\rho^*(\omega^{reg})$ extends to a regular (but not necessarily symplectic) 2-form on $\widehat{X}$. Once such $\widehat{X}$ exists, for any other projective resolution $\rho_1:\widehat{X}_1\to X$, the form $\rho_1^*(\omega^{reg})$ also extends to a regular 2-form, see [Be, (1.2)].

Now let X be an affine Poisson variety. We say that X is *conical* if there is an algebra grading $\mathbb{C}[X]=\bigoplus_{i=0}^{\infty}\mathbb{C}[X]_i$ and a positive integer d such that

- $\mathbb{C}[X]_0=\mathbb{C}$.
- $\{f,g\}\in\mathbb{C}[X]_{i+j-d}$ for any i,j and $f\in\mathbb{C}[X]_i, g\in\mathbb{C}[X]_j$.

By a *conical symplectic singularity* we mean a conical variety with symplectic singularities. Let us give two classical examples of conical symplectic singularities, see [Be, (2.5),(2.6)].

Example 2.1. Let V be a symplectic vector space and $\Gamma\subset\operatorname{Sp}(V)$ be a finite subgroup. Then V/Γ is a conical symplectic singularity with $d=2$.

Example 2.2. Let $\mathfrak{g}$ be the Lie algebra of an algebraic group. Then $\mathfrak{g}^*$ is a Poisson algebraic variety whose symplectic leaves are coadjoint orbits. Assume that $\mathfrak{g}$ is semisimple. Let $\mathbb{O}$ be a nilpotent orbit in $\mathfrak{g}^*$. The algebra $\mathbb{C}[\mathbb{O}]$ is finitely generated, in fact, it is the normalization of $\mathbb{C}[\overline{\mathbb{O}}]$. The variety $X:=\operatorname{Spec}(\mathbb{C}[\mathbb{O}])$ is a conical symplectic singularity with $d=1$.

We will need the definition and some properties of $\mathbb{Q}$-factorial terminalizations. Let $\tilde{X}$ be a normal algebraic variety. Recall that $\tilde{X}$ is called $\mathbb{Q}$-*factorial* if for any Weil divisor a nonzero integral multiple is Cartier.

Proposition 2.3 (Proposition 2.1 in [Lo10])**.** *Let X have symplectic singularities. Then there is a birational projective morphism $\rho:\tilde{X}\to X$, where $\tilde{X}$ has the following properties:*

(a) *$\tilde{X}$ is an irreducible, normal, Poisson variety (and hence has symplectic singularities).*
(b) *$\tilde{X}$ is $\mathbb{Q}$-factorial.*
(c) *$\tilde{X}$ has terminal singularities.*
(d) $\operatorname{codim}_{\tilde{X}}\tilde{X}^{sing}\geqslant 4$.
(e) *If X is, in addition, conical, then $\tilde{X}$ admits a $\mathbb{C}^\times$-action such that ρ is $\mathbb{C}^\times$-equivariant.*

Below we will say that $\tilde{X}$ is a $\mathbb{Q}$-*factorial terminalization* (or $\mathbb{Q}$-*terminalization*) of X.

Example 2.4. Consider the situation of Example 2.1. Suppose, first, that Γ does not contain *symplectic reflections*, i.e., elements γ with $\operatorname{rk}(\gamma-\operatorname{id})=2$. Then X itself satisfies

properties (a)-(e). An opposite extreme is when X has a symplectic resolution of singularities. This happens, for example, when Γ is a so called *wreath-product* group $\Gamma = S_n \ltimes \Gamma_1^n$, where Γ_1 is a finite subgroup of $\mathrm{SL}_2(\mathbb{C})$ and $V = \mathbb{C}^{2n}$.

$\mathbb{Q}$-terminalizations of the varieties $\mathrm{Spec}(\mathbb{C}[\mathbb{O}])$ will be considered in more detail in Section 4.3.

To finish let us provide an important property of a terminalization $\tilde{X}$.

Proposition 2.5. *If X is affine, then $\mathbb{C}[\tilde{X}^{reg}] = \mathbb{C}[\tilde{X}] = \mathbb{C}[X]$, $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for all $i > 0$, and $H^i(\tilde{X}^{reg}, \mathcal{O}_{\tilde{X}}) = 0$ for $i = 1, 2$.*

Proof. While this is classical (see, e.g., the proof of [N1, Lemma 12]), we provide a proof for reader's convenience.

Let ι denote the inclusion $\tilde{X}^{reg} \hookrightarrow \tilde{X}$. Since $\tilde{X}, X$ are singular symplectic, (a) of Proposition 2.3, they have rational singularities, see [Be, Proposition 2.3]. In particular, they are Cohen-Macaulay. Thanks to this, we have $\iota_* \mathcal{O}_{\tilde{X}^{reg}} = \mathcal{O}_{\tilde{X}}, R^i \iota_* \mathcal{O}_{\tilde{X}^{reg}}$ for $0 < i < \mathrm{codim}_{\tilde{X}} \tilde{X}^{sing} - 1$, i.e., by (d) of Proposition 2.3, for $i = 1, 2$. This reduces the proof to showing that $\mathbb{C}[\tilde{X}] = \mathbb{C}[X]$ and $H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) = 0$ for $i > 0$. These two claims follow because $\tilde{X} \to X$ is proper and birational and X has rational singularities. $\square$

2.2. **Filtered Poisson deformations.** Let X' be a Poisson variety.

We are going to recall three notions of Poisson deformations of X'.

We can talk about formal Poisson deformations of X'. Let $\widehat{B}$ be a quotient of $\mathbb{C}[[\hbar_1, \ldots, \hbar_k]]$ for some $k > 0$. By definition, a *formal Poisson deformation* of X' over $\widehat{B}$ a pair $(X'_{\widehat{B}}, \iota)$, where

- $X'_{\widehat{B}}$ is a flat formal Poisson scheme over the formal spectrum $\mathrm{Specf}(\widehat{B})$
- and ι is an isomorphism $X'_{\widehat{B}} \times_{\mathrm{Specf}(\widehat{B})} \mathrm{pt} \xrightarrow{\sim} X'$ of Poisson schemes.

An isomorphism of deformations $(X'^1_{\widehat{B}}, \iota^1), (X'^2_{\widehat{B}}, \iota^2)$ is an isomorphism of formal Poisson schemes $X'^1_{\widehat{B}} \xrightarrow{\sim} X'^2_{\widehat{B}}$ over $\mathrm{Specf}(\widehat{B})$ such that the induced isomorphism $X'^1_{\widehat{B}} \times_{\mathrm{Specf}(\widehat{B})} \mathrm{pt} \xrightarrow{\sim} X'^2_{\widehat{B}} \times_{\mathrm{Specf}(\widehat{B})} \mathrm{pt}$ intertwines ι^1 with ι^2.

Now suppose that X' is equipped with a $\mathbb{C}^\times$-action such that the Poisson bracket has degree $-d$ for some $d \in \mathbb{Z}_{>0}$. Let $\widehat{B}$ be the completion at 0 of a positively graded algebra. In this situation we can talk about *graded* formal Poisson deformation of X'. By definition, this is a formal deformation $(X'_{\widehat{B}}, \iota)$ together with an algebraic action of $\mathbb{C}^\times$ on $X'_{\widehat{B}}$ subject to the following properties:

(i) The action rescales the Poisson bracket by $t \mapsto t^{-d}$.
(ii) It is compatible with the action on $\widehat{B}$.
(iii) And ι is $\mathbb{C}^\times$-equivariant.

An isomorphism of graded formal deformations is defined similarly to that of formal deformations.

Let B be a positively graded algebra. We can introduce the notion of a *graded Poisson deformation* X'_B of X' similarly to the notion of a graded formal deformation. The difference is that now X'_B is an actual scheme over $\mathrm{Spec}(B)$. We note that from a graded deformation one can naturally get a graded formal deformation: replacing B with its completion at 0 and X'_B by the formal neighborhood of X' in X'_B to be denoted by $X'^{\wedge}_B$.

Proposition 2.6. *Let $\tilde{X}$ be a $\mathbb{Q}$-factorial terminalization of a conical symplectic singularity. Set $\mathfrak{P} := H^2(\tilde{X}^{reg}, \mathbb{C})$. Then there is a universal graded deformation $\tilde{X}_{\mathfrak{P}}$ of $\tilde{X}$ over $\mathfrak{P}$. The universal property is understood as follows: for any graded Poisson deformation $\tilde{X}_B$ over $\operatorname{Spec}(B)$ there is a unique $\mathbb{C}^\times$-equivariant morphism $\operatorname{Spec}(B) \to \mathfrak{P}$ and an isomorphism of graded deformations $\operatorname{Spec}(B) \times_{\mathfrak{P}} \tilde{X}_{\mathfrak{P}} \xrightarrow{\sim} \tilde{X}_B$ (which is not required to be unique).*

Proof. The proof is in several steps and is based on [N1].

Step 1. Namikawa proved that the Poisson deformation functor of $\tilde{X}$ is pro-representable, [N1, Theorem 14] and unobstructed [N1, Corollary 15]. This means that there is a universal formal deformation of $\tilde{X}$. This is the direct limit of the schemes denoted by X_n^{univ} in the discussion before [N1, Lemma 20]. Since the Poisson deformation functor is unobstructed, the base of the universal deformation is the formal neighborhood of 0 in the space of deformations of $\tilde{X}$ over the dual numbers. By [N1, Proposition 13], this space coincides with the similarly defined space for $\tilde{X}^{reg}$. By [N1, Lemma 12], the latter space coincides with $\mathfrak{P} = H^2(\tilde{X}^{reg}, \mathbb{C})$. We write $\mathfrak{P}^\wedge$ for the formal neighborhood of 0 in $\mathfrak{P}$ and $\tilde{X}^\wedge_{\mathfrak{P}}$ for the universal formal deformation of $\tilde{X}$.

Step 2. By [N1, Lemma 20], $\tilde{X}^\wedge_{\mathfrak{P}}$ is a graded formal deformation over $\mathfrak{P}^\wedge$. We claim that it is universal. Let $\widehat{B}$ be the completion of a positively graded algebra at 0 and $\tilde{X}_{\widehat{B}}$ be a graded formal deformation of $\tilde{X}$. By Step 1, we have a unique morphism $\operatorname{Specf}(\widehat{B}) \to \mathfrak{P}^\wedge$ and an isomorphism of formal Poisson deformations $\tilde{X}_{\widehat{B}} \xrightarrow{\sim} \tilde{X}^1_{\widehat{B}} := \operatorname{Specf}(\widehat{B}) \times_{\mathfrak{P}^\wedge} \tilde{X}^\wedge_{\mathfrak{P}}$. By the uniqueness of the former, $\operatorname{Specf}(\widehat{B}) \to \mathfrak{P}^\wedge$ is $\mathbb{C}^\times$-equivariant. So $\tilde{X}^1_{\widehat{B}}$ becomes a graded formal deformation. We need to prove that we can choose a $\mathbb{C}^\times$-equivariant Poisson isomorphism $\tilde{X}_{\widehat{B}} \xrightarrow{\sim} \tilde{X}^1_{\widehat{B}}$. The proof is standard but we provide it for reader's convenience.

Let $\mathfrak{m}$ denote the maximal ideal in $\widehat{B}$. Set $B_k := \widehat{B}/\mathfrak{m}^k$. We write $\tilde{X}_k, \tilde{X}^1_k$ for the base changes of $\tilde{X}_{\widehat{B}}, \tilde{X}^1_{\widehat{B}}$ to $\operatorname{Spec}(B_k)$. Note that for $k = 1$, both of these schemes are just $\tilde{X}$. It is enough to show that, for all $k > 0$, any $\mathbb{C}^\times$-equivariant isomorphism of Poisson deformations $\varphi_k : \tilde{X}_k \xrightarrow{\sim} \tilde{X}^1_k$ can be lifted to a $\mathbb{C}^\times$-equivariant isomorphism of deformations $\tilde{X}_{k+1} \xrightarrow{\sim} \tilde{X}^1_{k+1}$.

Note that a Poisson deformation of an arbitrary Poisson variety X' over an Artinian base B gives a sheaf of Poisson B-algebras on X' that deforms the structure sheaf: we take the structure sheaf of X'_B and sheaf theoretically pull it back to X'. We get a deformation of $\mathcal{O}_{X'}$ over B. Conversely, from such a sheaf, say $\mathcal{O}_{X',B}$, we can get a deformation of X' over $\operatorname{Spec}(B)$: we cover X' with open affine subsets, $X' = \bigcup U_i$, and glue the affine schemes $\operatorname{Spec}(\Gamma(U_i, \mathcal{O}_{X',B}))$ along their intersections $\operatorname{Spec}(\Gamma(U_i \cap U_j, \mathcal{O}_{X',B}))$. We get a Poisson deformation X'_B over B.

Denote the sheaves on $\tilde{X}$ corresponding to $\tilde{X}_{k+1}, \tilde{X}^1_{k+1}$ by $\mathcal{O}_{k+1}, \mathcal{O}^1_{k+1}$. We have a $\mathbb{C}^\times$-equivariant isomorphism of sheaves of Poisson algebras $\varphi_k : \mathcal{O}_k \xrightarrow{\sim} \mathcal{O}^1_k$. We know that there is a lift to an isomorphism $\mathcal{O}_{k+1} \xrightarrow{\sim} \mathcal{O}^1_{k+1}$. What we need to show that it can be chosen to be $\mathbb{C}^\times$-equivariant. Let $\mathsf{A}_{\tilde{X}}$ denote the set of lifts. This is an affine space with associated vector space $\operatorname{PDer}(\mathcal{O}_{\tilde{X}}) \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1}$, where PDer stands for the space of Poisson derivations. We need to show that $\mathbb{C}^\times$ has a fixed point on $\mathsf{A}_{\tilde{X}}$. Note that any affine space admits a canonical embedding into a vector space equipped with a linear function, say t, the image of the embedding is the level set $t = 1$. Let $\mathsf{V}_{\tilde{X}}$ be this space

for $\mathsf{A}_{\tilde{X}}$. It embeds as a subspace in the space of B_{k+1}-linear homomorphisms of sheaves $\mathrm{Hom}_{B_{k+1}}(\tilde{\mathcal{O}}_{k+1}, \tilde{\mathcal{O}}^1_{k+1})$ as follows. For $z \neq 0$, the level set $t = z$ embeds as the subset of all maps ψ such that $z^{-1}\psi$ is an algebra homomorphism giving φ_k after base change to B_k. The level set $t = 0$ embeds as $\mathrm{PDer}(\mathcal{O}_{\tilde{X}}) \otimes \mathfrak{m}^k/\mathfrak{m}^{k+1}$ (these maps vanish on $\mathfrak{m}\mathcal{O}_{k+1}$ and have image in $\mathfrak{m}^k\mathcal{O}^1_{k+1}$). We need to show that the action of $\mathbb{C}^\times$ on $\mathsf{V}_{\tilde{X}}$ is rational (i.e. comes from a grading), the existence of a fixed point in $\mathsf{A}_{\tilde{X}}$ will follow.

By a result of Sumihiro, [S], we can cover $\tilde{X}$ with open affine $\mathbb{C}^\times$-stable subsets $U_i, i = 1, \ldots, m$. Restricting sheaves $\mathcal{O}_{k+1}, \mathcal{O}^1_{k+1}$ to U_i we get graded Poisson flat B_{k+1}-algebras $\mathcal{O}_{k+1}(U_i), \mathcal{O}^1_{k+1}(U_i)$. We can consider the subspaces $\mathsf{V}_{U_i} \subset \mathrm{Hom}_{B_{k+1}}(\mathcal{O}_{k+1}(U_i), \mathcal{O}^1_{k+1}(U_i))$ defined similarly to $\mathsf{V}_{\tilde{X}}$. We have a $\mathbb{C}^\times$-equivariant embedding $\mathsf{V}_{\tilde{X}} \hookrightarrow \bigoplus_i \mathsf{V}_{U_i}$ by restriction. So it is enough to show that the $\mathbb{C}^\times$-action on V_{U_i} is rational.

The algebra $\mathcal{O}_{k+1}(U_i)$ is generated by homogeneous elements, say $f_1, \ldots, f_\ell$, of degrees $d_1, \ldots, d_\ell$. Note that every element of $\mathsf{V}_{U_i} \subset \mathrm{Hom}_{B_{k+1}}(\mathcal{O}_{k+1}(U_i), \mathcal{O}^1_{k+1}(U_i))$ is uniquely determined by the images of $f_1, \ldots, f_\ell$. This gives a $\mathbb{C}^\times$-equivariant embedding $\mathsf{V}_{U_i} \hookrightarrow \bigoplus_{j=1}^\ell \mathcal{O}^1_{k+1}(U_i)\langle -d_j\rangle$, where in the brackets we have a grading shift. The action of $\mathbb{C}^\times$ on the target of this embedding is manifestly rational. This finishes the proof of the claim that $\tilde{X}^\wedge_{\mathfrak{P}}$ is a universal graded formal deformation.

Step 3. By [N1, Lemma 22], the formal scheme $\tilde{X}^\wedge_{\mathfrak{P}}$ over $\mathfrak{P}^\wedge$ with an action of $\mathbb{C}^\times$ admits the algebraization: a graded Poisson deformation $\tilde{X}_{\mathfrak{P}}$ over $\mathfrak{P}$ such that the formal neighborhood of $\tilde{X}$ in $\tilde{X}_{\mathfrak{P}}$ is identified with $\tilde{X}^\wedge_{\mathfrak{P}}$. We claim that $\tilde{X}_{\mathfrak{P}}$ has the required universal property.

Indeed, let $\tilde{X}_B$ be another graded deformation. Then we have the graded formal deformation $\tilde{X}^\wedge_B$ over the completion $\widehat{B}$ of B. By Step 2, we have a unique $\mathbb{C}^\times$-equivariant morphism of formal schemes $\mathrm{Specf}(\widehat{B}) \to \mathfrak{P}^\wedge$ and an isomorphism of graded formal deformations

$$\mathrm{Specf}(\widehat{B}) \times_{\mathfrak{P}^\wedge} \tilde{X}^\wedge_{\mathfrak{P}} \xrightarrow{\sim} \tilde{X}^\wedge_B. \tag{2.1}$$

The morphism $\mathrm{Specf}(\widehat{B}) \to \mathfrak{P}^\wedge$ is induced by the unique $\mathbb{C}^\times$-equivariant morphism $\mathrm{Spec}(B) \to \mathfrak{P}$. The source of (2.1) is the formal neighborhood of $\tilde{X}$ in $\mathrm{Spec}(B) \times_{\mathfrak{P}} \tilde{X}_{\mathfrak{P}}$. The proof of [N1, Proposition A.5] shows that the algebraizations $\mathrm{Spec}(B) \times_{\mathfrak{P}} \tilde{X}_{\mathfrak{P}}, \tilde{X}_B$ are uniquely recovered from their formal neighborhoods of $\tilde{X}$ together with the $\mathbb{C}^\times$-action. This shows that (2.1) comes from a unique isomorphism of graded Poisson deformations $\mathrm{Spec}(B) \times_{\mathfrak{P}} \tilde{X}_{\mathfrak{P}} \xrightarrow{\sim} \tilde{X}_B$. □

Remark 2.7. In this remark we relate results of [N1] to those of [KV]. Kaledin and Verbitsky, [KV, Theorem 3.6], show that for a <u>smooth</u> symplectic variety X' with $H^i(X', \mathcal{O}_{X'}) = 0$ for $i = 1, 2$, there is a universal formal deformation with base being the formal neighborhood of 0 in $H^2(X', \mathbb{C})$. The vanishing condition above holds for $X' = \tilde{X}^{reg}$ thanks to Proposition 2.5.

Using the construction relating Poisson deformations over Artinian bases to sheaves (Step 2 of the proof of Proposition 2.6) we can relate the Poisson deformations of $\tilde{X}$ to those $\tilde{X}^{reg}$. Namely, given a deformation $\tilde{X}_B$ over an Artinian base $\mathrm{Spec}(B)$, we form the corresponding sheaf $\mathcal{O}_{\tilde{X},B}$, restrict it to $\tilde{X}^{reg}$ via the sheaf-theoretic pullback and take the corresponding deformation of $\tilde{X}^{reg}$. To get back we apply the pushforward for $\iota : \tilde{X}^{reg} \to \tilde{X}$. As we have pointed out in the proof of Proposition 2.5, $R^1\iota_*\mathcal{O}_{\tilde{X}^{reg}} = 0$. Any deformation of $\mathcal{O}_{\tilde{X}^{reg}}$ over B has a filtration with successive quotients $\mathcal{O}_{\tilde{X}^{reg}}$. An

easy argument using the long exact sequence in cohomology shows that ι_* sends any deformation of $\mathcal{O}_{\tilde{X}^{reg}}$ to a deformation of $\mathcal{O}_{\tilde{X}}$ over B. So we get a bijection between the deformations of $\mathcal{O}_{\tilde{X}^{reg}}, \mathcal{O}_{\tilde{X}}$ over B.

It follows that the Poisson deformation functors for $\tilde{X}$ and $\tilde{X}^{reg}$ are isomorphic. In particular, the existence of the universal formal deformation for $\tilde{X}$ implies that for $\tilde{X}^{reg}$ and vice versa.

We also note that in the setting of [KV] we can talk about graded formal deformations of X' if it comes with a $\mathbb{C}^\times$-action that rescales the Poisson bracket by $t \mapsto t^{-d}$. Similarly to Step 2 of Proposition 2.6, the universal deformation over the formal neighborhood of 0 in $H^2(X', \mathbb{C})$ is a universal graded formal deformation.

2.3. **Deformations from $\mathbb{Q}$-terminalizations.** Let A be a positively graded Poisson algebra with $\deg\{\cdot,\cdot\} = -d$. For example, we can take $A = \mathbb{C}[X]$, where X is a conical symplectic singularity.

By a *filtered Poisson deformation* of A we mean a pair $(\mathcal{A}^0, \iota)$, where

- $\mathcal{A}^0$ is a $\mathbb{Z}_{\geqslant 0}$-filtered Poisson algebra, $\mathcal{A}^0 = \bigcup_{i\in\mathbb{Z}} \mathcal{A}^0_{\leqslant i}$, such that $\{\mathcal{A}^0_{\leqslant i}, \mathcal{A}^0_{\leqslant j}\} \subset \mathcal{A}^0_{\leqslant i+j-d}$,
- and ι is an isomorphism $\operatorname{gr}\mathcal{A}^0 \xrightarrow{\sim} A$ of graded Poisson algebras.

We note that to give a filtered Poisson deformation of A is the same as to give a graded Poisson deformation of $\operatorname{Spec}(A)$ over $\operatorname{Spec}(\mathbb{C}[\hbar])$, where $\hbar$ has degree 1. Indeed, starting with $\mathcal{A}^0$ we can form its Rees algebra $R_\hbar(\mathcal{A}^0)$ and set $X_\hbar := \operatorname{Spec}(R_\hbar(\mathcal{A}^0))$. The isomorphism ι gives rise to an isomorphism $\operatorname{pt} \times_{\operatorname{Spec}(\mathbb{C}[\hbar])} X_\hbar \xrightarrow{\sim} \operatorname{Spec}(A)$ that makes $X_\hbar$ into a graded Poisson deformation of $\operatorname{Spec}(A)$. Conversely, starting with a graded Poisson deformation $X_\hbar$ we set $\mathcal{A}^0 := \mathbb{C}[X_\hbar]/(\hbar - 1)$ getting a filtered Poisson deformation of A.

We proceed to a construction of filtered Poisson deformations of $\mathbb{C}[X]$. Let $\tilde{X}$ stand for a $\mathbb{Q}$-factorial terminalization of X. We set $\mathfrak{P} := H^2(\tilde{X}^{reg}, \mathbb{C})$ and let $\tilde{X}_{\mathfrak{P}}$ be the universal graded deformation of $\tilde{X}$, see Proposition 2.6. Our primary goal in this section is to construct filtered Poisson deformations of $\mathbb{C}[X]$ out of $\tilde{X}_{\mathfrak{P}}$.

First, we are going to describe the structure of the space $\mathfrak{P}$ following [N4] and [N3].

Kaledin, [K1, Theorem 2.3], proved that X has finitely many symplectic leaves. Moreover, he proved that for any leaf $\mathcal{L} \subset X$, there is a formal slice to $\mathcal{L}$ in X: if we pick a point $x \in \mathcal{L}$ and let $X^{\wedge_x}, \mathcal{L}^{\wedge_x}$ denote the formal neighborhoods of x in X and $\mathcal{L}$, respectively, then there is an affine formal Poisson scheme Σ' with $X^{\wedge_x} \cong \mathcal{L}^{\wedge_x} \times \Sigma'$.

Let $\mathcal{L}_1, \ldots, \mathcal{L}_k$ be the codimension 2 symplectic leaves of X. The formal slice Σ'_i to $\mathcal{L}_i$ in X is an ADE type Kleinian singularity $(\mathbb{C}^2)^{\wedge_0}/\Gamma_i$ (here $\bullet^{\wedge_0}$ stands for the formal neighborhood at 0) so we can consider the Weyl group $\widehat{W}_i, i = 1, \ldots, k$ of that ADE type. We write Σ_i for $\mathbb{C}^2/\Gamma_i$, $\tilde{\Sigma}_i$ for its minimal resolution and set $\widehat{\mathfrak{P}}_i := H^2(\tilde{\Sigma}_i, \mathbb{C})$. The space $\widehat{\mathfrak{P}}_i$ is the Cartan space for $\widehat{W}_i$.

Choose a point $x \in \mathcal{L}_i$. The preimage of x in $\tilde{X}$ is the same as the preimage of $0 \in \Sigma_i$ in $\tilde{\Sigma}_i$, see the discussion in the beginning of [N3, Section 1]. So this preimage is the union of $\mathbb{P}^1$'s intersecting according to the Dynkin diagram of the root system associated to $\widehat{W}_i$. The fundamental group $\pi_1(\mathcal{L}_i)(= \pi_1(\mathcal{L}_i, x))$ acts on the preimage via the monodromy action. Hence it acts on $\widehat{\mathfrak{P}}_i, \widehat{W}_i$ by diagram automorphisms. Let $\mathfrak{P}_i \subset \widehat{\mathfrak{P}}_i, W_i \subset \widehat{W}_i$ denote the fixed points. Following Namikawa, see the discussion in the beginning of [N3, Section 1], define the Namikawa-Weyl group as $W(= W_X) := \prod_{i=1}^k W_i$.

