# TRAPPED SURFACE FORMATION FOR SPHERICALLY SYMMETRIC EINSTEIN-MAXWELL-CHARGED SCALAR FIELD SYSTEM WITH DOUBLE NULL FOLIATION 

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#### Abstract

In this paper, under spherical symmetry we prove a trapped surface formation criterion for the Einstein-Maxwell-charged scalar field system. We generalize an approach introduced by Christodoulou for studying the Einstein-scalar field. In appendix, with double null foliation we also reprove Christodoulou's result and for Minkowskian incoming characteristic initial data, we improve Christodoulou's bound.


## 1. Introduction

1.1. Motivation. In a series of papers [9]-[12], Christodoulou studied singularity formation for the Einstein-scalar field system:

$$
\begin{align*}
& \operatorname{Ric}_{\mu \nu}-\frac{1}{2} R g_{\mu \nu}=8 \pi T_{\mu \nu}, \\
& T_{\mu \nu}=\partial_{\mu} \phi \partial_{\nu} \phi-\frac{1}{2} g_{\mu \nu} \partial^{\sigma} \phi \partial_{\sigma} \phi . \tag{1.1}
\end{align*}
$$

Through these papers, Christodoulou proved in four steps that under spherical symmetry, the weak cosmic censorship conjecture holds. More precisely, Christodoulou proved that for (1.1) with large initial data, a so-called naked singularity may form; however, for generic initial data, these singularities are covered by a black hole region and are invisible for observers far away. These are celebrated results.

The Penrose diagram of a spherically symmetric gravitational collapse spacetime for (1.1) with generic initial data is as follows:


Here, $\Gamma$ is the center of symmetry-invariant under $S O(3)$, and $i^{+}, \mathcal{I}^{+}, i_{0}$ are timelike infinity, future null infinity, and spacelike infinity respectively. The
boundary of the causal past of $i^{+}$is $\mathcal{H}$, which is called the event horizon. $\mathcal{T}$ is the trapped region, where not even light can escape to $\mathcal{I}^{+}$. $\mathcal{A}$ is called the apparent horizon and it is the lower boundary of $\mathcal{T}$. $\mathcal{S}_{0}$ is the first singular point along $\Gamma$ and $\mathcal{S}$ is the singular boundary of $\mathcal{T}$.

A crucial step of Christodoulou's proof of the weak cosmic censorship conjecture is [9. There, Christodoulou established a sharp trapped surface ${ }^{1}$ formation criterion for (1.1). Christodoulou's original proof in [9] was based on a geometric Bondi coordinate system with a null frame.

However, at present, the double null foliation is a more popular choice of coordinate system. There have been many recent works published in general relativity using a double null foliation. In order to generalize Christodoulou's results in [9]-[12] to other matter models, here we adopt the double null foliation. In our paper, we will review Christodoulou's result in the setting of a double null foliation. Then, we will generalize his result to the Einstein-Maxwell-charged scalar field system.

Within the study of spherically symmetric systems, there are interesting results on formation of trapped surfaces and singularities for other matter models, e.g. Einstein-Vlasov studied by Andréasson [6], Andéasson-Rein [7], Moschidis [19], Einstein-Euler studied by Burtscher and LeFloch [8], Einsteinscalar field studied by Li-Liu [17], An-Zhang [4], An-Gajic [5], Einstein-null dust studied by Moschidis[20], Einstein-scalar field with positive cosmological constant by Costa [13]. For the Einstein-Maxwell-(real) scalar field system, we refer interested readers to [14, 15] by Dafermos, [18] by Luk and Oh on the recent development of proving strong cosmic censorship. And for the Einstein-Maxwell-charged scalar field system, we refer to [21]-23] by Van de Moortel.
1.2. The Main Result. We consider the characteristic initial value problem for (1.1) in the rectangular region.


We employ the double-null foliation with $u$ and $v$ as optical functions; that is, $g^{\alpha \beta} \partial_{\alpha} u \partial_{\beta} u=0$ and $g^{\alpha \beta} \partial_{\alpha} v \partial_{\beta} v=0$. Thus, we have $u=$ constant as the outgoing null hypersurface; $v=$ constant as the incoming null hypersurface.

Due to spherical symmetry, we have a central axis $\Gamma$. We prescribe initial data along the outgoing cone $u=u_{0}$ and the incoming cone $v=v_{1}$.
For the metric of the $3+1$-dimensional spacetime, we impose spherical symmetry and write it with double-null coordinates:

$$
\begin{equation*}
g_{\mu \nu} d x^{\mu} d x^{\nu}=-\Omega^{2}(u, v) d u d v+r^{2}(u, v)\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right) . \tag{1.2}
\end{equation*}
$$

[^0]In the above diagram every point $(u, v)$ represents a 2 -sphere $S_{u, v}$. The Hawking mass of such a 2 -sphere is defined as

$$
\begin{equation*}
m(u, v)=\frac{r}{2}\left(1+4 \Omega^{-2} \partial_{u} r \partial_{v} r\right) . \tag{1.3}
\end{equation*}
$$

Along $u=u_{0}$, we also define the initial mass input

$$
\eta_{0}:=\frac{m\left(u_{0}, v_{2}\right)-m\left(u_{0}, v_{1}\right)}{r\left(u_{0}, v_{2}\right)}, \text { and denote } \delta_{0}:=\frac{r\left(u_{0}, v_{2}\right)-r\left(u_{0}, v_{1}\right)}{r\left(u_{0}, v_{2}\right)} .
$$

Finally, let $u_{*}$ denote the value of $u \in\left[u_{0}, u_{*}\right]$ such that $r\left(u_{*}, v_{2}\right)=\frac{3 \delta_{0}}{1+\delta_{0}}$. $r\left(u_{0}, v_{2}\right)$.

Theorem 1.1. (Christodoulou [9] and reproved in appendix)
Define the function

$$
E(x):=\frac{x}{(1+x)^{2}}\left[\ln \left(\frac{1}{2 x}\right)+5-x\right] .
$$

Consider the system (1.1) with characteristic initial data along $u=u_{0}$ and $v=v_{1}$. For initial mass input $\eta_{0}$ along $u=u_{0}$, if the following lower bound holds:

$$
\eta_{0}>E\left(\delta_{0}\right)
$$

then a trapped surface $S_{u, v}$, with properties $\partial_{v} r(u, v)<0$ and $\partial_{u} r(u, v)<0$, forms in the region $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right] \subset \mathcal{R}$.

Remark 1. For $0<\delta_{0} \ll 1$, we can check that the order of the lower bound of $\eta_{0}, E\left(\delta_{0}\right)$, is of order $\delta_{0} \ln \left(\frac{1}{\delta_{0}}\right)$. Hence, if $\eta_{0} \gtrsim \delta_{0} \ln \left(\frac{1}{\delta_{0}}\right)$, a trapped surface is guaranteed to form within $\mathcal{R}$.

The above theorem is crucial for Christodoulou's final proof of the weak cosmic censorship in [12]. There, Christodoulou studied the first singular point formed in the evolution of (1.1): if that point is not covered by a trapped region, then a perturbation of the initial data would lead to the condition in Theorem 1.1 being satisfied. Hence, a trapped surface would form to cover that singular point.

We provide a reproof of Theorem 1.1 in the appendix. While Christodoulou's proof was written in Bondi coordinates, here we have rewritten it in a double null foliation. Double null foliations are widely used in studying both the exterior and interior regions of black holes for various matter models. Many results pertaining to spherical symmetry are also based on double null foliations. Hence, there is strong motivation to rewrite 9] with a double null foliation.

By strenghtening the hypothesis on the initial data in Theorem 1.1, in appendix we also improve Christodoulou's bound:


Theorem 1.2. Assume that Minkowskian data are prescribed along $v=v_{1}$ and require $\phi\left(u, v_{1}\right)=0$. Suppose that the following lower bound on $\eta_{0}$ holds:

$$
\eta_{0}>\frac{9}{2} \delta_{0}
$$

then there exist a MOTS or a trapped surface in $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right] \subset \mathcal{R}$, i.e. $\partial_{v} r \leq 0$ at some point in $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right]$.

Remark 2. Theorem 1.2 improves the almost scale critical result in Theorem 1.1, i.e., $\eta_{0}>\delta_{0} \ln \left(\frac{1}{\delta_{0}}\right)$ implies trapped surface formation, to a scale critical result, i.e., $\eta_{0}>\frac{9}{2} \delta_{0}$ implies trapped surface formation. In [2], the first author and Luk first noted that by prescribing Minkowskian data along $v=v_{1}$, for Einstein vacuum equations a scale-critical trapped surface formation criterion could be established $\sqrt[2]{ }$ For $a$ being a large universal constant, the corresponding requirement for $\eta_{0}$ is $\eta_{0} \geq \delta a$. For Einstein-scalar field system under spherical symmetry, Theorem 1.2 improves the large universal constant $a$ into a concrete number $9 / 2$.

The main result of our paper is the next theorem. We generalize the above results to the Einstein scalar field coupled with the electromagnetic field. More precisely, we consider the following Einstein-Maxwell-charged scalar field system:

$$
\begin{gathered}
R_{\mu \nu}-\frac{1}{2} g_{\mu \nu} R=8 \pi T_{\mu \nu} \\
T_{\mu \nu}=T_{\mu \nu}^{S F}+T_{\mu \nu}^{E M} \\
T_{\mu \nu}^{S F}=\frac{1}{2} D_{\mu} \phi\left(D_{\nu} \phi\right)^{\dagger}+\frac{1}{2} D_{\nu} \phi\left(D_{\mu} \phi\right)^{\dagger}-\frac{1}{2} g_{\mu \nu}\left(g^{\alpha \beta} D_{\alpha} \phi\left(D_{\beta} \phi\right)^{\dagger}\right), \\
T_{\mu \nu}^{E M}=\frac{1}{4 \pi}\left(g^{\alpha \beta} F_{\alpha \mu} F_{\beta \nu}-\frac{1}{4} g_{\mu \nu} F^{\alpha \beta} F_{\alpha \beta}\right) .
\end{gathered}
$$

Here, the Einstein scalar field is coupled to the electromagnetic field by the following form of the Maxwell equation:

$$
\nabla^{\nu} F_{\mu \nu}=2 \pi \mathfrak{e} i\left(\phi\left(D_{\mu} \phi\right)^{\dagger}-\phi^{\dagger} D_{\mu} \phi\right),
$$

where $D_{\mu}:=\partial_{\mu}+\mathfrak{e} i A_{\mu}$ is known as the gauge covariant derivative. Here $\mathfrak{e}$ is the coupling constant and $A_{\mu}$ is the electromagnetic potential. Using the gauge covariant derivative instead of the usual derivative ensures that the physical equations remain invariant under local $U(1)$ transformations on $\phi$.

[^1]Recall that under spherical symmetry, we have the ansatz (1.2). Using the $\Omega$ appearing in the ansatz, we define the charge $Q(u, v)$ contained in a sphere $S(u, v)$ to be $Q:=2 r^{2} \Omega^{-2} F_{u v}$.

