# On the local central limit theorem for interacting spin systems 

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#### Abstract

We prove the equivalence between integral and local central limit theorem for spin system interacting via an absolutely summable pair potential without any conditions on the temperature of the system.


## 1. Introduction

Since the pioneering work by Dobrushin and Tirozzi [3], a certain effort has been spent in statistical mechanics in order to understand conditions for the equivalence between the integral and the local central limit theorem in the framework of interacting discrete spin systems. This kind of results are very useful for instance in the context of the problem of the equivalence of the ensembles.
In [3] the authors proved the equivalence between integral and local central limit theorem for spin systems in $\mathbb{Z}^{d}$ with finite range interaction, while few years later Campanino, Capocaccia and Tirozzi [2] generalized the proof to long range (translational invariant) pair potential $J(x-y)$ satisfying the condition

$$
\begin{equation*}
\sum_{x \in \mathbb{Z}^{d}}|J(x)|^{1 / 2}<\infty . \tag{1.1}
\end{equation*}
$$

Recently Endo and Margarint, in [4], presented a similar proof in which the conditions on the decay of the pair potential $J(x-y)$ are conveniently weakened allowing $J(x)$ to be absolutely summable, but with the additional assumption that the temperature is sufficiently high. In [4] a remarkably large literature on the subject has been provided, and we refer the reader to that discussion.
In this paper we prove the equivalence between integral and local central limit theorem for spin systems interacting via an absolutely summable pair potential (which does not need to be translational invariant), without any further assumption on the temperature of the system. In order to achieve this result we make a judicious use of classical results concerning cluster and polymer expansion, together with a suitable decimation of the space as in [2]. In particular, we use the estimates on polymer activities based on tree graphs inequalities valid for pair potentials originally discussed in [9, 10], the recent generalization of the Penrose tree graph identity given in [11] and the results and estimates on the gas of non overlapping subsets discussed in $[5,1]$.
To our knowledge, although it has been known for a long time that the integral central limit theorem holds for spin systems interacting via an absolutely summable potential with some additional conditions (for instance FKG inequalities, see e.g. [7, 8]), there are no results in literature about the validity of the local central limit theorem with the same conditions.

## 2. The model

We work in $\mathbb{Z}^{d}$. In each site $x \in \mathbb{Z}^{d}$ we define a spin variable $s_{x}$ which we suppose for simplicity to take values in a bounded interval $I$ of $\mathbb{Z}$, (i.e. $s_{x}$ can take all value set $I=[m, m+1, \ldots, n-1, n]$
with $m, n \in \mathbb{Z}$ and $m<n$ ). However, all results obtained in this paper can be generalized straightforwardly for bounded spin systems in which the variable $s_{x}$ is lattice distributed with maximal span, see [3], [2] and [4]. We recall that random variable $s_{x}$ is lattice distributed if there are two real numbers $a$ and $h$ such that $s_{x}=a+m h$ with $m \in \mathbb{Z}$. The number $h$ is called the span of the distribution and $h$ is maximal if the same representation for $s_{x}$ cannot be obtained for any $a^{\prime}, h^{\prime} \in \mathbb{R}$ such that $h^{\prime}>h$ (see [6]).
We set $\sigma=\max _{s_{x} \in I}\left|s_{x}\right|$ and $|I|$ the cardinality of $I$. Let $\Omega$ denote the set of all spin configurations in $\mathbb{Z}^{d}$ and if $\Lambda \subset \mathbb{Z}^{d}$ then $\Omega_{\Lambda}$ is the set of all spin configurations in $\Lambda$. We denote by $s_{\Lambda}$ a generic configuration in $\Lambda$. Note that if $\Lambda$ is finite then $\Omega_{\Lambda}$ contains $|I|^{|\Lambda|}$ different configurations where $|I|$ denotes the cardinality of $I$. Given a boundary condition $\omega \in \Omega$ and given $\Lambda \subset \mathbb{Z}^{d}$ finite, the Hamiltonian $H_{\Lambda}^{\omega}\left(s_{\Lambda}\right)$ of the system in $\Lambda$ as the function from $\Omega_{\Lambda}$ to $\mathbb{R}$ given by

$$
\begin{equation*}
-H_{\Lambda}^{\omega}\left(s_{\Lambda}\right)=\sum_{\{x, y\} \in \Lambda} J_{x y} s_{x} s_{y}+\sum_{x \in \Lambda} \sum_{y \in \Lambda^{c}} J_{x y} s_{x} \omega_{y} \tag{2.1}
\end{equation*}
$$

(the minus sign is there just to accord to physicists' convention). We set $\Lambda^{c}=\mathbb{Z}^{d} \backslash \Lambda$ and

$$
\begin{equation*}
h_{x}^{\omega}\left(s_{x}\right)=\sum_{y \in \Lambda^{c}} J_{x y} s_{x} \omega_{y} . \tag{2.2}
\end{equation*}
$$

The pair potential $J_{x y} \in \mathbb{R}$ (with no definite sign) is supposed to be absolutely summable. Namely,

$$
\begin{equation*}
\sup _{x \in \mathbb{Z}^{d}} \sum_{y \neq x}\left|J_{x y}\right|=J<+\infty . \tag{2.3}
\end{equation*}
$$

The probability of a spin configuration $s_{\Lambda}$ in $\Lambda$ (Gibbs measure) is given by

$$
\begin{equation*}
\mathbb{P}_{\Lambda}^{\omega}\left(s_{\Lambda}\right)=\frac{e^{-H_{\Lambda}^{\omega}\left(s_{\Lambda}\right)}}{Z_{\Lambda}^{\omega}}=\frac{e^{\sum_{\{x, y\} \in \Lambda_{n}} J_{x y} s_{x} s_{y}+\sum_{x \in \Lambda_{n}} h_{x}^{\omega}\left(s_{x}\right)}}{Z_{\Lambda}^{\omega}} \tag{2.4}
\end{equation*}
$$

where

$$
\begin{equation*}
Z_{\Lambda}^{\omega}=\sum_{s_{\Lambda} \in \Omega_{\Lambda}} e^{\sum_{\{x, y\} \in \Lambda_{n}} J_{x y} s_{x} s_{y}+\sum_{x \in \Lambda_{n}} h_{x}^{\omega}\left(s_{x}\right)} \tag{2.5}
\end{equation*}
$$

is the partition function.
We further define the single spin probability distribution at the site $x \in \Lambda$ as

$$
\begin{equation*}
p_{x}^{\omega}\left(s_{x}\right)=\frac{e^{h_{x}^{\omega}\left(s_{x}\right)}}{Z_{x}^{\omega}} \tag{2.6}
\end{equation*}
$$

where $Z_{x}^{\omega}=\sum_{s_{x} \in I} e^{h_{x}^{\omega}\left(s_{x}\right)}$. Hereafter $E_{x}^{\omega}(\cdot)$ will denote hereafter the expectation w.r.t. the single spin probability measure $p_{x}^{\omega}\left(s_{x}\right)$. Note that, due to (2.3) we have, for all allowed values of $s_{x}$ and uniformly in $x$ and $\omega$ that $\left|h_{x}^{\omega}\left(s_{x}\right)\right| \leq J \sigma^{2}$. Therefore the single spin probability distribution satisfies, for any $s_{x}$ and any $\omega$, the lower bound

$$
\begin{equation*}
p_{x}^{\omega}\left(s_{x}\right) \geq \frac{e^{-2 J \sigma^{2}}}{|I|} \doteq \kappa(J, \sigma) . \tag{2.7}
\end{equation*}
$$

Let now $\Lambda_{n}$ be the cube of size $2 n+1$ centered at the origin. We set shortly $\mathbb{P}_{\Lambda_{n}}^{\omega}(\cdot)=\mathbb{P}_{n}^{\omega}(\cdot)$ and $\mathbb{E}_{\Lambda_{n}}^{\omega}(\cdot)=\mathbb{E}_{n}^{\omega}(\cdot)$ for the Gibbs measure on $\Omega_{\Lambda_{n}}$ and its expected value respectively. We further define the random variables

$$
S_{n}=\sum_{x \in \Lambda_{n}} s_{x}
$$

and

$$
\bar{S}_{n}=\frac{S_{n}-\mathbb{E}_{n}^{\omega}\left(S_{n}\right)}{\sqrt{D_{n}}}
$$

where $D_{n}$ is the variance of $S_{n}$.