Lemma 2.8. *We have* $\mathfrak{P} = H^2(X^{reg}, \mathbb{C}) \oplus \bigoplus_{i=1}^k \mathfrak{P}_i$.

Proof. Let X^1 denote the complement in X to the union of all symplectic leaves with codimension 4 or higher. So $X^1 = X^{reg} \sqcup \bigsqcup_{i=1}^k \mathcal{L}_i$. According to [N4, Proposition 4.2], $\mathfrak{P} = H^2(X^1, \mathbb{C}) \oplus \bigoplus_{i=1}^k \mathfrak{P}_i$. What we need to do therefore is to show that $H^2(X^1, \mathbb{C}) = H^2(X^{reg}, \mathbb{C})$.

Pick a point $x \in \mathcal{L}_i$. Let $\Sigma \subset X$ be a slice to $\mathcal{L}_i$ through x in the complex analytic category. Using Hamiltonian flows for suitable analytic functions in a neighborhood of x (and shrinking Σ if necessary) we can identify a complex analytic neighborhood of x with $B \times \Sigma$, where B is a neighborhood of x in $\mathcal{L}_i$. Since X is normal, Σ is also normal as a complex analytic space. It is 2-dimensional so it must be a Kleinian singularity of the same type as Σ_i.

Pick tubular neighborhoods $Y_1, \ldots, Y_k$ of $\mathcal{L}_1, \ldots, \mathcal{L}_k$ and set $Y_i^\times := Y_i \setminus \mathcal{L}_i$. So, by the previous paragraph, topologically Y_i is a locally trivial fibration over $\mathcal{L}_i$ with fiber D/Γ_i, where D is an open unit ball in $\mathbb{C}^2$. In particular, $H^j(Y_i, \mathbb{C}) = H^j(\mathcal{L}_i, \mathbb{C})$ for all j. Also note that Γ_i acts freely on $D^\times$. It follows that $H^j(D^\times/\Gamma_i, \mathbb{C}) = H^j(\mathbb{S}^3, \mathbb{C})^{\Gamma_i}$, in particular, this homology group is zero when $j = 1, 2$. It follows that $H^j(Y_i^\times, \mathbb{C}) = H^j(\mathcal{L}_i, \mathbb{C})$ for $j = 1, 2$. Hence the pullback homomorphism $H^j(Y_i, \mathbb{C}) \to H^j(Y_i^\times, \mathbb{C})$ is an isomorphism for $j = 1, 2$. The Mayer-Vietoris exact sequence for the covering $X^1 = X^{reg} \cup (\bigsqcup Y_i)$ gives

$$H^1(X^{reg}, \mathbb{C}) \oplus \bigoplus_{i=1}^k H^1(Y_i, \mathbb{C}) \to \bigoplus_{i=1}^k H^1(Y_i^\times, \mathbb{C}) \to H^2(X^1, \mathbb{C})$$
$$\to H^2(X^{reg}, \mathbb{C}) \oplus \bigoplus_{i=1}^k H^2(Y_i, \mathbb{C}) \to \bigoplus_{i=1}^k H^2(Y_i^\times, \mathbb{C}).$$

By the argument above in this paragraph the first and the last map restrict to isomorphisms $\bigoplus_{i=1}^k H^j(Y_i, \mathbb{C}) \to \bigoplus_{i=1}^j H^1(Y_i^\times, \mathbb{C})$. It follows that the pull-back map $H^2(X^1, \mathbb{C}) \to H^2(X^{reg}, \mathbb{C})$ is an isomorphism. □

Let us proceed to producing filtered Poisson deformations of $\mathbb{C}[X]$ from $\tilde{X}_{\mathfrak{P}}$.

Proposition 2.9. *The following is true:*

1. *The algebra* $\mathbb{C}[\tilde{X}_{\mathfrak{P}}]$ *is finitely generated and free as a module over* $\mathbb{C}[\mathfrak{P}]$. *Moreover, we have* $\mathbb{C}[\tilde{X}_{\mathfrak{P}}]/(\mathfrak{P}) = \mathbb{C}[X]$.
2. *The group* W *acts on* $\mathbb{C}[\tilde{X}_{\mathfrak{P}}]$ *by graded Poisson algebra automorphisms and the action is compatible with that on* $\mathfrak{P}$. *The induced action on* $\mathbb{C}[X]$ *is trivial.*

Proof. Let $\mathcal{X}$ be the flat deformation of X over $\mathfrak{P}/W$ established in [N4], before Theorem 5.5 there. Combining [N4, (14)] with [N3, Theorem 1.1] we get the following commutative diagram

$$\begin{array}{ccc} \tilde{X}_{\mathfrak{P}} & \longrightarrow & \mathcal{X} \\ \downarrow & & \downarrow \\ \mathfrak{P} & \longrightarrow & \mathfrak{P}/W \end{array}$$

Consider the induced morphism

$$\tilde{X}_{\mathfrak{P}} \to \mathfrak{P} \times_{\mathfrak{P}/W} \mathcal{X}. \tag{2.2}$$

It is projective by the construction. We claim that

(*) the target is normal and (2.2) is birational.

This claim implies that $\mathbb{C}[\tilde{X}_{\mathfrak{P}}] \xrightarrow{\sim} \mathbb{C}[\mathfrak{P} \times_{\mathfrak{P}/W} \mathcal{X}]$. This, in turn, implies (2). Also note that $\mathcal{X}$ is flat over $\mathfrak{P}/W$ by the construction in [N4, (5.4)] and carries a $\mathbb{C}^\times$-action compatible with that on $\mathfrak{P}/W$ because $\mathcal{X}$ is obtained by algebraization of a formal deformation. So $\mathbb{C}[\mathcal{X}]$ is a flat graded module over $\mathbb{C}[\mathfrak{P}]^W$. Hence it is free. So (*) implies (1) as well.

Let us prove that $\mathfrak{P} \times_{\mathfrak{P}/W} \mathcal{X}$ is normal. As we have seen in the proof of Proposition 2.5, the variety X is Cohen-Macaulay. Since $\mathfrak{P} \times_{\mathfrak{P}/W} \mathcal{X}$ is flat over $\mathfrak{P}$, deforms X and comes with a contracting $\mathbb{C}^\times$-action, the variety $\mathcal{X}$ is also Cohen-Macaulay. To show that it is normal we need to check that it is smooth outside of a codimension 2 locus. Equivalently, we need to show that the morphism $\mathfrak{P} \times_{\mathfrak{P}/W} \mathcal{X} \to \mathfrak{P}$ is smooth outside of a codimension 2 locus. The variety X is normal hence smooth outside of a codimension 2 locus. Thanks to the contracting $\mathbb{C}^\times$-action the same is true for the other fibers of $\mathfrak{P} \times_{\mathfrak{P}/W} \mathcal{X} \to \mathfrak{P}$. So the variety $\mathfrak{P} \times_{\mathfrak{P}/W} \mathcal{X}$ is smooth outside of a codim 2 locus. Hence it is normal.

Now we prove that (2.2) is birational. For a Zariski generic point $\lambda \in \mathfrak{P}$ the induced morphism of fibers $\tilde{X}_\lambda \to \mathcal{X}_{W\lambda}$ is an isomorphism, see [N4, Theorem 5.5,(c)]. This shows (2.2) is birational and completes the proof. □

Let us write $X_{\mathfrak{P}}$ for $\operatorname{Spec}(\mathbb{C}[\tilde{X}_{\mathfrak{P}}])$ (i.e., $\mathfrak{P} \times_{\mathfrak{P}/W} \mathcal{X}$ from the previous proof) and X_λ for the fiber of $X_{\mathfrak{P}} \to \mathfrak{P}$ over λ.

Now let us examine the situation when $X_\lambda, X_{\lambda'}$ give isomorphic filtered Poisson deformations.

Proposition 2.10. *We have $\mathbb{C}[X_\lambda] \cong \mathbb{C}[X_{\lambda'}]$ as filtered Poisson deformations of $\mathbb{C}[X]$ if and only if $\lambda' \in W\lambda$.*

This follows from results of [N3] (but also can be proved independently). We postpone the proof until the next section. Note also that our present setting is different from that of Proposition 2.6: there we considered varieties that are $\mathbb{Q}$-factorial terminal and Poisson deformations were parameterized by $H^2(?, \mathbb{C})$. Here X is, generally, not of this form.

Remark 2.11. Recall the decomposition $\mathfrak{P} = H^2(X^{reg}, \mathbb{C}) \oplus \bigoplus_{i=1}^k \mathfrak{P}_i$. We can write a deformation parameter λ as $(\lambda_0, \lambda_1, \ldots, \lambda_k)$ according to this decomposition. Then $\lambda_0 \in H^2(X^{reg}, \mathbb{C})$ controls the deformation of X^{reg}, while λ_i controls the deformation of the slice $\Sigma_i, i = 1, \ldots, k$ induced by the deformation of X. We will not use this claim in what follows.

2.4. **Universal deformation of X.** In [N4, Theorem 5.1], Namikawa has proved that the Poisson deformation functor of X is unobstructed. Here is variant of his result for graded Poisson deformations.

Proposition 2.12. *Recall that X is a conical symplectic singularity. There is a positively graded polynomial algebra B and a graded Poisson B-algebra A such that $\mathbb{C} \otimes_B A \xrightarrow{\sim} \mathbb{C}[X]$ with the following universal property:*

- *For any finitely generated positively graded algebra B' and a graded Poisson B'-algebra A' such that $\mathbb{C} \otimes_{B'} A' \xrightarrow{\sim} \mathbb{C}[X]$ there is a unique graded algebra homomorphism $B \to B'$ and a B'-linear Poisson graded algebra isomorphism $B' \otimes_B A \xrightarrow{\sim} A'$ intertwining the isomorphisms $\mathbb{C} \otimes_{B'} A' \xrightarrow{\sim} \mathbb{C}[X] \xrightarrow{\sim} \mathbb{C} \otimes_B A$.*

In the notation of [N4, (14)] *we have $B = \mathbb{C}[x_1, \ldots, x_d]$ and $A = \mathbb{C}[\mathcal{X}]$.*

Proof. By [N4, Theorem 5.1], the formal neighborhood of X in $\mathcal{X}$ is the universal formal Poisson deformation of X. Arguing as in Step 2 of the proof of Proposition 2.6, we see that this formal neighborhood is a universal graded formal Poisson deformation. Then an easy analog of Step 3 of that proof shows that $\mathcal{X}$ is a universal graded Poisson deformation. Translating to the language of algebras, we get the claim of this proposition. □

Corollary 2.13. *We have $B = \mathbb{C}[\mathfrak{P}]^W$ and $A = \mathbb{C}[X_{\mathfrak{P}}]^W$.*

Proof. By [N3, Theorem 1.1], $B = \mathbb{C}[\mathfrak{P}]^W$. (2.2) shows that $\mathbb{C}[\mathcal{X}]$ is $\mathbb{C}[X_{\mathfrak{P}}]^W$. □

Using this corollary we can give a proof of Proposition 2.10.

Proof of Proposition 2.10. Recall, see the discussion in the beginning of Section 2.3, that the filtered deformations of $\mathbb{C}[X]$ are in one-to-one correspondence with graded deformations of X over $\mathbb{C}[\hbar]$. In particular, $\mathbb{C}[X_\lambda]$ corresponds to the graded deformation $\mathbb{C} \times_{\mathfrak{P}/W} X_{\mathfrak{P}}/W$, where the morphism $\mathbb{C} \to \mathfrak{P}/W$ is given by $z \mapsto W(\lambda z^d)$. Thanks to the universal property of $X_{\mathfrak{P}}/W$, Proposition 2.12, Corollary 2.13, we see that an isomorphism $\mathbb{C}[X_\lambda] \cong \mathbb{C}[X_{\lambda'}]$ of filtered Poisson deformation is equivalent to the coincidence of the morphisms $\mathbb{C} \to \mathfrak{P}/W$ coming from λ and λ', which, in turn, is equivalent to $\lambda' \in W\lambda$. □

2.5. **Poisson derivations.** Here is the main result of this section.

Proposition 2.14. *Let, as before, X be a conical symplectic singularity. Then all Poisson derivations of $\mathbb{C}[X]$ are Hamiltonian.*

The crucial step in the proof is the following lemma (that is classical for symplectic resolutions) whose proof in the generality we need was communicated to me by Kaledin.

Lemma 2.15. *We have $H^1(\tilde{X}^{reg}, \mathbb{C}) = 0$.*

Proof. Let us write $\mathcal{O}^{an}$ for the sheaves of holomorphic functions on $\tilde{X}$ and its open subvarieties. We have the exponential exact sequence

$$H^0(\tilde{X}^{reg}, \mathcal{O}^{an}) \to H^0(\tilde{X}^{reg}, (\mathcal{O}^{an})^\times) \to H^1(\tilde{X}^{reg}, \mathbb{C}) \to H^1(\tilde{X}^{reg}, \mathcal{O}^{an})$$

First of all, we claim that $H^1(\tilde{X}^{reg}, \mathcal{O}^{an}) = 0$. Indeed, we have an exact sequence

$$H^1(\tilde{X}, \mathcal{O}^{an}) \to H^1(\tilde{X}^{reg}, \mathcal{O}^{an}) \to H^2_{\tilde{X}^{sing}}(\tilde{X}, \mathcal{O}^{an})$$

As in the algebraic situation, the first and the third terms are zero (see, e.g., [Ma1, Appendix]) and so $H^1(\tilde{X}^{reg}, \mathcal{O}^{an})$ is zero. So we get an exact sequence

$$H^0(\tilde{X}^{reg}, \mathcal{O}^{an}) \to H^0(\tilde{X}^{reg}, (\mathcal{O}^{an})^\times) \to H^1(\tilde{X}^{reg}, \mathbb{C}) \to 0. \tag{2.3}$$

Note that, by the Hartogs theorem, we have

$$H^0(\tilde{X}^{reg}, \mathcal{O}^{an}) \cong H^0(\tilde{X}, \mathcal{O}^{an}), H^0(\tilde{X}^{reg}, (\mathcal{O}^{an})^\times) \cong H^0(\tilde{X}, (\mathcal{O}^{an})^\times).$$

From the analog of (2.3) for $\tilde{X}$, we conclude that $H^1(\tilde{X}^{reg}, \mathbb{C}) \cong H^1(\tilde{X}, \mathbb{C})$. So we need to show that the latter space vanishes.

Let $\pi : \overline{X} \to \tilde{X}$ be a resolution of singularities. By [K2, Corollary 1.5], we have $R^1(\rho \circ \pi)_* \mathbb{C}_{\overline{X}} = 0$ (recall that ρ stands for the morphism $\tilde{X} \to X$) and $R^1\pi_* \mathbb{C}_{\overline{X}} = 0$. Since, obviously, $\pi_* \mathbb{C}_{\overline{X}} = \mathbb{C}_{\tilde{X}}$, we use the standard spectral sequence for the composition of derived functors to check that $R^1\rho_* \mathbb{C}_{\tilde{X}} = 0$. Since X is conical (and hence contractible), this implies that $H^1(\tilde{X}, \mathbb{C}) = 0$. □

Proof of Proposition 2.14. We will show that all Poisson vector fields on X lift to $\tilde{X}^{reg}$, then our claim will follow from Lemma 2.15.

Recall the open subvariety $X^1 \subset X$ from the proof of Lemma 2.8. Let $\tilde{X}^1$ be the preimage of X^1 in $\tilde{X}$. We claim that

$$\operatorname{codim}_{\tilde{X}} \tilde{X} \setminus \tilde{X}^1 \geqslant 2. \tag{2.4}$$

Assume the contrary. Let Z be an irreducible divisor inside $\tilde{X} \setminus \tilde{X}^1$. Let $D_1, \ldots, D_k$ denote the irreducible components of the preimage of $X^1 \setminus X^{reg}$ in $\tilde{X}$. Since $H^i(\tilde{X}^{reg}, \mathcal{O}^{an}) = 0$ for $i = 1, 2$, we see that $H^2(\tilde{X}^{reg}, \mathbb{C})$ coincides with the complexified analytic Picard group of $\tilde{X}^{reg}$. Let us write $[D_1], \ldots, [D_k], [Z]$ for the classes of the line bundles corresponding to these divisors in $H^2(\tilde{X}^{reg}, \mathbb{C})$. Recall, Lemma 2.8, that $H^2(\tilde{X}^{reg}, \mathbb{C}) = H^2(X^{reg}, \mathbb{C}) \oplus \operatorname{Span}([D_1], \ldots, [D_k])$. Since $[Z]$ projects to zero in $H^2(X^{reg}, \mathbb{C})$, it must lie in $\operatorname{Span}([D_1], \ldots, [D_k])$.

Thanks to [BCHM, Corollary 1.4.3], we can contract all divisors in $\tilde{X}$ but Z, we get a normal variety X' with a morphism $X' \to X$. We also have a rational map $\tilde{X} \to X'$ that is defined outside of a codimension 2 locus in $\tilde{X}$ because $\tilde{X}$ is proper over X. It is easy to see that Z defines a nontorsion class in the analytic Picard group of X'^{reg}. Indeed, if this class is torsion, then we have an analytic function on X' whose vanishing locus is exactly Z. It must be pulled back from an analytic function on X whose vanishing locus has complement of codimension at least 2 and is nonempty, which is impossible. Since Z defines a nontorsion class in the analytic Picard group of X'^{reg}, we apply pullback to $\tilde{X}$ to see that $[Z] \notin \operatorname{Span}([D_1], \ldots, [D_k])$, a contradiction. This shows (2.4).

Let ξ be a Poisson vector field on X, equivalently, on $\pi^{-1}(X^{reg}) \subset \tilde{X}^{reg}$. What we need to show is that ξ extends to a regular vector field on $\pi^{-1}(X^{\wedge_x})$, where x is a point in a codimension 2 leaf in X. This in turn will follow if we check that the restriction of ξ to $X^{\wedge_x}$ is Hamiltonian. But $X^{\wedge_x}$ is a symplectic quotient singularity. Note that any Poisson vector field on $(V/\Gamma)^{\wedge_0}$ lifts to a Γ-invariant Poisson vector field on $V^{\wedge_0}$. It follows that any Poisson vector field on $(V/\Gamma)^{\wedge_0}$ and hence on $X^{\wedge_x}$ is Hamiltonian. This completes the proof. □

3. Quantizations of symplectic singularities

3.1. **Quantizations and period maps.** This section is a quantum counterpart of Section 2.2.

Let A be a graded Poisson algebra with bracket of degree $-d$, where d is a positive integer. By a *filtered quantization* of A one means a pair $(\mathcal{A}, \iota)$, where

- $\mathcal{A}$ is a filtered associative algebra, $\mathcal{A} = \bigcup_{i \geqslant 0} \mathcal{A}_{\leqslant i}$, such that $[\mathcal{A}_{\leqslant i}, \mathcal{A}_{\leqslant j}] \subset \mathcal{A}_{\leqslant i+j-d}$,
- and ι is a graded Poisson algebra isomorphism $\operatorname{gr} \mathcal{A} \xrightarrow{\sim} A$.

By an isomorphism ψ of filtered quantizations $(\mathcal{A}, \iota), (\mathcal{A}', \iota')$ we mean a filtration preserving algebra isomorphism $\psi : \mathcal{A} \to \mathcal{A}'$ such that $\operatorname{gr} \psi$ intertwines ι, ι'.

Our goal is to classify the filtered quantizations of $\mathbb{C}[X]$, where X is a conical symplectic singularity. As with filtered Poisson deformations, we are going to produce quantizations of $\mathbb{C}[X]$ from those of $\tilde{X}$. We now explain what one means by a filtered quantization of $\tilde{X}$.

Let X' be a Poisson scheme equipped with a $\mathbb{C}^\times$-action that rescales the Poisson bracket by $t \mapsto t^{-d}$. We assume that every point in X' has a $\mathbb{C}^\times$-stable affine open neighborhood. By a result of Sumihiro, [S], this is the case when X' is normal. By a *filtered quantization*

$\mathcal{D}$ of X' we mean a sheaf of filtered associative algebras in the conical topology on X', where the filtration is complete and separated, together with an isomorphism $\iota : \operatorname{gr}\mathcal{D} \xrightarrow{\sim} \mathcal{O}_{X'}$ of graded Poisson algebras. We note that quantizations of $\mathbb{C}[X]$ are in a natural bijection with quantizations of X via taking global sections to get from X to $\mathbb{C}[X]$ and taking the microlocalization to get back, see the discussion on "Variations on formal microlocalization..." in [G, Section 1].

We can also talk about graded formal quantizations. These are pairs $(\mathcal{D}_\hbar, \iota)$. Here $\mathcal{D}_\hbar$ is a sheaf a $\mathbb{C}[[\hbar]]$-algebras in the Zariski topology on X that is flat over $\mathbb{C}[[\hbar]]$ and complete and separated in the $\hbar$-adic topology. We require that $[\mathcal{D}_\hbar, \mathcal{D}_\hbar] \subset \hbar^d \mathcal{D}_\hbar$, which gives rise to a Poisson bracket on $\mathcal{D}_\hbar/(\hbar)$. We also require that $\mathbb{C}^\times$ acts on $\mathcal{D}_\hbar$ by $\mathbb{C}$-algebra automorphisms with $t.\hbar = t\hbar$. For ι we take a $\mathbb{C}^\times$-equivariant Poisson isomorphism $\mathcal{D}_\hbar/(\hbar) \xrightarrow{\sim} \mathcal{O}_X$.

For a $\mathbb{C}^\times$-stable open affine subset $U \subset X'$ the algebra $\Gamma(U, \mathcal{D}_\hbar)$ acquires a $\mathbb{C}^\times$-action. The action on $\Gamma(U, \mathcal{D}_\hbar)/(\hbar^k) = \Gamma(U, \mathcal{D}_\hbar/(\hbar^k))$ is not required to be rational. However, we can replace the $\mathbb{C}^\times$-action on $\mathcal{D}_\hbar$ canonically to make the action on $\Gamma(U, \mathcal{D}_\hbar)/(\hbar^k)$ rational. Let us explain how to do this. Consider the subgroup $\mathbb{C}^\times_{fin} \subset \mathbb{C}^\times$ of finite order elements. It acts diagonalizably in any, not necessarily rational, representation of $\mathbb{C}^\times$. Note that $\Gamma(U, \mathcal{D}_\hbar)/(\hbar^k)$ is $\mathbb{C}^\times$-equivariantly filtered with successive quotients $\mathbb{C}[U]$. It follows that the characters of $\mathbb{C}^\times_{fin}$ in $\Gamma(U, \mathcal{D}_\hbar)/(\hbar^k)$ are restrictions of characters of $\mathbb{C}^\times$. There is a unique rational action of $\mathbb{C}^\times$ on $\Gamma(U, \mathcal{D}_\hbar)/(\hbar^k)$ extending the action of $\mathbb{C}^\times$, it is by algebra automorphisms. These actions glue to an action on $\mathcal{D}_\hbar$ as in the previous paragraph. Below we will always assume that the action of $\mathbb{C}^\times$ on $\Gamma(U, \mathcal{D}_\hbar)/(\hbar^k)$ is rational for all open affine subsets $U \subset X'$.

With this assumption there is a natural bijection between the sets of isomorphism classes of filtered quantizations and of graded formal quantizations. Namely, take a filtered quantization $\mathcal{D}$ of $\mathcal{O}_{X'}$. Form the Rees sheaf $R_\hbar(\mathcal{D})$ of $\mathcal{D}$ and complete it in the $\hbar$-adic topology. Then we extend this completion (that is still a sheaf in the conical topology) to the Zariski topology by localizing. We get a graded formal quantization in the sense of [Lo5, Section 2.2]. Conversely, given a graded formal quantization $\mathcal{D}_\hbar$ of X' we can restrict it to the conical topology. Inside this restriction we can consider the subsheaf $\mathcal{D}_{\hbar,fin}$ of $\mathbb{C}^\times$-finite sections. Because of the rationality assumption on the action of $\mathbb{C}^\times$ on $\mathcal{D}_\hbar$, we recover $\mathcal{D}_\hbar$ as the $\hbar$-adic completion of $\mathcal{D}_{\hbar,fin}$. The sheaf $\mathcal{D}_{\hbar,fin}/(\hbar - 1)$ comes with a natural filtration. This filtration is complete and separated because the $\hbar$-adic topology on $\mathcal{D}_\hbar$ is so. So the resulting sheaf is a filtered quantization of X'. It is straightforward to see that these procedures define mutually inverse bijections between the sets of isomorphism classes of filtered and of graded formal quantizations.

Now suppose X' is smooth and symplectic. In this case a filtered quantization $\mathcal{D}$ defines a class in $H^2(X', \mathbb{C})$ to be called the *period* of $\mathcal{D}$, see [BeKa, Section 4], where the case of formal quantizations was considered, and [Lo5, Section 2] that treats graded formal quantizations.

The following proposition should be thought of as a quantum version of Proposition 2.6 and Remark 2.7.

Proposition 3.1. *Assume that X' is smooth and symplectic and $H^i(X', \mathcal{O}_{X'}) = 0$ for $i = 1, 2$. Then the following claims hold:*

(1) *taking the period defines a bijection between the isomorphism classes of filtered quantizations and $\mathfrak{P} := H^2(X', \mathbb{C})$.*

(2) *Moreover, there is a "universal quantization" $\mathcal{D}'_{\mathfrak{P}}$. This a sheaf of filtered flat $\mathbb{C}[\mathfrak{P}]$-algebras in the conical topology with complete and separated filtration and an isomorphism $\operatorname{gr}\mathcal{D}'_{\mathfrak{P}}/(\mathfrak{P}) \xrightarrow{\sim} \mathcal{O}_{X'}$. The completion of $\operatorname{gr}\mathcal{D}'$ at $0 \in \mathfrak{P}$ is the deformation of the universal deformation of X', see Remark 2.7. Moreover, the quantization $\mathcal{D}'_\lambda$ corresponding to λ is obtained by specializing $\mathcal{D}'_{\mathfrak{P}}$ to λ.*

Proof. Both (1) and (2) were proved in [Lo5, Corollary 2.3.3] for the graded formal quantizations of X'. Let us elaborate on the proof of (2) in the setting of filtered quantizations. Take the universal graded formal deformation of X', see Remark 2.7, and its canonical quantization as in [Lo5, Corollary 2.3.3]. Pull it back to X' getting a sheaf of $\mathbb{C}[[\mathfrak{P},\hbar]]$-algebras on X' deforming $\mathcal{O}_{X'}$. We can convert it into a sheaf of filtered algebras as discussed before the proposition. The claim about the specializations of $\mathcal{D}'_{\mathfrak{P}}$ (in the language of graded formal quantizations) is [BPW, Proposition 3.1]. □

Now consider the variety $\tilde{X}$, a $\mathbb{Q}$-factorial terminalization of a conical symplectic singularity X, and set $\mathfrak{P} := H^2(\tilde{X}^{reg},\mathbb{C})$.