Theorem 1.3. (Main Theorem) Denoting the outgoing null hypersurface $u=$ $u_{0}$ by $C$ and the incoming null hypersurface $v=v_{1}$ by $\underline{C}$, we define

$$
\epsilon:=\sup _{C \cup \underline{C}} \frac{Q^{2}}{r^{2}}<1, \text { and } L:=\sup _{\underline{C}} r|\phi|^{2} .
$$

Let $\omega$ be any positive constant in $\left(0, \frac{2}{3}\right)$. Choose $v_{2}-v_{1}$ sufficiently small such that

$$
\begin{gather*}
\frac{9 \mathfrak{e}^{2}}{4(1-\epsilon)^{2}}\left(v_{2}-v_{1}\right)^{2}+\frac{12 \pi L \mathfrak{e}}{1-\epsilon}\left(v_{2}-v_{1}\right) \leq \frac{\omega}{4}  \tag{1.4}\\
\frac{45 \pi \mathfrak{e}^{2}\left(v_{2}-v_{1}\right)^{2}}{\pi(1-\epsilon)^{2}}+160 \pi \mathfrak{e}^{2} r\left(u_{0}, v_{2}\right) \frac{v_{2}-v_{1}}{1-\epsilon}\left|\phi_{1}\right|^{2} \leq 4 \omega . \tag{1.5}
\end{gather*}
$$

Further require that the initial data along $\underline{C}$ are not supercharged, i.e.

$$
\begin{equation*}
m\left(u, v_{1}\right) \geq|Q|\left(u, v_{1}\right) \tag{1.6}
\end{equation*}
$$

Denote

$$
g_{\omega}(x):=\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{1}{(1+x)^{2}}\left(\left(\frac{2^{1-\frac{\omega}{2}}}{\omega}+\frac{1}{2^{1+\frac{\omega}{2}}\left(1+\frac{\omega}{2}\right)}\right) x^{1-\frac{\omega}{2}}-\frac{2}{\omega} x-\frac{1}{1+\frac{\omega}{2}} x^{2}\right) .
$$

Assume the following lower bound on $\eta_{0}$ holds

$$
\eta_{0}>\max \left\{\frac{13 \epsilon}{\omega}+g_{\omega}\left(\delta_{0}\right), \frac{9}{2^{1+\frac{\omega}{2}}\left(1+\delta_{0}\right)^{2}} \delta_{0}^{1-\frac{\omega}{2}}+g_{\omega}\left(\delta_{0}\right)\right\},
$$


then a trapped surface is guaranteed to form in $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right] \subset \mathcal{R}$.

Remark 3. By comparing the order of the lower bounds (when $0<\delta_{0} \ll 1$ ) of $\eta_{0}$ in the hypothesis of the theorem, we can interprete the theorem as: If $\eta_{0} \gtrsim \delta_{0}^{1-\frac{\omega}{2}}+\frac{13 \epsilon}{\omega}$, a trapped surface forms in $\mathcal{R}$. Since $\omega$ could be chosen to be arbitrary small number in $\left(0, \frac{3}{2}\right)$, if we require that $\epsilon$ (upper bound
of $\frac{Q^{2}}{r^{2}}$ on $C \cup \underline{C}$ ) is small and satisfies $\frac{13 \epsilon}{\omega} \leq \delta_{0}^{1-\frac{\omega}{2}}$, our theorem is also an almost-scale-critical result.

Remark 4. Moreover, although we use the symbol $\epsilon$ to denote the upper bound of $\frac{Q^{2}}{r^{2}}$ on $C \cup \underline{C}, \epsilon$ is not necessarily to be small. In particular, we could choose $0<\delta_{0} \ll 1, \omega=\frac{1}{2}$ and require $\epsilon$ to be of size 1 . This is not in the perturbative regime of Christodoulou's result for Einstein-scalar field. Intuitively, for this case our Theorem 1.3 is saying: if the incoming mass contained between $v_{1}$ and $v_{2}$ is large enough to overcome the initial charge on $C \cup \underline{C}$, then we can guarantee the formation of a trapped surface.

Remark 5. For initial data along $v=v_{1}$, we require that the initial data are not super-charged, i.e.

$$
\begin{equation*}
m\left(u, v_{1}\right) \geq|Q|\left(u, v_{1}\right) \tag{1.7}
\end{equation*}
$$

It is natural to consider initial data, which are not-super-charged, otherwise there could be non-physical super-charged naked singularity prescribed along $v=v_{1}$. At the same time, (1.7) also implies an important inequality used in the proof of Proposition 3.3,

Lemma 1.4. Along $v=v_{1}$, condition (1.7) implies

$$
\begin{equation*}
\frac{m}{r}\left(u, v_{1}\right) \geq \frac{Q^{2}}{r^{2}}\left(u, v_{1}\right) \tag{1.8}
\end{equation*}
$$

Proof. Since there is no MOTS or trapped surface along $v=v_{1}$, we have

$$
\frac{2 m}{r}\left(u, v_{1}\right) \leq 1, \text { which gives } \frac{m}{r}\left(u, v_{1}\right) \leq \frac{1}{2} .
$$

Together with the non-super charged condition, we also have

$$
\begin{equation*}
\frac{|Q|}{r}\left(u, v_{1}\right) \leq \frac{m}{r}\left(u, v_{1}\right) \leq \frac{1}{2} . \tag{1.9}
\end{equation*}
$$

Then, we have

$$
\begin{aligned}
\frac{m}{r}\left(u, v_{1}\right)-\frac{Q^{2}}{r^{2}}\left(u, v_{1}\right) & \geq \frac{|Q|}{r}\left(u, v_{1}\right)-\frac{Q^{2}}{r^{2}}\left(u, v_{1}\right) \\
& \geq \frac{|Q|}{r}\left(1-\frac{|Q|}{r}\right)\left(u, v_{1}\right) \geq 0 .
\end{aligned}
$$

For the last inequality, we used (1.9).
The inequality (1.8) is crucial in proving Proposition 3.3, And all subsequent results in Section 3 depend on Proposition 3.3,

## 2. Preliminaries and Set-up

To study the problem of trapped surface formation, we need to choose a convenient coordinate system in which to express the Einstein field equations. We describe the double null coordinate system for spherically symmetric spacetimes in what follows.


Figure 1. Illustration of double null coordinate patch
Definition 2.1. A spacetime $(\mathcal{M}, g)$ is called spherically symmetric if $S O(3)$ acts on it by isometry, and the orbits of the group are (topological) 2-dimensional spheres $S$. We define the area-radius coordinate $r(S)$ such that $A=4 \pi r^{2}$, where $A$ is the area of the $S$ determined by the induced metric $\left.g\right|_{S} 3^{3}$.

Under the assumption of spherical symmetry, a spacetime can be represented by a two-dimensional diagram by considering only the quotient $\mathcal{M} / S$. Hence, a point on such diagram represents a 2 -sphere in spacetime. There is no loss in generality by assuming that the outgoing ( $v$ coordinate) and incoming ( $u$ coordinate) null geodesics make 45-degree angles with the horizontal and vertical axes. Furthermore, it is possible to bring the points at infinity to a finite region through a conformal transformation, so that we can visualize the entire spacetime in a finite region. Such a representation of a spacetime is called a Penrose diagram.
We now introduce the setup of the coordinate system, along with important points of interest on the Penrose diagram.
(1) Let $\Gamma$ denote the axis of symmetry of the spacetime.
(2) Fix a point $p$ on the penrose diagram. Label the incoming null geodesic intersecting $p$ by $\underline{C}$, and the outgoing null geodesic intersecting $p$ by $C$. On the actual spacetime, $C$ and $\underline{C}$ are therefore null hypersurfaces.
(3) Parametrize $\underline{C}$ with the variable $u$, and $C$ by the variable $v$. At the intersection of $\Gamma$ and $\underline{C}$, set $u=0$. Extend $C$ backwards until it intersects $\Gamma$. At the intersection of $\Gamma$ and $C$, we similarly set $v=0$. Fixing these values determines the coordinate of $p$, which we call $\left(u_{0}, v\right)$.
(4) In the domain of dependence of $C \cup \underline{C}$, we can now establish a coordinate system: through every point in the domain of dependence runs

[^2]an incoming and outgoing null geodesic emanating from $C$ and $\underline{C}$ respectively. Using the parameters $u$ and $v$ defined on $\underline{C}$ and $C$ gives us a coordinate for the point in question.
(5) Finally, let $\mathcal{D}\left(0, v_{1}\right)$ denote the region in spacetime bounded by $C, \underline{C}$ and $\Gamma$.

The construction above is illustrated in Figure 1. With respect to the double null coordinate system, the spherically symmetric metric can be expressed as

$$
\begin{equation*}
g=-\Omega^{2}(u, v) d u d v+r^{2}(u, v) d \theta^{2}+r^{2}(u, v) \sin ^{2} \theta d \phi^{2} . \tag{2.1}
\end{equation*}
$$

We now define several useful geometric quantities:

Definition 2.2. The Hawking mass $m(u, v)$ contained inside a sphere $S(u, v)$ is defined to be the quantity $\frac{r}{2}\left(1+4 \Omega^{-2} \partial_{u} r \partial_{v} r\right)$.

Definition 2.3. We define the charge $Q(u, v)$ contained in a sphere $S(u, v)$ to be

$$
\begin{equation*}
Q:=2 r^{2} \Omega^{-2} F_{u v} \tag{2.2}
\end{equation*}
$$

Note: $F_{u v}=\partial_{u} A_{v}-\partial_{v} A_{u}$, where $A$ is the electromagnetic potential.

Due to gauge freedom in the electromagnetic potential, we can impose the condition $A_{v} \equiv 0$. Hence the above definition becomes:

$$
F_{u v}=-\partial_{v} A_{u}
$$

By substituting the expression (2.1), (2.2) and (2.3) into the Einstein field equations and Maxwell equations, we arrive at the following system of equations with dynamical real-valued unknowns $r, A_{u}$ and $\Omega^{2}$, and complex-valued unknown $\phi$. For a more comprehensive explanation of these variables, we refer to [16], from which the following Einstein-Maxwell-charged scalar field system has been obtained.