Definition 1 The integral central limit theorem holds for the spin system under study if the following conditions are satisfied.

$$
\begin{gather*}
\lim _{n \rightarrow \infty} \frac{D_{n}}{\left|\Lambda_{n}\right|}=\alpha>0  \tag{2.8}\\
\lim _{n \rightarrow \infty} \mathbb{P}_{n}^{\omega}\left(\bar{S}_{n}<x\right)=\frac{1}{\sqrt{2 \pi}} \int_{-\infty}^{x} e^{-\frac{z^{2}}{2}} d z \tag{2.9}
\end{gather*}
$$

Definition 2 The local central limit theorem holds for the spin system under study if (2.8) holds and

$$
\begin{equation*}
\lim _{n \rightarrow \infty} \sup _{p}\left|\sqrt{D_{n}} \mathbb{P}_{n}^{\omega}\left(S_{n}=p\right)-\frac{e^{-\frac{z_{n}^{2}(p)}{2}}}{\sqrt{2 \pi}}\right|=0 \tag{2.10}
\end{equation*}
$$

where

$$
z_{n}(p)=\frac{p-\mathbb{E}_{n}^{\omega}\left(S_{n}\right)}{\sqrt{D_{n}}}
$$

The result of the present paper consists in the proof of the following theorem.
Theorem 1 Under the assumption (2.3), if the sequence of Gibbs measures $\mathbb{P}_{n}^{\omega}$ satisfies the Integral central limit theorem, then it satisfies also the local central limit theorem.

The rest of the paper is devoted to the proof of Theorem 1.

## 3. Preliminaries

The starting point is to observe that

$$
\begin{equation*}
\frac{1}{\sqrt{2 \pi}} e^{-\frac{z_{n}^{2}(p)}{2}}=\frac{1}{2 \pi} \int_{-\infty}^{\infty} e^{-i t z_{n}(p)} e^{-\frac{t^{2}}{2}} d t \tag{3.1}
\end{equation*}
$$

and

$$
\begin{equation*}
\sqrt{D_{n}} \mathbb{P}_{n}^{\omega}\left(S_{n}=p\right)=\frac{1}{2 \pi} \int_{-\pi \sqrt{D_{n}}}^{+\pi \sqrt{D_{n}}} \mathbb{E}_{n}^{\omega}\left(e^{i t \bar{S}_{n}}\right) e^{-i t z_{n}(p)} d t \tag{3.2}
\end{equation*}
$$

Equality (3.1) is a standard Gaussian integral while (3.2) holds in general for spin random variables $s_{x}$ which are lattice distributed (see above). We remind that in our case $p$ can take only integer values. Let us set

$$
G_{n}=2 \pi\left(\sqrt{D_{n}} \mathbb{P}_{n}^{\omega}\left(S_{n}=p\right)-\frac{1}{\sqrt{2 \pi}} e^{-\frac{z_{n}^{2}(p)}{2}}\right)
$$

Then, by (3.2) and (3.1) we have

$$
G_{n}=\int_{-\pi \sqrt{D_{n}}}^{+\pi \sqrt{D_{n}}} \mathbb{E}_{n}^{\omega}\left(e^{i t \overline{S_{n}}}\right) e^{-i t z_{n}(p)} d t-\int_{-\infty}^{\infty} e^{-i t z_{n}(p)} e^{-\frac{t^{2}}{2}} d t
$$

Let now $\delta<\pi$ and let $A<\delta \sqrt{D_{n}}$ a sufficiently large positive constant. Thus

$$
\begin{aligned}
\left|G_{n}\right| \leq & \int_{-A}^{+A}\left|\mathbb{E}_{n}^{\omega}\left(e^{i t \bar{S}_{n}}\right)-e^{-\frac{t^{2}}{2}}\right| d t+\int_{A<|t| \leq \delta \sqrt{D_{n}}}\left|\mathbb{E}_{n}^{\omega}\left(e^{i t \bar{S}_{n}}\right)\right| d t+\int_{\delta \sqrt{D_{n}}<|t| \leq \pi \sqrt{D_{n}}}\left|\mathbb{E}_{n}^{\omega}\left(e^{i t \bar{S}_{n}}\right)\right| d t \\
& +\int_{|t| \geq A} e^{-\frac{t^{2}}{2}} d t .
\end{aligned}
$$

Therefore

$$
\sup _{p}\left|\sqrt{D_{n}} \mathbb{P}_{n}^{\omega}\left(S_{n}=p\right)-\frac{e^{-\frac{z_{n}^{2}(p)}{2}}}{\sqrt{2 \pi}}\right| \leq \frac{1}{2 \pi}\left[I_{n}^{(1)}+I_{n}^{(2)}+I_{n}^{(3)}+I_{n}^{(4)}\right]
$$

where

$$
\begin{gathered}
I_{n}^{(1)}=\int_{-A}^{+A}\left|\mathbb{E}_{n}^{\omega}\left(e^{i t \bar{S}_{n}}\right)-e^{-\frac{t^{2}}{2}}\right| d t, \quad I_{n}^{(2)}=\int_{A<|t| \leq \delta \sqrt{D_{n}}}\left|\mathbb{E}_{n}^{\omega}\left(e^{i t \bar{S}_{n}}\right)\right| d t \\
I_{n}^{(3)}=\int_{\delta \sqrt{D_{n}}<|t| \leq \pi \sqrt{D_{n}}}\left|\mathbb{E}_{n}^{\omega}\left(e^{i t \bar{S}_{n}}\right)\right| d t, \quad I_{n}^{(4)}=\int_{|t| \geq A} e^{-\frac{t^{2}}{2}} d t .
\end{gathered}
$$

Now we have trivially that $I^{(4)}$ is as small as we please due to arbitrarily of $A$. Moreover, by the integral central limit theorem, i.e. (2.9), we also have that $\lim _{n \rightarrow \infty} I_{n}^{(1)}=0$. Therefore we need to bound $I_{n}^{(2)}$ and $I_{n}^{(3)}$ and prove that they go to zero as $n \rightarrow \infty$.
Observe that

$$
\left|\mathbb{E}_{n}^{\omega}\left(e^{i t \bar{S}_{n}}\right)\right|=\left\lvert\, \mathbb{E}_{n}^{\omega}\left(e^{\left.i t \frac{S_{n}}{\sqrt{D_{n}}}\right) \mid . ~}\right.\right.
$$

So, by the change of variables $\tau=t / \sqrt{D_{n}}$, and using that $\left|\mathbb{E}_{n}^{\omega}\left(e^{i t \frac{S_{n}}{\sqrt{D_{n}}}}\right)\right|$ is an even function of $t$, the integrals $I_{n}^{(2)}$ and $I_{n}^{(3)}$ can be written as

$$
\begin{equation*}
I_{n}^{(2)}=2 \sqrt{D_{n}} \int_{\frac{A}{\sqrt{D_{n}}}}^{\delta}\left|\mathbb{E}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right| d t, \quad I_{n}^{(3)}=2 \sqrt{D_{n}} \int_{\delta}^{\pi}\left|\mathbb{E}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right| d t \tag{3.3}
\end{equation*}
$$

We now define the decimation introduced in [2]. Let $r_{0} \in \mathbb{N}$ and define

$$
\mathbb{Z}^{d}\left(r_{0}\right)=\left\{\left(n_{1} r_{0}, \ldots, n_{d} r_{0}\right): n_{i} \in \mathbb{Z}\right\},
$$

i.e. $\mathbb{Z}^{d}\left(r_{0}\right)$ is a cubic sublattice of $\mathbb{Z}^{d}$ of step $r_{0}$. Let $\tilde{\Lambda}_{n}=\Lambda_{n} \cap \mathbb{Z}^{d}\left(r_{0}\right)$ and $\tilde{S}_{n}=\sum_{x \in \tilde{\Lambda}_{n}} s_{x}$. Then

$$
\mathbb{E}_{n}^{\omega}\left(e^{i t S_{n}}\right)=\mathbb{E}_{n}^{\omega}\left(\mathbb{E}_{n}^{\omega}\left(e^{i t S_{n}} \mid s_{\Lambda_{n}} \backslash \tilde{\Lambda}_{n} \text { fixed }\right)\right)=\mid \mathbb{E}_{n}^{\omega}\left(e^{i t S_{n}-\tilde{S}_{n}} \mathbb{E}_{n}^{\omega}\left(e^{i t \tilde{S}_{n}} \mid s_{\Lambda_{n}} \backslash \tilde{\Lambda}_{n} \text { fixed }\right)\right) \mid
$$

Thus

$$
\begin{equation*}
\left|\mathbb{E}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right| \leq \sup _{\omega \in \Omega_{\left(\Lambda_{n} \backslash \tilde{\Lambda}_{n}\right) \cup \Lambda_{n}^{c}}} \mid \mathbb{E}_{n}^{\omega}\left(e^{i t \tilde{S}_{n}} \mid s_{\Lambda_{n} \backslash \tilde{\Lambda}_{n}} \text { fixed }\right)\left|=\sup _{\omega \in \Omega_{\tilde{\Lambda}_{n}^{c}}}\right| \tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t \tilde{S}_{n}}\right) \mid . \tag{3.4}
\end{equation*}
$$