Corollary 3.2. *The following claims hold:*

(1) *The filtered quantizations of $\tilde{X}$ are classified by $\mathfrak{P}$. The filtered quantization $\mathcal{D}_\lambda$ corresponding to λ is the pushforward of the filtered quantization $\mathcal{D}'_\lambda$ of $\tilde{X}^{reg}$ corresponding to λ.*

(2) *Let $\mathcal{D}_{\mathfrak{P}}$ denote the pushforward of the universal quantization $\mathcal{D}'_{\mathfrak{P}}$ from $\tilde{X}^{reg}$ to $\tilde{X}$. Then $\mathcal{D}_\lambda$ is the specialization of $\mathcal{D}_{\mathfrak{P}}$ to $\lambda \in \mathfrak{P}$.*

Proof. As for Poisson deformations over Artinian bases in Remark 2.7, the sheaf theoretic pullback and pushforward define mutually inverse bijections between the sets of isomorphism classes of filtered quantizations of $\tilde{X}^{reg}$ and of $\tilde{X}$, compare to [BPW, Proposition 3.4]. Now we use (1) of Proposition 3.1 to establish (1).

Now we prove (2). Consider the sheaf $\mathcal{D}'_{\mathfrak{P},\hbar}$ as in the proof of (2) of Proposition 3.1. For any Artinian quotient B of $\mathbb{C}[[\mathfrak{P},\hbar]]$, the base change $B\otimes_{\mathbb{C}[[\mathfrak{P},\hbar]]}\mathcal{D}'_{\mathfrak{P},\hbar}$ is filtered with quotients isomorphic to $\mathcal{O}_{\tilde{X}^{reg}}$. So $R^1\iota_*(B\otimes_{\mathbb{C}[[\mathfrak{P},\hbar]]}\mathcal{D}'_{\mathfrak{P},\hbar})=0$. Since the filtration on $\mathcal{D}'_{\mathfrak{P},\hbar}$ induced by the maximal ideal in $\mathbb{C}[[\mathfrak{P}\hbar]]$ is complete and separated, we get $R^1\iota_*\mathcal{D}'_{\mathfrak{P},\hbar}=0$. It follows that $\iota_*\mathcal{D}'_{\mathfrak{P},\hbar}$ is a deformation of $\mathcal{O}_{\tilde{X}}$ over $\mathbb{C}[[\mathfrak{P},\hbar]]$. The sheaf $\iota_*\mathcal{D}_{\mathfrak{P}}$ of filtered algebras is obtained from $\iota_*\mathcal{D}'_{\mathfrak{P},\hbar}$ by passing to $\mathbb{C}^\times$-finite sections and taking the quotient by $\hbar-1$. Specializing this sheaf to λ we get a filtered quantization of $\tilde{X}$ whose restriction to $\tilde{X}^{reg}$ is $\mathcal{D}'_\lambda$. By the construction in (1), this is the quantization $\mathcal{D}_\lambda$. □

3.2. **Quantizations from $\mathbb{Q}$-terminalizations.** Now we will produce some filtered quantizations of $\mathbb{C}[X]$ following [BPW, Section 3] and state our main classification result.

Set $\mathcal{A}_\lambda := \Gamma(\mathcal{D}_\lambda), \mathcal{A}_{\mathfrak{P}} := \Gamma(\mathcal{D}_{\mathfrak{P}})$, where $\mathcal{D}_\lambda, \mathcal{D}_{\mathfrak{P}}$ were introduced in the previous section. The following is a quantum version of Proposition 2.9.

Proposition 3.3. *The following is true:*

(1) *The algebras $\mathcal{A}_\lambda, \mathcal{A}_{\mathfrak{P}}$ are filtered quantizations of $\mathbb{C}[X], \mathbb{C}[X_{\mathfrak{P}}]$, respectively. Moreover, $\mathcal{A}_\lambda$ is the specialization of $\mathcal{A}_{\mathfrak{P}}$ to λ.*

(2) *The group W acts on $\mathcal{A}_{\mathfrak{P}}$ by filtered algebra automorphisms so that the associated graded action on $\mathbb{C}[X_{\mathfrak{P}}]$ coincides with the action from Proposition 2.9. Moreover, the actions of W on $\mathcal{A}_{\mathfrak{P}}$ and on $\mathfrak{P}$ are compatible.*

Proof. The proof of (1) is standard, let us provide it for reader's convenience.

We start by proving the claim for $\mathcal{A}_\lambda$. Recall, Proposition 2.5, that $\mathbb{C}[\tilde{X}]=\mathbb{C}[X]$ and $H^i(\tilde{X},\mathcal{O}_{\tilde{X}})=0$ for $i>0$, in particular for $i=1$. Consider the $\hbar$-adic completion $\mathcal{D}_{\lambda,\hbar}$ of the Rees sheaf of $\mathcal{D}_\lambda$. This is a graded formal deformation of $\mathcal{O}_{\tilde{X}}$. As in the proof of (2) of Corollary 3.2, $H^1(\tilde{X},\mathcal{D}_{\lambda,\hbar})=0$. This, in turn, implies that $\Gamma(\mathcal{D}_{\lambda,\hbar})/(\hbar)\xrightarrow{\sim}\mathbb{C}[\tilde{X}]$. The algebra $\mathcal{A}_\lambda=\Gamma(\mathcal{D}_\lambda)$ is obtained from $\Gamma(\mathcal{D}_{\lambda,\hbar})$ by taking the $\mathbb{C}^\times$-finite sections and then passing to quotient by $\hbar-1$. It follows that $\operatorname{gr}\mathcal{A}_\lambda\xrightarrow{\sim}\mathbb{C}[\tilde{X}]$. This is an isomorphism of graded Poisson algebras. So $\mathcal{A}_\lambda$ is a quantization of $\mathbb{C}[\tilde{X}]$.

Now we prove that $\mathcal{A}_\mathfrak{P}$ is a quantization of $\mathbb{C}[X_\mathfrak{P}]$. Let $\mathfrak{m}$ denote the maximal ideal of 0 in $\mathbb{C}[\mathfrak{P}]$. For $k>0$, set $\tilde{X}_{\mathfrak{P},k}:=\operatorname{Spec}(\mathbb{C}[\mathfrak{P}]/\mathfrak{m}^k)\times_\mathfrak{P}\tilde{X}_\mathfrak{P}$. Since $H^i(\tilde{X},\mathcal{O}_{\tilde{X}})=0$ for all $i>0$, we use the long exact sequence in cohomology to show that $H^i(\tilde{X}_{\mathfrak{P},k},\mathcal{O}_{\tilde{X}_{\mathfrak{P},k}})=0$ for $i>0$. Thanks to the formal function theorem, the completions of $H^i(\tilde{X}_\mathfrak{P},\mathcal{O}_{\tilde{X}_\mathfrak{P}})$ vanish. Thanks to the contracting $\mathbb{C}^\times$-action on $\tilde{X}_\mathfrak{P}$, we conclude that $H^i(\tilde{X}_\mathfrak{P},\mathcal{O}_{\tilde{X}_\mathfrak{P}})=0$ for $i>0$. Now the proof that $\mathcal{A}_\mathfrak{P}$ is a filtered quantization of $\mathbb{C}[\tilde{X}_\mathfrak{P}]$ repeats the proof for $\mathcal{A}_\lambda$ in the previous paragraph.

Let us show that $\mathcal{A}_\lambda$ is the specialization of $\mathcal{A}_\mathfrak{P}$. Let $\mathfrak{P}_1\subset\mathfrak{P}$ be an affine space. We write $\mathcal{D}_{\mathfrak{P}_1,\hbar}$ for the completion of the Rees sheaf of the specialization $\mathcal{D}_{\mathfrak{P}_1}$ of $\mathcal{D}_\mathfrak{P}$. Similarly to the previous paragraph, the sheaf $\mathcal{D}_{\mathfrak{P}_1,\hbar}$ has no higher cohomology. Using the long exact sequences in cohomology and the descending induction on $\dim\mathfrak{P}_1$ one proves that $\Gamma(\mathcal{D}_{\mathfrak{P}_1,\hbar})$ is the specialization of $\Gamma(\mathcal{D}_{\mathfrak{P},\hbar})$. In particular, $\mathcal{A}_\lambda$ is the specialization of $\mathcal{A}_\mathfrak{P}$ to λ.

(2) is what is proved in the proof of [BPW, Proposition 3.10] (note that the statement of that proposition is formally weaker). The smooth locus $X_\mathfrak{P}^{reg}$ in $X_\mathfrak{P}$ is preserved by W. As argued in the proof of [BPW, Proposition 3.10], since the restriction of $\mathcal{D}_\mathfrak{P}$ to $X_\mathfrak{P}^{reg}$ is the canonical quantization of $X_\mathfrak{P}^{reg}$, the action of W on $\mathcal{O}_{X_\mathfrak{P}^{reg}}$ lifts to $\mathcal{D}_\mathfrak{P}|_{X_\mathfrak{P}^{reg}}$ and hence to $\Gamma(\mathcal{D}_\mathfrak{P}|_{X_\mathfrak{P}^{reg}})$. And, as argued in the proof of [BPW, Proposition 3.10], the global sections coincide with $\mathcal{A}_\mathfrak{P}$. By the construction, the action of W on $\operatorname{gr}\mathcal{A}_\mathfrak{P}$ coincides with the action on $\mathbb{C}[\tilde{X}_\mathfrak{P}]$. □

The following is one of the main results of this paper.

Theorem 3.4. *Any filtered quantization of $\mathbb{C}[X]$ is isomorphic to $\mathcal{A}_\lambda$ for some λ. Moreover, $\mathcal{A}_\lambda,\mathcal{A}_{\lambda'}$ are isomorphic as filtered quantizations if and only if $\lambda'\in W\lambda$.*

In fact, the algebra $\mathcal{A}_\mathfrak{P}^W$ enjoys a universal property similar to that of $\mathbb{C}[X_\mathfrak{P}]^W$. This property is the subject of the next proposition, which is a more technical version of Theorem 3.4 and also implies that theorem.

Proposition 3.5. *Let B' be a finitely generated commutative positively graded algebra and A' be a graded Poisson flat B'-algebra such that $\mathbb{C}\otimes_{B'}A'=\mathbb{C}[X]$. Further, let $\mathcal{A}'$ be a B'-algebra that is a filtered quantization of A' such that the structure map $B'\to\mathcal{A}'$ is a filtered algebra homomorphism whose associated graded map is $B'\to A'$. Then there is a unique filtered algebra homomorphism $\mathbb{C}[\mathfrak{P}]^W\to B'$ with the following properties:*

1. *The associated graded of this homomorphism comes from the universal property of $\mathbb{C}[X_\mathfrak{P}]^W/\mathbb{C}[\mathfrak{P}]^W$ (see Proposition 2.12 and Corollary 2.13).*
2. *We have a B'-linear isomorphism $B'\otimes_{\mathbb{C}[\mathfrak{P}]^W}\mathcal{A}_\mathfrak{P}^W\xrightarrow{\sim}\mathcal{A}'$ of filtered quantizations of A'.*

Proposition 3.5 follows from Proposition 3.12 and Lemma 3.15 that will be stated and proved later.

Our last task for this section is to explain how we can recover the parameter $W\lambda$ from a filtered quantization $\mathcal{A}_\lambda$. In order to do this, we will need to recall the construction and properties of quantum slice algebras following [Lo9, Section 3.2].

Let $x \in \mathcal{L}_i$. Then we can form the completions $\mathbb{C}[X]^{\wedge_x}, \mathbb{C}[T_x\mathcal{L}_i]^{\wedge_0}, \mathbb{C}[\Sigma_i]^{\wedge_0}$. By [K1, Theorem 2.3], we have an isomorphism

$$\mathbb{C}[X]^{\wedge_x} \cong \mathbb{C}[T_x\mathcal{L}_i]^{\wedge_0}\widehat{\otimes}\mathbb{C}[\Sigma_i]^{\wedge_0}. \tag{3.1}$$

One can lift (3.1) to a quantum level. Form the Rees algebra $R_\hbar(\mathcal{A}_\lambda)$ of $\mathcal{A}_\lambda$ and consider the completion $R_\hbar(\mathcal{A}_\lambda)^{\wedge_x}$ at the maximal ideal of $R_\hbar(\mathcal{A}_\lambda)$ pulled back from the maximal ideal of x in $\mathbb{C}[X]$ under $R_\hbar(\mathcal{A}_\lambda) \twoheadrightarrow \mathbb{C}[X]$. Then, by [Lo9, Lemma 3.3], we have the following decomposition

$$R_\hbar(\mathcal{A}_\lambda)^{\wedge_x} \cong \mathbb{A}_\hbar(T_x\mathcal{L}_i)^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}R_\hbar(\mathcal{A}^i_{\lambda_i})^{\wedge_0}, \tag{3.2}$$

lifting (3.1), where $\mathbb{A}_\hbar$ stands for the homogenized Weyl algebra of a symplectic vector space.

Proposition 3.6. *In a quantization parameter $\lambda = (\lambda_0, \ldots, \lambda_k)$, the component λ_0 is the period of the quantization of X^{reg} given by restricting $\mathcal{A}_\lambda$. The orbit $W_i\lambda_i$ is uniquely recovered from the formal quantization $R_\hbar(\mathcal{A}^i_{\lambda_i})^{\wedge_0}$ of $\mathbb{C}[\Sigma_i]^{\wedge_0}$.*

Proof. The claim about λ_0 is a consequence of the following general claim: if X' is such as in Proposition 3.1, $U \subset X'$ is an open $\mathbb{C}^\times$-stable subvariety and $\mathcal{D}'$ is the filtered quantization of X', then the period of the restriction $\mathcal{D}'|_U$ is the pullback of the period of $\mathcal{D}'$ to U. This general claim is a direct consequence of the construction of the period in [BeKa, Section 4]. To get the characterization of λ_0 we notice that the restriction of $\mathcal{A}_\lambda$ to X^{reg} coincides with the restriction of the quantization $\mathcal{D}'_\lambda$ of $\tilde{X}^{reg}$ to X^{reg}. This is because the morphism $\tilde{X} \to X$ is an isomorphism over X^{reg}.

Now we prove the claim about $W_i\lambda_i$ for $i > 0$. The proof is in two steps.

Step 1. First of all, we claim that if $R_\hbar(\mathcal{A}^i_{\lambda_i})^{\wedge_0}$ and $R_\hbar(\mathcal{A}^i_{\lambda'_i})^{\wedge_0}$ are isomorphic as formal quantizations of $\mathbb{C}[\Sigma'_i]$, then $\mathcal{A}^i_{\lambda_i}$ and $\mathcal{A}^i_{\lambda'_i}$ are isomorphic as filtered quantizations of $\mathbb{C}[\Sigma_i]$. This will follow if we prove the following claim

- any $\mathbb{C}[[\hbar]]$-algebra isomorphism $R_\hbar(\mathcal{A}^i_{\lambda_i})^{\wedge_0} \xrightarrow{\sim} R_\hbar(\mathcal{A}^i_{\lambda'_i})^{\wedge_0}$ that is the identity modulo $\hbar$ is the composition of a $\mathbb{C}[[\hbar]]$-algebra automorphism, say α, of $R_\hbar(\mathcal{A}^i_{\lambda'_i})^{\wedge_0}$ that is the identity modulo $\hbar$ and a $\mathbb{C}[[\hbar]]$-algebra isomorphism $R_\hbar(\mathcal{A}^i_{\lambda_i})^{\wedge_0} \xrightarrow{\sim} R_\hbar(\mathcal{A}^i_{\lambda'_i})^{\wedge_0}$ intertwining the $\mathbb{C}^\times$-actions.

Consider the group $\mathcal{G}$ of algebra automorphisms of $R_\hbar(\mathcal{A}^i_{\lambda'_i})^{\wedge_0}$ that rescale $\hbar$. This is a pro-algebraic group. As such it is the semidirect product of a reductive algebraic group and pro-unipotent pro-algebraic group. The projection from the reductive part to the automorphism group of $\mathbb{C}[\Sigma_i]'$ is injective. It follows that every two embeddings of $\mathbb{C}^\times$ into $\mathcal{G}$ that give the dilation action on Σ'_i are conjugate by an element that is the identity on Σ'_i. We apply this to the $\mathbb{C}^\times$-actions on $R_\hbar(\mathcal{A}^i_{\lambda'_i})^{\wedge_0}$ coming from $R_\hbar(\mathcal{A}^i_{\lambda_i}), R_\hbar(\mathcal{A}^i_{\lambda'_i})$ and let α be a conjugating element. This establishes the claim above.

Step 2. Now suppose $\mathcal{A}^i_{\lambda_i} \xrightarrow{\sim} \mathcal{A}^i_{\lambda'_i}$, an isomorphism of filtered quantizations (meaning that the associated graded isomorphism intertwines the isomorphisms with $\mathbb{C}[\Sigma_i]$). By Theorem 3.4, we have $\widehat{W_i}\lambda'_i = \widehat{W_i}\lambda_i$. Recall, Section 2.3, that $\widehat{W_i}$ is the Weyl group of the

same ADE type as Σ_i and $W_i = \widehat{W}_i^{\Xi}$ for a suitable group Ξ of diagram automorphisms. So it is enough to show that for $\lambda_i, \lambda_i' \in \widehat{\mathfrak{P}}_i^{\Xi}$, the equality $\widehat{W}_i\lambda_i' = \widehat{W}_i\lambda_i$ implies $W_i\lambda_i' = W_i\lambda_i$.

Pick fundamental weights $\varpi_j, j \in J$, for $\widehat{W}_i$ and let C denote the Weyl chamber for $\widehat{W}_i$ spanned by the weights ϖ_j. The group Ξ acts on J. Then $C^{\Xi} = C \cap \widehat{\mathfrak{P}}_i^{\Xi}$ is spanned by $\sum_{k\in\Xi j} \varpi_j$. This is a fundamental chamber for $W_i = \widehat{W}_i^{\Xi}$. Note that the locus $\{\lambda_i \in \widehat{\mathfrak{P}}_i^{\Xi} | \widehat{W}_i\lambda_i \cap \widehat{\mathfrak{P}}_i^{\Xi} = W_i\lambda_i\}$ is the union of finitely many vector subspaces defined over $\mathbb{Q}$. So in the proof it is enough to assume that λ_i, λ_i' are real. Conjugating them by elements of $\widehat{W}_i^{\Xi}$ we can assume that they lie in C^{Ξ}, hence in C. Since they are $\widehat{W}_i$-conjugate, they must coincide. □

3.3. **Scheme Y.** We start by constructing a finite type affine scheme Y over $\mathbb{C}$ together with an action of $\mathbb{C}^\times \ltimes U$, where U is a unipotent group. This scheme will, in a sense, parameterize deformations of $\mathbb{C}[X]$ compatible with the Poisson bracket, and the group action will correspond to isomorphisms of deformations.

Definition 3.7. Let R be a finitely generated commutative $\mathbb{C}$-algebra with 1. By a *deformation datum* (over R) on $R[X]$ we mean a pair $(*, \langle\cdot,\cdot\rangle)$ of R-bilinear maps $R[X]\otimes_R R[X] \to R[X]$ satisfying the following condition:

(i) $*$ is an associative product such that $1 \in R[X]$ is a unit and $f * g - fg \in \bigoplus_{k<i+j} R[X]_k$ for any i, j and $f \in R[X]_i, g \in R[X]_j$ (here, as usual, the subscript stands for the graded component of that degree).
(ii) $\langle\cdot,\cdot\rangle$ is a skew-symmetric bracket on $R[X]$ such that $\langle f, g\rangle - \{f, g\} \in \bigoplus_{k<i+j-d} R[X]_k$ for i, j, f, g as in (i).
(iii) There is $z \in R$ such that $f * g - g * f = z\langle f, g\rangle$. Note that (ii) implies that z is unique.
(iv) We have $\langle f*g, h\rangle = f*\langle g, h\rangle + \langle f, h\rangle*g$ for all $f, g, h \in \mathbb{C}[X]$ (the Leibniz identity) and also the Jacobi identity for $\langle\cdot,\cdot\rangle$.

Clearly, if $R = \mathbb{C}$ and $z = 0$, then $*$ is a commutative product and $\langle\cdot,\cdot\rangle$ is a Poisson bracket so that $(\mathbb{C}[X], *, \langle\cdot,\cdot\rangle)$ defines a filtered Poisson deformation of $\mathbb{C}[X]$. If, on the other hand, $z = 1$, then $\langle\cdot,\cdot\rangle$ is recovered from $*$, and $(\mathbb{C}[X], *)$ is a filtered quantization of $\mathbb{C}[X]$.

Definition 3.8. By an isomorphism of deformation data $(*, \langle\cdot,\cdot\rangle), (*', \langle\cdot,\cdot\rangle')$ we mean an R-linear map $\varphi : R[X] \to R[X]$ with the following properties:

(I) $\varphi(f) - f \in \bigoplus_{i<k} R[X]_i$ for any k and $f \in R[X]_k$.
(II) φ intertwines $*$ with $*'$, as well as $\langle\cdot,\cdot\rangle$ with $\langle\cdot,\cdot\rangle'$.

Clearly, isomorphic deformation data correspond to isomorphic filtered Poisson deformations (for $z = 0$) and quantizations (for $z = 1$).

Now we proceed to constructing Y. For $n \in \mathbb{Z}_{\geqslant 0}$, set $V_{\leqslant n} := \bigoplus_{i=0}^n \mathbb{C}[X]_i$. Let $f_1, \ldots, f_k$ be a minimal set of homogeneous generators of $\mathbb{C}[X]$ and let m be the maximum of the degrees of the generators f_i. Further, let $G_1, \ldots, G_\ell$ be a minimal set of homogeneous (with respect to the grading on $\mathbb{C}[X]$) relations between the generators $f_1, \ldots, f_k$. Consider $\deg G_i$, the degree of G_i with respect to $f_1, \ldots, f_k$ and set

$$e := 1 + \max(2, \deg G_1, \ldots, \deg G_\ell).$$

Set $T := \operatorname{Hom}(\bigoplus_{i=1}^e V_{\leqslant m}^{\otimes i}, V_{\leqslant me}) \oplus \operatorname{Hom}(V_{\leqslant m}^{\otimes 2}, V_{\leqslant 2m-d})$. A deformation pair defines an element (α, β) of T, where α comes from the iterated product $*$ restricted to $V_{\leqslant m}$ and β comes from the bracket $\langle \cdot, \cdot \rangle$ restricted to $V_{\leqslant m}$.

We will realize Y as a closed subscheme in T. First, note that conditions (i)-(iii) give polynomial equations on T. For example, by (ii) we have $\langle \cdot, \cdot \rangle \neq 0$. Then (iii) means that, for $f, g \in V_{\leqslant m}$, the elements $f * g - g * f$ and $\langle f, g \rangle$ are proportional, which results in polynomial equations on α, β. Let Y^1 denote the subscheme defined by these polynomial equations. Note that, by the construction, z can be viewed as an element of $\mathbb{C}[Y^1]$. Indeed, both $(f, g) \mapsto f * g - g * f$ and $(f, g) \mapsto \langle f, g \rangle$ give linear maps $V_{\leqslant m}^{\otimes 2} \mapsto V_{\leqslant 2m-d}$. The locus, say Z, in $\operatorname{Hom}_{\mathbb{C}}(V_{\leqslant m}^{\otimes 2}, V_{\leqslant 2m-d})^2$, where two linear maps, say φ_1, φ_2, are proportional is an algebraic subscheme, on its open subscheme Z^0 where $\varphi_2 \neq 0$, the ratio φ_1/φ_2 is a regular function. Now, by the construction, we have a morphism $Y^1 \to Z^0$, the element $z \in \mathbb{C}[Y^1]$ is the pullback of the regular function φ_1/φ_2 (note that β is nonzero thanks to (ii) in Definition 3.7).

Now consider a finite type commutative algebra R and an algebra homomorphism $(\alpha, \beta) : \mathbb{C}[Y^1] \to R$ (where the meaning of α, β as before: α corresponds to $*$ and β corresponds to $\langle \cdot, \cdot \rangle$). We construct the unital associative algebra

$$\mathcal{A}_{(\alpha,\beta)} := (R \otimes T(V_{\leqslant me}))\,/(x - \alpha(x), \alpha(y) - \alpha^{opp}(y) - z\beta(y)), \tag{3.3}$$

where x runs over $\bigoplus_{i=0}^e V_{\leqslant m}^{\otimes i}$, y runs over $V_{\leqslant m}^{\otimes 2}$ and we write α^{opp} for $\alpha|_{V_{\leqslant m}^{\otimes 2}} \circ \sigma$, where σ is the permutation of tensor factors.

The R-algebra $\mathcal{A}_{(\alpha,\beta)}$ comes with a filtration induced from $V_{\leqslant me}$ and $\deg R = 0$. We have a natural epimorphism

$$R \otimes \mathbb{C}[X] \to \operatorname{gr} \mathcal{A}_{(\alpha,\beta)}. \tag{3.4}$$

Lemma 3.9. *The condition that (3.4) is an isomorphism is equivalent to a system of polynomial equations on α, β. These equations are independent of R.*

Proof. The condition that (3.4) is an isomorphism is equivalent to

(*) For each $j \geqslant 0$, the filtered piece $(\mathcal{A}_{(\alpha,\beta)})_{\leqslant j}$ is a free R-module of rank $\dim V_{\leqslant j}$. Note that $(\mathcal{A}_{(\alpha,\beta)})_{\leqslant j}$ is automatically the quotient of a free R-module of rank $\dim V_{\leqslant j}$ – because (3.4) is an epimorphism.