$$
\begin{gather*}
r \partial_{v} \partial_{u} r+\partial_{v} r \partial_{u} r=-\frac{\Omega^{2}}{4}\left(1-\frac{Q^{2}}{r^{2}}\right),  \tag{2.4}\\
r^{2} \partial_{u} \partial_{v} \log \Omega=-2 \pi r^{2}\left(D_{u} \phi\left(\partial_{v} \phi\right)^{\dagger}+\partial_{v} \phi\left(D_{u} \phi\right)^{\dagger}\right)-\frac{1}{2} \Omega^{2} \frac{Q^{2}}{r^{2}}+\frac{1}{4} \Omega^{2}+\partial_{u} \partial_{v} r,  \tag{2.5}\\
\partial_{u}\left(\Omega^{-2} \partial_{u} r\right)=-4 \pi r \Omega^{-2} D_{u} \phi\left(D_{u} \phi\right)^{\dagger},  \tag{2.6}\\
\partial_{v}\left(\Omega^{-2} \partial_{v} r\right)=-4 \pi r \Omega^{-2} \partial_{v} \phi\left(\partial_{v} \phi\right)^{\dagger},  \tag{2.7}\\
r \partial_{u} \partial_{v} \phi+\partial_{u} r \partial_{v} \phi+\partial_{v} r \partial_{u} \phi+\mathfrak{e} i \Psi(A)=0,  \tag{2.8}\\
\Psi(A)=A_{u} \partial_{v}(r \phi)-\frac{\Omega^{2}}{4} \frac{Q}{r} \phi,  \tag{2.9}\\
Q=-2 r^{2} \Omega^{-2} \partial_{v} A_{u},  \tag{2.10}\\
\partial_{u} Q=2 \pi \mathfrak{e} i r^{2}\left(\phi\left(D_{u} \phi\right)^{\dagger}-\phi^{\dagger} D_{u} \phi\right)=4 \pi \mathfrak{e r} r^{2} \operatorname{Im}\left(\phi^{\dagger} D_{u} \phi\right),  \tag{2.11}\\
\partial_{v} Q=2 \pi \mathfrak{e} i r^{2}\left(\phi\left(\partial_{v} \phi\right)^{\dagger}-\phi^{\dagger} \partial_{v} \phi\right)=4 \pi \mathfrak{e} r^{2} \operatorname{Im}^{\prime}\left(\phi^{\dagger} \partial_{v} \phi\right), \tag{2.12}
\end{gather*}
$$

where $D_{u}:=\partial_{u}+i \mathfrak{e} A_{u}$, and $\mathfrak{e}$ is the coupling constant between the scalar and electromagnetic field. It is worth noting that to reduce the above system into that of an uncharged scalar field, it suffices to set $\mathfrak{e}=0$. Also, we can combine (2.8), (2.9) and (2.10), which gives us:

$$
\begin{align*}
& r \partial_{u} \partial_{v} \phi+\partial_{u} r \partial_{v} \phi+\partial_{v} r \partial_{u} \phi+\mathfrak{e} i A_{u} \partial_{v}(r \phi)-\mathfrak{e} i \frac{\Omega^{2}}{4} \frac{Q}{r} \phi=0 \\
& \quad \Longrightarrow \partial_{v}\left(r \partial_{u} \phi\right)+\partial_{u} r \partial_{v} \phi+\mathfrak{e} i\left(A_{u} \partial_{v}(r \phi)+r \phi \partial_{v} A_{u}\right)=-\mathfrak{e} i \frac{Q \phi \Omega^{2}}{4 r} \\
& \Longrightarrow \partial_{v}\left(r \partial_{u} \phi\right)+\partial_{u} r \partial_{v} \phi+\mathfrak{e} i \partial_{v}\left(r \phi A_{u}\right) \\
& =\partial_{v}\left(r D_{u} \phi\right)+\partial_{v} \phi \partial_{u} r=-\mathfrak{e} i \frac{Q \phi \Omega^{2}}{4 r} . \tag{2.13}
\end{align*}
$$

The above system (2.4)-(2.13) is subject to initial conditions. There are two types of initial conditions to be considered:
(1) The first type of initial conditions are derived from geometrical considerations and are independent of the physical scenario. On the center of symmetry $\Gamma$, we must have $r=0$. In addition, by the spherical symmetry assumption, as we consider points infinitesimally close to the center, its incoming null geodesics essentially become outgoing (in the opposite direction). Hence we require that $\partial_{v} r\left(u_{0}, 0\right)=-\partial_{u} r\left(u_{0}, 0\right)$. The evolution of $r$ in the spacetime is then determined by equations (2.4), (2.6), and (2.7).

On $C \cup \underline{C}$, we set $\Omega^{2}=1$. This amounts to fixing a normalization for the coordinate system. The evolution of $\Omega^{2}$ in the coordinate patch $\left[u_{0}, 0\right] \times\left[v_{1}, \infty\right)$ is then given by equation (2.5).
(2) The second type of initial conditions are those derived from quantities such as the scalar field $\phi$ and electromagnetic potential $A_{u}$. We can prescribe initial data of $\phi$ freely on $C \cup \underline{C}$, which will completely determine its first derivatives as $C$ and $\underline{C}$ are characteristic hypersurfaces.

The electromagnetic potential $A_{u}$ along outgoing null hypersurfaces can be determined through equation (2.10) up to an arbitrary constant, which is in turn determined by (2.12). For completeness, it is worth mentioning that there is no loss in generality in letting $A_{u}=0$ along $\Gamma$ due to gauge freedom, although we will not make use of this fact.

Using the above system of equations, we can compute the derivatives of the Hawking mass:

$$
\begin{align*}
\partial_{u} m= & \partial_{u}\left(\frac{r}{2}\left(1+4 \Omega^{-2} \partial_{u} r \partial_{v} r\right)\right) \\
= & \frac{\partial_{u} r}{2}+2 \partial_{u} r \Omega^{-2} \partial_{u} r \partial_{v} r+2 r \partial_{u}\left(\Omega^{-2} \partial_{u} r\right) \partial_{v} r+2 r \Omega^{-2} \partial_{u} r \partial_{u} \partial_{v} r \\
= & \frac{\partial_{u} r}{2}+2 \partial_{u} r \Omega^{-2} \partial_{u} r \partial_{v} r-8 \pi r^{2} \Omega^{-2} \partial_{v} r\left|\partial_{u} \phi\right|^{2} \\
& +2 \Omega^{-2} \partial_{u} r\left(-\frac{\Omega^{2}}{4}\left(1-\frac{Q^{2}}{r^{2}}\right)-\partial_{v} r \partial_{u} r\right) \\
= & -8 \pi r^{2} \Omega^{-2} \partial_{v} r\left|D_{u} \phi\right|^{2}+\frac{Q^{2} \partial_{u} r}{2 r^{2}},  \tag{2.14}\\
\partial_{v} m= & \partial_{v}\left(\frac{r}{2}\left(1+4 \Omega^{-2} \partial_{u} r \partial_{v} r\right)\right) \\
= & \frac{\partial_{v} r}{2}+2 \partial_{v} r \Omega^{-2} \partial_{u} r \partial_{v} r+2 r \partial_{v}\left(\Omega^{-2} \partial_{v} r\right) \partial_{u} r+2 r \Omega^{-2} \partial_{v} r \partial_{u} \partial_{v} r \\
= & \frac{\partial_{u} r}{2}+2 \partial_{u} r \Omega^{-2} \partial_{u} r \partial_{v} r-8 \pi r^{2} \Omega^{-2} \partial_{u} r\left|\partial_{v} \phi\right|^{2} \\
& +2 \Omega^{-2} \partial_{v} r\left(-\frac{\Omega^{2}}{4}\left(1-\frac{Q^{2}}{r^{2}}\right)-\partial_{v} r \partial_{u} r\right) \\
= & -8 \pi r^{2} \Omega^{-2} \partial_{u} r\left|\partial_{v} \phi\right|^{2}+\frac{Q^{2} \partial_{v} r}{2 r^{2}} . \tag{2.15}
\end{align*}
$$

Finally, we define what is meant by a trapped surface.
Definition 2.4. A trapped surface $S$ in a spherically symmetric spacetime is a point $(u, v)$ on the Penrose diagram (which represents a sphere) such that $\partial_{u} r(u, v)<0$ and $\partial_{v} r(u, v)<0$. If $\partial_{v} r(u, v)=0$, we call $(u, v)$ a marginally outer trapped surface (MOTS).

In the following, we will only focus on a narrow strip of the double null coordinate patch $\left[u_{0}, 0\right] \times\left[v_{1}, v_{2}\right]$, for some $v_{2}>v_{1}$. We are going to give conditions under which trapped surface formation is guaranteed in this strip. We introduce:


Figure 2. Problem setup on Penrose Diagram

$$
\begin{gather*}
r_{i}(u):=r\left(u, v_{i}\right), \quad m_{i}(u):=m\left(u, v_{i}\right), \quad i=1,2 \\
\delta(u):=\frac{r_{2}(u)}{r_{1}(u)}-1, \quad \delta_{0}:=\delta\left(u_{0}\right) \\
\eta(u):=\frac{2\left(m_{2}(u)-m_{1}(u)\right)}{r_{2}(u)}, \quad \eta_{0}:=\eta\left(u_{0}\right) \\
x(u):=\frac{r_{2}(u)}{r_{2}\left(u_{0}\right)} \tag{2.16}
\end{gather*}
$$

See Figure 2 for an illustration. Henceforth, any dynamical quantity (except for $u$ and $v$ ) with the subscript $\{1,2\}$ shall be treated as a function of $u$ with $v=v_{i}, i=\{1,2\}$ fixed. Furthermore, denote the region $\left[u_{0}, 0\right] \times\left[v_{1}, v_{2}\right]$ by $\mathcal{R}$.

## 3. A Trapped Surface formation criterion for the Complex Scalar Field

3.1. Outline. Before giving the complete proof, we briefly describe the main ideas.
(1) First, we prove that $r(u, v)$ is decreasing with respect to $u$ in Lemma 3.1, hence the dimensionless length scale $x(u):=\frac{r_{2}(u)}{r_{2}\left(u_{0}\right)}$ decreases as $u$ increases, and $x\left(u_{0}\right)=1$.
(2) Then we employ a proof-by-contradiction argument: assuming that $\mathcal{D}\left(0, v_{1}\right) \cup \mathcal{R}$ does not have a trapped surface, we derive an inequality for $\eta$ in terms of $x$ in the region $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right]$. We further show that $\frac{d \eta}{d x}$ is bounded from above, i.e. $\frac{d \eta}{d u}$ is bounded from below, and therefore we get a lower bound on $\eta\left(u_{*}\right)$.
If this lower bound is greater than 1, i.e., $\eta\left(u_{*}\right)=\frac{2\left(m_{2}-m_{1}\right)}{r_{2}}\left(u_{*}\right)>1$, it implies $\frac{2 m_{2}}{r_{2}}\left(u_{*}\right)>1$ and this means $S\left(u_{*}, v_{2}\right)$ is a trapped surface.

Since $\left(u_{*}, v_{2}\right)$ is a point in $\mathcal{R}$, the above gives us the desired contradiction.
The key of above arguments is to bound $\frac{d \eta}{d u}$. A direct computation gives:

$$
\begin{equation*}
\frac{d \eta}{d x}=-\frac{\eta}{x}-\frac{16 \pi \partial_{v} r_{2} \Omega_{2}^{-2}}{x \partial_{u} r_{2}}\left(r_{2}^{2}\left|D_{u} \phi_{2}\right|^{2}-\frac{\Omega_{1}^{-2} \partial_{v} r_{1}}{\Omega_{2}^{-2} \partial_{v} r_{2}} r_{1}^{2}\left|D_{u} \phi_{1}\right|^{2}\right)+\frac{Q_{2}^{2}}{x r_{2}^{2}} \tag{3.1}
\end{equation*}
$$

We show in Lemma 3.4 that the non-supercharged assumption along $v=v_{1}$ allows us to show that $Q_{2}^{2} / r_{2}^{2}$ remains bounded by $\eta$, plus a small error term. If $\eta$ is large enough compared to $Q_{2}^{2} / r_{2}^{2}$, then the error can be absorbed into $\eta$. With the control of $Q_{2}^{2} / r_{2}^{2}$ in terms of $\eta$, we further have

$$
\begin{gather*}
\Theta^{2}:=\left(r_{2}\left|D_{u} \phi_{2}\right|-r_{1} \mid D_{u} \phi_{1}\right)^{2} \lesssim \frac{-\partial_{u} r_{2}}{8 \pi \Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right)  \tag{3.2}\\
\text { and } \frac{\Omega_{2}^{-2} \partial_{v} r_{2}}{\Omega_{1}^{-2} \partial_{v} r_{1}}(u) \lesssim e^{-\eta(u)} . \tag{3.3}
\end{gather*}
$$

We can substitute (3.2) and (3.3) into (3.1), to obtain

$$
\begin{equation*}
\frac{d \eta}{d x} \lesssim-\frac{\eta}{x}\left(1-\frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}}\right)+\frac{1}{x} \frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}} . \tag{3.4}
\end{equation*}
$$

Integrating this will give us a lower bound on $\eta(u)$ for $u \in\left[u_{0}, u_{*}\right]$. This ensures that for $u \in\left[u_{0}, u^{\prime}\right], \eta(u)$ is always large enough compared to $\frac{Q_{2}^{2}}{r_{2}^{2}}$, so that the differential inequality is always valid in a neighbourhood of $u^{\prime}$, and hence the domain of validity of (3.4) can be extended to the whole $\left[u_{0}, u_{*}\right]$. Finally, the inequality also shows that $\eta\left(u_{*}\right)>1$, a contradiction to the no-trapped-surfaces assumption. Hence the initial assumption that $\mathcal{D}\left(0, v_{1}\right) \cup \mathcal{R}$ has no trapped surfaces cannot be true and this completes the argument.
3.2. Proof of Theorem 1.3. To begin the proof proper, we first give a (negative) upper bound for $\partial_{u} r$ in the region $\left[u_{0}, 0\right] \times\left[v_{1}, \infty\right)$ as promised. The following lemma is the analog of Lemma 4.1 in the uncharged case. This has two important consequences described in the remarks.