Here above $\tilde{\Lambda}_{n}^{c}=\mathbb{Z}^{d} \backslash \tilde{\Lambda}_{n}=\left(\Lambda_{n} \backslash \tilde{\Lambda}_{n}\right) \cup \Lambda_{n}^{c}$ and $\tilde{\mathbb{E}}_{n}^{\omega}$ is the expectation w.r.t. the measure

$$
\begin{equation*}
\tilde{\mathbb{P}}_{n}^{\omega}\left(s_{\Lambda_{n}}\right)=\frac{e^{\sum_{\{x, y\} \in \tilde{\Lambda}_{n}} J_{x y} s_{x} s_{y}+\sum_{x \in \tilde{\Lambda}_{n}} h_{x}^{\omega}\left(s_{x}\right)}}{Z_{n}^{\omega}} \tag{3.5}
\end{equation*}
$$

where now

$$
h_{x}^{\omega}\left(s_{x}\right)=\sum_{y \in \tilde{\Lambda}_{n}^{c}} J_{x y} s_{x} \omega_{y}
$$

and

$$
Z_{n}^{\omega}=\sum_{s_{\tilde{\Lambda}_{n}} \in \Omega_{\tilde{\Lambda}_{n}}} e^{-\sum_{\{x, y\} \in \tilde{\Lambda}_{n}} J_{x y} s_{x} s_{y}+\sum_{x \in \tilde{\Lambda}_{n}} h_{x}^{\omega}\left(s_{x}\right)} .
$$

We set

$$
\begin{equation*}
J_{r_{0}}=\sup _{x \in \mathbb{Z}^{d}\left(r_{0}\right)} \sum_{\substack{y \in \mathbb{Z}^{d}\left(r_{0}\right) \\ y \neq x}}\left|J_{x y}\right| \tag{3.6}
\end{equation*}
$$

Note that $J_{r_{0}}$ can be done as small as we please by taking $r_{0}$ sufficiently large. Moreover We now state a key lemma from which Theorem 1 follows as an immediate corollary.

Lemma 2 Let $\kappa(J, \sigma)$ as in (2.7), let $\delta, C$ and $c$ the positive numbers given by

$$
\begin{equation*}
\delta=\frac{\kappa(J, \sigma)}{12 \sigma}, \quad C=\sigma^{2} \frac{\kappa(J, \sigma)}{4}, \quad c=\kappa(J, \sigma) \sin ^{2}(\delta / 2) \tag{3.7}
\end{equation*}
$$

and let $r_{0}$ be chosen such that

$$
\begin{equation*}
e^{\frac{J_{r_{0}}}{2}} J_{r_{0}}^{\frac{1}{2}} \leq \min \left\{\frac{[\kappa(J, \sigma)]^{\frac{3}{2}}}{96 \sqrt{2} \sigma^{3} e^{2}}, \frac{e^{-\frac{5 c}{4}}\left(e^{\frac{c}{4}}-1\right)}{(1+\delta \sigma) e \sigma^{2}}\right\}, \tag{3.8}
\end{equation*}
$$

then
(a) For any $t \in(0, \delta]$

$$
\begin{equation*}
\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right| \leq e^{-\frac{C}{2}\left|\tilde{\Lambda}_{n}\right| t^{2}} \tag{3.9}
\end{equation*}
$$

(b) For any $t \in(\delta, \pi]$

$$
\begin{equation*}
\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t \tilde{S}_{n}}\right)\right| \leq e^{-\frac{c}{2}\left|\tilde{\Lambda}_{n}\right|} \tag{3.10}
\end{equation*}
$$

Assuming Lemma 2 , Theorem 1 follows straightforwardly. Indeed, by (3.9), (3.10) and using (3.4), the integrals $I_{n}^{(2)}$ and $I_{n}^{(3)}$ given in (3.3) can be bounded as

$$
\begin{gather*}
I_{n}^{(2)} \leq 2 \sqrt{\frac{D_{n}}{\left|\tilde{\Lambda}_{n}\right|}} \int_{A \sqrt{\frac{\tilde{\Lambda}_{n} \mid}{D_{n}}}}^{\delta \sqrt{\left|\tilde{\Lambda}_{n}\right|}} e^{-\frac{C}{2} \tau^{2}} d \tau,  \tag{3.11}\\
I_{n}^{(3)} \leq 2 \sqrt{D_{n}}(\pi-\delta) e^{-\frac{c}{2}\left|\tilde{\Lambda}_{n}\right|} . \tag{3.12}
\end{gather*}
$$

The r.h.s. of (3.11) goes to zero as $n \rightarrow \infty$ due to (2.8) and the arbitrariness of $A$ while the r.h.s. of (3.12) goes to zero as $n \rightarrow \infty$ due to (2.8).
The rest of the paper is devoted to the proof of Lemma 2.

## 4. Proof of Lemma 2, part (a)

Recalling the definition of the single spin distribution (2.6), the expectation $\tilde{\mathbb{E}}_{n}^{\omega}(\cdot)$ w. r. t. the measure (3.5) can be written as

$$
\begin{equation*}
\tilde{\mathbb{E}}_{n}^{\omega}(\cdot)=\frac{\sum_{s_{\tilde{\Lambda}_{n}} \in \Omega_{\tilde{\Lambda}_{n}}} e^{-\sum_{\{x, y\} \in \tilde{\Lambda}_{n}} J_{x y} s_{x} s_{y}}(\cdot) \prod_{x \in \tilde{\Lambda}_{n}} p_{x}^{\omega}\left(s_{x}\right)}{\sum_{s_{\tilde{\Lambda}_{n}} \in \Omega_{\tilde{\Lambda}_{n}}} e^{-\sum_{\{x, y\} \in \tilde{\Lambda}_{n}} J_{x y} s_{x} s_{y}} \prod_{x \in \tilde{\Lambda}_{n}} p_{x}^{\omega}\left(s_{x}\right)} . \tag{4.1}
\end{equation*}
$$

Therefore we can write

$$
\begin{equation*}
\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t \tilde{S}_{n}}\right)=\frac{\Xi_{n}^{\omega}(t)}{\Xi_{n}^{\omega}(0)} \tag{4.2}
\end{equation*}
$$

where

$$
\begin{equation*}
\Xi_{n}(t)=\sum_{s_{\tilde{\Lambda}_{n}} \in \Omega_{\tilde{\Lambda}_{n}}} \prod_{\{x, y\} \subset \tilde{\Lambda}_{n}} e^{J_{x y} s_{x} s_{y}} \prod_{x \in \tilde{\Lambda}_{n}} e^{i t s_{x}} \prod_{x \in \tilde{\Lambda}_{n}} p_{x}^{\omega}\left(s_{x}\right) . \tag{4.3}
\end{equation*}
$$

Proposition 3 The function $\Xi_{n}^{\omega}(t)$ can be rewritten as the $t$-dependent grand canonical partition function of a gas of non overlapping subsets of $\tilde{\Lambda}_{n}$. Namely the following identity holds.

$$
\begin{equation*}
\Xi_{\tilde{\Lambda}_{n}}^{\omega}(t)=1+\sum_{\substack{k \geq 1}} \sum_{\substack{\left\{R_{1}, \ldots, R_{k}\right\}: R_{i} \subset \tilde{\Lambda}_{n} \\ R_{i} \neq \emptyset, R_{i} \cap R_{j}=\emptyset}} \prod_{i=1}^{k} \xi_{t}\left(R_{i}\right) \tag{4.4}
\end{equation*}
$$

with activities given by

$$
\xi_{t}(R)= \begin{cases}\sum_{s_{R} \in \Omega_{R}} \sum_{g \in G_{R}} \sum_{S \subset R} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right) \prod_{x \in S}\left(e^{i t s_{x}}-1\right) \prod_{x \in R} p_{x}^{\omega}\left(s_{x}\right) & \text { if }|R| \geq 2  \tag{4.5}\\ \sum_{s_{x} \in I}\left(e^{i t s_{x}}-1\right) p_{x}^{\omega}\left(s_{x}\right) & \text { if } R=\{x\}\end{cases}
$$

where $G_{R}$ denotes the set of all connected graphs with vertex set $R$ and $E_{g}$ denotes the edge set of $g \in G_{R}$.