Let I denote the kernel of $R \otimes T(V_{\leqslant me}) \to \mathcal{A}_{(\alpha,\beta)}$. The ideal I comes with the R-module map

$$R \otimes \left((\bigoplus_{i=0}^{e} V_{\leqslant m}^{\otimes i}) \oplus V_{\leqslant m}^{\otimes 2} \right) \to R \otimes T(V_{\leqslant me}), (x, y) \mapsto x - \alpha(x) + \alpha(y) - \alpha^{opp}(y) - z\beta(y).$$

The image generates I as an $R \otimes T(V_{\leqslant me})$-bimodule. Consider the filtration on the $R \otimes T(V_{\leqslant me})$-bimodule I induced from the natural filtration on $R \otimes \big((\bigoplus_{i=0}^e V_{\leqslant m}^{\otimes i}) \oplus V_{\leqslant m}^{\otimes 2}\big)$. The associated graded of I contains the ideal of relations of $R \otimes \mathbb{C}[X]$ (including the commutativity relations). The condition (*) is equivalent to the following condition

(**) For each $j \geqslant 0$, we have that $R \otimes T(V_{\leqslant me})_{\leqslant j}/I_{\leqslant j}$ is a free module of rank $\dim V_{\leqslant j}$ (while, a priori, the former is the quotient of a free module of the given rank).

The quotient $R \otimes T(V_{\leqslant me})_{\leqslant j}/I_{\leqslant j}$ is the cokernel of a matrix whose entries are polynomials in the entries of α and β. For an arbitrary pair (α, β), the matrix is such that the cokernel is the quotient of a free R-module of rank $\dim V_{\leqslant j}$. So the claim that the

cokernel is a free module of that rank is equivalent to the vanishing of all minors of a specified size. This gives polynomial conditions on (α, β) that are independent of R. $\square$

Let $Y^2 \subset Y^1$ denote the (scheme-theoretic) vanishing locus of polynomial equations from Lemma 3.9. For any algebra homomorphism $(\alpha, \beta) : \mathbb{C}[Y^2] \to R$ we get the filtered associative algebra $\mathcal{A}_{(\alpha,\beta)}$ with $R \otimes \mathbb{C}[X] \xrightarrow{\sim} \operatorname{gr} A_{(\alpha,\beta)}$. There is at most one bracket $\langle \cdot, \cdot \rangle$ on $\mathcal{A}_{(\alpha,\beta)}$ satisfying (iv) whose pullback to $V_{\leqslant m}^{\otimes 2}$ coincides with β. Such a bracket then automatically satisfies (ii) and (iii). Note that (iv) translates into a collection on polynomial equations on α and β. Let $Y \subset Y^2$ denote the closed subscheme defined by these equations.

By the construction, Y represents the functor of taking deformation data: to give a deformation datum over R is the same thing as to give an R-point of Y. Denote this set of points by $Y(R)$.

Now we proceed to group actions on Y. Define the unipotent group U. We take the subgroup of $\mathrm{GL}(V_{\leqslant me})$ consisting of all linear maps $\Phi : V_{\leqslant me} \to V_{\leqslant me}$ with $\deg(\Phi(f) - f) < \deg f$ for all $f \in V_{\leqslant me}$. We have an induced action on T that preserves the defining ideal of Y as well as z.

Also define an action of $\mathbb{C}^\times$ on $V_{\leqslant me}$ by $t.f := t^{-\deg f} f$ for a homogeneous element $f \in V_{\leqslant me}$. Then $\mathbb{C}^\times$ normalizes $U \subset \mathrm{GL}(V_{\leqslant me})$ and also preserves the ideal of Y. Recall that z can be viewed as an element of $\mathbb{C}[Y^1]$. Hence z defines an element of $\mathbb{C}[Y]$ (which is a quotient of $\mathbb{C}[Y^1]$), equivalently, a morphism $Y \to \mathbb{C}$.

Note that, by the construction, we have the following property:

(♠) The morphism $Y \to \mathbb{C}$ given by z is U-invariant and $\mathbb{C}^\times$-equivariant, $t \in \mathbb{C}^\times$ acts on $\mathbb{C}$ by multiplication by t^{-d}. Moreover, $\mathbb{C}[Y], \mathbb{C}[U]$ are positively graded with respect to the $\mathbb{C}^\times$-action.

We write $U(R)$ for the group of R-points of U. Note that $U(R)$ acts on $Y(R)$. We write $U_{\mathrm{Spec}(R)}$ for the constant group scheme over $\mathrm{Spec}(R)$ with fiber U. So $U(R)$ is the group of sections $\mathrm{Spec}(R) \hookrightarrow U_{\mathrm{Spec}(R)}$.

3.4. **Generating maps and the algebras $\mathcal{A}_Y, \mathcal{A}_{\mathbb{Y}}$.** Now we need another concept: that of a generating map. Let B be a finitely generated commutative $\mathbb{C}$-algebra (but not, a priori, a $\mathbb{C}[Y]$-algebra) and let $\mathcal{A}_B$ be a filtered B-algebra with

- B in degree 0,
- an isomorphism $\operatorname{gr} \mathcal{A}_B \cong B \otimes \mathbb{C}[X]$ of graded Poisson algebras.
- a B-linear bracket $\langle \cdot, \cdot \rangle$ satisfying (ii)-(iv) of Definition 3.7 for some $z \in B$,

Definition 3.10. By a *generating map* for $\mathcal{A}_B$ we mean a filtered B-module map $B \otimes V_{\leqslant me} \to \mathcal{A}_B$ that becomes the inclusion $B \otimes V_{\leqslant me} \to B \otimes \mathbb{C}[X]$ after passing to the associated graded algebra.

If B' is a B-algebra, then $\mathcal{A}_{B'} := B' \otimes_B \mathcal{A}_B$ inherits a generating map from $\mathcal{A}_B$.

Note that any generating map is injective and the image of $B \otimes V_{\leqslant m}$ generates $\mathcal{A}_B$ (hence the name). Denote the set of generating maps by $\mathsf{Gen}(\mathcal{A}_B)$. It comes with a $U(B)$-action. There is at least one generating map, and if we fix it, we get a $U(B)$-equivariant bijection $U(B) \xrightarrow{\sim} \mathsf{Gen}(\mathcal{A}_B)$.

Let us explain a connection between generating maps and deformation data. Every generating map for $\mathcal{A}_B$ gives rise to a deformation datum over B: by restricting the

product and the bracket from $\mathcal{A}_B$ to $B\otimes V_{\leqslant me}$. So we get a map of sets

$$\mathsf{Gen}(\mathcal{A}_B)\to Y(B). \tag{3.5}$$

This map is $U(B)$-equivariant.

We also need a certain automorphism group. The space $B\otimes(\bigoplus_{i=0}^{d-1}\mathbb{C}[X]_i)$ is a nilpotent Lie subalgebra of $\mathcal{A}_B$ with respect to $\langle\cdot,\cdot\rangle$. So we can consider the Lie algebra

$$\mathfrak{h}_B:=(B\otimes(\bigoplus_{i=0}^{d-1}\mathbb{C}[X]_i))/B\otimes\mathbb{C}[X]_0$$

($\mathfrak{h}$ for Hamiltonian). Consider the corresponding unipotent group scheme $H_{\mathrm{Spec}(B)}$ over $\mathrm{Spec}(B)$. Note that we have a group scheme monomorphism $H_{\mathrm{Spec}(B)}\hookrightarrow U_{\mathrm{Spec}(B)}$ via $\exp(a)\mapsto\exp(\langle a,\cdot\rangle)$ for $a\in\mathfrak{h}_B$.

The following lemma explains the meaning of $H_{\mathrm{Spec(B)}}$.

Lemma 3.11. *The group scheme over* $\mathrm{Spec}(B)$ *of filtered B-algebra automorphisms of $\mathcal{A}_B$ that are the identity on* $\operatorname{gr}\mathcal{A}_B$ *coincides with* $H_{\mathrm{Spec}(B)}$.

Proof. Let ψ denote an automorphism of $\mathcal{A}_B$ as in the statement of the lemma. Then $\ln(\psi)$ is well-defined and is a B-linear derivation of $\mathcal{A}_B$ that is zero on $\operatorname{gr}\mathcal{A}_B$. We need to show that $\ln(\psi)=\langle a,\cdot\rangle$ for $a\in\mathfrak{h}_B$. Note that $\ln(\psi)$ gives rise to a homogeneous negative degree Poisson B-linear derivation of $\operatorname{gr}\mathcal{A}_B=B\otimes\mathbb{C}[X]$, the top degree term of $\ln(\psi)$. Proposition 2.14 implies that such a derivation is Hamiltonian. Since the Poisson center of $B\otimes\mathbb{C}[X]$ is B, we see that the top degree term of $\ln(\psi)$ takes the form $\{f,\cdot\}$ for a unique homogeneous element $f\in B\otimes\bigoplus_{i=1}^{d-1}\mathbb{C}[X]_i$. This implies the analogous statement for $\ln(\psi)$ itself finishing the proof of the lemma. □

Now let R be a finitely generated $\mathbb{C}[Y]$-algebra, let us write (α,β) for the corresponding homomorphism $\mathbb{C}[Y]\to R$. We get the algebra $\mathcal{A}_{(\alpha,\beta)}$ defined in (3.3). The algebra corresponding to the identity automorphism of $\mathbb{C}[Y]$ will be denoted by $\mathcal{A}_Y$. Note that the algebra $\mathcal{A}_Y$ is graded and comes with an action of U by $\mathbb{C}$-algebra automorphisms that is compatible with the action of U on $\mathbb{C}[Y]$. Also $\mathcal{A}_Y$ admits a $\mathbb{Z}_{\geqslant 0}$-filtration with $\mathbb{C}[Y]$ in degree 0 such that $\operatorname{gr}\mathcal{A}_Y\cong\mathbb{C}[Y]\otimes\mathbb{C}[X]$ as a graded Poisson algebra (the Poisson bracket on $\operatorname{gr}\mathcal{A}_Y$ comes from $\langle\cdot,\cdot\rangle$). Note that this filtration does not come from the grading on $\mathcal{A}_Y$.

Every algebra $\mathcal{A}_{(\alpha,\beta)}$ defined by (3.3) comes with a canonical generating map that gives the deformation datum specified by (α,β). This deformation map gives an identification $\mathsf{Gen}(\mathcal{A}_{\alpha,\beta})\cong U(R)$, hence a map $U(R)\to Y(R)$. This map is functorial in R and hence comes from a morphism of schemes $U_Y=U\times Y\to Y\times Y$. This morphism is nothing else but the graph of the action of U on Y. So the preimage of the diagonal is the automorphism group scheme H_Y. To simplify the notation below we are going to denote the preimage of the diagonal by $\mathbb{Y}$. Note that, by construction, $\mathbb{Y}\subset U\times Y$ is U-stable for the action given by $u.(u_1,y)=(uu_1u^{-1},uy)$ and $\mathbb{C}^\times$-stable.

We write $\mathcal{A}_{\mathbb{Y}}$ for the algebra $\mathbb{C}[\mathbb{Y}]\otimes_{\mathbb{C}[Y]}\mathcal{A}_Y$. This algebra comes with a generating map. Namely, consider the diagonal action of U on the $\mathbb{C}[U\times Y]$-algebra $\mathbb{C}[U]\otimes\mathcal{A}_Y$. There is a unique generating map for this algebra that is $U\times U$-equivariant, where the first action of U on itself from the left and the second action is diagonal, and whose fiber over $1\in U$ is the natural generating map for $\mathcal{A}_Y$. Then we get a generating map for $\mathcal{A}_{\mathbb{Y}}$ by base change from $\mathbb{C}[U\times Y]$ to its quotient $\mathbb{C}[\mathbb{Y}]$. Note that this generating map is U-equivariant. A

way to think about this generating map is that at the points of the diagonal $\{1\} \times Y \subset \mathbb{Y}$ we get the natural generating map for $\mathcal{A}_Y$ and then we extend this map by the action of H_Y. The following is a universal property of $\mathcal{A}_{\mathbb{Y}}$ and its generating map.

Proposition 3.12. *Let $\mathcal{A}_B$ be as in the beginning of the section. Fix a generating map for $\mathcal{A}_B$. Then there is a unique algebra homomorphism $\mathbb{C}[\mathbb{Y}] \to B$ and a unique filtered B-algebra isomorphism*

$$\mathcal{A}_B \xrightarrow{\sim} B \otimes_{\mathbb{C}[\mathbb{Y}]} \mathcal{A}_{\mathbb{Y}}$$

that intertwines the brackets, the generating maps, and the isomorphisms $\operatorname{gr} \mathcal{A}_B, \operatorname{gr}(B \otimes_{\mathbb{C}[\mathbb{Y}]} \mathcal{A}_{\mathbb{Y}}) \xrightarrow{\sim} B \otimes \mathbb{C}[X]$.

Proof. As was mentioned before, the choice of a generating map for $\mathcal{A}_B$ gives rise to a point in $Y(B)$, or, equivalently, an algebra homomorphism $\mathbb{C}[Y] \to B$. This homomorphism equips $\mathcal{A}_B$ with another generating map, which is pulled back from $\mathcal{A}_Y$. This new map does not need to coincide with the initial one, but gives the same deformation datum. Since $\mathsf{Gen}(\mathcal{A}_B)$ is a torsor over $U(B)$, a choice of two generating maps for $\mathcal{A}_B$ gives a morphism $\operatorname{Spec}(B) \to U_Y$. Since the generating maps give the same deformation data, the morphism factors through $\operatorname{Spec}(B) \to \mathbb{Y}$. By the construction, we have a filtered algebra isomorphism

$$\mathcal{A}_B \xrightarrow{\sim} B \otimes_{\mathbb{C}[\mathbb{Y}]} \mathcal{A}_{\mathbb{Y}}$$

with the required properties. It is unique: the only automorphism of $\mathcal{A}_B$ that fixes a generating map is the identity. □

3.5. **Structure of $\mathbb{Y}$.** Our goal here is to describe the structure of $\mathbb{Y}$ (in fact, of its very close relative) and use this to give a proof of Proposition 3.5.

Let $\mathcal{A}^W_{\mathfrak{P},\hbar}$ denote the Rees algebra of the filtered algebra $\mathcal{A}^W_{\mathfrak{P}}$. This is a graded algebra over $\mathbb{C}[\mathfrak{P}]^W[\hbar]$, where $\mathbb{C}[\mathfrak{P}]^W \subset \mathcal{A}^W_{\mathfrak{P},\hbar}$ is graded with $\mathfrak{P}^*$ of degree d and the degree of $\hbar$ is 1.

Corollary 3.13. *There is a $\mathbb{C}^\times$-equivariant scheme morphism $\psi : \mathfrak{P}/W \times \mathbb{C} \to \mathbb{Y}$ such that $\psi^*(z) = \hbar^d$ and there is a $\mathbb{C}[\mathfrak{P}/W, \hbar]$-algebra isomorphism $\mathcal{A}^W_{\mathfrak{P},\hbar} \cong \mathbb{C}[\mathfrak{P}/W, \hbar] \otimes_{\mathbb{C}[\mathbb{Y}]} \mathcal{A}_{\mathbb{Y}}$.*

Proof. This is because $\mathcal{A}^W_{\mathfrak{P},\hbar}$ comes with a generating map (as any other deformation). This generating map can be chosen to be $\mathbb{C}^\times$-equivariant. The bracket $\langle \cdot, \cdot \rangle$ equals $[\cdot, \cdot]/\hbar^d$. These observations together with Proposition 3.12 imply the claim of the corollary. □

Now we describe $\mathbb{Y}_0$, the scheme theoretic fiber of $\mathbb{Y}$ at $z = 0$. Corollary 3.13 yields an induced scheme morphism $\psi_0 : \mathfrak{P}/W \to \mathbb{Y}_0$.

Proposition 3.14. *The U-equivariant morphism $U \times \mathfrak{P}/W \to \mathbb{Y}_0$ extending ψ_0 is an isomorphism.*

Proof. The proof is in several steps.

Step 1. Consider the algebra $\mathbb{C}[U] \otimes \mathbb{C}[X_{\mathfrak{P}}]^W$ over $\mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W$. It comes with a generating map produced as follows. We pick a generating map in the fiber over $1 \in U$ and then extend it to $\mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W \otimes V_{\leqslant me} \to \mathbb{C}[U] \otimes \mathbb{C}[X_{\mathfrak{P}}]^W$ so that corresponding scheme morphism $U \times X_{\mathfrak{P}}/W \to U \times \mathfrak{P}/W \times V^*_{\leqslant me}$ is U-equivariant. Here in the target U acts by $u.(u', p, \alpha) = (uu', p, u.\alpha)$. Compare with the description of the generating map for $\mathcal{A}_{\mathbb{Y}}$ in the previous section.

Step 2. Now let B be a positively graded algebra and let $\mathcal{A}_B$ be a graded Poisson algebra deforming $\mathbb{C}[X]$. Then we have a $\mathbb{C}^\times$-equivariant generating map $B \otimes V_{\leqslant me} \to \mathcal{A}_B$. We

claim that there is a unique morphism of schemes $\operatorname{Spec}(B) \to U \times \mathfrak{P}/W$ and a Poisson algebra isomorphism $\mathcal{A}_B \xrightarrow{\sim} B \otimes_{\mathbb{C}[U]\otimes\mathbb{C}[\mathfrak{P}/W]} (\mathbb{C}[U] \otimes \mathbb{C}[X_{\mathfrak{P}}]^W)$ intertwining the generating maps.

By the universal property of $\mathbb{C}[X_{\mathfrak{P}}]^W$, see Proposition 2.12, there is a unique graded algebra homomorphism $\mathbb{C}[\mathfrak{P}]^W \to B$ and a graded Poisson $\mathbb{C}[B]$-algebra isomorphism $\mathcal{A}_B \cong B \otimes_{\mathbb{C}[\mathfrak{P}]^W} \mathbb{C}[X_{\mathfrak{P}}]^W$ that is the identity modulo the augmentation ideal in B. Since the generating maps form a torsor over $U(B)$, we further see that there is a unique homomorphism $\mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W \to B$ and a unique isomorphism $\mathcal{A}_B \xrightarrow{\sim} B \otimes_{\mathbb{C}[U]\otimes\mathbb{C}[\mathfrak{P}/W]} (\mathbb{C}[U] \otimes \mathbb{C}[X_{\mathfrak{P}}]^W)$ intertwining the generating maps.

Step 3. By Proposition 3.12, there is a unique scheme morphism $\iota_1 : U \times \mathfrak{P}/W \to \mathbb{Y}_0$ and a unique Poisson $\mathbb{C}[U \times \mathfrak{P}/W]$-algebra isomorphism $\mathbb{C}[U \times \mathfrak{P}/W] \otimes_{\mathbb{C}[\mathbb{Y}_0]} \mathcal{A}_{\mathbb{Y}_0} \to \mathbb{C}[U] \otimes \mathbb{C}[X_{\mathfrak{P}}]^W$ intertwining the generating maps. On the other hand, $\mathcal{A}_{\mathbb{Y}_0}$ is positively graded. So, by Step 2, there is a unique scheme morphism $\iota_2 : \mathfrak{P}/W \times U \to \mathbb{Y}_0$ and a unique Poisson algebra isomorphism $\mathcal{A}_{\mathbb{Y}_0} \xrightarrow{\sim} \mathbb{C}[\mathbb{Y}_0] \otimes_{\mathbb{C}[U\times\mathfrak{P}/W]} (\mathbb{C}[U] \otimes \mathbb{C}[X_{\mathfrak{P}}]^W)$ intertwining the generating maps. So the morphisms ι_1, ι_2 are mutually inverse. □

Now we will describe the structure of a slight modification of $\mathbb{Y}$. Set

$$\mathbb{Y}' := \operatorname{Spec}(\mathbb{C}[\mathbb{Y}][\hbar]/(z - \hbar^d)).$$

Lemma 3.15. *We have a $\mathbb{C}^\times \times U$-equivariant isomorphism $U \times \mathfrak{P}/W \times \operatorname{Spec}(\mathbb{C}[\hbar]) \xrightarrow{\sim} \mathbb{Y}'$ of schemes over $\operatorname{Spec}(\mathbb{C}[\hbar])$. It gives rise to a $\mathbb{C}[\mathbb{Y}']$-linear and $U \times \mathbb{C}^\times$-linear algebra isomorphism $\mathbb{C}[\hbar] \otimes_{\mathbb{C}[z]} \mathcal{A}_{\mathbb{Y}} \xrightarrow{\sim} \mathbb{C}[U] \otimes \mathcal{A}^W_{\mathfrak{P},\hbar}$.*

Proof. Consider the algebra $\mathbb{C}[U] \otimes \mathcal{A}^W_{\mathfrak{P},\hbar}$. It comes with a generating map defined as in Step 1 of the proof of Proposition 3.14. This generating map is $\mathbb{C}^\times$-equivariant by the construction. So we get a unique graded algebra homomorphism $\mathbb{C}[\mathbb{Y}] \to \mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W[\hbar]$ and a unique graded algebra isomorphism

$$\mathbb{C}[U] \otimes \mathcal{A}^W_{\mathfrak{P},\hbar} \xrightarrow{\sim} \left(\mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W[\hbar]\right) \otimes_{\mathbb{C}[\mathbb{Y}]} \mathcal{A}_Y$$

that intertwines the brackets $\langle \cdot, \cdot \rangle$ and the generating maps. The homomorphism $\mathbb{C}[\mathbb{Y}] \to \mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W[\hbar]$ maps z to $\hbar^d$ and so extends to a graded $\mathbb{C}[\hbar]$-algebra homomorphism $\mathbb{C}[\mathbb{Y}'] \to \mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W[\hbar]$. At $\hbar = 0$, it specializes to the isomorphism $\mathbb{C}[\mathbb{Y}_0] \xrightarrow{\sim} \mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W$. Note that both $\mathbb{C}[\mathbb{Y}']$ and $\mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W[\hbar]$ are positively graded. It follows that $\mathbb{C}[\mathbb{Y}'] \xrightarrow{\sim} \mathbb{C}[U] \otimes \mathbb{C}[\mathfrak{P}]^W[\hbar]$. □

Proof of Proposition 3.5. Let $B', \mathcal{A}'$ be as in the statement of the proposition. Consider the Rees algebras $B'_\hbar := R_\hbar(B'), \mathcal{A}'_\hbar := R_\hbar(\mathcal{A}')$. Take $z := \hbar^d$. Thanks to Proposition 3.12, for each choice of a generating map for $\mathcal{A}'_\hbar$, we get a unique $\mathbb{C}[z]$-linear homomorphism $\mathbb{C}[\mathbb{Y}] \to B'_\hbar$ and a unique $B'_\hbar$-linear isomorphism $B'_\hbar \otimes_{\mathbb{C}[\mathbb{Y}]} \mathcal{A}_{\mathbb{Y}} \to \mathcal{A}'_\hbar$ intertwining the generating maps. Thanks to the uniqueness, these homomorphisms are graded. Different choices of generating maps for $\mathcal{A}'_\hbar$ result in an action of $U(B'_\hbar)$ and hence do not change the homomorphism $\mathbb{C}[\mathfrak{P}]^W[\hbar] \to B'_\hbar$. Hence there is a unique $\mathbb{C}[\hbar]$-linear graded algebra homomorphism $\mathbb{C}[\mathfrak{P}]^W[\hbar] \to B'_\hbar$ and a graded $B'_\hbar$-algebra isomorphism $B'_\hbar \otimes_{\mathbb{C}[\mathfrak{P}]^W[\hbar]} \mathcal{A}^W_{\mathfrak{P},\hbar} \xrightarrow{\sim} \mathcal{A}'_\hbar$. We specialize at $\hbar = 1$ and arrive at the claim of the proposition. □

3.6. **Spherical symplectic reflection algebras.** Let us consider the case of a symplectic quotient singularity $X = V/\Gamma$. Let $\pi : V \to V/\Gamma$ denote the quotient morphism. Recall that by a symplectic reflection in Γ we mean an element $s \in \Gamma$ with $\operatorname{rk}(s - \operatorname{id}) = 2$. To a symplectic reflection s we assign the subgroup $\Gamma^s \subset \Gamma$, the pointwise stabilizer of V^s,

and the quotient $\Xi^s := N_\Gamma(\Gamma^s)/\Gamma^s$. The codimension 2 symplectic leaves are in one-to-one correspondence with the conjugacy classes of the subgroups Γ^s. The leaf corresponding to Γ^s is of the form $\{v \in V | \Gamma_v = \Gamma^s\}/\Xi^s$. So we see that the fundamental group is Ξ^s. From here we deduce that the irreducible components of $\pi^{-1}(\mathcal{L})$ are labelled by the non-trivial $N_\Gamma(\Gamma^s)$-conjugacy classes in Γ^s.

The following lemma is proved in [Bel, Lemma 2.4].

Lemma 3.16. *We have $H^2(X^{reg}, \mathbb{C}) = 0$.*

From Lemma 3.16 and the preceding paragraph we conclude that the dimension of $\mathfrak{P}$ coincides with the number of conjugacy classes of symplectic reflections in Γ, this result was obtained in [Bel, Theorem 1.3].

There is a way to deform $\mathbb{C}[V]^\Gamma$ discovered by Etingof and Ginzburg, [EG]. Namely, we first deform the smash-product algebra $\mathbb{C}[V]\#\Gamma$. Let $t \in \mathbb{C}$ and c be a Γ-invariant function $S \to \mathbb{C}$, where S is the set of all symplectic reflections. Let ω denote the symplectic form on V. For $s \in S$, we write ω_s for the rank 2 form on V whose kernel coincides with V^s and whose restriction to $\operatorname{im}(s-1)$ coincides with the restriction of ω. Then we can form the algebra $H_{t,c}$ (known as a symplectic reflection algebra) by

$$H_{t,c} = T(V)\#\Gamma / \left(u \otimes v - v \otimes u - t\omega(u,v) - \sum_{s \in S} c(s)\omega_s(u,v)s | u, v \in V \right).$$

This is a filtered deformation of $\mathbb{C}[V]\#\Gamma$, see [EG, Theorem 1.3]. Now take the averaging idempotent $e \in \mathbb{C}\Gamma$. We can form the so called *spherical subalgebra* $eH_{t,c}e \subset H_{t,c}$ that is a filtered associative algebra with unit e. It is a filtered deformation of $\mathbb{C}[V]^\Gamma$ that induces the Poisson bracket $t\omega^{-1}$ on $\mathbb{C}[V]^\Gamma$, see the proof of [EG, Theorem 1.6]. So we get a filtered quantization when $t = 1$.