Lemma 3.1. $\partial_{u} r \leq-\frac{1-\epsilon}{2} \Omega^{2}$ everywhere in $\mathcal{D}\left(0, v_{1}\right) \cup\left(\left[u_{0}, 0\right] \times\left[v_{1}, \infty\right)\right)$.
Proof. Rewrite (2.4) as

$$
\partial_{v}\left(r \partial_{u} r\right)=-\frac{\Omega^{2}}{4}\left(1-\frac{Q^{2}}{r^{2}}\right) .
$$

Applying the assumption that $\epsilon<1$ and $\Omega^{2}=1$ on $C$, the following inequalities hold on $C$ :

$$
-\frac{1}{4} \leq \partial_{v}\left(r \partial_{u} r\right)\left(u_{0}, v\right) \leq-\frac{1}{4}+\frac{\epsilon}{4}
$$

Integrating both sides and dividing by $r$ :

$$
\begin{equation*}
-\frac{v}{4 r\left(u_{0}, v\right)} \leq \partial_{u} r\left(u_{0}, v\right) \leq-\frac{v(1-\epsilon)}{4 r\left(u_{0}, v\right)} . \tag{3.5}
\end{equation*}
$$

For the first inequality of (3.5) at $v=0$, we get

$$
\begin{equation*}
-\frac{1}{4 \partial_{v} r\left(u_{0}, 0\right)} \leq \partial_{u} r\left(u_{0}, 0\right)=-\partial_{v} r\left(u_{0}, 0\right) \Longrightarrow \partial_{v} r\left(u_{0}, 0\right) \leq \frac{1}{2} \tag{3.6}
\end{equation*}
$$

Since $\Omega^{2}=1$ on $C$, (2.7) gives us $\partial_{v} \partial_{v} r \leq 0$, i.e. $r$ is concave with respect to $v$. Combining this with the fact that $r\left(u_{0}, 0\right)=v\left(u_{0}, 0\right)=0$, we have:

$$
\frac{r}{v}\left(u_{0}, v\right) \leq \partial_{v} r\left(u_{0}, 0\right) .
$$

Substituting this into the second inequality of (3.5), followed by applying (4.8), we get

$$
\partial_{u} r\left(u_{0}, v\right) \leq-\frac{1-\epsilon}{4 \partial_{v} r\left(u_{0}, 0\right)} \leq-\frac{1-\epsilon}{2}
$$

By (2.6), $\Omega^{-2} \partial_{u} r$ is decreasing along incoming null geodesics. Hence for a general point in $\mathcal{D}\left(0, v_{1}\right) \cup\left(\left[u_{0}, 0\right] \times\left[v_{1}, \infty\right)\right)$, we have $\Omega^{-2} \partial_{u} r \leq-\frac{1-\epsilon}{2}$.
Remark 6. Under the assumption of no trapped surfaces, $m(u, v) \geq 0$ for all $(u, v) \in \mathcal{D}\left(0, v_{1}\right) \cup\left(\left[u_{0}, 0\right] \times\left[v_{1}, \infty\right)\right)$

Proof. Given any point $(u, v) \in \mathcal{R}$, we can extend the outgoing null geodesic backwards until it intersects $\Gamma$ at some coordinate $\left(u, v_{c}\right)$, so that $r\left(u, v_{c}\right)=0$. Using (2.15), we have

$$
\partial_{v} m=-8 \pi r^{2} \Omega^{-2} \partial_{u} r\left|\partial_{v} \phi\right|^{2}+\frac{Q^{2} \partial_{v} r}{2 r^{2}} .
$$

Since $\partial_{u} r \leq 0$ by Lemma 3.1, and $\partial_{v} r>0$ by the no trapped surface assumption, we get $\partial_{v} m \geq 0$. Combining with the fact that $m\left(u, v_{c}\right)=0$, we obtain the desired result.
3.3. Estimates for $Q, r$. In this section we will bound $Q$ in terms of the Hawking mass in $\mathcal{R}$, and show that $\partial_{u} \partial_{v} r \leq 0$ under appropriate conditions. Obtaining a bound on $Q$ will require a bound on $\frac{r_{2}}{r_{1}}$, which in turn requires a bound on $Q$. Hence we will develop these bounds using a bootstrap argument.
Proposition 3.2. Fix $0<\omega<\frac{2}{3}$. Choose $v_{2}-v_{1}$ sufficiently small satisfying (1.4) and (1.5). Let $v_{1}<v_{a} \leq v_{2}$. Assume that $\frac{r_{2}(u)}{r_{1}(u)} \leq \frac{3}{2}$, i.e., $\delta(u) \leq \frac{1}{2}$ for $u \in\left[u_{0}, 0\right]$, and that $\mathcal{R}:=\left[u_{0}, 0\right] \times\left[v_{1}, v_{2}\right]$ is free of trapped surfaces. Then the following inequality holds:

$$
\frac{Q_{a}^{2}(u)}{r_{a}^{2}(u)} \leq \frac{\omega}{4} \eta_{a}(u)+\frac{2 Q_{1}^{2}(u)}{r_{1}^{2}(u)}
$$

where

$$
\eta_{a}(u):=\frac{2\left(m_{a}(u)-m_{1}(u)\right)}{r_{a}} .
$$

Over here the subscript $a$ indicates a quantity evaluated at the point ( $u, v_{a}$ ).
Proof. We write $Q_{a}$ as the integral of its derivative:

$$
\begin{align*}
Q_{a}^{2} & =\left(\int_{v_{1}}^{v_{a}} \partial_{v} Q d v+Q_{1}\right)^{2} \leq 2\left(\int_{v_{1}}^{v_{a}} \partial_{v} Q d v\right)^{2}+2 Q_{1}^{2} \\
& \leq 2\left(\int_{v_{1}}^{v_{a}} 4 \pi \mathfrak{e}|\phi|\left|\partial_{v} \phi\right| r^{2} d v\right)^{2}+2 Q_{1}^{2}, \text { by applying (2.12) } \\
& \leq 32 \pi^{2} \mathfrak{e}^{2} \int_{v_{1}}^{v_{a}} r^{2}|\phi|^{2} d v \cdot \int_{v_{1}}^{v_{a}} r^{2}\left|\partial_{v} \phi\right|^{2} d v+2 Q_{1}^{2} . \tag{3.7}
\end{align*}
$$

The second integral in the previous line can be bounded:

$$
\begin{align*}
& \int_{v_{1}}^{v_{a}} r^{2}\left|\partial_{v} \phi\right|^{2} d v=\int_{v_{1}}^{v_{a}} \frac{-8 \pi r^{2} \Omega^{-2} \partial_{u} r\left|\partial_{v} \phi\right|^{2}}{-8 \pi \Omega^{-2} \partial_{u} r} d v \\
\leq & \frac{1}{4 \pi(1-\epsilon)} \int_{v_{1}}^{v_{a}} \partial_{v} m-\frac{Q^{2} \partial_{v} r}{2 r^{2}} d v \leq \frac{m_{a}-m_{1}}{4 \pi(1-\epsilon)}, \tag{3.8}
\end{align*}
$$

where we applied Lemma 3.1 in the second last inequality to pull out the term $\Omega^{-2} \partial_{u} r$ in the denominator, and used the assumption that $\partial_{v} r \geq 0$ for the last inequality. Next, we bound the first integral:

$$
\begin{aligned}
\int_{v_{1}}^{v_{a}} r^{2}|\phi|^{2} d v & =\int_{v_{1}}^{v_{a}}\left(r^{2}\left|\int_{v_{1}}^{v} \partial_{v^{\prime}} \phi d v^{\prime}+\phi_{1}\right|^{2}\right) d v \\
& \leq 2 r_{a}^{2} \int_{v_{1}}^{v_{a}}\left[\left(\int_{v_{1}}^{v} \partial_{v^{\prime}} \phi d v^{\prime}\right)^{2}+\left|\phi_{1}\right|^{2}\right] d v \\
& \leq 2 r_{a}^{2} \int_{v_{1}}^{v_{a}}\left[\left(v-v_{1}\right) \int_{v_{1}}^{v}\left|\partial_{v^{\prime}} \phi\right|^{2} d v^{\prime}+\left|\phi_{1}\right|^{2}\right] d v \\
& \leq 2 r_{a}^{2} \int_{v_{1}}^{v_{a}}\left[\left(v-v_{1}\right) \int_{v_{1}}^{v}\left(\frac{-8 \pi r^{2} \Omega^{-2} \partial_{u} r\left|\partial_{v} \phi\right|^{2}}{-8 \pi r^{2} \Omega^{-2} \partial_{u} r}\right) d v^{\prime}+\left|\phi_{1}\right|^{2}\right] d v \\
& \leq \frac{2 r_{a}^{2}\left(v_{a}-v_{1}\right)}{r_{1}^{2}} \frac{1}{8 \pi} \frac{2}{1-\epsilon} \int_{v_{1}}^{v_{a}} \int_{v_{1}}^{v} \partial_{v^{\prime}} m d v^{\prime} d v+2 r_{a}^{2} \int_{v_{1}}^{v_{a}}\left|\phi_{1}\right|^{2} d v
\end{aligned}
$$

by Lemma 3.1

$$
\begin{align*}
& \leq \frac{v_{a}-v_{1}}{2 \pi(1-\epsilon)}\left(\frac{r_{a}}{r_{1}}\right)^{2} \int_{v_{1}}^{v_{a}}\left(m_{a}-m_{1}\right) d v+2 r_{a}^{2}\left(v_{a}-v_{1}\right)\left|\phi_{1}\right|^{2} \\
& \leq \frac{v_{a}-v_{1}}{2 \pi(1-\epsilon)}\left(\frac{r_{a}}{r_{1}}\right)^{2}\left(v_{a}-v_{1}\right)\left(m_{a}-m_{1}\right)+2 r_{a}^{2}\left(v_{a}-v_{1}\right)\left|\phi_{1}\right|^{2} . \tag{3.9}
\end{align*}
$$