Proof. The identity (4.4) is obtained via the standard Mayer trick on both the one point terms of the form $e^{i t s_{x}}$ and the two point terms of the form $e^{J_{x y} s_{x} s_{y}}$. Namely in the r.h.s. of (4.3) write $\prod_{x \in \tilde{\Lambda}_{n}} e^{i t s_{x}}=\prod_{x \in \tilde{\Lambda}_{n}}\left[\left(e^{i t s_{x}}-1\right)+1\right]$ and $\prod_{\{x, y\} \subset \tilde{\Lambda}_{n}} e^{J_{x y} s_{x} s_{y}}=\prod_{\{x, y\} \subset \tilde{\Lambda}_{n}}\left[\left(e^{J_{x y} s_{x} s_{y}}-1\right)+1\right]$ and develop the products.
In the next section we will make use of the following well know expression (see e.g. [5]) of the logarithm of $\Xi_{n}^{\omega}(t)$.

$$
\begin{equation*}
\ln \Xi_{n}^{\omega}(t)=\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\left(R_{1}, \ldots, R_{k}\right) \in \mathcal{P}_{n}^{k}} \Phi^{T}\left(R_{1}, \ldots, R_{k}\right) \prod_{i=1}^{k} \xi_{t}\left(R_{i}\right) \tag{4.6}
\end{equation*}
$$

where $\mathcal{P}_{n}=\left\{R \subset \tilde{\Lambda}_{n}:|R| \geq 1\right\}$ and

$$
\phi^{T}\left(R_{1}, \ldots, R_{k}\right)= \begin{cases}1 & \text { if } k=1  \tag{4.7}\\ \sum_{g \in G_{k}} \prod_{\{i, j\} \in E_{g}}\left(e^{-V_{h . c}\left(R_{i}, R_{j}\right)}-1\right) & \text { if } k \geq 2\end{cases}
$$

with $V_{h . c}\left(R_{i}, R_{j}\right)=+\infty$ if $R_{i} \cap R_{j} \neq \emptyset$ and $V_{h . c}\left(R_{i}, R_{j}\right)=0$ if $R_{i} \cap R_{j}=\emptyset$ and $G_{k}$ denotes the set of all connected graphs with vertex set $\{1,2, \ldots, k\} \doteq[k]$.

### 4.1 Estimate on $\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t \tilde{S}_{n}}\right)\right|$ when $t \in(0, \delta]$ with $\delta$ given in (3.7)

By Proposition 3,

$$
\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right|=\left|\exp \left\{\ln \Xi_{\tilde{\Lambda}_{n}}^{\omega}(t)-\ln \Xi_{\tilde{\Lambda}_{n}}^{\omega}(0)\right\}\right|=\exp \left\{\Re\left(\ln \Xi_{\tilde{\Lambda}_{n}}^{\omega}(t)-\ln \Xi_{\tilde{\Lambda}_{n}}^{\omega}(0)\right)\right\} .
$$

Then, by (4.6) we have that

$$
\begin{equation*}
\Re\left(\ln \Xi_{\tilde{\Lambda}_{n}}^{\omega}(J, t)-\ln \Xi_{\tilde{\Lambda}_{n}}^{\omega}(0)\right)=\sum_{k \geq 1} \frac{1}{k!} \sum_{\left(R_{1}, \ldots, R_{k}\right) \in \mathcal{P}_{n}^{k}} \Phi^{T}\left(R_{1}, \ldots, R_{k}\right) \Re\left[\prod_{i=1}^{k} \xi_{t}\left(R_{i}\right)-\prod_{i=1}^{k} \xi_{0}\left(R_{i}\right)\right] . \tag{4.8}
\end{equation*}
$$

Set

$$
\begin{equation*}
F_{k}(t)=\prod_{i=1}^{k} \xi_{t}\left(R_{i}\right)-\prod_{i=1}^{k} \xi_{0}\left(R_{i}\right) \tag{4.9}
\end{equation*}
$$

It is not difficult (but tedious) to check that $\left.\Re \frac{d F_{k}(t)}{d t}\right|_{t=0}=0$. Moreover, since we also have that $F_{k}(0)=0$, by the Taylor remainder theorem, we can conclude that there exists a $0<\theta<t<\delta$ such that

$$
\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right|=\exp \left\{\left.\frac{t^{2}}{2} \sum_{k \geq 1} \frac{1}{k!} \sum_{\left(R_{1}, \ldots, R_{k}\right) \in \mathcal{P}_{n}^{k}} \Phi^{T}\left(R_{1}, \ldots, R_{k}\right) \Re \frac{d^{2}}{d t^{2}}\left[\prod_{i=1}^{k} \xi_{t}\left(R_{i}\right)\right]\right|_{t=\theta}\right\}
$$

where $\mathcal{P}_{n}=\left\{R \subset \tilde{\Lambda}_{n}:|R| \geq 1\right\}$. Let

$$
G(\theta)=\left.\sum_{k \geq 1} \frac{1}{k!} \sum_{\left(R_{1}, \ldots, R_{k}\right) \in \mathcal{P}_{n}^{k}} \Phi^{T}\left(R_{1}, \ldots, R_{k}\right) \frac{d^{2}}{d t^{2}}\left[\prod_{i=1}^{k} \xi_{t}\left(R_{i}\right)\right]\right|_{t=\theta}
$$

We can write

$$
G(\theta)=G_{1}(\theta)+G_{2}(\theta)+G_{3}(\theta)+G_{4}(\theta)
$$

where

$$
\begin{gather*}
G_{1}(\theta)=\left.\sum_{x \in \tilde{\Lambda}_{n}} \Phi^{T}(\{x\}) \frac{d^{2}}{d t^{2}} \xi_{t}(\{x\})\right|_{t=\theta},  \tag{4.10}\\
G_{2}(\theta)=\left.\sum_{x \in \tilde{\Lambda}_{n}} \frac{1}{2} \Phi^{T}(\{x\},\{x\}) \frac{d^{2}}{d t^{2}} \xi_{t}^{2}(\{x\})\right|_{t=\theta},  \tag{4.11}\\
G_{3}(\theta)=\left.\sum_{x \in \tilde{\Lambda}_{n}} \sum_{k \geq 3} \frac{1}{k!} \Phi^{T}(\underbrace{\{x\}, \ldots,\{x\}}_{k \text { times }}) \frac{d^{2}}{d t^{2}} \xi_{t}^{k}(\{x\})\right|_{t=\theta},  \tag{4.12}\\
G_{4}(\theta)=\left.\sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{\left(R_{1}, \ldots, R_{k}\right) \in \mathcal{P}_{n}^{k} \\
\exists z_{i}\left|R_{i}\right| \geq 2}} \Phi^{T}\left(R_{1}, \ldots, R_{k}\right) \frac{d^{2}}{d t^{2}}\left[\prod_{i=1}^{k} \xi_{t}\left(R_{i}\right)\right]\right|_{t=\theta}, \tag{4.13}
\end{gather*}
$$

so that

$$
\begin{aligned}
\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right| & =\exp \left\{\frac{t^{2}}{2} \Re\left(G_{1}(\theta)+G_{2}(\theta)+G_{3}(\theta)+G_{4}(\theta)\right)\right\} \\
& \leq \exp \left\{\frac{t^{2}}{2}\left(\Re G_{1}(\theta)+\Re G_{2}(\theta)+\left|G_{3}(\theta)\right|+\left|G_{4}(\theta)\right|\right)\right\} .
\end{aligned}
$$

In order to control $G_{i}(\theta)(i=1,2,3,4)$ we need to evaluate $\frac{d}{d t} \xi_{t}(\{x\})$ and $\frac{d^{2}}{d t^{2}} \xi_{t}(\{x\})$. Recalling the definition of $\xi_{t}(\{x\})$ given in (4.5), we have

$$
\begin{equation*}
\frac{d}{d t} \xi_{t}(\{x\})=i \sum_{s_{x} \in I} s_{x} e^{i t s_{x}} p_{x}^{\omega}\left(s_{x}\right), \quad \frac{d^{2}}{d t^{2}} \xi_{t}(\{x\})=-\sum_{s_{x} \in I} p_{x}^{\omega}\left(s_{x}\right) s_{x}^{2} e^{i t s_{x}} \tag{4.14}
\end{equation*}
$$

whence

$$
\left.\Re \frac{d^{2}}{d t^{2}} \xi_{t}(\{x\})\right|_{t=\theta}=-E_{x}^{\omega}\left(s_{x}^{2} \cos \left(\theta s_{x}\right)\right)
$$

where we recall that $E_{x}^{\omega}(\cdot)$ is the expectation w.r.t. the single spin probability measure $p_{x}^{\omega}\left(s_{x}\right)$. Moreover, when $t<\delta$,

$$
\begin{equation*}
\left|\xi_{t}(\{x\})\right| \leq \delta \sigma, \quad\left|\frac{d \xi_{t}(\{x\})}{d t}\right| \leq \sigma, \quad\left|\frac{d^{2}}{d t^{2}} \xi_{t}(\{x\})\right| \leq \sigma^{2} \tag{4.15}
\end{equation*}
$$

Bounding $\Re G_{1}(\theta)$
Due to (3.7) and (2.7), $\delta<\frac{1}{12 \sigma}$ ), we have that surely $\cos \left(\theta s_{x}\right) \geq \frac{7}{8}$ for any $\theta<\delta$. Then

$$
-\Re G_{1}(\theta)=E_{x}^{\omega}\left(s_{x}^{2} \cos \left(\theta s_{x}\right)\right)=\sum_{s_{x} \in I} p_{x}^{\omega}\left(s_{x}\right) s_{x}^{2} \cos \left(\theta s_{x}\right) \geq \frac{e^{-2 J \sigma^{2}} \sigma^{2}}{2|I|}=\frac{7 \sigma^{2}}{8} \kappa(J, \sigma)
$$

where $\kappa(J, \sigma)$ is the positive number defined in (2.7). This bound implies that

$$
\begin{equation*}
\Re G_{1}(\theta) \leq-\frac{7 \sigma^{2}}{8} \kappa(J, \sigma)\left|\tilde{\Lambda}_{n}\right| . \tag{4.16}
\end{equation*}
$$