Let $\mathfrak{c}$ denote the space of Γ-invariant functions $S \to \mathbb{C}$, this is the space of parameters c. The space $\mathfrak{c}$ is in an affine bijection with $\mathfrak{P}$. Namely, we can split S into the union $S_1 \sqcup S_2 \sqcup \ldots \sqcup S_k$, where S_i stands for the symplectic reflections in Γ_i. Consequently, $\mathfrak{c}$ splits into the direct sum $\bigoplus_{i=1}^k \mathfrak{c}_i$. Let $\widehat{\mathfrak{c}}_i$ denote the space of Γ_i-invariant functions $\Gamma_i \setminus \{1\} \to \mathbb{C}$. It comes with a natural action of Ξ_i. The space $\mathfrak{c}_i$ embeds as the Ξ_i-invariants into $\widehat{\mathfrak{c}}_i$.

An affine isomorphism between $\widehat{\mathfrak{c}}_i$ and $\widehat{\mathfrak{P}}_i$ in the form we need was explained in [Lo5, Section 6.2]. Namely, define the element $C_i \in \mathbb{C}\Gamma_i$ by $|\Gamma_i|^{-1}\left(1 + \sum_{\gamma \in \Gamma_i} c(\gamma)\gamma_i\right)$. Let $N_1, \ldots, N_{r_i}$ denote the nontrivial irreducible representations of Γ_i. Then we send an element $\{c(\gamma)\}|_{\gamma \in \Gamma_i} \in \widehat{\mathfrak{c}}_i$ to $\sum_{j=1}^{r_i} (\operatorname{tr}_{N_j} C_i)\varpi_j$, where $\varpi_1, \ldots, \varpi_{r_i}$ are the fundamental weights in $\widehat{\mathfrak{P}}_i$. Note that this isomorphism $\widehat{\mathfrak{c}}_i \xrightarrow{\sim} \widehat{\mathfrak{P}}_i$ is Ξ_i-equivariant.

Let us denote the resulting affine isomorphism $\mathfrak{c} \xrightarrow{\sim} \mathfrak{P}$ by ι.

The following proposition generalizes [Lo5, Theorem 6.2.1].

Proposition 3.17. *We have an isomorphism $eH_{1,c}e \xrightarrow{\sim} \mathcal{A}_{\iota(c)}$ of filtered quantizations of $\mathbb{C}[V]^\Gamma$.*

In particular, every filtered quantization of $\mathbb{C}[V]^\Gamma$ is a spherical symplectic reflection algebra (see [Bod, Ho, Le] for various special cases of this result). An analog of this result for $t = 0$ was obtained by Bellamy in [Bel, Corollary 1.6].

Proof of Proposition 3.17. Thanks to Theorem 3.4, we already know that $eH_{1,c}e \cong \mathcal{A}_\lambda$ for some $\lambda \in \mathfrak{P}$ and we need to show that $\lambda = \iota(c)$. Consider the Rees algebra $R_\hbar(eH_{1,c}e)$

and its completion $R_\hbar(eH_{1,c}e)^{\wedge_x}$ at a point $x \in \mathcal{L}_i$. According to [Lo3, Theorem 1.2.1], we get an isomorphism of formal quantizations

$$R_\hbar(eH_{1,c}e)^{\wedge_x} \cong \mathbb{A}_\hbar(V^{\Gamma_i})^{\wedge_0} \widehat{\otimes}_{\mathbb{C}[[\hbar]]} R_\hbar(e^i H^i_{1,c_i} e^i)^{\wedge_0}.$$

Here the notation is as follows. We write c_i for the projection of c to $\mathfrak{c}_i$. The notation $\mathbb{A}_\hbar(V^{\Gamma_i})$ is for the Rees algebra of the Weyl algebra of the symplectic vector space V^{Γ_i}. The notation H^i_{1,c_i} is for the SRA associated to $(\Gamma_i, V/V^{\Gamma_i})$ and e^i is the averaging idempotent in Γ_i.

By Lemma 2.15, $\mathfrak{P}_0 = \{0\}$. So, by Proposition 3.6, we reduce to proving that, in the notation of that proposition, we have an isomorphism of filtered quantizations $e^i H^i_{1,c_i} e^i \cong \mathcal{A}^i_{\lambda_i}$. This follows from [Lo5, Theorem 6.2.2]. □

This proposition establishes an isomorphism $eH_{1,c}e \cong eH_{1,c'}e$ of quantizations when $\iota(c)$ and $\iota(c')$ are W-conjugate. In the next section we will determine when $eH_{1,c}e, eH_{1,c'}e$ are isomorphic as filtered algebras.

Remark 3.18. One application of Proposition 3.17 is to construct shift $H_{1,c+\psi}$-$H_{1,c}$-bimodules $\mathsf{S}_{c,\psi}$, where ψ is an integral element of $\mathfrak{c}$, compare with [BC, Lo8]. For a fixed ψ and a Zariski generic c, the bimodule $\mathsf{S}_{c,\psi}$ gives a Morita equivalence between $H_{1,c}$ and $H_{1,c+\psi}$, which can be established similarly to [Lo8, Corollary 3.5] using the fact that the algebra $H_{1,c}$ is simple for a Weil generic c, [Lo3, Theorem 4.2.1]. We do not provide details in the present paper, see [Lo10] instead.

3.7. **Automorphisms and isomorphisms.** Here we are going to study the relationship between three different objects:

(1) The reductive part of the automorphism group of the graded Poisson algebra $\mathbb{C}[X]$ (note that this group is algebraic). The reductive part will be denoted by $\mathcal{G}$.
(2) Filtered Poisson algebra isomorphisms $\mathbb{C}[X_\lambda] \xrightarrow{\sim} \mathbb{C}[X_{\lambda'}]$.
(3) Filtered algebra isomorphisms $\mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{\lambda'}$.

In what follows we always assume that d is even: we can always rescale the action of $\mathbb{C}^\times$ on X to achieve this.

Note that $\mathcal{G}$ acts on the set of isomorphism classes of filtered Poisson deformations (resp, quantizations) by replacing the isomorphism $\iota : \operatorname{gr}\mathcal{A} \xrightarrow{\sim} \mathbb{C}[X]$ with $g \circ \iota$, for $g \in \mathcal{G}$. So we have two, a priori different, actions of $\mathcal{G}$ on $\mathfrak{P}/W$ viewed as the space of parameters for filtered Poisson deformations and for filtered quantizations. These actions will be called Poisson and quantum below. Note that the universal properties for the algebras $\mathbb{C}[X_{\mathfrak{P}}]^W$ (Corollary 2.13) and $\mathcal{A}^W_{\mathfrak{P}}$ (Proposition 3.5) yield $\mathcal{G}$-actions on $\mathbb{C}[X_{\mathfrak{P}}]^W$ (by graded Poisson algebra automorphisms) and on $\mathcal{A}^W_{\mathfrak{P}}$ (by filtered algebra automorphisms). Let us explain why this is the case for quantizations. Any element $g \in \mathcal{G}$ gives rise to an automorphism $\tilde{g} : \mathcal{A}^W_{\mathfrak{P}} \to \mathcal{A}^W_{\mathfrak{P}}$ by the universal property. The automorphism $\tilde{g}$ is defined uniquely up to composing with a $\mathbb{C}[\mathfrak{P}]^W$-linear automorphism of $\mathcal{A}^W_{\mathfrak{P}}$ that gives the identity on $\mathbb{C}[X]$. The group of such automorphisms is easily seen to be unipotent. So we can choose $\tilde{g}$ so that the map $g \mapsto \tilde{g}$ is a group homomorphism.

These actions preserve the subalgebras $\mathbb{C}[\mathfrak{P}]^W$ and induce the Poisson and quantum actions on $\mathfrak{P}/W$. Note that the quantum action on $\mathbb{C}[\mathfrak{P}]^W$ is filtration preserving and the Poisson action on $\mathbb{C}[\mathfrak{P}]^W$ is obtained from the quantum action by passing to the associated graded action.

Lemma 3.19. *We have a filtered Poisson algebra isomorphism $\mathbb{C}[X_\lambda] \xrightarrow{\sim} \mathbb{C}[X_{\lambda'}]$ (resp., filtered associative algebra isomorphism $\mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{\lambda'}$) if and only if $W\lambda, W\lambda'$ lie in the same $\mathcal{G}$-orbit for the Poisson (resp., quantum) action on $\mathfrak{P}/W$.*

Proof. The if part follows from the preceding discussion. So we need to show, say, that if $\mathcal{A}_\lambda, \mathcal{A}_{\lambda'}$ are isomorphic as filtered algebras, then $W\lambda, W\lambda'$ are in the same $\mathcal{G}$-orbit.

An isomorphism $\mathcal{A} \to \mathcal{A}'$ of filtered (associative/Poisson) algebras induces an isomorphism $\operatorname{gr}\mathcal{A} \to \operatorname{gr}\mathcal{A}'$ of graded Poisson algebras. So $W\lambda, W\lambda'$ lie in the same orbit for the group of graded Poisson automorphisms of $\mathbb{C}[X]$. We need to show that the unipotent radical of this group acts trivially on $\mathfrak{P}/W$. Let g be an element in the unipotent radical. Since it is unipotent, $\ln(g)$ makes sense. It is a graded Poisson algebra derivation of $\mathbb{C}[X]$. By Proposition 2.14, it is inner. So it is the Poisson bracket with a degree d element in $\mathbb{C}[X]$. Lift this element to an element $a \in (\mathcal{A}_{\mathfrak{P}}^W)_{\leqslant d}$. Since $\ln(g)$ is nilpotent, so is $[a, \cdot]$. So we can integrate $[a, \cdot]$ to a $\mathbb{C}[\mathfrak{P}]^W$-linear filtered algebra isomorphism of $\mathcal{A}_{\mathfrak{P}}^W$ lifting g. From here we deduce that g acts trivially on $\mathbb{C}[\mathfrak{P}]^W$. □

We are now going to show that the Poisson and quantum actions are the same. In the proof we will need a lemma describing the group of graded Poisson automorphisms of $\mathbb{C}[V]^\Gamma$. Namely, let V be a symplectic vector space and Γ be a finite group of its linear symplectomorphisms. Set $\Theta := N_{\mathrm{Sp}(V)}(\Gamma)/\Gamma$. This groups acts on $\mathbb{C}[V]^\Gamma$ faithfully.

Lemma 3.20. *The group $\mathcal{G}$ of graded Poisson automorphisms of $\mathbb{C}[V]^\Gamma$ coincides with Θ.*

Proof. We have an inclusion $\Theta \hookrightarrow \mathcal{G}$. Note that V^0, the free locus for the Γ-action, is the simply-connected cover of $V^0/\Gamma = (V/\Gamma)^{reg}$. The Galois group of this cover is Γ. Every automorphism of $\mathbb{C}(V/\Gamma) = \mathbb{C}(V)^\Gamma$ lifts to an automorphism of $\mathbb{C}(V)$. The lift is unique up to composing with an element of Γ. An automorphism of V^0/Γ therefore lifts to an automorphism of V^0. So the $\mathcal{G}$-action on V^0/Γ lifts to an action of an extension $\tilde{\mathcal{G}}$ of $\mathcal{G}$ by Γ on V^0 by automorphisms.

The action of $\tilde{\mathcal{G}}$ on V^0 extends to an action on V because $\operatorname{codim}_V(V \setminus V^0) \geqslant 2$. It commutes with the dilating $\mathbb{C}^\times$ and preserves the symplectic form. So it is via a group homomorphism $\tilde{\mathcal{G}} \to \mathrm{Sp}(V)$. Also it descends to V/Γ and so normalizes Γ. We deduce that $\tilde{\mathcal{G}} \subset N_{\mathrm{Sp}(V)}(\Gamma)$ and hence $\mathcal{G} \subset \Theta$. □

Proposition 3.21. *The Poisson and quantum actions coincide. Moreover, both are trivial on $\mathcal{G}^\circ$.*

Proof. Recall that the Poisson action on $\mathbb{C}[\mathfrak{P}]^W$ is the associated graded action for the quantum action. So we need to prove that the quantum action on $\mathbb{C}[\mathfrak{P}]^W$ is by graded algebra automorphisms and is trivial on $\mathcal{G}^\circ$.

Take $g \in \mathcal{G}$. Recall, Lemma 2.8, that $\mathfrak{P}/W = H^2(X^{reg}, \mathbb{C}) \times \prod_{i=1}^k \mathfrak{P}_i/W_i$. We claim that

(*) the projection $\mathfrak{P}/W \twoheadrightarrow H^2(X^{reg}, \mathbb{C})$ intertwines the action of g with the usual linear action on $H^2(X^{reg}, \mathbb{C})$.

Indeed, by Proposition 3.6, the image of $W\lambda$ under the projection to $H^2(X^{reg}, \mathbb{C})$ is the period of the restriction of $\mathcal{A}_\lambda$ to X^{reg}. The period map is functorial by the construction in [BeKa, Section 4]. This implies (*). Note that (*) in particular implies that the action of $\mathcal{G}^\circ$ on the factor $H^2(X^{reg}, \mathbb{C})$ of $\mathfrak{P}/W$ is trivial.

Recall that Σ_i denotes a slice to the symplectic leaf $\mathcal{L}_i, \Sigma_i = \mathbb{C}^2/\Gamma_i$, and Ξ_i is the group of diagram automorphisms of the corresponding Dynkin diagram coming from the action of $\pi_1(\mathcal{L}_i)$.

The element g permutes the symplectic leaves (we denote the corresponding permutation of $\{1,\ldots,k\}$ again by g). We have $\Gamma_i \cong \Gamma_{g(i)}$ and $\Xi_i \cong \Xi_{g(i)}$. Fix some identifications. Then g gives rise to automorphisms of Γ_i and Ξ_i.

Fix a parameter $\lambda \in \mathfrak{P}$ and a point $x \in \mathcal{L}_i$. Let λ' be a representative of $g(W\lambda)$. So g gives rise to a filtered isomorphism $\mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{\lambda'}$. Recall, (3.2), the decompositions

$$R_\hbar(\mathcal{A}_\lambda)^{\wedge_x} \cong \mathbb{A}_\hbar(T_x\mathcal{L}_i)^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}R_\hbar(\mathcal{A}^i_{\lambda_i})^{\wedge_0}, R_\hbar(\mathcal{A}_{\lambda'})^{\wedge_{gx}} \cong \mathbb{A}_\hbar(T_{gx}\mathcal{L}_{gi})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}R_\hbar(\mathcal{A}^{gi}_{\lambda'_{gi}})^{\wedge_0}.$$

So g gives an isomorphism of the right hand sides in these decompositions:

$$\alpha : \mathbb{A}_\hbar(T_x\mathcal{L}_i)^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}R_\hbar(\mathcal{A}^i_{\lambda_i})^{\wedge_0} \xrightarrow{\sim} \mathbb{A}_\hbar(T_{gx}\mathcal{L}_{gi})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}R_\hbar(\mathcal{A}^{gi}_{\lambda'_{gi}})^{\wedge_0}. \tag{3.6}$$

We claim that we can compose α with a $\mathbb{C}[[\hbar]]$-linear automorphism of the target so that the composition is $\mathbb{C}^\times$-equivariant. Let γ, γ' denote the actions of $\mathbb{C}^\times$ on the source and the target of (3.6) and let γ_0, γ'_0 denote the actions modulo $\hbar$, note that the corresponding isomorphism α_0 can be viewed as an automorphism.

We argue similarly to Step 1 of the proof of Proposition 3.6. We can modify α so that $\alpha\gamma\alpha^{-1}$ and γ' commute. In particular, $\alpha_0\gamma_0\alpha_0^{-1}$ preserves

$$(T_{gx}\mathcal{L}_{gi})^* \subset \mathbb{A}_\hbar(T_{gx}\mathcal{L}_{gi})^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}R_\hbar(\mathcal{A}^{gi}_{\lambda'_{gi}})^{\wedge_0}/(\hbar)$$

by the degree reasons: this is the subspace of elements of degree $d/2$. So it preserves its Poisson centralizer, $R_\hbar(\mathcal{A}^{gi}_{\lambda'_{gi}})^{\wedge_0}/(\hbar)$. Consider the maximal ideal $\mathfrak{m}$ in that algebra so that $\mathfrak{m}/\mathfrak{m}^2$ is 3-dimensional. Since α_0 is an automorphism, the eigenvalues of $\alpha_0\gamma_0\alpha_0^{-1}$ and γ'_0 on $\mathfrak{m}/\mathfrak{m}^2$ coincide. Combining this with the observation that these actions commute and rescale the bracket by $t \mapsto t^{-d}$, we can use an easy case by case argument to see that they coincide on $\mathfrak{m}/\mathfrak{m}^2$. So the two actions on the target of (3.6) differ by a pro-unipotent family of automorphisms and hence are conjugate by a pro-unipotent automorphism. Modifying α accordingly, we get a $\mathbb{C}^\times$-equivariant isomorphism in (3.6).

The conclusion is that we have a filtered algebra isomorphism $\mathbb{A}(T_x\mathcal{L}_i) \otimes \mathcal{A}^i_{\lambda_i} \xrightarrow{\sim} \mathbb{A}(T_{gx}\mathcal{L}_{gi}) \otimes \mathcal{A}^{gi}_{\lambda'_{gi}}$.

Note that the algebras above are filtered quantizations of the symplectic quotient singularity $T_x\mathcal{L}_i \times \mathbb{C}^2/\Gamma_i$. The space of quantization parameters is $\widehat{\mathfrak{P}}_i/\widehat{W}_i$. The map $\widehat{\mathfrak{P}}_i/\widehat{W}_i \to \widehat{\mathfrak{P}}_i/\widehat{W}_i$ coming from g is in the image of the action of the group Θ from Lemma 3.20. This group coincides with $N_{\mathrm{SL}_2}(\Gamma_i)/\Gamma_i$. The discussion of the isomorphism $\widehat{\mathfrak{c}}_i \xrightarrow{\sim} \widehat{\mathfrak{P}}_i$ in Section 3.6 shows that the action of $N_{\mathrm{SL}_2}(\Gamma_i)/\Gamma_i$ on $\widehat{\mathfrak{P}}_i$ is by linear automorphisms. So the automorphism of $\widehat{\mathfrak{P}}_i/\widehat{W}_i$ induced by g is $\mathbb{C}^\times$-equivariant.

Now we show that the automorphism of $\mathfrak{P}_i/W_i$ induced by g is $\mathbb{C}^\times$-equivariant. Step 2 of the proof of Proposition 3.6 shows that the map $\mathfrak{P}_i/W_i \to \widehat{\mathfrak{P}}_i/\widehat{W}_i$ is injective. It also intertwines the actions of g. It follows that the action of g on $\mathfrak{P}_i/W_i$ is $\mathbb{C}^\times$-equivariant. This finishes the proof of the claim that g acts on $\mathfrak{P}/W$ by a $\mathbb{C}^\times$-equivariant automorphism.

The claim that the action of $\mathcal{G}^\circ$ on $\mathfrak{P}/W$ is trivial follows from the observation that the action of $(N_{\mathrm{SL}_2}(\Gamma_i)/\Gamma_i)^\circ$ on $\widehat{\mathfrak{P}}_i$ is trivial. □

Corollary 3.22. *The filtered Poisson algebras $\mathbb{C}[X_\lambda], \mathbb{C}[X_{\lambda'}]$ are isomorphic if and only if the filtered associative algebras $\mathcal{A}_\lambda, \mathcal{A}_{\lambda'}$ are. So we have a bijection between*

- *Filtered Poisson deformations of* $\mathbb{C}[X]$ *viewed up to a filtered Poisson algebra isomorphism,*
- *and filtered quantizations of* $\mathbb{C}[X]$ *viewed up to a filtered algebra isomorphism.*

Proof. The first claim follows from Proposition 3.21. Recall that both the filtered Poisson deformations and the filtered quantizations are classified by $\mathfrak{P}/W$ (Proposition 2.9 and Corollary 2.13 for Poisson deformations and Theorem 3.4 for quantizations). Up to filtered algebra isomorphisms both are classified by the orbits of the group $\mathcal{G}$ on $\mathfrak{P}/W$. □

Let us get back to spherical symplectic reflection algebras.

Proposition 3.23. *We have a filtered algebra isomorphism* $eH_{1,c}e \cong eH_{1,c'}e$ *if and only if* Wc *and* Wc' *are* Θ*-conjugate.*

Proof. This follows from Lemma 3.19 and Lemma 3.20. □

Remark 3.24. We will need a relative version of results of this section. Let $G \subset \mathcal{G}$ be a connected reductive subgroup. By Proposition 3.21, G acts trivially on $\mathfrak{P}/W$, hence it acts on each $\mathcal{A}_\lambda, \mathcal{A}_\lambda^0$ by filtered algebra automorphisms. As in Lemma 3.19, we have a G-equivariant filtered algebra isomorphism $\mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{\lambda'}$ if and only if $W\lambda, W\lambda'$ are in the same $Z_{\mathcal{G}}(G)$-orbit. The same holds for filtered Poisson deformations. Proposition 3.21 implies that there is a G-equivariant filtered algebra isomorphism $\mathcal{A}_\lambda \xrightarrow{\sim} \mathcal{A}_{\lambda'}$ if and only if there is a G-equivariant filtered Poisson algebra isomorphism $\mathcal{A}_\lambda^0 \xrightarrow{\sim} \mathcal{A}_{\lambda'}^0$. Also note that the stabilizer of $W\lambda$ in $Z_{\mathcal{G}}(G)$ acts on $\mathcal{A}_\lambda$ and $\mathcal{A}_\lambda^0$ by G-equivariant filtered algebra automorphisms.

4. Birational induction and sheets

From now on, G is a connected reductive algebraic group over $\mathbb{C}$ with Lie algebra $\mathfrak{g}$.

4.1. **Lusztig-Spaltenstein induction.** We use the notation of Section 1.4. Recall that $\mathfrak{l} \subset \mathfrak{g}$ is a Levi subalgebra, $P \subset G$ is a parabolic subgroup such that $\mathfrak{p} := \mathrm{Lie}(P)$ has Levi $\mathfrak{l}$, $\mathbb{O}' \subset \mathfrak{l}^*$ is a nilpotent orbit, and $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. Form the variety $G \times^P (\xi + \overline{\mathbb{O}}' + \mathfrak{p}^\perp)$, the homogeneous bundle over G/P with fiber $\xi + \overline{\mathbb{O}}' + \mathfrak{p}^\perp$. This variety maps to $\mathfrak{g}^*$. Let π denote the map onto its image, to be called the *generalized Springer map*. Let $\mathbb{O}_\xi$ denote the open orbit in $\operatorname{im}\pi$, it is known to be independent of the choice of P. Note that, similarly to [LS, Theorem 1.3], $(\xi + \overline{\mathbb{O}}' + \mathfrak{p}^\perp) \cap \mathbb{O}_\xi$ is a single P-orbit.

Lemma 4.1. *The open* G*-orbit in* $G \times^P (\xi + \overline{\mathbb{O}}' + \mathfrak{p}^\perp)$ *depends only on* $(\mathfrak{l}, \mathbb{O}', \xi)$*, not on the choice of* P*.*

In the proof we will need the following construction, also to be used later. Let $\mathbb{O}$ denote a nilpotent orbit in $\mathfrak{g}^*$. Pick an element $\chi \in \mathbb{O}$ and let Q be a maximal reductive subgroup of the stabilizer G_χ. Also we can pick an invariant form on $\mathfrak{g}$ and identify $\mathfrak{g} \cong \mathfrak{g}^*$. Let $e \in \mathfrak{g}$ be the element corresponding to χ. Include e into an $\mathfrak{sl}_2$-triple e, h, f. We can assume that h, f are Q-stable, then $Q = Z_G(e,h,f)$. The component group Q/Q° is identified with the G-equivariant fundamental group of $\mathbb{O}$, to be denoted by $A(\mathbb{O})$.

Consider the Slodowy slice $e + \mathfrak{z}_{\mathfrak{g}}(f)$ and let S denote its image in $\mathfrak{g}^*$. Then S is a transversal slice to $\mathbb{O}$. Note that it is a Poisson variety with a Q-action. We also have a contracting $\mathbb{C}^\times$-action on S: let $\gamma : \mathbb{C}^\times \to G$ denote the one-parameter subgroup corresponding to h, then we define a $\mathbb{C}^\times$-action on S via $t.s := t^{-2}\gamma(t)s$. This action contracts S to χ. The proofs can be found e.g. in [GG, Section 2].

Proof of Lemma 4.1. The proof is in several steps.

Step 1. The claim easily reduces to the case when ξ is central (by replacing G with the stabilizer G_ξ). We can then assume that $\xi = 0$. Let $\mathbb{O}$ denote the open orbit in $\operatorname{im}\pi$. Let $X' := \operatorname{Spec}(\mathbb{C}[\mathbb{O}'])$. Set $\check{X} = G \times^P ((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^* \times X' \times \mathfrak{p}^\perp)$. The open orbit in $\check{X}$ coincides with that in $G \times^P (\overline{\mathbb{O}}' + \mathfrak{p}^\perp)$.

The variety $\check{X}$ naturally maps to $\mathfrak{g}^*$ and also to $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. The morphism to $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is flat, G-invariant and $\mathbb{C}^\times$-equivariant. Note that

$$\dim \check{X} = \dim(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^* + \dim \mathbb{O}. \tag{4.1}$$

Step 2. Consider the preimage $\check{S}$ of S in $\check{X}$ and the restriction of $\check{X} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ to $\check{S}$. Our goal is to prove that this restriction is finite and étale.