Substituting (3.8) and (3.9) back into (3.7) and rearranging, we get

$$
Q_{a}^{2} \leq \frac{4 \mathfrak{e}^{2}\left(v_{a}-v_{1}\right)^{2}}{(1-\epsilon)^{2}}\left(\frac{r_{a}}{r_{1}}\right)^{2}\left(m_{a}-m_{1}\right)^{2}+\frac{16 \pi \mathfrak{e}^{2}}{1-\epsilon} r_{a}^{2}\left(m_{a}-m_{1}\right)\left(v_{a}-v_{1}\right)\left|\phi_{1}\right|^{2}+2 Q_{1}^{2}
$$

Dividing both sides by $r_{a}^{2}$, we get

$$
\begin{aligned}
\frac{Q_{a}^{2}}{r_{a}^{2}} & \leq \frac{\mathfrak{e}^{2}\left(v_{a}-v_{1}\right)^{2}}{(1-\epsilon)^{2}}\left(\frac{r_{a}}{r_{1}}\right)^{2} \eta_{a}^{2}+\frac{8 \pi \mathfrak{e}^{2}\left(v_{a}-v_{1}\right)}{1-\epsilon}\left(\frac{r_{a}}{r_{1}}\right)\left(r_{1}\left|\phi_{1}\right|^{2}\right) \eta_{a}+2 \frac{Q_{1}^{2}}{r_{a}^{2}} \\
& \leq\left(\frac{9 \mathfrak{e}^{2}}{4(1-\epsilon)^{2}}\left(v_{a}-v_{1}\right)^{2} \eta_{a}+\frac{12 \pi L \mathfrak{e}^{2}}{1-\epsilon}\left(v_{a}-v_{1}\right)\right) \eta_{a}+2 \frac{Q_{1}^{2}}{r_{1}^{2}}, \quad \text { since } \frac{r_{a}}{r_{1}} \leq \frac{3}{2} \\
& \leq\left(\frac{9 \mathfrak{e}^{2}}{4(1-\epsilon)^{2}}\left(v_{a}-v_{1}\right)^{2}+\frac{12 \pi L \mathfrak{e}^{2}}{1-\epsilon}\left(v_{a}-v_{1}\right)\right) \eta_{a}+2 \frac{Q_{1}^{2}}{r_{1}^{2}}
\end{aligned}
$$

where in the last inequality we have used the fact that $\eta_{a}<1$. This is because the no trapped surface or MOTS assumption gives:

$$
\eta_{a} \leq \frac{2\left(m_{a}-m_{1}\right)}{r_{a}} \leq \frac{2 m_{a}}{r_{a}}<1 .
$$

Hence by the assumption (1.4) on $v_{2}-v_{1}$, we have $\frac{Q_{a}^{2}}{r_{a}^{2}} \leq \frac{\omega}{4} \eta_{a}+2 \epsilon$.
We wish to get rid of the $\epsilon$ term in the upper bound given by the previous proposition. This will be done in Lemma 3.4. For that, we will need the next proposition which is the equivalent of Proposition 4.2 in the uncharged case. This proposition is proven using a bootstrap argument.

Proposition 3.3. Assume the initial data along $\underline{C}$ is not super-charged, and $\mathcal{R}$ is free of trapped surfaces. Then $\partial_{u} \partial_{v} r \leq 0$ in $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right]$ and $\delta(u):=$ $\frac{r_{2}}{r_{1}}-1 \leq \frac{1}{2}$ for $u \in\left[u_{0}, u_{*}\right]$, where $u_{*}$ is defined such that $x\left(u_{*}\right)=\frac{3 \delta_{0}}{1+\delta_{0}}$.
Proof. Let $x^{\prime}:=\inf \left\{\left.x \in\left[\frac{3 \delta_{0}}{1+\delta_{0}}, 1\right] \right\rvert\, \delta(y) \leq \frac{1}{2}\right.$ holds for $\left.y \in[x, 1]\right\}$. We will aim to show that $x^{\prime}=\frac{3 \delta_{0}}{1+\delta_{0}}$. This proves the claim that $\delta(u) \leq \frac{1}{2}$ for $u \in\left[u_{0}, u_{8}\right]$, as $x$ is monotonically decreasing with respect to $u$.

Since we have $\delta\left(x^{\prime}\right) \leq \frac{1}{2}$, it follows that $\frac{r_{2}\left(x^{\prime}\right)}{r_{1}\left(x^{\prime}\right)} \leq \frac{3}{2}$ and hence we can apply Proposition 3.2. Thus, for every $x \in\left[x^{\prime}, 1\right], v_{a} \in\left[v_{1}, v_{2}\right]$, we have

$$
\begin{aligned}
\frac{Q^{2}}{r^{2}}\left(x, v_{a}\right) & \leq \frac{\omega}{4} \eta_{a}+\frac{2 Q_{1}^{2}}{r^{2}} \leq \eta_{a}+\frac{2 Q_{1}^{2}}{r^{2}} \\
& =\frac{2 m_{a}}{r}-\frac{2 m_{1}}{r}+\frac{2 Q_{1}^{2}}{r^{2}}=\frac{2 m_{a}}{r}-\frac{2}{r}\left(m_{1}-\frac{Q_{1}^{2}}{r}\right) \\
& =\frac{2 m_{a}}{r}-\frac{2}{r}\left(m_{1}-\frac{Q_{1}^{2}}{r_{1}}\right) .
\end{aligned}
$$

By the non-supercharged assumption, (1.8) from Remark 5 tells us that $m_{1}-$ $\frac{Q_{1}^{2}}{r_{1}} \geq 0$. Hence we have

$$
\frac{Q^{2}}{r^{2}}\left(x, v_{a}\right) \leq \frac{2 m}{r}\left(x, v_{a}\right) .
$$

Now we rewrite (2.4) into the following equivalent form:

$$
\begin{equation*}
\partial_{u} \partial_{v} r=-\frac{\Omega^{2}}{4 r}\left(\frac{2 m}{r}-\frac{Q^{2}}{r}\right) . \tag{3.10}
\end{equation*}
$$

Since we have just shown that $\frac{2 m}{r} \geq \frac{Q^{2}}{r}$ for $x \in\left[x^{\prime}, 1\right]$, it follows that $\partial_{u} \partial_{v} r \leq 0$ in the region $\left[u_{0}, u^{\prime}\right] \times\left[v_{1}, v_{2}\right]$.

Integrating with respect to $u$, we get:

$$
\partial_{v} r(u)-\partial_{v} r\left(u_{0}\right) \leq 0 \Longrightarrow \partial_{v} r(u) \leq \partial_{v} r\left(u_{0}\right)
$$

Integrating the last inequality above with respect to $v$, we obtain

$$
r_{2}(u)-r_{1}(u) \leq r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right), \quad \text { for all } u \in\left[u_{0}, u^{\prime}\right] .
$$

Here $u^{\prime}$ is defined so that $x\left(u^{\prime}\right)=x^{\prime}$. We can use this to derive a bound for $\delta(u)$ :

$$
\begin{aligned}
\delta(u)=\frac{r_{2}}{r_{1}}-1 & =\frac{r_{2}-r_{1}}{r_{2}-\left(r_{2}-r_{1}\right)} \leq \frac{r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right)}{r_{2}(u)-\left(r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right)\right)} \\
& \leq \frac{\delta_{0}}{\frac{r_{2}(u)}{r_{1}\left(u_{0}\right)}-\delta_{0}}=\frac{\delta_{0}}{x(u)\left(1+\delta_{0}\right)-\delta_{0}}, \quad \text { for all } u \in\left[u_{0}, u^{\prime}\right] .
\end{aligned}
$$

Hence, if $x^{\prime}>\frac{3 \delta_{0}}{1+\delta_{0}}$, we have $\delta\left(x^{\prime}\right)<\frac{\delta_{0}}{3 \delta_{0}-\delta_{0}}=\frac{1}{2}$. By the continuity of the function $\delta(x)$, there exist some $x^{\prime \prime}<x^{\prime}$ such that $\delta(x)<\frac{1}{2}$ for all $x \in\left[x^{\prime \prime}, 1\right]$, which is a contradiction to infimum property of $x^{\prime}$. Therefore, we must have $x^{\prime}=\frac{3 \delta_{0}}{1+\delta_{0}}$.

Remark 7. Since all subsequent Lemmas and Propositions depend on Proposition 3.2, the non-supercharged hypothesis is necessary for all of them.

Recall that $\epsilon:=\sup _{\underline{C}} \frac{Q^{2}}{r^{2}}<1$. Combining Proposition 3.2 and Proposition 3.3, we get the following lemma:

Lemma 3.4. Assume that the initial data along $\underline{C}$ is not super-charged and that $\mathcal{R}$ is free of trapped surfaces. Then for every $\left(u, v_{a}\right) \in\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right]$, we have the following estimate for the charge:

$$
\frac{Q_{a}^{2}}{r_{a}^{2}} \leq \frac{\omega}{4} \eta_{a}+2 \epsilon .
$$

Furthermore, if $\eta_{a} \geq \frac{8 \epsilon}{\omega}$, then $\frac{Q_{a}^{2}}{r_{a}^{2}} \leq \frac{\omega}{2} \eta_{a}$.
Proof. The first part of the lemma is almost proven: Since the hypothesis of this lemma satisfies that of Proposition 3.3, we have $\delta(u) \leq \frac{3}{2}$ for all $u \in$ [ $u_{0}, u_{*}$ ], which is the hypothesis of Proposition [3.2. This gives us the first part of the Lemma. The second part follows from a computation. $\eta_{a} \geq \frac{8 \epsilon}{\omega}$ implies that $2 \epsilon \leq \frac{\omega}{4} \eta_{a}$. Hence,

$$
\frac{Q_{a}^{2}}{r_{a}^{2}} \leq \frac{\omega}{4} \eta_{a}+2 \epsilon \leq \frac{\omega}{4} \eta_{a}+\frac{\omega}{4} \eta_{a}=\frac{\omega}{2} \eta_{a} .
$$

3.4. Estimates for $D_{u} \phi, \partial_{v} r$. In this section, we will prove two lemmas which hold in the region $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right]$ under the premise that the region is free of trapped surfaces. These are the equivalents of Lemmas 4.3 and 4.4 in the uncharged case.

Lemma 3.5. Define $\Theta:=r_{2}\left|D_{u} \phi_{2}\right|-r_{1}\left|D_{u} \phi_{1}\right|$. Suppose that the initial data along $\underline{C}$ is not super-charged and $\mathcal{D}\left(0, v_{1}\right) \cup \mathcal{R}$ is free of trapped surfaces. If $\eta \geq \frac{8 \epsilon}{\omega}$, then

$$
\Theta(u)^{2} \leq\left(1+\frac{\omega}{2}\right) \frac{-\partial_{u} r_{2}}{8 \pi \Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right)(u)
$$

for all $u \in\left[u_{0}, u_{*}\right]$.