## $\underline{\text { Bounding } \Re G_{2}(\theta)}$

We need to evaluate $\frac{d^{2}}{d t^{2}} \xi_{t}^{2}(\{x\})$ appearing in the term $G_{2}(\theta)$. We have, recalling (4.14)

$$
\frac{d^{2}}{d t^{2}} \xi_{t}^{2}(\{x\})=2\left[\left(\frac{d \xi_{t}(\{x\})}{d t}\right)^{2}+\xi_{t}(\{x\}) \frac{d^{2} \xi_{t}(\{x\})}{d t^{2}}\right]=-2\left[\left(E_{x}^{\omega}\left(s_{x} e^{i t s_{x}}\right)\right)^{2}+\xi_{t}(\{x\}) E_{x}^{\omega}\left(s_{x}^{2} e^{i t s_{x}}\right)\right] .
$$

Now observe that

$$
\Re\left(E_{x}^{\omega}\left(s_{x} e^{i t s_{x}}\right)\right)^{2}=\frac{\sqrt{2}}{2}\left[\left(E _ { x } ^ { \omega } \left(s_{x}\left(\cos \left(s_{x} t-\pi / 4\right)\right)\left(E_{x}^{\omega}\left(s_{x}\left(\cos \left(s_{x} t+\pi / 4\right)\right)\right)\right]\right.\right.\right.
$$

and $\cos (x \pm \pi / 4)$ is greater than 0 when $|x|<\pi / 4$. So, since $\delta<\frac{1}{12 \sigma}$, we have that surely

$$
E_{x}^{\omega}\left(s_{x}\left(\cos \left(s_{x} t \pm \pi / 4\right)\right)>0\right.
$$

for any $t<\delta$. In conclusion we have that

$$
\Re\left(\frac{d \xi_{t}(\{x\})}{d t}\right)^{2}<0
$$

Thus, recalling (4.15), we have, when $\theta<t \leq \delta$ that

$$
\left.\Re \frac{d^{2}}{d t^{2}} \xi_{t}^{2}(\{x\})\right|_{t=\theta} \leq 2\left|\xi_{t}(\{x\})\right|\left|\frac{d^{2} \xi_{t}(\{x\})}{d t^{2}}\right| \leq 2 \delta \sigma^{3}
$$

which implies that

$$
\begin{equation*}
\Re G_{2}(\theta) \leq 2 \delta \sigma^{3}\left|\tilde{\Lambda}_{n}\right| . \tag{4.17}
\end{equation*}
$$

Bounding $\left|G_{3}(\theta)\right|$
In order to bound To bound $\left|G_{3}(\theta)\right|$ we will use the following well known identity (see e.g. [5]).

$$
\begin{equation*}
\Phi^{T}(\underbrace{\{x\}, \ldots,\{x\}}_{k \text { times }})=(-1)^{k-1}(k-1)!. \tag{4.18}
\end{equation*}
$$

We have, for $k \geq 3$

$$
\frac{d^{2}}{d t^{2}} \xi_{t}^{k}(\{x\})=k(k-1) \xi_{t}^{k-2}(\{x\})\left(\frac{d}{d t} \xi_{t}(\{x\})\right)^{2}+k \xi_{t}^{k-1}(\{x\}) \frac{d^{2}}{d t^{2}} \xi_{t}(\{x\})
$$

Hence, by (4.15), we have, for any $t<\delta$,

$$
\left|\frac{d^{2}}{d t^{2}} \xi_{t}^{k}(\{x\})\right| \leq k(k-1)(\delta \sigma)^{k-2} \sigma^{2}+k(\delta \sigma)^{k-1} \sigma^{2} .
$$

Therefore, since $\theta<\delta<\frac{1}{12 \sigma}$, using (4.18), we have that surely

$$
\begin{equation*}
\left|G_{3}(\theta)\right| \leq \sigma^{2}\left(\sum_{k \geq 3}(\delta \sigma)^{k-2}[(k-1)+(\delta \sigma)]\right)\left|\tilde{\Lambda}_{n}\right| \leq \frac{5}{2} \delta \sigma^{3}\left|\tilde{\Lambda}_{n}\right| . \tag{4.19}
\end{equation*}
$$

Collecting the bounds obtained above for $\Re G_{1}(\theta), \Re G_{2}(\theta)$ and $\left|G_{2}(\theta)\right|$ we have that

$$
\Re G_{1}(\theta)+\Re G_{2}(\theta)+\left|G_{3}(\theta)\right| \leq-\left|\tilde{\Lambda}_{n}\right| \sigma^{2}\left[\frac{7}{8} \kappa(J, \sigma)-\frac{9}{2} \delta \sigma\right]
$$

and recalling that $\delta=\frac{\kappa(J, \sigma)}{12 \sigma}$, we can conclude that as soon as $\theta<\delta$ the following inequality holds.

$$
\Re G_{1}(\theta)+\Re G_{2}(\theta)+\left|G_{3}(\theta)\right| \leq-\left|\tilde{\Lambda}_{n}\right| \sigma^{2} \frac{\kappa(J, \sigma)}{2} .
$$

Bounding $\left|G_{4}(\theta)\right|$
We start by observing that

$$
\begin{equation*}
\frac{d^{2}}{d t^{2}}\left[\prod_{i=1}^{k} \xi_{t}\left(R_{i}\right)\right]=\sum_{i=1}^{k} \frac{d^{2} \xi_{t}\left(R_{i}\right)}{d t^{2}} \prod_{\substack{j \in[k] \\ j \neq i}} \xi_{t}\left(R_{j}\right)+\sum_{i=1}^{k} \sum_{\substack{j \in[k] \\ j \neq i}} \frac{d \xi_{t}\left(R_{i}\right)}{d t} \frac{d \xi_{t}\left(R_{j}\right)}{d t} \prod_{\substack{l \in[k] \\ s \neq i, j}} \xi_{t}\left(R_{l}\right) . \tag{4.20}
\end{equation*}
$$

We thus need an estimate of $\left|\xi_{t}(R)\right|,\left|\frac{d \xi_{t}(R)}{d t}\right|$ and $\left|\frac{d^{2} \xi_{t}(R)}{d t^{2}}\right|$ when $|R| \geq 2$,.
Let us define

$$
w_{0}(R)=\left\{\begin{array}{ll}
(1+\delta \sigma)^{|R|} & \sum_{s_{R} \in \Omega_{R}} \prod_{x \in R} p_{x}^{\omega}\left(s_{x}\right)\left|\sum_{g \in G_{R}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right|  \tag{4.21}\\
\text { if }|R| \geq 2 \\
\delta \sigma & \text { if }|R|=1
\end{array} .\right.
$$

We have, for $|R| \geq 2$ and for any $t<\delta$

$$
\begin{align*}
\left|\frac{d \tilde{\xi}_{t}(R)}{d t}\right| & \left.=\left|\sum_{s_{R} \in \Omega_{R}} \sum_{g \in G_{R}} \sum_{\substack{S \subset R \\
S \neq \emptyset}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right) \frac{d}{d t}\left[\prod_{x \in S}\left(e^{i t s_{x}}-1\right)\right]\right| \prod_{x \in R} p_{x}^{\omega}\left(s_{x}\right) \right\rvert\, \\
& \leq \sum_{s_{R} \in \Omega_{R}} \sum_{\substack{S \subset R \\
S \neq \emptyset}}\left|\sum_{g \in G_{R}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right| \sum_{\substack{x \in S}}\left|s_{x}\right|\left[\prod_{\substack{y \in S \\
y \neq x}}\left|e^{i t s_{y}}-1\right|\right] \prod_{x \in R} p_{x}^{\omega}\left(s_{x}\right) \\
& \leq \sigma \sum_{s_{R} \in \Omega_{R}} \sum_{\substack{S \subset R \\
S \neq \emptyset}}|S|(\delta \sigma)^{|S|-1}\left|\sum_{g \in G_{R}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right| \prod_{x \in R} p_{x}^{\omega}\left(s_{x}\right)  \tag{4.22}\\
& \leq \sigma|R|(1+\delta \sigma)^{|R|} \sum_{s_{R} \in \Omega_{R}}\left|\sum_{g \in G_{R}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right| \prod_{x \in R} p_{x}^{\omega}\left(s_{x}\right) \\
& =\sigma|R| w_{0}(R)
\end{align*}
$$