We start by showing $\dim \check{S} = \dim(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. The slice S is transverse to all G-orbits it intersects, see, e.g., [GG, Section 2.2]. Equivalently, the action morphism $G \times S \to \mathfrak{g}^*$ is smooth of relative dimension $\dim S$. Since being smooth is stable under pullback, the action morphism $G \times \check{S} \to \check{X}$ is also smooth of relative dimension $\dim S$. It follows that

$$\dim \check{S} = \dim \check{X} + \dim S - \dim \mathfrak{g} = (\dim(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^* + \dim \mathbb{O}) + \dim S - \dim \mathfrak{g} = \dim(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*.$$

The first equality follows from (4.1), and the second follows because $\dim \mathfrak{g} = \dim S + \dim \mathbb{O}$.

Step 3. The variety $\check{S}$ inherits actions of $\mathbb{C}^\times$ and Q from S. The morphism $\check{S} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is Q-invariant and $\mathbb{C}^\times$-equivariant by the construction. We claim that the $\mathbb{C}^\times$-action on $\check{S}$ is contracting. Indeed, the morphism $\check{X} \to \mathfrak{g}^* \times (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is projective. It follows that the morphism $\check{S} \to S \times (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is projective. This morphism is $\mathbb{C}^\times$-equivariant. The $\mathbb{C}^\times$-action on $S \times (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ contracts $S \times (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ to a point. Hence the action on $\check{S}$ is contracting as well.

Step 4. We claim that the morphism $\check{S} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is unramified. Thanks to the contracting $\mathbb{C}^\times$-action we need to show that the fiber over zero is a reduced finite scheme. This fiber coincides with the scheme theoretic preimage of S in $G \times^P (X' \times \mathfrak{p}^\perp)$. The morphism $G \times^P (X' \times \mathfrak{p}^\perp) \to \overline{\mathbb{O}}$ is étale over $\mathbb{O}$. The slice S intersects $\mathbb{O}$ transversally at the single point, χ, and does not intersects the boundary of $\mathbb{O}$. It follows that the scheme theoretic preimage of S in $G \times^P (X' \times \mathfrak{p}^\perp)$ coincides with the preimage of χ in the open orbit of $G \times^P (X' \times \mathfrak{p}^\perp)$. This and the claim that the intersection of S with $\mathbb{O}$ is transversal show the claim in the beginning of the step. Note that the preimage of S in $G \times^P (X' \times \mathfrak{p}^\perp)$ is a single Q/Q°-orbit.

Step 5. We have seen in Steps 2-4 that $\dim \check{S} = \dim(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ and the morphism $\check{S} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is $\mathbb{C}^\times$-equivariant and etale. Since the $\mathbb{C}^\times$-action on $\check{S}$ is contracting (Step 3) and the preimage of $0 \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is finite, the morphism $\check{S} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is finite. The group Q acts on $\check{S}$ and the map $\check{S} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is Q-invariant. It follows that Q° acts on $\check{S}$ trivially.

Step 6. As we have pointed out in Step 4, the preimage of χ in $\check{S}$ is a single Q/Q°-orbit, that is the preimage of χ in the open orbit in $G \times^P (X' \times \mathfrak{p}^\perp)$. Since the morphism $\check{S} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is étale and finite, the fiber of $\check{S}$ over any point of $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is the same Q/Q°-orbit. Also note that the fiber of $\check{X} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ over a generic point $\xi_1 \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is independent of the choice of P: this fiber is $G \times^L (\xi_1 \times X')$. The morphisms $G \times^L (\xi_1 \times X') \to \mathfrak{g}^*, (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ are also independent of the choice of P. So the fiber of $\check{S} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ over such a point is independent of the choice of P. This is also the fiber over χ. The claim of the lemma follows. $\square$

Below we write H for the stabilizer G_x.

Recall, Definition 1.2, that $(\mathfrak{l}, \mathbb{O}', \xi)$ is called an induction datum. When $\xi = 0$ we omit it and say that $(\mathfrak{l}, \mathbb{O}')$ is an induction datum. We also can talk about birationally induced and birationally rigid orbits. More generally, we say that the open orbit in $G \times^P (\xi + \overline{\mathbb{O}}' + \mathfrak{p}^\perp)$, an equivariant cover of a coadjoint orbit, is *birationally induced* from $(\mathfrak{l}, \mathbb{O}', \xi)$. Finally, recall, Definition 1.2, that if π is birational and $\mathbb{O}'$ is birationally rigid, then we say that $(\mathfrak{l}, \mathbb{O}', \xi)$ is a *birationally minimal induction datum.*

We have the following properties of birationally minimal induction data.

Proposition 4.2. *For any fixed $\mathfrak{l}$ and birationally rigid $\mathbb{O}' \subset \mathfrak{l}$, the set $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}$ of all $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ such that $(\mathfrak{l}, \mathbb{O}', \xi)$ is birationally minimal, is the complement to a finite union of subspaces. Also this locus is independent of the choice of P.*

Proof. The independence of P directly follows from Lemma 4.1.

Now let $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. We can naturally embed $\mathfrak{l}^*$ into $\mathfrak{g}^*$ because there is a unique $\mathfrak{l}$-invariant complement to $\mathfrak{l}$ in $\mathfrak{g}$. Hence we can view ξ as an element of $\mathfrak{g}^*$. Set $\tilde{\mathfrak{l}} := \mathfrak{g}_\xi$, this is a Levi subalgebra. Clearly, $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}$ if and only if the induction from $(\mathfrak{l}, \mathbb{O}')$ to $\tilde{\mathfrak{l}}$ is birational. So the inclusion $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}$ depends only on $\tilde{\mathfrak{l}}$. There is a finite number of choices of $\tilde{\mathfrak{l}}$. So in order to complete the proof of the proposition, we only need to show that if the induction from $(\mathfrak{l}, \mathbb{O}')$ is birational, then so is the induction from $(\mathfrak{l}, \xi + \mathbb{O}')$ for any $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. In order to see this consider the varieties $\check{S} \subset \check{X}$ from the proof of Lemma 4.1. The claim that the induction from $(\mathfrak{l}, \mathbb{O}')$ is birational is equivalent to the condition that the preimage of $0 \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ in $\check{S}$ is a single point. According to Step 5 of that proof, the morphism $\check{S} \to (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ is finite and etale. It follows that it is an isomorphism. In particular, the preimage of S in $G \times^P (\xi \times X' \times \mathfrak{p}^\perp)$ is a single point for each $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. So the morphism from the open orbit in $G \times^P (\xi \times X' \times \mathfrak{p}^\perp)$ to $\mathfrak{g}^*$ must be generically injective. Equivalently, the induction from $(\mathfrak{l}, \xi + \mathbb{O}')$ is birational. $\square$

4.2. **Sheets and birational sheets.** Recall that by a *sheet* in $\mathfrak{g}$ we mean an irreducible component of $\{x \in \mathfrak{g} | \dim Gx = d\}$ for some d. Each sheet contains a single nilpotent orbit. The rigid nilpotent orbits are precisely the orbits $\mathbb{O}$ such that there is only one sheet containing $\mathbb{O}$ and this sheet is $\mathbb{O} + (\mathfrak{g}/[\mathfrak{g},\mathfrak{g}])^*$, see, e.g., [Bor, BoKr2]. In general, the sheets are indexed by the pairs $(\mathfrak{l}, \mathbb{O}')$, where $\mathbb{O}'$ is a rigid orbit in $\mathfrak{l}$: the corresponding sheet consists of orbits induced from $(\mathfrak{l}, \mathbb{O}', \xi)$ for $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$.

Let Z be a sheet and $\mathbb{O}$ be the unique nilpotent orbit contained in Z. It turns out that the action of G on Z admits a geometric quotient. Recall the subgroup $Q \subset G$ and an affine subspace $S \subset \mathfrak{g}^*$ from Section 4.1. The following is [Ka, Theorems 0.3,0.4]. Note that the component group of Q is $A(\mathbb{O})$.

Proposition 4.3. *The following is true:*

1. *The action of Q° on $Z \cap S$ is trivial and $A(\mathbb{O})$ permutes the components of $Z \cap S$ transitively.*
2. *The variety $(Z \cap S)/A(\mathbb{O})$ is the geometric quotient for the action of G on Z, in particular, it is a categorical quotient and each fiber of $Z \to (Z \cap S)/A(\mathbb{O})$ is a single G-orbit.*

In general, sheets do not behave well: they may intersect, may fail to be smooth, etc. *Birational sheets* to be introduced now do not intersect and are very often smooth (and

always smooth up to a bijective normalization). However, there are birational sheets that do not contain nilpotent orbits.

Pick a Levi subalgebra $\mathfrak{l}$ and a birationally rigid nilpotent orbit $\mathbb{O}' \subset \mathfrak{l}$. Let $Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ denote the set of all orbits birationally induced from $(\mathfrak{l}, \mathbb{O}', \xi)$, where $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}$.

To $(\mathfrak{l}, \mathbb{O}')$ we assign a finite group $W(\mathfrak{l}, \mathbb{O}')$ acting on $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ as follows. The group $N_G(L)$ naturally acts on the set of nilpotent orbits in $\mathfrak{l}^*$. Let $N_G(L, \mathbb{O}')$ denote the stabilizer of $\mathbb{O}'$. We set $W(\mathfrak{l}, \mathbb{O}') := N_G(L, \mathbb{O}')/L$. Note that this finite group naturally acts on $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. It follows from Proposition 4.2 (namely from the independence of P part) that $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}$ is $W(\mathfrak{l}, \mathbb{O}')$-stable.

The following theorem describes basic properties of birational sheets.

Theorem 4.4. *The following is true:*

(1) *Any coadjoint orbit is induced from a unique (up to G-cojugacy) birationally minimal induction datum. In particular, we have $\mathfrak{g} = \bigsqcup_{(\mathfrak{l},\mathbb{O}')} Z^{bir}_{\mathfrak{l},\mathbb{O}'}$.*
(2) *$Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ is a locally closed subvariety of $\mathfrak{g}$. Its normalization is smooth and the morphism from the normalization to $Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ is bijective.*
(3) *A geometric quotient for the G-action on $Z^{bir}_{\mathfrak{l},\mathbb{O}}$ exists. The normalization of the quotient is $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}/W(\mathfrak{l}, \mathbb{O}')$. This is a smooth variety. The morphism from the normalization to the quotient is bijective.*

This theorem will be proved in Section 4.5.

4.3. **Structure of $\mathbb{Q}$-terminalizations.** In order to prove Proposition 4.2 and Theorem 4.4 we will need to examine the structure of $\mathbb{Q}$-terminalizations. Results of this section have been already obtained by Namikawa, [N2], for classical types, and Fu, [F], for exceptional types. Their proofs used case-by-case arguments, while our proof is conceptual.

Our main result in this section is the following proposition.

Proposition 4.5. *Let $\mathbb{O}$ be a birationally rigid nilpotent orbit and let $X := \operatorname{Spec}(\mathbb{C}[\mathbb{O}])$. Then the following claims hold:*

(1) *$\mathfrak{P} = \{0\}$, equivalently (Proposition 2.12), $\mathbb{C}[X]$ has no nontrivial filtered Poisson deformations.*
(2) *X is $\mathbb{Q}$-factorial and terminal, and $H^2(X^{reg}, \mathbb{C}) = \{0\}$.*

Proof. In the proof we can assume that G is semisimple and simply connected. We also can assume that the action of G on $\mathbb{O}$ is faithful, equivalently, $\mathbb{O}$ projects nontrivially to all direct summands of $\mathfrak{g}$.

First, we show that (1) implies (2) (in fact, they are equivalent but we do not need this). By [N4, Theorem 5.5(c)], for a generic element $\lambda \in \mathfrak{P}$, the morphism $\tilde{X}_\lambda \to X_\lambda$ is an isomorphism. In particular, if $\mathfrak{P} = \{0\}$, then $\tilde{X} \xrightarrow{\sim} X$, hence X is $\mathbb{Q}$-factorial and terminal. The claim that $\mathfrak{P} = \{0\} \Rightarrow H^2(X^{reg}, \mathbb{C}) = \{0\}$ follows from Lemma 2.8.

It remains to prove that if $\mathbb{O}$ is birationally rigid, then (1) holds, i.e., $\mathfrak{P} = \{0\}$. The proof of this is in several steps.

Step 1. Here we consider the situation when $\mathbb{O}$ is an arbitrary nilpotent orbit. We consider the deformation $X_{\mathfrak{P}}$. We claim that we have a Hamiltonian G-action on $X_{\mathfrak{P}}$ with moment map deforming that on X.

Consider the degree 1 component $\mathbb{C}[X_{\mathfrak{P}}]_1$. Note that $\mathfrak{g} \hookrightarrow \mathbb{C}[X]_1$ and $\mathbb{C}[X_{\mathfrak{P}}]_1 = \mathbb{C}[X]_1 \oplus \mathfrak{P}$. Also note that since $d = 1$, $\mathbb{C}[X_{\mathfrak{P}}]_1$ is a Lie algebra with respect to $\{\cdot,\cdot\}$ and the embedding $\mathfrak{g} \hookrightarrow \mathbb{C}[X]_1$ is an inclusion of Lie algebras. So we get an extension of $\mathfrak{g}$ by

the abelian Lie algebra $\mathfrak{P}$. Therefore $\mathfrak{g}$ canonically splits. Note also that $\mathbb{C}[X_{\mathfrak{P}}]$ acquires a $\mathbb{C}[\mathfrak{P}]$-linear action of $\mathfrak{g}$ via $\{\cdot,\cdot\}$. This action preserves the grading, hence it is locally finite. Its weights are the same as in $\mathbb{C}[X]$. Hence it integrates to G. So we get the required Hamiltonian action. Let $\mu_{\mathfrak{P}} : X_{\mathfrak{P}} \to \mathfrak{g}^*$ denote the moment map and μ_λ be the restriction of $\mu_{\mathfrak{P}}$ to X_λ.

We remark that $\mu_{\mathfrak{P}}$ is W_X-invariant, equivalently, the image of $\mathfrak{g}$ in $\mathbb{C}[X_{\mathfrak{P}}]$ is W_X-invariant. Recall that W_X acts on $\mathbb{C}[X_{\mathfrak{P}}]$ by graded Poisson automorphisms. The subspace $\mathbb{C}[X_{\mathfrak{P}}]_1$ is W_X-stable and the projection $\mathbb{C}[X_{\mathfrak{P}}]_1 \twoheadrightarrow \mathbb{C}[X]_1$ is W_X-equivariant. The action on $\mathbb{C}[X_{\mathfrak{P}}]_1$ is by Lie algebra automorphisms. It follows that the preimage of $\mathfrak{g}$ in $\mathbb{C}[X_{\mathfrak{P}}]_1$ is W_X-stable. Since the action is by Lie algebra automorphisms and $\mathfrak{g}$ splits canonically, $\mathfrak{g}$ is W_X-stable. And since the projection of $\mathfrak{g}$ to $\mathbb{C}[X]_1$ is injective, the action of W_X on $\mathfrak{g}$ is trivial. So $\mu_{\mathfrak{P}}$ is indeed W_X-invariant.

Step 2. We still assume that $\mathbb{O}$ is an arbitrary nilpotent orbit. We claim that, for every $\lambda \in \mathfrak{P}$, the G-action on X_λ has an open orbit. There is an open G-orbit on $X = X_0$. The locus of $x \in X$ with $\dim Gx \geqslant \dim X$ is open and $\mathbb{C}^\times$-stable. Since the action of $\mathbb{C}^\times$ is contracting, there is an open orbit in every fiber X_λ. So μ_λ is generically finite and $\mu_\lambda(X_\lambda)$ is the closure of a single orbit to be denoted by $\mathbb{O}_\lambda$. Since μ_0 is finite and the action of $\mathbb{C}^\times$ on $X_{\mathfrak{P}}$ is contracting, μ_λ is finite for all λ. Note that $\overline{\mathbb{O}} \subset \overline{\mathbb{C}^\times \mathbb{O}_\lambda}$ and $\dim \mathbb{O}_\lambda = \dim \mathbb{O}$.

Step 3. By Step 2, $X_{\mathfrak{P}}/\!/G = \mathfrak{P}$. It follows that $\mu_{\mathfrak{P}}$ induces a morphism $\mathfrak{P} \to \mathfrak{g}^*/\!/G$. We claim that if $\mathfrak{P} \neq \{0\}$, then the image is different from $\{0\}$. Indeed, otherwise $\mathbb{O}_\lambda = \mathbb{O}$ for all λ. Moreover, since $X_{\mathfrak{P}}$ is a flat deformation of X, we see that the open orbit in X_λ is $\mathbb{O}$ and not its proper cover. We get the map $X_{\mathfrak{P}} \to \mathfrak{P} \times \overline{\mathbb{O}}$ induced by $\mu_{\mathfrak{P}}$. This map is finite and birational so it is the normalization. Also it is Poisson and $\mathbb{C}^\times$-equivariant. So it lifts to a $\mathbb{C}^\times$-equivariant Poisson isomorphism $X_{\mathfrak{P}} \xrightarrow{\sim} X \times \mathfrak{P}$. Since $X_{\mathfrak{P}}/W_X$ is a universal deformation, this implies $\mathfrak{P} = \{0\}$. Contradiction. We see that if $\mathfrak{P} \neq \{0\}$, then the image of $X_{\mathfrak{P}}$ in $\mathfrak{g}$ contains non-nilpotent elements.

Step 4. Now assume that $\mathbb{O}$ is birationally rigid. Pick a Zariski generic $\lambda \in \mathfrak{P}$. Let ξ be the semisimple part of an element in $\mathbb{O}_\lambda$, $\mathfrak{l}$ be the stabilizer of ξ in $\mathfrak{g}$ and $\mathbb{O}'$ be the nilpotent orbit in $\mathfrak{l}$ such that $\xi + \mathbb{O}' \subset \mathbb{O}_\lambda$. We claim that $\mathbb{O}$ is birationally induced from $(\mathfrak{l}, \mathbb{O}')$. Namely, let X' be the normalization of $\overline{\mathbb{O}'}$ and $\tilde{X}'$ be its $\mathbb{Q}$-terminalization. Consider the variety $\tilde{X}^1_{\mathbb{C}\xi} := G \times^P (\mathbb{C}\xi \times \tilde{X}' \times \mathfrak{p}^\perp)$. Here $\mathfrak{p}$ be a parabolic subalgebra with Levi subalgebra $\mathfrak{l}$ and P is the corresponding subgroup. The variety $\tilde{X}^1_{\mathbb{C}\xi}$ is a normal Poisson $\mathbb{C}\xi$-scheme and has a Hamiltonian G-action. Its algebra of regular functions is finite over $\mathbb{C}[\mathfrak{g}^*]$ hence is finitely generated. Let $X^1_{\mathbb{C}\xi} := \operatorname{Spec}(\mathbb{C}[\tilde{X}^1_{\mathbb{C}\xi}])$. Note that the fiber of $X^1_{\mathbb{C}\xi}$ over ξ is the normalization of $\overline{\mathbb{O}}_\lambda$. The fiber over 0 is $X^1 := \operatorname{Spec}(\mathbb{C}[\tilde{X}^1])$, where $\tilde{X}^1 = G \times^P (\tilde{X}' \times \mathfrak{p}^\perp)$. Note that we get a finite G-equivariant morphism $X^1 \twoheadrightarrow \overline{\mathbb{O}}$ that factors through $X^1 \twoheadrightarrow X$.

Since the morphism $X_{\mathfrak{P}} \to \mathfrak{P}$ is G-invariant and flat, for any irreducible G-module V, we have that $\operatorname{Hom}_G(V, \mathbb{C}[X_{\mathfrak{P}}])$ is flat over $\mathbb{C}[\mathfrak{P}]$. As was mentioned in the beginning of Step 3, $\mathbb{C}[X_{\mathfrak{P}}]^G \xrightarrow{\sim} \mathbb{C}[\mathfrak{P}]$. So $\operatorname{Hom}_G(V, \mathbb{C}[X_{\mathfrak{P}}])$ is also finitely generated $\mathbb{C}[\mathfrak{P}]$-module, hence is projective. It follows that $\mathbb{C}[X] \cong \mathbb{C}[X_\lambda]$ as G-modules. Analogously, $\mathbb{C}[X^1] \cong \mathbb{C}[X^1_\lambda]$. We have G-equivariant finite dominant morphisms $X^1 \twoheadrightarrow X$ and $X_\lambda \twoheadrightarrow X^1_\lambda$, the latter is induced by the moment map μ_λ. Hence $\mathbb{C}[X^1] \cong \mathbb{C}[X]$ as G-modules. We conclude that $X^1 \xrightarrow{\sim} X$. This contradicts $\mathbb{O}$ being birationally rigid and completes the proof of the proposition. $\square$

Corollary 4.6. *Let $\mathbb{O}$ be a nilpotent orbit. Then the following claims are true.*

(1) *There is a unique (up to G-conjugacy) birationally minimal induction datum $(\mathfrak{l}, \mathbb{O}')$ for $\mathbb{O}$.*
(2) *The variety $\tilde{X} := G \times^P (X' \times \mathfrak{p}^\perp)$ (where X' stands for $\operatorname{Spec}(\mathbb{C}[\mathbb{O}'])$) is a $\mathbb{Q}$-terminalization of X.*

Proof. We start by proving (2). We can find some minimal birational induction datum $(\mathfrak{l}, \mathbb{O}')$ for $\mathbb{O}$. The variety X' is $\mathbb{Q}$-factorial terminal by Proposition 4.5. Hence so is $\tilde{X}$. Hence it is a $\mathbb{Q}$-terminalization of X.

Let us prove (1). We need to prove that $(\mathfrak{l}, \mathbb{O}')$ is unique up to conjugacy. Note that the morphism $X_{\mathfrak{P}}/W_X \twoheadrightarrow \mathfrak{P}/W_X$ is uniquely recovered from X thanks to Proposition 2.12 and does not depend on the choice of the $\mathbb{Q}$-terminalization $\tilde{X} = G \times^P (X' \times \mathfrak{p}^\perp)$, i.e., the choices of P and $(\mathfrak{l}, \mathbb{O}')$. For $\lambda \in \mathfrak{P}$, the moment map $X_\lambda \to \mathfrak{g}^*$ is also independent of these choices. According to Step 4 of the proof Proposition 4.5, we recover $(\mathfrak{l}, \mathbb{O}')$ as follows. Consider the fiber X_λ of $X_{\mathfrak{P}}$ over a Zariski generic point $\lambda \in \mathfrak{P}$. Let $\mathbb{O}_\lambda$ denote the open orbit in the image of X_λ in $\mathfrak{g}^*$ and let $x \in \mathbb{O}_\lambda$. Then $\mathfrak{l}$ is the stabilizer of the semisimple part of x and and $\mathbb{O}'$ is the L-orbit of the nilpotent part of x. $\square$

4.4. **Computation of Weyl groups.** In this section we will get some information on the Namikawa-Weyl group W_X for $X := \operatorname{Spec}(\mathbb{C}[G/H])$, where G/H is the open G-orbit in $\tilde{X} = G \times^P (X' \times \mathfrak{p}^\perp)$. Here $X' = \operatorname{Spec}(\mathbb{C}[\mathbb{O}'])$ for a birationally rigid nilpotent orbit $\mathbb{O}' \subset \mathfrak{l}^*$.

Our main result is as follows.

Proposition 4.7. *Assume G is semisimple. The following claims are true:*

(1) *We have $\mathfrak{P} = (\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^*$.*
(2) *The universal deformation $\tilde{X}_{\mathfrak{P}}$ of $\tilde{X}$ over $\mathfrak{P}$ (Proposition 2.6) is isomorphic to $\check{X} := G \times^P (\mathfrak{P} \times X' \times \mathfrak{p}^\perp)$ as Poisson scheme over $\mathfrak{P}$ with a $\mathbb{C}^\times$-action.*
(3) *The Namikawa-Weyl group $W = W_X$ of X is a normal subgroup in $W(\mathfrak{l}, \mathbb{O}')$. Moreover, the quotient $W(\mathfrak{l}, \mathbb{O}')/W$ is isomorphic to the group A of G-equivariant Poisson automorphisms of X.*

The group A is naturally identified with $N_{Z_G(x)}(H)/H$, where $x \in G/H$. In particular, if $H = Z_G(x)$, we see that $W = W(\mathfrak{l}, \mathbb{O}')$. A formally weaker result (where $\mathbb{O}' = \{0\}$) was obtained in [N3, Section 2]. Note also that A coincides with the group of graded G-equivariant automorphisms of $\mathbb{C}[X]$.

Proof of Proposition 4.7. To compute $\mathfrak{P}$ we note that since $H^i(X'^{reg}, \mathbb{C}) = \{0\}$ for $i = 1, 2$ (see Proposition 4.5) and $H^1(G/P, \mathbb{C}) = 0$, we get $H^2(\tilde{X}^{reg}, \mathbb{C}) = H^2(G/P, \mathbb{C}) = (\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^*$.

Now we prove (2). Note that $\check{X}$ is a graded deformation of $\tilde{X}$ in the sense of Section 2.2. By Proposition 2.6, there is a unique linear map $(\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^* \to \mathfrak{P}$ and a $\mathbb{C}^\times$-equivariant Poisson isomorphism $\check{X} \xrightarrow{\sim} (\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^* \times_{\mathfrak{P}} \tilde{X}_{\mathfrak{P}}$. We need to show that $(\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^* \to \mathfrak{P}$ is an isomorphism. Thanks to (1) it is enough to show it is injective. Assume the contrary: let $\lambda \in (\mathfrak{l}/[\mathfrak{l}, \mathfrak{l}])^*$ go to 0. This yields an isomorphism $G \times^P (X' \times \mathfrak{p}^\perp) \xrightarrow{\sim} G \times^P (\{\lambda\} \times X' \times \mathfrak{p}^\perp)$. In particular, we have an embedding $G/P \hookrightarrow G \times^P (\{\lambda\} \times X' \times \mathfrak{p}^\perp)$. The target is a homogeneous bundle over the affine variety G/G_λ with fiber $G_\lambda \times^{G_\lambda \cap P} (\{\lambda\} \times X' \times \mathfrak{p}^\perp)$. Any morphism from G/P to G/G_λ maps G/P to a single point because G/P is projective and connected while G/G_λ is affine. So we have a closed embedding $G/P \hookrightarrow G_\lambda \times^{G_\lambda \cap P}$

$(\{\lambda\} \times X' \times \mathfrak{p}^\perp)$. The latter variety is a bundle over the projective variety $G_\lambda/(G_\lambda \cap P)$ whose fiber is affine. So the composition

$$G/P \hookrightarrow G_\lambda \times^{G_\lambda \cap P} (\{\lambda\} \times X' \times \mathfrak{p}^\perp) \twoheadrightarrow G_\lambda/(G_\lambda \cap P)$$

is a finite morphism. The latter is impossible because $\dim G/P > \dim G_\lambda/(G_\lambda \cap P)$. This contradiction finishes the proof of the claim that the linear map $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^* \to \mathfrak{P}$ is an isomorphism and hence the proof of (2).