Proof. By integrating equation (2.13), we get

$$
\begin{align*}
\Theta^{2} & =\left(r_{2}\left|D_{u} \phi_{2}\right|-r_{1}\left|D_{u} \phi_{1}\right|\right)^{2} \\
& \leq\left|r_{2} D_{u} \phi_{2}-r_{1} D_{u} \phi_{1}\right|^{2}=\left|\int_{v_{1}}^{v_{2}}-\partial_{u} r \partial_{v} \phi-i \mathfrak{e} \frac{Q \phi \Omega^{2}}{4 r} d v\right|^{2} \\
& \leq(1+\kappa)\left|\int_{v_{1}}^{v_{2}}-\partial_{u} r\right| \partial_{v} \phi|d v|^{2}+\left(1+\frac{1}{\kappa}\right) \mathfrak{e}^{2}\left|\int_{v_{1}}^{v_{2}} \frac{Q \phi \Omega^{2}}{4 r} d v\right|^{2}, \text { for any } \kappa>0 \\
& \leq \frac{1+\kappa}{8 \pi} \int_{v_{1}}^{v_{2}}-8 \pi r^{2} \partial_{u} r \Omega^{-2}\left|\partial_{v} \phi\right|^{2} d v \int_{v_{1}}^{v_{2}}-\frac{\partial_{u} r}{r^{2} \Omega^{-2}} d v+\left(1+\frac{1}{\kappa}\right) \mathfrak{e}^{2}\left|\int_{v_{1}}^{v_{2}} \frac{Q \phi \Omega^{2}}{4 r} d v\right|^{2}, \tag{3.11}
\end{align*}
$$

where we used Holder's inequality for the last inequality.
We bound the first summand like how we did in Lemma 4.3. We bound the first integral of the first term:

$$
\begin{align*}
\int_{v_{1}}^{v_{2}}-8 \pi r^{2} \partial_{u} r \Omega^{-2}\left|\partial_{v} \phi\right|^{2} d v & =\int_{v_{1}}^{v_{2}} \partial_{v} m-\frac{Q^{2} \partial_{v} r}{2 r^{2}} d v \\
& \leq \int_{v_{1}}^{v_{2}} \partial_{v} m d v, \text { since } \partial_{v} r>0 \\
& =m_{2}-m_{1} . \tag{3.12}
\end{align*}
$$

To bound the second integral of the first term, we apply Proposition 3.3 to get $\partial_{v} \partial_{u} r \leq 0$, and hence $\partial_{u} r \geq \partial_{u} r_{2}$. Also, equation (2.7) implies that $\Omega_{2}^{-2} \partial_{v} r_{2} \leq \Omega^{-2} \partial_{v} r$. Combining these two pieces of information, we have

$$
\begin{align*}
\int_{v_{1}}^{v_{2}}-\frac{\partial_{u} r}{r^{2} \Omega^{-2}} d v & =\int_{r_{1}}^{r_{2}}-\frac{\partial_{u} r}{r^{2} \Omega^{-2} \partial_{v} r} d r \\
& \leq-\partial_{u} r_{2} \int_{r_{1}}^{r_{2}} \frac{1}{r^{2} \Omega^{-2} \partial_{v} r} d r \\
& \leq \frac{-\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}} \int_{r_{1}}^{r_{2}} \frac{1}{r^{2}} d r \\
& =\frac{\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}}\left(\frac{1}{r_{2}}-\frac{1}{r_{1}}\right) \tag{3.13}
\end{align*}
$$

Substituting (3.12) and (3.13) back into (3.11) gives us:
$\Theta(u)^{2} \leq(1+\kappa) \frac{\partial_{u} r_{2}}{8 \pi \Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{2}}-\frac{1}{r_{1}}\right)+\left(1+\frac{1}{\kappa}\right) \mathfrak{e}^{2}\left|\int_{v_{1}}^{v_{2}} \frac{Q \phi \Omega^{2}}{4 r} d v\right|^{2}$.

To bound the remaining integral, we apply the Cauchy-Schwarz inequality and Lemma 3.4:

$$
\begin{align*}
\left|\int_{v_{1}}^{v_{2}} \frac{Q \phi \Omega^{2}}{4 r} d v\right|^{2} & =\left|\frac{1}{4} \int_{v_{1}}^{v_{2}} \frac{Q}{r^{\frac{3}{2}} \Omega^{-2}} r^{\frac{1}{2}} \phi d v\right|^{2} \\
& \leq \frac{1}{16} \int_{v_{1}}^{v_{2}} \frac{Q^{2}}{r^{2}} \frac{1}{r} \frac{1}{\Omega^{-2}} \frac{1}{\Omega^{-2}} d v \cdot \int_{v_{1}}^{v_{2}} r|\phi|^{2} d v \\
& \leq \frac{1}{16} \int_{r_{1}}^{r_{2}} \frac{\omega}{2} \eta_{a} \frac{1}{r} \frac{1}{\Omega^{-2} \partial_{v} r} \frac{-\partial_{u} r}{-\Omega^{-2} \partial_{u} r} d r \int_{v_{1}}^{v_{2}} r|\phi|^{2} d v . \tag{3.15}
\end{align*}
$$

We bound the first integral: by Proposition 3.3 we have $\partial_{u} \partial_{v} r \leq 0$, which implies $\partial_{u} r_{2} \leq \partial_{u} r$. And with $\Omega_{2}^{-2} \partial_{v} r_{2} \leq \Omega^{-2} \partial_{v} r$ by (2.7), we derive

$$
\begin{align*}
\int_{r_{1}}^{r_{2}} \frac{\omega}{2} \eta_{a} \frac{1}{r} \frac{1}{\Omega^{-2} \partial_{v} r} \frac{-\partial_{u} r}{-\Omega^{-2} \partial_{u} r} d r & \leq \frac{-\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}} \int_{r_{1}}^{r_{2}} \frac{\omega}{2} \frac{2\left(m_{a}-m_{1}\right)}{r_{a}} \frac{1}{r} \frac{1}{-\Omega^{-2} \partial_{u} r} d r \\
& \leq \frac{-2 \omega}{1-\epsilon} \frac{\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}} \int_{r_{1}}^{r_{2}}\left(m_{2}-m_{1}\right) \frac{1}{r^{2}} d r \\
& \text { by Lemma } \\
& =\frac{-2 \omega}{1-\epsilon} \frac{\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}^{-2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right) \\
& =-\frac{2 \omega}{1-\epsilon} \frac{\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right) . \tag{3.16}
\end{align*}
$$

Now we bound the second integral:

$$
\begin{aligned}
& \int_{v_{1}}^{v_{2}} r|\phi|^{2} d v=\int_{v_{1}}^{v_{2}} r\left|\int_{v_{1}}^{v^{\prime}} \partial_{v} \phi d v+\phi_{1}\right|^{2} d v^{\prime} \\
\leq & 2 \int_{v_{1}}^{v_{2}}\left(r\left|\int_{v_{1}}^{v^{\prime}} \partial_{v} \phi d v\right|^{2}+r\left|\phi_{1}\right|^{2}\right) d v^{\prime} \\
\leq & 2 \int_{v_{1}}^{v_{2}}\left(r\left(v^{\prime}-v_{1}\right) \int_{v_{1}}^{v^{\prime}}\left|\partial_{v} \phi\right|^{2} d v+r\left|\phi_{1}\right|^{2}\right) d v^{\prime} \\
\leq & 2 r_{2} \int_{v_{1}}^{v_{2}}\left(\left(v_{2}-v_{1}\right) \int_{v_{1}}^{v^{\prime}} \frac{-8 \pi \Omega^{-2} r^{2} \partial_{u} r\left|\partial_{v} \phi\right|^{2}}{-8 \pi \Omega^{-2} \partial_{u} r r^{2}} d v+\left|\phi_{1}\right|^{2}\right) d v^{\prime} \\
\leq & 2 r_{2} \int_{v_{1}}^{v_{2}}\left(\frac{2\left(v_{2}-v_{1}\right)}{(1-\epsilon) 8 \pi r_{1}^{2}} \int_{v_{1}}^{v^{\prime}} \partial_{v} m d v+\left|\phi_{1}\right|^{2}\right) d v^{\prime} \\
= & 2 r_{2} \int_{v_{1}}^{v_{2}}\left(\frac{v_{2}-v_{1}}{(1-\epsilon) 4 \pi r_{1}^{2}}\left(m_{2}-m_{1}\right)+\left|\phi_{1}\right|^{2}\right) d v^{\prime} \\
\leq & \frac{r_{2}^{2}}{r_{1}^{2}} \frac{\left(v_{2}-v_{1}\right)^{2}}{4 \pi(1-\epsilon)} \frac{2\left(m_{2}-m_{1}\right)}{r_{2}}+2 r_{2}\left(v_{2}-v_{1}\right)\left|\phi_{1}\right|^{2} .
\end{aligned}
$$

Using the no trapped surface or MOTS assumption, we have $\frac{2\left(m_{2}-m_{1}\right)}{r_{2}}=\eta<1$. Using Lemma 3.3, $\frac{r_{2}}{r_{1}} \leq \frac{3}{2}$. Hence we have:

$$
\begin{equation*}
\int_{v_{1}}^{v_{2}} r|\phi|^{2} d v \leq \frac{9\left(v_{2}-v_{1}\right)^{2}}{16 \pi(1-\epsilon)}+2 r_{2}\left(u_{0}\right)\left(v_{2}-v_{1}\right)\left|\phi_{1}\right|^{2} \leq \omega \frac{1-\epsilon}{320 \mathfrak{e}^{2} \pi}, \tag{3.17}
\end{equation*}
$$

where we use the assumption in (1.5).
Substituting (3.16) and (3.17) back into (3.15), we get

$$
\left|\int_{v_{1}}^{v_{2}} \frac{Q \phi \Omega^{2}}{4 r} d v\right|^{2} \leq-\frac{\omega^{2}}{160 \mathfrak{e}^{2} \pi} \frac{\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right) .
$$

Now, set $\kappa=\frac{\omega}{4}$ in (3.14) and utilize the inequality above:

$$
\begin{aligned}
\Theta^{2} \leq & \left(1+\frac{\omega}{4}\right) \frac{-\partial_{u} r_{2}}{8 \pi \Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right) \\
& +\left(1+\frac{4}{\omega}\right) \frac{\omega^{2}}{160 \pi} \frac{-\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right) \\
= & \left(1+\frac{\omega}{4}+\frac{\omega}{20}(4+\omega)\right) \frac{-\partial_{u} r_{2}}{8 \pi \Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right) .
\end{aligned}
$$

Using the assumption that $\omega<\frac{2}{3}$, we have $\omega+4<5$, and therefore

$$
\begin{aligned}
\Theta^{2} & \leq\left(1+\frac{\omega}{4}+\frac{5 \omega}{20}\right) \frac{-\partial_{u} r_{2}}{8 \pi \Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right) \\
& =\left(1+\frac{\omega}{2}\right) \frac{-\partial_{u} r_{2}}{8 \pi \Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{1}}-\frac{1}{r_{2}}\right) .
\end{aligned}
$$