and

$$
\begin{align*}
&\left|\frac{d^{2} \tilde{\xi}_{t}(R)}{d t^{2}}\right|= \left.\left|\sum_{s_{R} \in \Omega_{R}} \sum_{g \in G_{R}} \sum_{\substack{S \subset R \\
S \neq \emptyset}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right) \frac{d^{2}}{d t^{2}}\left[\prod_{x \in S}\left(e^{i t s_{x}}-1\right)\right]\right| t\left|\prod_{t=\theta}\right| p_{x \in R}^{\omega}\left(s_{x}\right) \right\rvert\, \\
& \leq \sum_{s_{R} \in \Omega_{R}} \prod_{x \in R} p_{x}^{\omega}\left(s_{x}\right)\left|\sum_{g \in G_{R}} \sum_{\substack{\begin{subarray}{c}{S R R \\
S \neq \emptyset} }}\end{subarray}} \prod_{\substack{x, y\} \in E_{g}}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right|\left[\sum_{x \in S} s_{x}^{2} \prod_{\substack{y \in S \\
y \neq x}}\left|e^{i t s_{y}}-1\right|\right. \\
&\left.\quad+\sum_{\substack{x, y \in S \\
x \neq y}}\left|s_{x} s_{y}\right| \prod_{\substack{z \in S \\
z \neq x, y}}\left|e^{i t s_{z}}-1\right|\right]  \tag{4.23}\\
& \leq \sigma^{2} w_{0}(R)\left|\sum_{\substack{S \subset R \\
S \neq \emptyset}}\left(|S|(\delta \sigma)^{|S|-1}+\left(|S|(|S|-1)(\delta \sigma)^{|S|-2}\right)\right]\right| \\
&= \sigma^{2} w_{0}(R)| | R \mid\left((1+\delta \sigma)^{|R|-1}+(|R|-1)(1+\delta \sigma)^{|R|-2}\right) \\
& \leq|R|^{2} \sigma^{2} w_{0}(R)
\end{align*}
$$

Therefore we have, for any $t<\delta$

$$
\left|\frac{d \tilde{\xi}_{t}(R)}{d t}\right|=\left\{\begin{array}{ll}
\sigma|R| w_{0}(R) & \text { if }|\mathrm{R}| \geq 2 \\
\frac{1}{\delta} w_{0}(R) & \text { if }|\mathrm{R}|=1
\end{array}, \quad\left|\frac{d^{2} \tilde{\xi}_{t}(R)}{d t^{2}}\right|=\left\{\begin{array}{ll}
\sigma^{2}|R|^{2} w_{0}(R) & \text { if }|\mathrm{R}| \geq 2 \\
\frac{\sigma}{\delta} w_{0}(R) & \text { if }|\mathrm{R}|=1
\end{array} .\right.\right.
$$

Since we are not interested in optimal bounds we can (very roughly) bound for any non empty $R \subset \tilde{\Lambda}_{n}$ and for any $t<\delta$

$$
\left|\frac{d \tilde{\xi}_{t}(R)}{d t}\right| \leq \frac{\sigma|R|}{\delta} w_{0}(R), \quad\left|\frac{d^{2} \tilde{\xi}_{t}(R)}{d t^{2}}\right| \leq \frac{\sigma^{2}}{\delta^{2}}|R|^{2} w_{0}(R)
$$

Using the two bounds above and recalling (4.20) we have

$$
\begin{aligned}
\left.\left|\frac{d^{2}}{d t^{2}}\left[\prod_{i=1}^{k} \xi_{t}\left(R_{i}\right)\right]\right|_{t=0} \right\rvert\, & \leq \frac{\sigma^{2}}{\delta^{2}}\left(\sum_{i=1}^{k}\left|R_{i}\right|^{2}+\sum_{\substack{(i, j) \in[k]^{2} \\
i \neq j}}\left|R_{i}\right|\left|R_{j}\right|\right) \prod_{i=1}^{k} w_{0}\left(R_{j}\right) \\
& =\frac{\sigma^{2}}{\delta^{2}}\left(\sum_{i=1}^{k}\left|R_{i}\right|\right)^{2} \prod_{i=1}^{k} w_{0}\left(R_{j}\right) \\
& \leq \frac{\sigma^{2}}{\delta^{2}} \prod_{i=1}^{k}\left[w_{0}\left(R_{i}\right) e^{\left|R_{i}\right|}\right] \\
& =\frac{\sigma^{2}}{\delta^{2}} \prod_{i=1}^{k} w_{1}\left(R_{i}\right) .
\end{aligned}
$$

where we have denoted shortly $w_{1}(R)=w_{0}(R) e^{|R|}$ for $R \subset \tilde{\Lambda}_{n}$. Now we can bound $\left|G_{4}(\theta)\right|$, when $\theta<\delta$, as follows.

$$
\begin{aligned}
\left|G_{4}(\theta)\right| & =\frac{\sigma^{2}}{\delta^{2}} \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{\left(R_{1}, \ldots, R_{k}\right) \in \mathcal{P}_{n}^{k} \\
\exists i:\left|R_{i}\right| \geq 2}}\left|\Phi^{T}\left(R_{1}, \ldots, R_{k}\right)\right| \prod_{i=1}^{k} w_{1}\left(R_{i}\right) \\
& =\frac{\sigma^{2}}{\delta^{2}} \sum_{\substack{R \in \mathcal{P}_{n} \\
|R| \geq 2}} \sum_{k \geq 1} \frac{1}{k!} \sum_{\substack{\left(R_{1}, \ldots, R_{k}\right) \in \mathcal{P}_{n}^{k} \\
\exists i: R_{i}=R}}\left|\Phi^{T}\left(R_{1}, \ldots, R_{k}\right)\right| \prod_{i=1}^{k} w_{1}\left(R_{i}\right) \\
& \leq \frac{\sigma^{2}}{\delta^{2}} \sum_{\substack{R \in \mathcal{P}_{n} \\
|R| \geq 2}} w_{1}(R) \Pi_{R}\left(\boldsymbol{w}_{1}\right)
\end{aligned}
$$

where $\Pi_{R}\left(\boldsymbol{w}_{1}\right)$ is the positive term series (see [5])

$$
\Pi_{R}\left(\boldsymbol{w}_{1}\right)=\sum_{k=0}^{\infty} \frac{1}{k!} \sum_{\left(R_{1}, \ldots, R_{k}\right) \in \mathcal{P}_{n}^{k}}\left|\phi^{T}\left(R, R_{1}, \ldots, R_{k}\right)\right| w_{1}\left(R_{1}\right) \cdots w_{1}\left(R_{k}\right)
$$

According to the standard cluster expansion theory of gas of non overlapping subsets (see [5] and [1]), denoting

$$
\begin{equation*}
w_{1}^{(k)}=\sup _{x \in \tilde{\Lambda}_{n}} \sum_{\substack{R \subset \tilde{\Lambda}_{n} \\ x \in R,|R|=k}} w_{1}(R) \tag{4.24}
\end{equation*}
$$

if for some $a>0$,

$$
\begin{equation*}
\sum_{n \geq 1} w_{1}^{(k)} e^{a k} \leq e^{a}-1 \tag{4.25}
\end{equation*}
$$

then

$$
\begin{equation*}
\Pi_{R}\left(\boldsymbol{w}_{1}\right) \leq e^{a|R|} \tag{4.26}
\end{equation*}
$$

Hence using (4.26) and (4.24) we get that

$$
\left|G_{4}(\theta)\right| \leq \frac{\sigma^{2}}{\delta^{2}} \sum_{\substack{R \subset \tilde{\Lambda}_{n} \\|R| \geq 2}} w_{1}(R) e^{a|R|} \leq \frac{\sigma^{2}}{\delta^{2}}\left|\tilde{\Lambda}_{n}\right| \sum_{k \geq 2} w_{1}^{(k)} e^{a k}
$$

To bound $w_{1}^{(k)}$ when $k \geq 2$ we can use the methods based on tree graph inequality first originally introduced in [9] and recently generalized in [11]. Observe now that by assumption (2.3), the pair potential $J_{x y}$ is stable. Namely, assumption (2.3) implies that for any finite $R \subset \mathbb{Z}^{d}\left(r_{0}\right)$ it holds

$$
\sum_{\{x, y\} \subset R} J_{x y} s_{x} s_{y} \geq-\frac{|R|}{2} J_{r_{0}} \sigma^{2}
$$

where $J_{r_{0}}$ is the positive number defined in (3.6).