(3) will be proved in several steps.

Step 1. Note that, for a Zariski generic $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$, the orbits $\mathbb{O}_\xi$ and $\mathbb{O}_{\xi'}$ induced from $(\mathfrak{l},\mathbb{O}',\xi)$ and $(\mathfrak{l},\mathbb{O}',\xi')$, respectively, coincide (here $\xi' \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$) if and only if $\xi' \in W(\mathfrak{l},\mathbb{O}')\xi$. It follows that, for a Zariski generic $\lambda \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$, the equality $\mu_\lambda(X_\lambda) = \mu_{\lambda'}(X_{\lambda'})$ implies $\lambda' \in W(\mathfrak{l},\mathbb{O}')\lambda$. But the moment map $\mu_{\mathfrak{P}}$ is W-invariant, see Step 1 of the proof of Proposition 4.5. It follows that $W \subset W(\mathfrak{l},\mathbb{O}')$.

Step 2. We are going to produce a group homomorphism $W(\mathfrak{l},\mathbb{O}') \to A$. Pick a Zariski generic element $\lambda \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$. Consider the deformation $X_{\mathbb{C}\lambda}$ of X over $\mathbb{C}\lambda$. It comes with the morphism $X_{\mathbb{C}\lambda} \to \mathbb{C}\lambda \times_{\mathfrak{g}^*//G} \overline{\mathbb{C}^\times\mathbb{O}_\lambda}$ that is a normalization morphism. Note that $w \in W(\mathfrak{l},\mathbb{O}')$ defines a $\mathbb{C}^\times$-equivariant morphism

$$\mathbb{C}\lambda \times_{\mathfrak{g}^*//G} \overline{\mathbb{C}^\times\mathbb{O}_\lambda} \xrightarrow{\sim} \mathbb{C}w\lambda \times_{\mathfrak{g}^*//G} \overline{\mathbb{C}^\times\mathbb{O}_{w\lambda}}$$

and hence a $\mathbb{C}^\times$-equivariant isomorphism $X_{\mathbb{C}\lambda} \to X_{\mathbb{C}w\lambda}$. This isomorphism is G-equivariant and intertwines the moment maps, hence it is Poisson. Specializing to 0, we get an element of A to be denoted by $a_{w,\lambda}$. Note that the group A is finite. So varying λ, we get the same element $a_{w,\lambda}$, we will write a_w for $a_{w,\lambda}$. Since $a_{w_1w_2,\lambda} = a_{w_1,w_2\lambda}a_{w_2,\lambda}$ by the construction, we see that $w \mapsto a_w$ is a group homomorphism.

Step 3. Let us show that $W \subset W(\mathfrak{l},\mathbb{O}')$ is the kernel of the homomorphism $w \mapsto a_w$. By the previous step, w induces a filtered Poisson algebra isomorphism $\mathbb{C}[X_\lambda] \to \mathbb{C}[X_{w\lambda}]$ such that the induced automorphism of $\mathbb{C}[X]$ is a_w. So $w \in W$ if and only if the isomorphism $\mathbb{C}[X_\lambda] \xrightarrow{\sim} \mathbb{C}[X_{w\lambda}]$ is that of filtered deformations if and only if $a_w = 1$.

Step 4. Let us prove that the homomorphism $W(\mathfrak{l},\mathbb{O}') \to A$ is surjective. Recall that A acts on $\mathbb{C}[\mathfrak{P}]^W$ by graded algebra automorphisms and on $\mathbb{C}[X_{\mathfrak{P}}]^W$ by graded Poisson algebra automorphisms. Under this action, $\mathfrak{g} \subset \mathbb{C}[X_{\mathfrak{P}}]^W$ stays fixed. It follows that if $W\lambda, W\lambda'$ are A-conjugate, then $\mu_\lambda(X_\lambda) = \mu_{\lambda'}(X_{\lambda'})$.

Also if $a \in A$ acts trivially on $\mathbb{C}[\mathfrak{P}]^W$, then it acts trivially on $\mathfrak{P}/W \times_{\mathfrak{g}^*//G} \mathfrak{g}^*$ and hence on its normalization $X_{\mathfrak{P}}/W$. Therefore $a = 1$.

In particular, we can take a Zariski generic λ and use Step 1 to see that if a sends $W\lambda$ to $W\lambda'$, then λ, λ' are $W(\mathfrak{l},\mathbb{O}')$-conjugate. Since A acts faithfully on $\mathfrak{P}/W$, this implies the surjectivity of $W(\mathfrak{l},\mathbb{O}') \to A$. □

Below we will need to relate Weyl groups for inductions to $\mathfrak{g}$ and to some Levi subalgebra of $\mathfrak{g}$. Namely, take a Levi $\mathfrak{l} \subset \mathfrak{g}$ and a birationally rigid nilpotent orbit $\mathbb{O}' \subset \mathfrak{l}^*$. Let $\underline{\mathfrak{g}}$ denote a Levi subalgebra of $\mathfrak{g}$ containing $\mathfrak{l}$. Let $\underline{\mathbb{O}}$ be the nilpotent orbit in $\underline{\mathfrak{g}}^*$ induced from $(\mathfrak{l},\mathbb{O}')$ and assume that the induction is birational. Let $\underline{X}$ denote the normalization of $\overline{\underline{\mathbb{O}}}$ and let $X := \operatorname{Spec}(\mathbb{C}[G \times^P (X' \times \mathfrak{p}^\perp)])$.

Lemma 4.8. *The group $W_{\underline{X}}$ is contained in the pointwise stabilizer of $(\underline{\mathfrak{g}}/[\underline{\mathfrak{g}},\underline{\mathfrak{g}}])^* \subset (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ in W_X.*

Proof. In the proof we will need a slightly different construction of the homomorphism $W(\mathfrak{l},\mathbb{O}') \to A$. Pick an element $w \in W(\mathfrak{l},\mathbb{O}')$ and its lift g to $N_G(\mathfrak{l},\mathbb{O}')$. Set $P^1 := gPg^{-1}$,

this is a parabolic subgroup of G with Levi subgroup L. Consider the universal deformations of $\mathbb{Q}$-factorial terminalizations associated to P, P^1, i.e., $\tilde{X}_{\mathfrak{P}} := G\times^P ((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*\times X'\times \mathfrak{p}^\perp), \tilde{X}^1_{\mathfrak{P}} := G\times^{P^1} ((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*\times X'\times \mathfrak{p}^{1,\perp})$. Note that g gives rise to an isomorphism of varieties $\tilde{X}_{\mathfrak{P}}\to \tilde{X}^1_{\mathfrak{P}}$. It is not an isomorphism of schemes over $\mathfrak{P}$, rather the induced map $\mathfrak{P}\to\mathfrak{P}$ is w. Passing to the spectra of the rings of regular functions we get an isomorphism $X_{\mathfrak{P}}\to X_{\mathfrak{P}}$. Its specialization to 0 coincides with a_w: in order to see this we can restrict to $\mathbb{C}\lambda\subset\mathfrak{P}$ for a Zariski generic $\lambda\in\mathfrak{P}$, we recover the construction from Step 2 of the proof of Proposition 4.7.

Now we need to produce an embedding of $W_{\underline{X}}$ into the pointwise stabilizer of $(\underline{\mathfrak{g}}/[\underline{\mathfrak{g}},\underline{\mathfrak{g}}])^*$ inside W_X. Thanks to our assumption that the induction from L to $\underline{G}$ is birational, we have $W_{\underline{X}} = \underline{W}(\mathfrak{l},\mathbb{O}')$, where $\underline{W}(\mathfrak{l},\mathbb{O}')$ is the analog of $W(\mathfrak{l},\mathbb{O}')$ for $\underline{G}$. Note that $\underline{W}(\mathfrak{l},\mathbb{O}')$ coincides with the pointwise stabilizer of $(\underline{\mathfrak{g}}/[\underline{\mathfrak{g}},\underline{\mathfrak{g}}])^*$ in $W(\mathfrak{l},\mathbb{O}')$. So we need to prove that $\underline{W}(\mathfrak{l},\mathbb{O}')$ maps trivially to A.

Set $\underline{X} := \mathbb{C}[\underline{\mathbb{O}}]$ and let $\underline{\mathfrak{P}} := ((\mathfrak{l}\cap[\underline{\mathfrak{g}},\underline{\mathfrak{g}}])/[\mathfrak{l},\mathfrak{l}])^*$. Every element $w\in\underline{W}(\mathfrak{l},\mathbb{O}')$ defines an automorphism of $\underline{X}_{\underline{\mathfrak{P}}}$ that is the identity on $\underline{X}$. Now choose P such that $\overline{P} := P\underline{G}$ is a parabolic subgroup in G. Construct $P^1, \tilde{X}^1_{\mathfrak{P}}$ from w. The morphisms $\tilde{X}_{\mathfrak{P}}, \tilde{X}^1_{\mathfrak{P}}\to X_{\mathfrak{P}}$ factor through $G\times^{\overline{P}} ((\underline{\mathfrak{g}}/[\underline{\mathfrak{g}},\underline{\mathfrak{g}}])^*\times \underline{X}_{\underline{\mathfrak{P}}}\times\overline{\mathfrak{p}}^\perp)$. An element $g\in N_{\underline{G}}(\mathfrak{l},\mathbb{O}')$ lifting w defines an automorphism of the latter variety that is the identity on $G\times^{\overline{P}} ((\underline{\mathfrak{g}}/[\underline{\mathfrak{g}},\underline{\mathfrak{g}}])^*\times\underline{X}\times\overline{\mathfrak{p}}^\perp)$ because $w\in W_{\underline{X}}$. It follows that the induced automorphism of X is the identity finishing the proof. □

In the proof of Theorem 4.4 in the next section we will see that in Lemma 4.8 we actually have an equality.

4.5. **Consequences.** In this section we prove Theorem 4.4.

We start with a criterium for $\mathbb{C}[X_\xi],\mathbb{C}[X_{\xi'}]$ with $\xi,\xi'\in(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}$ to be isomorphic as filtered algebras.

Lemma 4.9. *Let $\xi,\xi'\in(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}$. Then the following are equivalent:*

1. *$\mathbb{C}[X_\xi],\mathbb{C}[X_{\xi'}]$ are G-equivariantly isomorphic as filtered Poisson algebras.*
2. *The G-orbits induced from $(\mathfrak{l},\mathbb{O}',\xi),(\mathfrak{l},\mathbb{O}',\xi')$ coincide.*
3. *$\xi'\in W(\mathfrak{l},\mathbb{O}')\xi$.*

Proof. Clearly (3) implies (2). Let $\mathbb{O}_\xi,\mathbb{O}_{\xi'}$ denote the orbits induced from $(\mathfrak{l},\mathbb{O}',\xi)$ and $(\mathfrak{l},\mathbb{O}',\xi')$. Note that $\mathbb{C}[X_\xi]\cong\mathbb{C}[\mathbb{O}_\xi]$ (a G-equivariant Poisson isomorphism) because $\xi\in(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}$ and similarly $\mathbb{C}[X_{\xi'}]\cong\mathbb{C}[\mathbb{O}_{\xi'}]$. The argument of Step 4 of the proof of Proposition 4.5 shows that if $\mathbb{O}_\xi=\mathbb{O}_{\xi'}$, then, in the notation there, we have a $\mathbb{C}^\times$-equivariant isomorphism $X^1_{\mathbb{C}\xi}\cong X^1_{\mathbb{C}\xi'}$. Since $\mathbb{C}[X_\xi]$ is the specialization of $\mathbb{C}[X^1_{\mathbb{C}\xi}]$ at 1 – and similarly for $\mathbb{C}[X_{\xi'}]$, the resulting isomorphism $\mathbb{C}[X_\xi]\cong\mathbb{C}[X_{\xi'}]$ is G-equivariant and of filtered algebras. So (2) implies (1).

Let us show that (1) implies (3). The group of the graded G-equivariant Poisson automorphisms of $\mathbb{C}[X]$ is A. The implication (1)⇒(3) now follows from Remark 3.24. □

Proof of Theorem 4.4. The proof is in several steps.

Step 1. Let us prove (1). Clearly, the birational sheets cover $\mathfrak{g}$. So we need to prove that an orbit determines a birationally minimal induction datum uniquely up to G-conjugacy (which, in particular, implies that the birational sheets do not intersect). Recall that if an orbit Gy with $y\in\mathfrak{g}^*$ is induced from $(\mathfrak{l},\mathbb{O}',\xi)$, then ξ is G-conjugate to y_s. This allows

to reduce the proof to the claim that every nilpotent orbit is birationally induced from a unique birationally minimal induction datum. This is (1) of Corollary 4.6.

Step 2. To prove (2) and (3) we first need to establish the following claim:

(*) The action of $A = W(\mathfrak{l}, \mathbb{O}')/W_X$ on $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}/W_X$ is free.

Indeed, let $\xi \in (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}$. Let $\underline{G}$ stand for the stabilizer of ξ in G. Let $\underline{\mathbb{O}}$ denote the nilpotent orbit in $\underline{\mathfrak{g}}$ induced from $(\mathfrak{l}, \mathbb{O}')$. Set $\underline{X} = \operatorname{Spec}(\mathbb{C}[\underline{\mathbb{O}}])$. By Lemma 4.8, we have $W_{\underline{X}} \subset W_{X,\xi}$. On the other hand, the group $\underline{W}(\mathfrak{l}, \mathbb{O}')$, the analog of $W(\mathfrak{l}, \mathbb{O}')$ for $\underline{\mathfrak{g}}$, coincides with the stabilizer $(W(\mathfrak{l}, \mathbb{O}'))_\xi$. But the induction from $(\mathfrak{l}, \mathbb{O}')$ to $\underline{\mathbb{O}}$ is birational, so by Proposition 4.7, we have $W_{\underline{X}} = \underline{W}(\mathfrak{l}, \mathbb{O}')$. Together with the equality $\underline{W}(\mathfrak{l}, \mathbb{O}') = (W(\mathfrak{l}, \mathbb{O}'))_\xi$, this shows (*).

Step 3. Now let us prove (3). Let $X^0_{\mathfrak{P}}$ denote the locus in $\tilde{X}_{\mathfrak{P}}$ consisting of orbits of maximal dimension, it embeds into $X_{\mathfrak{P}}$. Recall, (1) and (2) of Proposition 4.7, that $\mathfrak{P} = (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$ and $\tilde{X}_{\mathfrak{P}} = G \times^P (\mathfrak{P} \times X' \times \mathfrak{p}^\perp)$. As in the proof of Lemma 4.1, consider the preimage $\check{S} \subset \tilde{X}_{\mathfrak{P}}$ of the Slodowy slice S to $\mathbb{O}$ under $\mu_{\mathfrak{P}} : \tilde{X}_{\mathfrak{P}} = G \times^P (\mathfrak{P} \times X' \times \mathfrak{p}^\perp) \to \mathfrak{g}^*$. Clearly, $\check{S} \subset X^0_{\mathfrak{P}}$. The morphism $\mu_{\mathfrak{P}} : \check{S} \to S$ is still proper. As we have seen in the proof of Lemma 4.1, see Steps 4 and 5, the morphism $\check{S} \to \mathfrak{P}$ is finite and étale.

As we have seen in the proof of Lemma 4.8, we have a $W(\mathfrak{l}, \mathbb{O}')$-action on $X_{\mathfrak{P}}$. Since $X_{\mathfrak{P}} \to \mathfrak{g}^*$ is W_X-invariant and $S \subset \mathfrak{g}^*$ is Q-stable, we see that $\check{S} \subset X^0_{\mathfrak{P}} \subset X_{\mathfrak{P}}$ is $W(\mathfrak{l}, \mathbb{O}')$-stable. By Proposition 4.3, every G-orbit in Z intersects S in a single $A(\mathbb{O})$-orbit. From here we deduce that the projection $\check{S} \twoheadrightarrow \mathfrak{P}$ induces an isomorphism $\check{S}/W(\mathfrak{l}, \mathbb{O}') \xrightarrow{\sim} \mathfrak{P}/W(\mathfrak{l}, \mathbb{O}')$.

Note that $Z^{bir}_{\mathfrak{l},\mathbb{O}'} \cap S$ coincides with the image of $\check{S} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'})$ under $\mu_{\mathfrak{P}}$, where we write π for the projection $\tilde{X}_{\mathfrak{P}} \to \mathfrak{P}$. This image is the complement of a closed subset in the image of the proper morphism $\mu_{\mathfrak{P}}|_{\check{S}}$. It follows that $Z^{bir}_{\mathfrak{l},\mathbb{O}'} \cap S$ is a locally closed subvariety. The proper epimorphism $\check{S} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}) \twoheadrightarrow Z^{bir}_{\mathfrak{l},\mathbb{O}'}/G$ factors through $(\check{S} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}))/W(\mathfrak{l}, \mathbb{O}')$. Moreover, it follows from Lemma 4.9 that if the images of two points from $\check{S} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'})$ in $Z^{bir}_{\mathfrak{l},\mathbb{O}'}/G$ coincide, then the points are $W(\mathfrak{l}, \mathbb{O}')$-conjugate. So the morphism $(\check{S} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}))/W(\mathfrak{l}, \mathbb{O}') \twoheadrightarrow Z^{bir}_{\mathfrak{l},\mathbb{O}'}/G$ is also injective. Therefore it is a bijective normalization morphism. The variety $\left(\check{S} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'})\right)/W(\mathfrak{l}, \mathbb{O}') = (\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}/W(\mathfrak{l}, \mathbb{O}')$ is smooth by (*) in Step 2. This proves (3).

Step 4. Let us prove (2). Note that $X^0_{\mathfrak{P}} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}) = \mu^{-1}_{\mathfrak{P}}(Z^{bir}_{\mathfrak{l},\mathbb{O}'})$. The subvariety $X^0_{\mathfrak{P}}$ is the union of orbits of maximal dimension in $X_{\mathfrak{P}}$. Since the action of $W(\mathfrak{l}, \mathbb{O}')$ on $X_{\mathfrak{P}}$ commutes with that of G, we see that $W(\mathfrak{l}, \mathbb{O}')$ preserves $X^0_{\mathfrak{P}}$. By Lemma 4.9, the induced morphism $(X^0_{\mathfrak{P}} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}))/W(\mathfrak{l}, \mathbb{O}') \to Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ is injective. It is also proper so it is finite. It remains to prove that $(X^0_{\mathfrak{P}} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'}))/W(\mathfrak{l}, \mathbb{O}')$ is smooth. This will follow if we show that $W(\mathfrak{l}, \mathbb{O}')$ acts on $X^0_{\mathfrak{P}} \cap \pi^{-1}((\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^{*reg}_{\mathbb{O}'})$ as a group generated by reflections (by a reflection in this case we mean an automorphism whose fixed locus is a divisor). We have seen in Step 2 that $(W(\mathfrak{l}, \mathbb{O}'))_\xi = W_{\underline{X}}$ is a reflection group (in its action on $(\mathfrak{l}/[\mathfrak{l},\mathfrak{l}])^*$). So it remains to check that $(W(\mathfrak{l}, \mathbb{O}'))_\xi$ fixes X_ξ pointwise. Recall, Step 2, that $W_{X,\xi} = (W(\mathfrak{l}, \mathbb{O}'))_\xi$. Assume that an element $w \in W_{X,\xi}$ acts on X_ξ nontrivially. The action preserves the filtration on $\mathbb{C}[X_\xi]$ and is the identity on the associated graded. But a finite group of automorphisms cannot contain a non-unit element with these properties. This finishes the proof of (2). $\square$

Remark 4.10. In fact, in most cases, the birational sheets as well as their quotients are smooth. Namely, $Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ is smooth provided $\mathbb{O}$ (the orbit induced from $(\mathfrak{l},\mathbb{O}')$) is not one of the seven orbits in [PT, Table 0] and is not induced from one of these orbits. In particular, the birational sheets in classical Lie algebras are always smooth. Let us sketch a proof of the smoothness.

First, let us consider the birational sheet $Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ containing $\mathbb{O}$. Let $Z(\mathbb{O})$ denote the union of sheets containing $\mathbb{O}$. One can show that $Z^{bir}_{\mathfrak{l},\mathbb{O}'}\cap S=(Z(\mathbb{O})\cap S)^{A(\mathbb{O})}$. Using techniques of [PT, Section 5] (where an analogous result was proved in the quantum case), one shows that $(Z(\mathbb{O})\cap S)^{A(\mathbb{O})}$ is an affine space provided $\mathbb{O}$ is not one of the orbits in Table 0 in the introduction of *loc.cit.*. So we see that $Z^{bir}_{\mathfrak{l},\mathbb{O}'}\cap S=Z^{bir}_{\mathfrak{l},\mathbb{O}'}/G$ is smooth. But, for $s\in S\cap Z^{bir}_{\mathfrak{l},\mathbb{O}'}$, we have $T_sZ^{bir}_{\mathfrak{l},\mathbb{O}'}=T_sGs\oplus T_s(Z^{bir}_{\mathfrak{l},\mathbb{O}'}\cap S)$. It follows that $Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ is smooth.

Now consider the case when $Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ does not contain a nilpotent orbit. Pick $y\in Z^{bir}_{\mathfrak{l},\mathbb{O}'}$. Let $\underline{G}=G_{y_s}$ and let $\underline{Z}^{bir}_{\mathfrak{l},\mathbb{O}'}$ be the birational sheet in $\underline{\mathfrak{g}}$ corresponding to $\mathfrak{l},\mathbb{O}'$, this birational sheet contains y_n. Then we have an étale morphism $G\times^{\underline{G}}(y_s+\underline{Z}^{bir}_{\mathfrak{l},\mathbb{O}'})\to Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ with y lying in the image. We deduce from the previous paragraph that both $Z^{bir}_{\mathfrak{l},\mathbb{O}'}$ and $Z^{bir}_{\mathfrak{l},\mathbb{O}'}/G$ are smooth provided $\mathbb{O}$ is not induced from one of the seven orbits in [PT, Table 0].

Remark 4.11. One can ask for an intrinsic characterization of birational sheets. As in Step 4 of the proof of Proposition 4.5 we see that for a pair $\mathbb{O}_1,\mathbb{O}_2$ of G-orbits lying in the same birational sheet, the G-modules $\mathbb{C}[\mathbb{O}_1],\mathbb{C}[\mathbb{O}_2]$ are isomorphic. We conjecture that the converse is also true: if the G-modules $\mathbb{C}[\mathbb{O}_1],\mathbb{C}[\mathbb{O}_2]$ are isomorphic as G-modules, then $\mathbb{O}_1,\mathbb{O}_2$ lie in the same birational sheet.

5. W-algebras and Orbit method

5.1. **W-algebras.** We start by recalling (finite) W-algebras that were originally defined by Premet in [P1], although we will follow an approach from [Lo1]. Throughout the section G is a semisimple group.

Pick a nilpotent orbit $\mathbb{O}\subset\mathfrak{g}^*$. Let $\chi\in\mathbb{O}$. Recall that S stands for a Slodowy slice to $\mathbb{O}$ in χ. It is acted on by $Q\times\mathbb{C}^\times$, where Q is a maximal reductive subgroup of G_χ.

We recall a filtered associative algebra $\mathcal{W}$ equipped with a Hamiltonian Q-action. Namely, consider the universal enveloping algebra $\mathcal{U}=U(\mathfrak{g})$ with its standard PBW filtration $\mathcal{U}=\bigcup_{i\geqslant 0}\mathcal{U}_{\leqslant i}$. It will be convenient for us to double the filtration and set $F_i\mathcal{U}:=\mathcal{U}_{\leqslant[i/2]}$. Form the Rees algebra $\mathcal{U}_\hbar:=\bigoplus_i(F_i\mathcal{U})\hbar^i$. The quotient $\mathcal{U}_\hbar/(\hbar)$ coincides with $S(\mathfrak{g})=\mathbb{C}[\mathfrak{g}^*]$. Consider the completion $\mathcal{U}_\hbar^{\wedge_\chi}$ in the topology induced by the preimage of the maximal ideal of χ. The space $V:=T_\chi\mathbb{O}$ is symplectic. So we can form the homogenized Weyl algebra $\mathbb{A}_\hbar$ of V, i.e., $\mathbb{A}_\hbar$ is the Rees algebra of the usual Weyl algebra $\mathbb{A}(V)$. We consider the completion $\mathbb{A}_\hbar^{\wedge_0}$ in the topology induced by the maximal ideal of $0\in V$. Both $\mathcal{U}_\hbar^{\wedge_\chi}$ and $\mathbb{A}_\hbar^{\wedge_0}$ come equipped with actions of $Q\times\mathbb{C}^\times$. The action of Q on $\mathcal{U}_\hbar^{\wedge_\chi},\mathbb{A}_\hbar^{\wedge_0}$ is induced from the natural actions of Q on $\mathfrak{g}$ and V, respectively. The group $\mathbb{C}^\times$ acts on $\mathfrak{g}^*$ via $t.\alpha:=t^{-2}\gamma(t)\alpha$, where $\gamma:\mathbb{C}^\times\to G$ is the one-parameter subgroup associated to the element h as in Section 4.1. The group $\mathbb{C}^\times$ naturally acts on V. Finally, we set $t.\hbar:=t\hbar$, this defines $\mathbb{C}^\times$-actions on $\mathcal{U}_\hbar^{\wedge_\chi},\mathbb{A}_\hbar^{\wedge_0}$ by topological algebra automorphisms that commute with the Q-actions.