Lemma 3.6. Assume that $\eta \geq \frac{8 \epsilon}{\omega}$, the initial data along $\underline{C}$ is not supercharged, and $\mathcal{D}\left(0, v_{1}\right) \cup \mathcal{R}$ is free of trapped surfaces. Then

$$
\frac{\Omega_{2}^{-2} \partial_{v} r_{2}}{\Omega_{1}^{-2} \partial_{v} r_{1}}(u) \leq e^{-\left(1-\frac{\omega}{2}\right) \eta(u)}
$$

for all $u \in\left[u_{0}, u_{*}\right]$.
Proof. Dividing both sides of equation (2.6) by $\Omega^{-2} \partial_{v} r$ and integrating from $v_{1}$ to $v_{2}$, we get

$$
\ln \left|\Omega_{2}^{-2} \partial_{v} r_{2}\right|-\ln \left|\Omega_{1}^{-2} \partial_{v} r_{1}\right|=\ln \left(\frac{\Omega_{2}^{-2} \partial_{v} r_{2}}{\Omega_{1}^{-2} \partial_{v} r_{1}}\right)=-4 \pi \int_{v_{1}}^{v_{2}} \frac{r\left|\partial_{v} \phi\right|^{2}}{\partial_{v} r} d v
$$

By equation (2.15) and the definition of the Hawking mass, we have

$$
\frac{1}{r-2 m}\left(\partial_{v} m-\frac{Q^{2} \partial_{v} r}{2 r^{2}}\right)=\frac{-8 \pi r^{2} \Omega^{-2} \partial_{u} r}{-4 r \Omega^{-2} \partial_{u} r \partial_{v} r}=\frac{2 \pi r\left|\partial_{v} \phi\right|^{2}}{\partial_{v} r} .
$$

Hence for any $u \in\left[u_{0}, u_{*}\right]$,

$$
\begin{aligned}
& \ln \left(\frac{\Omega_{2}^{-2} \partial_{v} r_{2}}{\Omega_{1}^{-2} \partial_{v} r_{1}}\right)=-2 \int_{v_{1}}^{v_{2}} \frac{1}{r-2 m}\left(\partial_{v} m-\frac{Q^{2} \partial_{v} r}{2 r^{2}}\right) d v \\
& \leq-2 \int_{v_{1}}^{v_{2}} \frac{1}{r}\left(\partial_{v} m-\frac{Q^{2} \partial_{v} r}{2 r^{2}}\right) d v \leq \frac{-2}{r_{2}} \int_{v_{1}}^{v_{2}}\left(\partial_{v} m-\frac{Q^{2} \partial_{v} r}{2 r^{2}}\right) d v \\
& =-\frac{2\left(m_{2}-m_{1}\right)}{r_{2}}+\frac{2}{r_{2}} \int_{r_{1}}^{r_{2}} \frac{Q^{2}}{2 r^{2}} d r=-\eta+\frac{2}{r_{2}} \int_{r_{1}}^{r_{2}} \frac{Q^{2}}{2 r^{2}} d r .
\end{aligned}
$$

By Lemma 3.4, we have

$$
\frac{Q(u, v)^{2}}{r(u, v)^{2}} \leq \frac{\omega}{2} \frac{2\left(m(u, v)-m\left(u, v_{1}\right)\right)}{r(u, v)}
$$

Hence,

$$
\begin{aligned}
\frac{2}{r_{2}} \int_{r_{1}}^{r_{2}} \frac{Q^{2}}{2 r^{2}} d r & \leq \frac{\omega}{2 r_{2}} \int_{r_{1}}^{r_{2}} \frac{2\left(m(u, v)-m\left(u, v_{1}\right)\right)}{r(u, v)} d r \\
& \leq \omega \cdot \frac{m_{2}-m_{1}}{r_{2}} \ln \left(\frac{r_{2}}{r_{1}}\right) \leq \frac{\omega}{2} \ln \left(\frac{3}{2}\right) \eta \leq \frac{\omega}{2} \eta
\end{aligned}
$$

Combining the above estimates, we get:

$$
\ln \left(\frac{\Omega_{2}^{-2} \partial_{v} r_{2}}{\Omega_{1}^{-2} \partial_{v} r_{1}}\right) \leq-\left(1-\frac{\omega}{2}\right) \eta .
$$

Exponentiating both sides of the above inequality gives us the desired result.


Figure 3. Idea of proof of Lemma 3.7 and Theorem 1.3
3.5. The proof of Theorem 1.3. We are finally ready to prove a lower bound on $\frac{d \eta}{d u}$. The presence of charge case poses some difficulties not present in the uncharged case. This is because in order to apply Lemma 3.4, 3.5 and 3.6, we need to ensure that the $\eta>\frac{8 \epsilon}{\omega}$ assumption always hold to get a lower bound on $\frac{d \eta}{d u}$. On the other hand, we exactly need this lower bound on $\frac{d \eta}{d u}$ to prove the assumption that $\eta>\frac{8 \epsilon}{\omega}$. Hence we need to do this using a bootstrap argument again.

We will in fact prove something a little stronger: we show that $\eta(u) \geq \frac{12 \epsilon}{\omega}$ for $u \in\left[u_{0}, u_{*}\right]$. We use a bootstrap argument to prove this in Lemma 3.7. Assuming that the differential inequality holds for all $u \in\left[u_{0}, u^{\prime}\right]$, where $u^{\prime}<u_{*}$, then $\eta(u) \geq \frac{12 \epsilon}{\omega}$ holds in a slightly larger region as well.

Lemma 3.7. Assume that the region $\mathcal{D}\left(0, v_{1}\right) \cup \mathcal{R}$ is free of trapped surfaces and the initial data along $\underline{C}$ is not super-charged. Then if $\eta_{0} \geq \frac{13 \epsilon}{\omega}+g_{\omega}\left(\delta_{0}\right)$, we have $\eta(x) \geq \frac{12 \epsilon}{\omega}$ for all $x(u) \in\left[\frac{3 \delta_{0}}{1+\delta_{0}}, 1\right]$. Over here, $g_{\omega}(x)$ is defined as:

$$
\begin{equation*}
g_{\omega}(x):=\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{1}{(1+x)^{2}}\left(\left(\frac{2^{1-\frac{\omega}{2}}}{\omega}+\frac{1}{2^{1+\frac{\omega}{2}}\left(1+\frac{\omega}{2}\right)}\right) x^{1-\frac{\omega}{2}}-\frac{2}{\omega} x-\frac{1}{1+\frac{\omega}{2}} x^{2}\right) . \tag{3.18}
\end{equation*}
$$

Proof. We define $u^{\prime}:=\sup \left\{u \in\left[u_{0}, u_{*}\right] \left\lvert\, \eta(s) \geq \frac{12 \epsilon}{\omega}\right.\right.$ for all $\left.s \in\left[u_{0}, u\right]\right\}$. We are going to show that $u^{\prime}=u_{*}$, where $x\left(u_{*}\right)=\frac{3 \delta_{0}}{1+\delta_{0}}$. A sketch of this is provided in figure 3.

We calculate $\frac{d \eta}{d x}$. The following computations holds for all $u \in\left[u_{0}, u_{*}\right]$ :

$$
\begin{aligned}
& \frac{d \eta}{d x}=\frac{d \eta}{d u} / \frac{d x}{d u}=\frac{r_{2}\left(u_{0}\right)}{\partial_{u} r_{2}}\left(-\frac{2 \partial_{u} r_{2}}{r_{2}^{2}}\left(m_{2}-m_{1}\right)+\frac{2}{r_{2}} \partial_{u}\left(m_{2}-m_{1}\right)\right) \\
= & -\frac{\eta}{x}+\frac{2}{x \partial_{u} r_{2}}\left(-8 \pi r_{2}^{2} \Omega_{2}^{-2} \partial_{v} r_{2}\left|D_{u} \phi_{2}\right|^{2}+8 \pi r_{1}^{2} \Omega_{1}^{-2} \partial_{v} r_{1}\left|D_{u} \phi_{1}\right|^{2}+\frac{Q_{2}^{2} \partial_{u} r_{2}}{2 r_{2}^{2}}-\frac{Q_{1}^{2} \partial_{u} r_{1}}{2 r_{1}^{2}}\right) \\
\leq & -\frac{\eta}{x}-\frac{16 \pi \partial_{v} r_{2} \Omega_{2}^{-2}}{x \partial_{u} r_{2}}\left(r_{2}^{2}\left|D_{u} \phi_{2}\right|^{2}-\frac{\Omega_{1}^{-2} \partial_{v} r_{1}}{\Omega_{2}^{-2} \partial_{v} r_{2}} r_{1}^{2}\left|D_{u} \phi_{1}\right|^{2}\right)+\frac{Q_{2}^{2}}{x r_{2}^{2}},
\end{aligned}
$$

$$
\begin{equation*}
\text { where we used that } \frac{Q_{1}^{2} \partial_{u} r_{1}}{x r_{1}^{2} \partial_{u} r_{2}} \geq 0 \tag{3.19}
\end{equation*}
$$

Now we focus our attention on the region $\left[u_{0}, u^{\prime}\right]$. Since $\eta \geq \frac{12 \epsilon}{\omega} \geq \frac{8 \epsilon}{\omega}$ in $\left[x^{\prime}, 1\right]$, we can use Lemma (3.6) to bound the factor in the second term:

$$
\begin{aligned}
r_{2}^{2}\left|D_{u} \phi_{2}\right|^{2}-\frac{\Omega_{1}^{-2} \partial_{v} r_{1}}{\Omega_{2}^{-2} \partial_{v} r_{2}} r_{1}^{2}\left|D_{u} \phi_{1}\right|^{2} & \leq r_{2}^{2}\left|D_{u} \phi_{2}\right|^{2}-e^{\eta\left(1-\frac{\omega}{2}\right)} r_{1}^{2}\left|D_{u} \phi_{1}\right|^{2} \\
& =\Theta^{2}+2 \Theta\left|D_{u} \phi_{1}\right| r_{1}+\left(1-e^{\eta\left(1-\frac{\omega}{2}\right)}\right) r_{1}^{2}\left|D_{u} \phi_{1}\right|^{2}
\end{aligned}
$$

The last expression, being a quadratic in $\Theta$, can be bounded by a monic quadratic polynomial in $\Theta$ :

$$
\begin{align*}
\Theta^{2}+2 \Theta\left|D_{u} \phi_{1}\right| r_{1}+\left(1-e^{\eta\left(1-\frac{\omega}{2}\right)}\right) r_{1}^{2}\left|D_{u} \phi_{1}\right|^{2} & \leq\left(1+\frac{1}{e^{\eta\left(1-\frac{\omega}{2}\right)}-1}\right) \Theta^{2} \\
& \leq\left(1+\frac{1}{\eta\left(1-\frac{\omega}{2}\right)}\right) \Theta^{2} \tag{3.20}
\end{align*}
$$

where we used the fact that $\eta\left(1-\frac{\omega}{2}\right) \geq 0$ in the second inequality. Then (3.20) combined with (3.19) gives:

$$
\frac{d \eta}{d x} \leq-\frac{\eta}{x}-\frac{16 \pi \partial_{v} r_{2} \Omega_{2}^{-2}}{x \partial_{u} r_{2}}\left(1+\frac{1}{\eta\left(1-\frac{\omega}{2}\right)}\right) \Theta^{2}+\frac{Q_{2}^{2}}{x r_{2}^{2}}
$$

Applying Lemma 3.5, we have

$$
\begin{equation*}
\frac{d \eta}{d x} \leq-\frac{\eta}{x}+\frac{\eta}{x}\left(1+\frac{\omega}{2}\right)\left(1+\frac{1}{\eta\left(1-\frac{\omega}{2}\right)}\right)\left(\frac{r_{2}}{r_{1}}-1\right)+\frac{Q_{2}^{2}}{x r_{2}^{2}} \tag{3.21}
\end{equation*}
$$

Using Proposition 3.3, we get
$\delta(u)=\frac{r_{2}(u)-r_{1}(u)}{r_{2}(u)-\left(r_{2}(u)-r_{1}(u)\right)} \leq \frac{r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right)}{r_{2}(u)-\left(r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right)\right)}=\frac{\delta_{0}}{x(u)\left(1+\delta_{0}\right)-\delta_{0}}$.