Therefore, following [11], we can bound, for any $R \subset \tilde{\Lambda}_{n}$

$$
\begin{aligned}
\left|\sum_{g \in G_{R}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right| & \leq e^{\frac{|R|}{2} J_{r_{0}} \sigma^{2}} \sum_{\tau \in T_{R}} \prod_{\{x, y\} \in E_{g}}\left(1-e^{-\left|J_{x y} s_{x} s_{y}\right|}\right) \\
& \leq e^{\frac{|R|}{2} J_{r_{0}} \sigma^{2}} \sigma^{2|R|-2} \sum_{\tau \in T_{R}} \prod_{\{x, y\} \in E_{\tau}}\left|J_{x y}\right|
\end{aligned}
$$

where $T_{R}$ is the set of trees (connected graphs with no loops) with vertex set $R$. Hence

$$
\begin{align*}
w_{1}^{(k)} & \leq(1+\delta \sigma)^{k} e^{k} e^{\frac{k}{2} J_{r_{0}} \sigma^{2}} \sigma^{2 k-2} \sup _{x \in \tilde{\Lambda}_{n}} \sum_{\tau \in T_{R}} \sum_{\substack{R \subset \tilde{\Lambda}_{n} \\
x \in R,|R|=k}} \prod_{\{x, y\} \in E_{\tau}}\left|J_{x y}\right| \\
& =(1+\delta \sigma)^{k} e^{k} e^{\frac{k}{2} J_{r_{0}} \sigma^{2}} \sigma^{2 k-2} \sup _{x \in \tilde{\Lambda}_{n}} \sum_{\tau \in T_{k}} \frac{1}{k-1)!} \sum_{\substack{\left(x_{1}, \ldots, x_{k}\right) \in \tilde{\Lambda}_{n}^{k} \\
x_{1}=x, x_{i} \neq x_{j}}} \prod_{\{i, j\} \in E_{\tau}}\left|J_{x_{i} x_{j}}\right| \tag{4.27}
\end{align*}
$$

where now $T_{n}$ denotes the set of trees with vertex set $\{1,2, \ldots, n\}$. Using (3.6) it is standard to check that

$$
\sum_{\substack{\left(x_{1}, \ldots, x_{k}\right) \in \tilde{\Lambda}_{n}^{k} \\ x_{1}=x, x_{i} \neq x_{j}}} \prod_{\{i, j\} \in E_{\tau}}\left|J_{x_{i} x_{j}}\right| \leq J_{r_{0}}^{k-1}, \quad \forall \tau \in T_{n}
$$

and using also that $\sum_{\tau \in T_{k}} 1=k^{k-2}$ (Cayley formula) we get, for $k \geq 2$

$$
\begin{align*}
w_{1}^{(k)} & \leq e^{k}(1+\delta \sigma)^{k} e^{\frac{k J_{r_{0}} \sigma^{2}}{2}} \sigma^{2 k-2} J_{r_{0}}^{k-1} \frac{k^{k-2}}{(k-1)!}  \tag{4.28}\\
& \leq\left[2 e^{2} e^{\frac{J_{r_{0}} \sigma^{2}}{2}} \sigma^{2} J_{r_{0}}^{\frac{1}{2}}\right]^{k}
\end{align*}
$$

where in the last line above we have used that $\frac{k^{k-2}}{(k-1)!} \leq e^{k}$ and that $1+\delta \sigma<2$. So, setting

$$
\nu\left(r_{0}\right)=2 e^{2} e^{\frac{J_{r_{0}} \sigma^{2}}{2}} \sigma^{2} J_{r_{0}}^{\frac{1}{2}}, \quad \varepsilon\left(\delta, r_{0}\right)=\min \left\{e \delta \sigma, \nu\left(r_{0}\right)\right\}
$$

we get that, for all $k \geq 1, w_{1}^{(k)} \leq\left[\varepsilon\left(\delta, r_{0}\right)\right]^{k}$. It is now easy to check that condition (4.25) is satisfied taking $a=\ln 2$ and $\varepsilon\left(\delta, r_{0}\right) \leq \frac{1}{4}$, i.e., since by hypothesis $\delta \sigma<\frac{1}{12}$ so that $e \delta \sigma<1 / 4$, condition (4.25) holds if $\nu\left(r_{0}\right) \leq \frac{1}{4}$. Therefore, for $r_{0}$ such that $\nu\left(r_{0}\right) \leq \frac{1}{4}$, we have

$$
\left|G_{4}(\theta)\right|=\frac{\sigma^{2}}{\delta^{2}}\left|\tilde{\Lambda}_{n}\right| \sum_{k \geq 2} 2^{k}\left[\nu\left(r_{0}\right)\right]^{k} \leq \frac{8 \sigma^{2}\left|\tilde{\Lambda}_{n}\right|}{\delta^{2}} \nu^{2}\left(r_{0}\right)
$$

In conclusion, when $\delta=\delta(J, \sigma)$ and $\nu\left(r_{0}\right) \leq \frac{1}{4}$ we get that

$$
\Re G_{1}(\theta)+\Re G_{2}(\theta)+\left|G_{3}(\theta)\right|+\left|G_{4}(\theta)\right| \leq-\left|\tilde{\Lambda}_{n}\right| \sigma^{2}\left[\frac{\kappa(J, \sigma)}{2}-\frac{8 \nu^{2}\left(r_{0}\right)}{\delta^{2}}\right]
$$

Therefore, recalling the definition of $\delta(J, \sigma)$ given in the statement of Lemma 2, as soon as

$$
\begin{equation*}
8 \nu^{2}\left(r_{0}\right) \leq \frac{\kappa^{3}(J, \sigma)}{4(12)^{2} \sigma^{2}} \tag{4.29}
\end{equation*}
$$

we get that

$$
\Re\left(G_{1}(\theta)+G_{2}(\theta)+G_{3}(\theta)+G_{4}(\theta)\right) \leq-\left|\tilde{\Lambda}_{n}\right| \sigma^{2} \frac{\kappa(J, \sigma)}{4} .
$$

Note that (4.29) is surely satisfied if

$$
\begin{equation*}
e^{\frac{J_{r_{0}}}{2}} J_{r_{0}}^{\frac{1}{2}} \leq \frac{\kappa^{3 / 2}(J, \sigma)}{96 \sqrt{2} e^{2} \sigma^{3}} . \tag{4.30}
\end{equation*}
$$

Therefore if (4.30) holds then Part (a) of Lemma 2 is proved.

## 5. Proof of Lemma 2, part (b)

In order to prove part (b) of Lemma 2 we first state and demonstrate a preliminary bound regarding the single spin probability distribution $p_{x}^{\omega}\left(s_{x}\right)$ introduced in (2.6). Recalling that $E_{x}^{\omega}(\cdot)$ denotes the expected value w.r.t. the probability distribution $p_{x}^{\omega}\left(s_{x}\right)$, the following proposition holds.

Proposition 4 Let $\delta$ and $c$ as in (3.7), and let $t \in[\delta, 2 \pi-\delta]$, then, uniformly in $\omega$ we have that

$$
\begin{equation*}
\left|E_{x}^{\omega}\left(e^{i t s_{x}}\right)\right| \leq e^{-c} \tag{5.1}
\end{equation*}
$$

Proof. We have

$$
\begin{aligned}
\left|E_{x}^{\omega}\left(e^{i t s_{x}}\right)\right| & =\left[\left(\sum_{s_{x} \in I} p_{x}^{\omega}\left(s_{x}\right) \cos \left(s_{x} t\right)\right)^{2}+\left(\sum_{s_{x} \in I} p_{x}^{\omega}\left(s_{x}\right) \sin \left(s_{x} t\right)\right)^{2}\right]^{\frac{1}{2}} \\
& =\left[\sum_{\left(s_{x}, s_{x}^{\prime}\right) \in I^{2}} p_{x}^{\omega}\left(s_{x}\right) p_{x}^{\omega}\left(s_{x}^{\prime}\right) \cos \left[\left(s_{x}-s_{x}^{\prime}\right) t\right]\right]^{\frac{1}{2}} \\
& \leq \exp \left\{\frac{1}{2}\left[\left(\sum_{\left(s_{x}, s_{x}^{\prime}\right) \in I^{2}} p_{x}^{\omega}\left(s_{x}\right) p_{x}^{\omega}\left(s_{x}^{\prime}\right) \cos \left(\left(s_{x}-s_{x}^{\prime}\right) t\right)\right)-1\right]\right\} \\
& =\exp \left\{-\sum_{\left(s_{x}, s_{x}^{\prime}\right) \in I^{2}} p_{x}^{\omega}\left(s_{x}\right) p_{x}^{\omega}\left(s_{x}^{\prime}\right) \sin ^{2}\left(\frac{\left(s_{x}-s_{x}^{\prime}\right) t}{2}\right)\right\} \\
& \leq \exp \left\{-\kappa^{2}(J, \sigma) \sin ^{2}\left(\frac{t}{2}\right)\right\} \\
& \leq \exp \left\{-\kappa^{2}(J, \sigma) \sin ^{2}\left(\frac{\delta}{2}\right)\right\}
\end{aligned}
$$

where in the first inequality we have used that $x \leq e^{\frac{1}{2}\left(x^{2}-1\right)}$ for $x>0$, in the second inequality we have used the bound given in (2.7) and the last inequality follows from the assumption that $t \in[\delta, 2 \pi-\delta]$.