It was checked in [Lo1, Section 3.3], see also [Lo6, Section 2.2] that there is a $Q\times\mathbb{C}^\times$-equivariant $\mathbb{C}[\hbar]$-linear embedding $\mathbb{A}_\hbar^{\wedge_0}\hookrightarrow\mathcal{U}_\hbar^{\wedge_\chi}$ such that we have the decomposition

$$\mathcal{U}_\hbar^{\wedge_\chi}\cong\mathbb{A}_\hbar^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathcal{W}'_\hbar, \tag{5.1}$$

where we write $\mathcal{W}'_\hbar$ for the centralizer of $\mathbb{A}_\hbar^{\wedge_0}$ in $\mathcal{U}_\hbar^{\wedge_\chi}$. The algebra $\mathcal{W}'_\hbar$ comes with an action of $Q\times\mathbb{C}^\times$. Let us write $\mathcal{W}_\hbar$ for the $\mathbb{C}^\times$-finite part of $\mathcal{W}'_\hbar$, then $\mathcal{W}'_\hbar$ is naturally identified with the completion $\mathcal{W}_\hbar^{\wedge_\chi}$. Set $\mathcal{W}:=\mathcal{W}_\hbar/(\hbar-1)$. This is a filtered algebra with a Hamiltonian Q-action that does not depend on the choice of the embedding $\mathbb{A}_\hbar^{\wedge_0}\hookrightarrow\mathcal{U}_\hbar^{\wedge_\chi}$ up to an isomorphism preserving the filtration and the action. See [Lo6, Section 2.1]. The associated graded algebra $\operatorname{gr}\mathcal{W}$ coincides with $\mathbb{C}[S]$, where S is the Slodowy slice.

5.2. **Restriction functor for HC bimodules.** By a G-equivariant Harish-Chandra $\mathcal{U}$-bimodule (or $(\mathcal{U},G)$-module) we mean a finitely generated $\mathcal{U}$-bimodule $\mathcal{B}$ such that the adjoint $\mathfrak{g}$-action is locally finite and integrates to an action of G. We can also introduce the notion of a Q-equivariant HC $\mathcal{W}$-bimodule, see [Lo2, Section 2.5]. We write $\mathrm{HC}^G(\mathcal{U}),\mathrm{HC}^Q(\mathcal{W})$ for the categories of equivariant HC bimodules.

In [Lo2, Sections 3.3,3.4], we have constructed an exact functor $\bullet_\dagger:\mathrm{HC}^G(\mathcal{U})\to\mathrm{HC}^Q(\mathcal{W})$. Let us recall the construction of the functor. Pick a G-equivariant HC bimodule $\mathcal{B}$ and equip it with a good filtration compatible with the filtration $\mathrm{F}_i\mathcal{U}$. So the Rees $\mathbb{C}[\hbar]$-module $\mathcal{B}_\hbar:=R_\hbar(\mathcal{B})$ is a G-equivariant $\mathcal{U}_\hbar$-bimodule. Consider the completion $\mathcal{B}_\hbar^{\wedge_\chi}$ in the χ-adic topology. This is a $Q\times\mathbb{C}^\times$-equivariant $\mathcal{U}_\hbar^{\wedge_\chi}$-bimodule (the action of Q is Hamiltonian, while the action of $\mathbb{C}^\times$ is not). As was checked in [Lo2, Proposition 3.3.1], $\mathcal{B}_\hbar^{\wedge_\chi}=\mathbb{A}_\hbar^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\underline{\mathcal{B}}'_\hbar$, where $\underline{\mathcal{B}}'_\hbar$ is the centralizer of $\mathbb{A}_\hbar^{\wedge_0}$. So $\underline{\mathcal{B}}'_\hbar$ is a $Q\times\mathbb{C}^\times$-equivariant $\mathcal{W}_\hbar^{\wedge_\chi}$-bimodule. One can show that $\underline{\mathcal{B}}'_\hbar$ coincides with the completion of its $\mathbb{C}^\times$-finite part $\underline{\mathcal{B}}_\hbar$. We set $\mathcal{B}_\dagger:=\underline{\mathcal{B}}_\hbar/(\hbar-1)$. This is an object in $\mathrm{HC}^Q(\mathcal{W})$ that comes equipped with a good filtration. This filtration depends on the choice of the filtration on $\mathcal{B}$, while $\mathcal{B}_\dagger$ itself does not.

Let us list properties of the functor $\bullet_\dagger$ established in [Lo2, Sections 3.3,3.4]. See [Lo2, Theorem 1.3.3, Lemma 3.3.2, Proposition 3.4.1].

Lemma 5.1. *The following is true:*

(1) $\mathcal{U}_\dagger=\mathcal{W}$.
(2) $\bullet_\dagger$ *is an exact functor.*
(3) $\bullet_\dagger$ *is a monoidal functor.*
(4) $\operatorname{gr}\mathcal{B}_\dagger$ *(with respect to the filtration above) coincides with the pull-back of* $\operatorname{gr}\mathcal{B}$ *to* S.
(5) *In particular,* $\bullet_\dagger$ *maps the category* $\mathrm{HC}^G_{\overline{\mathbb{O}}}(\mathcal{U})$ *of all HC bimodules supported on* $\overline{\mathbb{O}}$ *to the category* $\mathrm{HC}^Q_{fin}(\mathcal{W})$ *of all finite dimensional* Q*-equivariant* $\mathcal{W}$*-bimodules. Further,* $\bullet_\dagger$ *annihilates* $\mathrm{HC}^G_{\partial\mathbb{O}}(\mathcal{U})$.

Recall that by a Dixmier algebra one means an associative algebra $\mathcal{A}$ together with a rational Hamiltonian action of G such that the corresponding quantum comoment map $\mathcal{U}\to\mathcal{A}$ makes $\mathcal{A}$ a HC bimodule. Since $\bullet_\dagger$ is a monoidal functor it sends a Dixmier algebra to an algebra $\mathcal{A}_\dagger$ that comes with a homomorphism $\mathcal{W}\to\mathcal{A}_\dagger$. Moreover, a good algebra filtration on $\mathcal{A}$ gives rise to an algebra filtration on $\mathcal{A}_\dagger$. An isomorphism in (4) of the lemma is that of algebras.

The functor $\bullet_\dagger:\mathrm{HC}^G_{\overline{\mathbb{O}}}(\mathcal{U})\to\mathrm{HC}^Q_{fin}(\mathcal{W})$ has the right adjoint $\bullet^\dagger:\mathrm{HC}^Q_{fin}(\mathcal{W})\to\mathrm{HC}^G_{\overline{\mathbb{O}}}(\mathcal{U})$, [Lo2, Proposition 3.4.1,(4)]. We will need the construction of the functor $\bullet^\dagger$ from [Lo2, Sections 3.3,3.4] below so let us recall it.

Pick $\underline{\mathcal{B}}\in\mathrm{HC}^Q_{fin}(\mathcal{W})$ and equip it with a Q-stable $\mathcal{W}$-bimodule filtration (for example, we can just take the trivial one). Then form the Rees bimodule $\underline{\mathcal{B}}_\hbar$ and the Q-equivariant $\mathcal{U}_\hbar^{\wedge_\chi}$-bimodule $\mathcal{B}'_\hbar=\mathbb{A}_\hbar^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\underline{\mathcal{B}}_\hbar^{\wedge_\chi}$. We can equip $\mathcal{B}'_\hbar$ with a $\mathfrak{g}$-module structure via $\xi.b:=$

$\frac{1}{\hbar^2}[\xi, b]$. Let $(\mathcal{B}'_\hbar)_{\mathfrak{g}-fin}$ denote the $\mathfrak{g}$-finite part of $\mathcal{B}'_\hbar$. This module is $\mathbb{C}^\times$-stable and we can twist a $\mathbb{C}^\times$-action (see [Lo2, Section 3.3] for details) to get one commuting with $\mathfrak{g}$. Let $(\mathcal{B}'_\hbar)_{fin}$ denote the $\mathbb{C}^\times$-finite part in $(\mathcal{B}'_\hbar)_{\mathfrak{g}-fin}$, this is a graded $\mathcal{U}_\hbar$-bimodule. It follows from [Lo2, Lemma 3.3.3] that this bimodule is finitely generated. Set $\underline{\mathcal{B}}^\diamondsuit = (\mathcal{B}'_\hbar)_{fin}/(\hbar - 1)$. This is a HC $\mathcal{U}$-bimodule supported on $\overline{\mathbb{O}}$ that comes with a natural filtration. Note that $\underline{\mathcal{B}}^\diamondsuit$ has no sub-bimodules supported on $\partial\mathbb{O}$ by the construction.

Also note that Q/Q° naturally acts on $\underline{\mathcal{B}}^\diamondsuit$ by filtered $\mathcal{U}$-bimodule automorphisms. We set $\underline{\mathcal{B}}^\dagger := (\underline{\mathcal{B}}^\diamondsuit)^{Q/Q^\circ}$, this gives a right adjoint functor of interest. We note that both the kernel and the cokernel of the adjunction homomorphism $\mathcal{B} \to (\mathcal{B}_\dagger)^\dagger$ are supported on $\partial\mathbb{O}$, [Lo2, Proposition 3.4.1,(5)].

Now assume A is a finite dimensional algebra with a homomorphism $\mathcal{W} \to \mathsf{A}$ and a compatible action of Q such that the composition $\mathfrak{q} \to \mathcal{W} \to \mathsf{A}$ is a quantum comoment map for the Q-action. The construction recalled above shows that both $\mathsf{A}^\diamondsuit$ and $\mathsf{A}^\dagger$ have Dixmier algebra structures. Moreover, for a Q-stable filtration on A such that the homomorphism $\mathcal{W} \to \mathsf{A}$ is that of filtered algebras, we get natural filtrations on $\mathsf{A}^\diamondsuit, \mathsf{A}^\dagger$. These are good algebra filtrations.

Now let us investigate what happens with $\mathcal{B}_\dagger$ and $(\mathcal{B}_\dagger)^\dagger$, where $\mathcal{B}$ is a quantization of $\mathbb{C}[G/H]$, where G/H is an equivariant cover of $\mathbb{O}$. Recall that G_χ/H is the fiber of $G/H \to \mathbb{O}$ over χ, it is finite.

Lemma 5.2. *The following is true:*

1. *As a filtered algebra $\mathcal{B}_\dagger$ is Q-equivariantly isomorphic to the algebra of functions $\mathbb{C}[G_\chi/H]$ (with the trivial filtration).*
2. *We have $\operatorname{gr}(\mathcal{B}_\dagger)^\dagger = \mathbb{C}[G/H]$ and the natural homomorphism $\mathcal{B} \to (\mathcal{B}_\dagger)^\dagger$ is an isomorphism of filtered algebras.*

Proof. Let us prove (1). That $\mathcal{B}_\dagger$ is an algebra with a natural filtration follows directly from the discussion following Lemma 5.1. The isomorphism $\operatorname{gr}\mathcal{B}_\dagger \cong \mathbb{C}[G_\chi/H]$ follows from (4) of Lemma 5.1. This is a graded algebra isomorphism, where $\mathbb{C}[G_\chi/H]$ is in degree 0. This shows $\mathcal{B}_\dagger \cong \mathbb{C}[G_\chi/H]$.

Let us now prove (2). Let $\mathcal{I}$ be an H-stable codimension 1 ideal of $\mathcal{W}$. We can form the algebra $\mathsf{A} = \bigoplus_{h\in G_\chi/H} \mathcal{W}/h.\mathcal{I}$. For example, $\mathcal{B}_\dagger$ is of this form by (1). Consider the algebra $\mathsf{A}^\dagger$. The algebra $\operatorname{gr}\mathsf{A}^\dagger$ is the quotient of the $\mathfrak{g}$-finite part of $\mathbb{A}_\hbar^{\wedge_0}\widehat{\otimes}_{\mathbb{C}[[\hbar]]}\mathsf{A}_\hbar^{\wedge_\chi}$ by the ideal generated by $\hbar$. This embeds into the $\mathfrak{g}$-finite part of $(\mathbb{A}_\hbar^{\wedge_0}/(\hbar)) \otimes \operatorname{gr}\mathsf{A}$. By [Lo2, Proposition 3.2.2,3.2.3], the latter coincides with $\mathbb{C}[G/H]$.

Now consider the natural homomorphism $\mathcal{B} \to (\mathcal{B}_\dagger)^\dagger$. This is a homomorphism of filtered algebras. The kernel is supported on $\partial\mathbb{O}$ by Lemma 5.1. Note however that $\mathcal{B}$ has no zero divisors and so is prime. It follows from [BoKr1, Corollar 3.6] that the kernel is zero. So $\mathcal{B} \hookrightarrow (\mathcal{B}_\dagger)^\dagger$, an inclusion of filtered algebras. Together with the inclusion $\operatorname{gr}(\mathcal{B}_\dagger)^\dagger \hookrightarrow \mathbb{C}[G/H] = \operatorname{gr}\mathcal{B}$, this implies that $\mathcal{B} \hookrightarrow (\mathcal{B}_\dagger)^\dagger$ is an isomorphism of filtered algebras. □

5.3. **Orbit method.** Let $\mathbb{O}_1$ be an adjoint orbit corresponding to a birationally minimal induction datum $(\mathfrak{l}, \mathbb{O}', \xi)$. By Theorem 4.4, $(\mathfrak{l}, \mathbb{O}', \xi)$ is recovered uniquely from $\mathbb{O}_1$ up to G-conjugacy. Set $X := \operatorname{Spec}(\mathbb{C}[G \times^P (X' \times \mathfrak{p}^\perp)])$, where, as usual, $X' := \operatorname{Spec}(\mathbb{C}[\mathbb{O}'])$. Let $\mathcal{A}_\xi$ denote the quantization of $\mathbb{C}[X]$ with quantization parameter ξ. It is uniquely determined by $\mathbb{O}_1$ as a filtered algebra with a G-action. Indeed, $(\mathfrak{l}, \mathbb{O}', \xi)$ is determined uniquely up to G-conjugacy from $\mathbb{O}_1$, see (1) of Theorem 4.4. Then we can use Corollary

4.9 to see that $\mathbb{C}[X_\xi]$ is uniquely determined up to a G-equivariant isomorphism of filtered Poisson algebras. Remark 3.24 then shows that $\mathcal{A}_\xi$ is uniquely determined up to a G-equivariant isomorphism of filtered algebras.

The algebra $\mathcal{A}_\xi$ comes with a quantum comoment map $\mathfrak{g} \to \mathcal{A}_\xi$. The classical comoment map $\mathfrak{g} \to \mathbb{C}[X]$ lifts to a quantum comoment map $\mathfrak{g} \to \mathcal{A}_\xi$, the argument is similar to Step 1 of the proof of Proposition 4.5.

We will write $\mathcal{A}(\mathbb{O}_1)$ for $\mathcal{A}_\xi$, $\mathcal{J}(\mathbb{O}_1)$ for the kernel of $\mathcal{U} \to \mathcal{A}(\mathbb{O}_1)$. We note that $\mathcal{J}(\mathbb{O}_1)$ is primitive. Indeed, the intersection of $\mathcal{J}(\mathbb{O}_1)$ with the center of $\mathcal{U}$ is a maximal ideal in the center because $(\mathcal{U}/\mathcal{J}(\mathbb{O}_1))^G \hookrightarrow \mathcal{A}_\xi^G$ and $\dim \mathcal{A}_\xi^G = \dim \mathbb{C}[X]^G = 1$. Further, the algebra $\mathcal{A}_\xi$ has no zero divisors because its associated graded is $\mathbb{C}[X]$, which has that property. It follows that the ideal $\mathcal{J}(\mathbb{O}_1)$ is completely prime, hence prime. It is known, see, for example, [J, Section 7.3], that a prime ideal in $\mathcal{U}$ whose intersection with the center is a maximal ideal is primitive.

The following is our version of the Orbit method.

Theorem 5.3. *The following is true.*

1. *If there is a G-equivariant algebra isomorphism $\mathcal{A}(\mathbb{O}_1) \xrightarrow{\sim} \mathcal{A}(\mathbb{O}_2)$, then $\mathbb{O}_1 = \mathbb{O}_2$.*
2. *Moreover, assume that $\mathfrak{g}$ is classical. If $\mathcal{J}(\mathbb{O}_1) = \mathcal{J}(\mathbb{O}_2)$, then $\mathbb{O}_1 = \mathbb{O}_2$.*

We note that (1) is a weaker version of [V2, Conjecture 3.9] – we only deal with coadjoint orbits and not with their covers[2].

Proof. Let $(\mathfrak{l}_1, \mathbb{O}_1', \xi_1), (\mathfrak{l}_2, \mathbb{O}_2', \xi_2)$ be the birationally minimal induction data giving $\mathbb{O}_1, \mathbb{O}_2$. Let χ_1, χ_2 be points in the coadjoint orbits induced from $(\mathfrak{l}_1, \mathbb{O}_1, 0), (\mathfrak{l}_2, \mathbb{O}_2, 0)$ and let $H_i \subset G_{\chi_i}$ be the finite index subgroups produced from $(\mathfrak{l}_i, \mathbb{O}_i, 0), i = 1, 2$.

The proofs of (1),(2) are in several steps.

Step 1. If $\mathcal{J}(\mathbb{O}_1) = \mathcal{J}(\mathbb{O}_2)$, then the nilpotent orbits induced from $(\mathfrak{l}_i, \mathbb{O}_i')$ are the same. Indeed, the closure of the orbit induced from $(\mathfrak{l}_i, \mathbb{O}_i')$ is the associated variety of $\mathcal{J}(\mathbb{O}_i)$. Now note that $\mathcal{J}(\mathbb{O}_i)$ coincides with the left annihilator of the HC bimodule $\mathcal{A}(\mathbb{O}_i)$. So if $\mathcal{A}(\mathbb{O}_1) \xrightarrow{\sim} \mathcal{A}(\mathbb{O}_2)$, then $\mathcal{J}(\mathbb{O}_1) = \mathcal{J}(\mathbb{O}_2)$. Therefore we can assume that the nilpotent orbits induced from $(\mathfrak{l}_i, \mathbb{O}_i)$ are the same. Let us write $\mathbb{O}$ for this common orbit and χ for χ_i.

Step 2. Assuming $\mathcal{A}(\mathbb{O}_1) \cong \mathcal{A}(\mathbb{O}_2)$, a G-equivariant algebra isomorphism, let us show that H_1, H_2 are conjugate in G_χ. Consider the W-algebra $\mathcal{W}$ corresponding to $\mathbb{O}$ and the corresponding restriction functor $\bullet_\dagger$. By (1) of Lemma 5.2, $\mathcal{A}(\mathbb{O}_i)_\dagger = \mathbb{C}[G_\chi/H_i]$. This implies H_1 and H_2 are conjugate. We can conjugate them and assume they coincide.

Step 3. So if we have a G-equivariant algebra isomorphism $\mathcal{A}(\mathbb{O}_1) \xrightarrow{\sim} \mathcal{A}(\mathbb{O}_2)$, then $H_1 = H_2, \mathcal{J}(\mathbb{O}_1) = \mathcal{J}(\mathbb{O}_2)$. Set $X := \operatorname{Spec}(\mathbb{C}[G/H_i])$. Let us prove (1). The algebra $\mathcal{A}(\mathbb{O}_i)_\dagger$ together with a homomorphism $\mathcal{W} \to \mathcal{A}(\mathbb{O}_i)_\dagger$ can be described as follows. Pick an H_i-stable codimension 1 ideal $\mathcal{I} \subset \mathcal{W}$ containing $\mathcal{J}(\mathbb{O}_i)_\dagger$. Form the G_χ/G_χ°-homogeneous bundle of algebras over G_χ/H_i with fiber $\mathcal{W}/\mathcal{I}$, denote it by A_i. By the construction, $\mathcal{A}(\mathbb{O}_i)_\dagger = \mathsf{A}_i$. Since $H_1 = H_2, \mathcal{J}(\mathbb{O}_1) = \mathcal{J}(\mathbb{O}_2)$, we have a Q-equivariant isomorphism of algebras $\mathsf{A}_1 \cong \mathsf{A}_2$. From (2) of Lemma 5.2 we deduce that $\mathcal{A}(\mathbb{O}_1) \cong \mathcal{A}(\mathbb{O}_2)$, a G-equivariant isomorphism of filtered algebras. Remark 3.24 implies that we have a G-equivariant filtered Poisson algebra isomorphism $\mathbb{C}[X_{\xi_1}] \cong \mathbb{C}[X_{\xi_2}]$. We use Lemma 4.9 to conclude that $\mathbb{O}_1 = \mathbb{O}_2$.

[2]In the forthcoming paper [LMM] we prove the full conjecture using the same method

Step 4. Let us proceed to proving part (2). We only need to check that $H_1 = H_2$. We will prove a more general claim: we have $H_1 = H_2$ provided $A(\mathbb{O})$ is abelian, which is always the case for classical types.

So suppose that $H_1 \neq H_2$. Let $\mathcal{I}$ be an H_1H_2-stable codimension 1 ideal in $\mathcal{W}$ containing $\mathcal{J}(\mathbb{O}_i)_\dagger$. Let $\Gamma_i := H_i/H_1 \cap H_2, \Gamma := (H_1H_2)/(H_1 \cap H_2)$. Consider the G_χ/G_χ°-equivariant algebra A that is the homogeneous bundle over $Z_G(e)/(H_1 \cap H_2)$ with fiber $\mathcal{W}/\mathcal{I}$ over 1. The group Γ acts on A by G_χ/G_χ°-equivariant filtered algebra automorphisms fixing the image of $\mathcal{W}$. Moreover, $\mathcal{A}(\mathbb{O}_i)_\dagger = \mathsf{A}^{\Gamma_i}$. So the group Γ also acts on $\mathsf{A}^\dagger$ by filtered algebra automorphisms fixing the image of $U(\mathfrak{g})$. Similarly to (2) of Lemma 5.2, we have G-equivariant filtered algebra isomorphisms $\mathcal{A}(\mathbb{O}_i) \cong (\mathsf{A}^\dagger)^{\Gamma_i}$. In particular, we see that the group Γ/Γ_i acts on $\mathcal{A}(\mathbb{O}_i)$ by filtered algebra automorphisms lifting the action of this group on $\mathbb{C}[G/H_i]$. This means that the parameter $W_i\xi_i$ of the quantization $\mathcal{A}(\mathbb{O}_i)$ is stable under the action of Γ/Γ_i (where we write W_i for the Namikawa-Weyl group of $\mathbb{C}[G/H_i]$). It follows from Remark 3.24 that the filtered deformation $\mathbb{C}[X_{\xi_i}]$ of $\mathbb{C}[G/H_i]$ also carries an action of Γ/Γ_i by G-equivariant filtered Poisson algebra automorphisms. Since G is semisimple, the moment map $X_{\xi_i} \to \mathfrak{g}^*$ is unique hence is preserved by Γ/Γ_i. This contradicts the assumption that the moment map $X_{\xi_i} \to \overline{\mathbb{O}}_i$ is birational and completes the proof. □

Remark 5.4. There are 12 orbits in exceptional Lie algebras with noncommutative $A(\mathbb{O})$: it can be equal to S_3 (10 orbits, the easiest example is the subregular orbit for G_2), S_4 (for a single orbit in type F_4) or S_5 (for a single orbit in type E_8). We have checked that the conclusion of (2) is still true for most of these orbits and we do not know what happens for the rest.

5.4. **Toward description of the image.** An interesting question is to describe the image of the map $\mathfrak{g}^*/G \hookrightarrow \operatorname{Prim}(\mathfrak{g})$. We will sketch the conjectural results.

Note that the orbits lying in the union $Z(\mathbb{O})$ of all sheets containing $\mathbb{O}$ get mapped to $\operatorname{Prim}_{\mathbb{O}}(\mathfrak{g})$, the set of primitive ideals whose associated variety is $\overline{\mathbb{O}}$. Let $\mathcal{W}$ denote the W-algebra corresponding to $\mathbb{O}$. Recall, [Lo2, Section 1.2], that $\operatorname{Prim}_{\mathbb{O}}(\mathfrak{g})$ is the quotient of $\operatorname{Irr}_{fin}(\mathcal{W})$, the set of isomorphism classes of finite dimensional irreducible modules over $\mathcal{W}$, by the action of $A(\mathbb{O})$.

By Lemma 5.2, the kernel $U(\mathfrak{g}) \to \mathcal{A}_\xi$, where $\mathcal{A}_\xi$ is a quantization of $\mathbb{C}[G/H]$ corresponds to a one-dimensional $\mathcal{W}$-module. When $A(\mathbb{O})$ is abelian, the argument of Step 4 of the proof of Theorem 5.3 implies that any such kernel lies in the image of $Z(\mathbb{O})/G \hookrightarrow \operatorname{Prim}_{\mathbb{O}}(\mathfrak{g})$. When $A(\mathbb{O})$ is not abelian, then this does not need to be the case, one gets a counter-example for the subregular orbit in G_2.

So suppose $A(\mathbb{O})$ is abelian. Now let us impose another assumption on $\mathbb{O}$: we assume that $\mathbb{O}$ is not one of the six bad orbits in [P2, Introduction] and is not induced from such an orbit. Recall that the six orbits (all in exceptional Lie algebras) are characterized by the property that $\mathbb{O} \subsetneq \operatorname{Spec}(\mathbb{C}[\mathbb{O}])^{reg}$, see the tables in [FJLS].

Results from [PT] imply that, under our assumption on $\mathbb{O}$, every *multiplicity free* (in the terminology of [PT]) primitive ideal in $\operatorname{Prim}_{\mathbb{O}}(\mathfrak{g})$ arises as the kernel of $U(\mathfrak{g}) \to \mathcal{A}_\xi$, where $\mathcal{A}_\xi$ is a suitable quantization of $\mathbb{C}[\mathbb{O}]$. Conversely, any such kernel is a multiplicity free primitive ideal.

From now on let us suppose that $\mathfrak{g}$ is classical. We conjecture that the image of the injective map $Z(\mathbb{O})/G \to \operatorname{Prim}_{\mathbb{O}}(\mathfrak{g})$ from Theorem 5.3 coincides with the subset of

all primitive ideals corresponding to the one-dimensional representations of $\mathcal{W}$. This conjecture constitutes a right statement of the Orbit method for classical Lie algebras[3].

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