We can prove Part (b) of Lemma 2. Recalling that

$$
\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t \tilde{S}_{n}}\right)\right|=\frac{\left|\Xi_{n}(t)\right|}{\left|\Xi_{n}(0)\right|}
$$

with $\Xi_{n}(t)$ given in (4.3), let us we apply the Mayer trick only to the factor $\prod_{\{x, y\} \subset \tilde{\Lambda}_{n}} e^{J_{x y} s_{x} s_{y}}$. We get

$$
\begin{aligned}
\Xi_{n}(t)= & \sum_{g \in \mathcal{G}_{\tilde{\Lambda}_{n}}} \sum_{\tilde{\Lambda}_{n} \in \Omega_{\tilde{\Lambda}_{n}}} \prod_{x \in \tilde{\Lambda}_{n}} e^{i t s_{x}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right) \prod_{x \in \tilde{\Lambda}_{n}} p_{x}^{\omega}\left(s_{x}\right) \\
= & \sum_{g \in \mathcal{G}_{\tilde{\Lambda}_{n}}}\left(\sum_{s_{S_{g} \in \Omega_{S_{g}}}} \prod_{x \in S_{g}} e^{i t s_{x}} p_{x}^{\omega}\left(s_{x}\right) \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right) \prod_{x \in \tilde{\Lambda}_{n} \backslash S_{g}} E_{x}^{\omega}\left(e^{i t s_{x}}\right) \\
= & e^{-c\left|\tilde{\Lambda}_{n}\right|} \sum_{g \in \mathcal{G}_{\tilde{\Lambda}_{n}}}\left(e^{c\left|S_{g}\right|} \sum_{s_{S_{g}} \in \Omega_{S_{g}}} \prod_{x \in S_{g}} e^{i t s_{x}} p_{x}^{\omega}\left(s_{x}\right) \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right) \\
& \quad \times \prod_{x \in \tilde{\Lambda}_{n} \backslash S_{g}}\left(e^{c} E_{x}^{\omega}\left(e^{i t s_{x}}\right)\right)
\end{aligned}
$$

where $\mathcal{G}_{\tilde{\Lambda}_{n}}$ is the set of all graphs (either connected or not connected) with vertex set $\tilde{\Lambda}_{n}$ and $S_{g}=\cup_{\{x, y\} \in E_{g}}\{x, y\}$. Now let

$$
\Xi_{n}^{c}(t)=\sum_{g \in \mathcal{G}_{\tilde{\Lambda}_{n}}}\left(e^{c\left|S_{g}\right|} \sum_{\sigma_{S_{g}} \in \Omega_{S_{g}}} \prod_{x \in S_{g}} e^{i t s_{x}} p_{x}^{\omega}\left(s_{x}\right) \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right)
$$

Similarly to Proposition 3 we have the identity

$$
\begin{equation*}
\Xi_{n}^{c}(t)=1+\sum_{k \geq 1} \sum_{\substack{\left\{R_{1}, \ldots, R_{k}\right\}: R_{i} \subset \bar{\Lambda}_{n} \\\left|R_{i}\right| \geq 2, R_{i} \cap R_{j}=\emptyset}} \prod_{i=1}^{k} \xi_{t}^{c}\left(R_{i}\right) \tag{5.2}
\end{equation*}
$$

where now

$$
\xi_{t}^{c}\left(R_{i}\right)=e^{c|R|} \sum_{\sigma_{\tilde{\Lambda}_{n}} \in \Omega_{R}} \prod_{x \in R} p_{x}^{\omega}\left(s_{x}\right) e^{i t s_{x}} \sum_{g \in G_{R}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right) .
$$

Using Proposition 4, we have that for any $t \in[\delta, 2 \pi-\delta]$

$$
\left|e^{c} E_{x}\left(e^{i t s_{x} \mid \omega}\right)\right|=1
$$

so that we get

$$
\left|\Xi_{n}(t)\right| \leq e^{-c\left|\tilde{\Lambda}_{n}\right|}\left|\Xi_{n}^{c}(t)\right|
$$

and thus $\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right| \leq e^{-c\left|\tilde{\Lambda}_{n}\right|}\left|\frac{\Xi_{n}^{c}(t)}{\Xi_{n}^{0}(0)}\right|$ where of course $\Xi_{n}^{0}(0)=\left.\Xi_{n}^{c}(t)\right|_{c=0, t=0}$. Therefore

$$
\begin{align*}
\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right| & =e^{-c\left|\tilde{\Lambda}_{n}\right|} e^{\Re\left(\ln \Xi_{n}^{c}(t)-\ln \Xi_{n}^{0}(0)\right)} \\
& \leq e^{-c\left|\tilde{\Lambda}_{n}\right|} e^{e^{\left|\ln \Xi_{n}^{c}(t)\right|+\left|\ln \Xi_{n}^{0}(0)\right|}}  \tag{5.3}\\
& \leq e^{-c\left|\tilde{\Lambda}_{n}\right|} e^{e^{\left.|\ln \Xi|\right|_{n} ^{c}(t)+|\ln \Xi|_{n}^{0}(0) \mid}}
\end{align*}
$$

where in the last line $|\ln \Xi|_{\tilde{\Lambda}_{n}}^{c}(t)$ denotes the positive term series

$$
\begin{equation*}
|\ln \Xi|_{\tilde{\Lambda}_{n}}^{c}(t)=\sum_{k=1}^{\infty} \frac{1}{k!} \sum_{\left(R_{1}, \ldots, R_{k}\right) \in \mathcal{P}_{n}^{k}}\left|\Phi^{T}\left(R_{1}, \ldots, R_{k}\right)\right| \prod_{i=1}^{k}\left|\xi_{t}^{c}\left(R_{i}\right)\right| . \tag{5.4}
\end{equation*}
$$

Now, recalling definition (4.21) and setting $w_{c}(R)=e^{c|R|} w_{0}(R)$, we have that

$$
\left|\xi_{t}^{c}(R)\right| \leq e^{c|R|} \sum_{\sigma_{\bar{\Lambda}_{n}} \in \Omega_{R}} \prod_{x \in R} p_{x}^{\omega}\left(s_{x}\right)\left|\sum_{g \in G_{R}} \prod_{\{x, y\} \in E_{g}}\left(e^{J_{x y} s_{x} s_{y}}-1\right)\right|=e^{c|R|} w_{0}(R) \doteq w_{c}(R)
$$

while

$$
\left|\xi_{0}^{0}(R)\right| \leq w_{0}(R) \leq w_{c}(R)
$$

Therefore, recalling definition (5.4) we have $|\ln \Xi|_{n}^{0}(0) \leq|\ln \Xi|_{n}^{c}(t)$ so that

$$
\begin{equation*}
\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right| \leq e^{-c\left|\tilde{\Lambda}_{n}\right|} e^{e^{2|\ln E| n} n_{n}^{c}(t)} \tag{5.5}
\end{equation*}
$$

Now, according to standard theory of gas of non overlapping subsets (see [1,5]), if for some $a>0$ the following inequality holds

$$
\begin{equation*}
\sum_{k \geq 2} w_{c}^{(k)} e^{a k} \leq e^{a}-1 \tag{5.6}
\end{equation*}
$$

where $w_{c}^{(k)}$ as in the r.h.s. of (4.24) with $w_{c}(R)$ in place of $w_{1}(R)$, the positive term series $\left|\ln \Xi{ }_{n}^{c}(t)\right|$ is bounded above by $a\left|\tilde{\Lambda}_{n}\right|$. So if we choose $a=\frac{c}{4}$ we get that

$$
\left|\tilde{\mathbb{E}}_{n}^{\omega}\left(e^{i t S_{n}}\right)\right| \leq e^{-\frac{c}{2}\left|\tilde{\Lambda}_{n}\right|}
$$

Now, recalling (4.28) with $c$ in place of 1 , we have the bound

$$
w_{c}^{(k)} \leq\left[(1+\delta \sigma) e^{\frac{J_{r_{0}} \sigma^{2}}{2}} e^{1+c} \sigma^{2} J_{r_{0}}^{\frac{1}{2}}\right]^{k} .
$$

Therefore the condition (5.6) (with $a=\frac{c}{4}$ ) is surely satisfied if

$$
\sum_{k \geq 1}\left[(1+\delta \sigma) e^{\frac{J_{r_{0}} \sigma^{2}}{2}} e^{1+c} \sigma^{2} J_{r_{0}}^{\frac{1}{2}} e^{\frac{c}{4}}\right]^{k} \leq e^{\frac{c}{4}}-1,
$$

i.e. if

$$
\begin{equation*}
e^{\frac{J_{r_{0}} \sigma^{2}}{2}} J_{r_{0}}^{\frac{1}{2}} \leq \frac{e^{-\frac{5 c}{4}}\left(e^{\frac{c}{4}}-1\right)}{(1+\delta \sigma) e \sigma^{2}} \tag{5.7}
\end{equation*}
$$

Therefore if (5.7) holds, Part (b) of Lemma 2 is proved.
In conclusion if (3.8) holds, then both statements (a) and (b) of Lemma 2 are satisfied.
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