Combining with (3.21), and using Lemma (3.4) to bound the term involving $Q$, we obtain

$$
\begin{aligned}
\frac{d \eta}{d x} & \leq \eta\left(\left(1+\frac{\omega}{2}\right) \frac{\delta_{0}}{x^{2}\left(1+\delta_{0}\right)-x \delta_{0}}-\frac{1}{x}\right)+\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{1}{x} \frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}}+\frac{Q_{2}^{2}}{x r_{2}^{2}} \\
& \leq \eta\left(\left(1+\frac{\omega}{2}\right) \frac{\delta_{0}}{x^{2}\left(1+\delta_{0}\right)-x \delta_{0}}-\frac{1}{x}\right)+\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{1}{x} \frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}}+\frac{\eta}{x} \frac{\omega}{2} \\
& =-\frac{\eta}{x}\left(1-\frac{\omega}{2}-\left(1+\frac{\omega}{2}\right) \frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}}\right)+\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{1}{x} \frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}} .
\end{aligned}
$$

Defining $g(x):=1-\frac{\omega}{2}-\left(1+\frac{\omega}{2}\right) \frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}}$ and $f(x):=\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}}$, we obtain the following differential inequality which holds for all $x \in\left[x^{\prime}, 1\right]$ :

$$
\frac{d \eta}{d x}+\eta \frac{g(x)}{x}-\frac{f(x)}{x} \leq 0
$$

To solve this differential inequality, we multiply by an integrating factor to get:

$$
\begin{aligned}
& \frac{d}{d x}\left(e^{-\int_{x}^{1} \frac{g(s)}{s} d s} \eta(x)\right)-e^{-\int_{x}^{1} \frac{g(s)}{s} d s} \frac{f(x)}{x} \leq 0 \\
\Longrightarrow & {\left[e^{-\int_{t}^{1} \frac{g(s)}{s} d s} \eta(t)\right]_{t=x}^{t=1} \leq \int_{x}^{1} e^{-\int_{t}^{1} \frac{g(s)}{s} d s} \frac{f}{t} d t+C, }
\end{aligned}
$$

where $C$ can be chosen to be any value which makes the inequality hold at the initial point $x=1$. Also denote $G(x):=\int_{x}^{1} \frac{g(s)}{s} d s$ and $F(x):=\int_{x}^{1} e^{-G(s)} \frac{f}{s} d s$. In this notation, we get

$$
\eta_{0}-e^{-G(x)} \eta(x) \leq F(x)+C
$$

Since $\eta_{0}=\left.\eta(x)\right|_{x=1}$ by definition, and $F(1)=G(1)=0$, setting $C=0$ makes the inequality tight. Hence, in the interval $\left[x^{\prime}, 1\right]$, we conclude that the following inequality holds:

$$
\begin{equation*}
\eta_{0}-e^{-G(x)} \eta(x) \leq F(x) \tag{3.22}
\end{equation*}
$$

Now we compute explicit expressions for $G(x)$ and $F(x)$ :

$$
\begin{aligned}
G(x) & =\int_{x}^{1} \frac{1-\frac{\omega}{2}}{s}-\frac{1+\frac{\omega}{2}}{s} \frac{\delta_{0}}{s\left(1+\delta_{0}\right)-\delta_{0}} d s \\
& =\int_{x}^{1} \frac{1-\frac{\omega}{2}}{s}+\frac{1+\frac{\omega}{2}}{s}-\frac{\left(1+\frac{\omega}{2}\right)\left(1+\delta_{0}\right)}{s\left(1+\delta_{0}\right)-\delta_{0}} d s \\
& =\left.\ln \left(\frac{s^{2}}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}\right)\right|_{x} ^{1}=\ln \left(\frac{\left(x\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}{x^{2}}\right)
\end{aligned}
$$

$$
\begin{aligned}
& F(x)=\int_{x}^{1} \frac{s^{2}}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}} \frac{f}{s} d s=\int_{x}^{1} \frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{\delta_{0} s}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{2+\frac{\omega}{2}}} d s \\
& =\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{\delta_{0}}{1+\delta_{0}} \int_{x}^{1} \frac{s\left(1+\delta_{0}\right)-\delta_{0}}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{2+\frac{\omega}{2}}}+\frac{\delta_{0}}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{2+\frac{\omega}{2}}} d s \\
& =\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{\delta_{0}}{1+\delta_{0}} \int_{x}^{1} \frac{1}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}+\frac{\delta_{0}}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{2+\frac{\omega}{2}}} d s \\
& =\left.\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{\delta_{0}}{\left(1+\delta_{0}\right)^{2}}\left[-\frac{2}{\omega} \frac{1}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{\frac{\omega}{2}}}-\frac{1}{1+\frac{\omega}{2}} \frac{\delta_{0}}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}\right]\right|_{x} ^{1} \\
& =\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{\delta_{0}}{\left(1+\delta_{0}\right)^{2}}\left(\frac{2}{\omega} \frac{1}{\left(x\left(1+\delta_{0}\right)-\delta_{0}\right)^{\frac{\omega}{2}}}-\frac{2}{\omega}+\frac{1}{1+\frac{\omega}{2}} \frac{\delta_{0}}{\left(x\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}-\frac{\delta_{0}}{1+\frac{\omega}{2}}\right) .
\end{aligned}
$$

Observe that $F(x)$ is monotonically decreasing and hence obtains its maximum at $x=\frac{3 \delta_{0}}{1+\delta_{0}}$ on the interval $\left[\frac{3 \delta_{0}}{1+\delta_{0}}, 1\right]$. Therefore,

$$
\begin{aligned}
F(x) \leq F\left(\frac{3 \delta_{0}}{1+\delta_{0}}\right) & =\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{\delta_{0}}{\left(1+\delta_{0}\right)^{2}}\left(\frac{2^{1-\frac{\omega}{2}}}{\omega} \delta_{0}^{-\frac{\omega}{2}}-\frac{2}{\omega}+\frac{1}{2^{1+\frac{\omega}{2}}\left(1+\frac{\omega}{2}\right)} \delta_{0}^{-\frac{\omega}{2}}-\frac{\delta_{0}}{1+\frac{\omega}{2}}\right) \\
& =\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{1}{\left(1+\delta_{0}\right)^{2}}\left(\left(\frac{2^{1-\frac{\omega}{2}}}{\omega}+\frac{1}{2^{1+\frac{\omega}{2}}\left(1+\frac{\omega}{2}\right)}\right) \delta_{0}^{1-\frac{\omega}{2}}-\frac{2}{\omega} \delta_{0}-\frac{1}{1+\frac{\omega}{2}} \delta_{0}^{2}\right) \\
& =g_{\omega}\left(\delta_{0}\right) .
\end{aligned}
$$

Substituting the expressions for $F(x)$ and $G(x)$ into (3.22), for all $x \in\left[x^{\prime}, 1\right]$, we get

$$
\begin{align*}
& \eta(x) \geq e^{G(x)}\left(\eta_{0}-F(x)\right) \\
\geq & \frac{\left(x\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}{x^{2}} \\
& \cdot\left(\eta_{0}-\frac{1+\frac{\omega}{2}}{1-\frac{\omega}{2}} \frac{1}{\left(1+\delta_{0}\right)^{2}}\left(\left(\frac{2^{1-\frac{\omega}{2}}}{\omega}+\frac{1}{2^{1+\frac{\omega}{2}}\left(1+\frac{\omega}{2}\right)}\right) \delta_{0}^{1-\frac{\omega}{2}}-\frac{2}{\omega} \delta_{0}-\frac{1}{1+\frac{\omega}{2}} \delta_{0}^{2}\right)\right) \\
= & \frac{\left(x\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}{x^{2}} \cdot\left(\eta_{0}-g_{\omega}\left(\delta_{0}\right)\right) . \tag{3.23}
\end{align*}
$$

For the last identity, we use the definition of $g_{\omega}(x)$ in (3.18).
Since $\omega<\frac{2}{3}$ implies that $\frac{x^{2}}{\left(x\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}$ is monotonically increasing, we get:

$$
\sup _{x \in\left[x_{*}, 1\right]} \frac{x^{2}}{\left(x\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}=1
$$

Combining this with the hypothesis that

$$
\eta_{0}>\frac{13 \epsilon}{\omega}+g_{\omega}\left(\delta_{0}\right)
$$

we obtain the inequality:

$$
\eta_{0}>\frac{13 \epsilon}{\omega}+g_{\omega}\left(\delta_{0}\right) \geq \frac{13 \epsilon}{\omega} \frac{x^{2}}{\left(x\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}+g_{\omega}\left(\delta_{0}\right)
$$

for $x \in\left[x^{\prime}, 1\right]$. Substituting the above into (3.23) gives us:

$$
\eta(x) \geq \frac{13 \epsilon}{\omega}, \quad \text { for all } x \in\left[x^{\prime}, 1\right]
$$

However, by the continuity of $\eta(x)$, we can find $x^{\prime \prime}<x^{\prime}$ such that $\eta(x) \geq \frac{12 \epsilon}{\omega}$ for all $x \in\left[x^{\prime \prime}, 1\right]$, i.e. $\eta(u) \geq \frac{12 \epsilon}{\omega}$ for all $u \in\left[u_{0}, u^{\prime \prime}\right]$, contradicting the supremum property of $u^{\prime}$.

We are now ready to prove the main theorem of this paper.
Proof. (Theorem 1.3) We prove the theorem by contradiction. Suppose that $\mathcal{R}$ contains no trapped surfaces or MOTS, in particular $\partial_{v} r_{2}>0$ for $u \in\left[u_{0}, u_{*}\right]$. Then Lemma 3.5 applies and (3.22) holds for $x \in\left[x_{*}, 1\right]$. Rearranging (3.22) gives us

$$
\eta_{0} \leq e^{-G(x)} \eta(x)+F(x)<e^{-G(x)}+F(x), \text { for all } x \in\left[x_{*}, 1\right]
$$

where we used the assumption that $\eta(x)=\frac{2\left(m_{2}-m_{1}\right)}{r_{2}} \leq \frac{2 m_{2}}{r_{2}}<1$ in the second inequality. In particular, by letting $x=x_{*}=\frac{3 \delta_{0}}{1+\delta_{0}}$, we get:

$$
\begin{aligned}
e^{-G\left(x_{*}\right)}+F\left(x_{*}\right) & \leq \frac{x_{*}^{2}}{\left(x_{*}\left(1+\delta_{0}\right)-\delta_{0}\right)^{1+\frac{\omega}{2}}}+g_{\omega}\left(\delta_{0}\right) \\
& =\frac{9}{2^{1+\frac{\omega}{2}}\left(1+\delta_{0}\right)^{2}} \delta_{0}^{1-\frac{\omega}{2}}+g_{\omega}\left(\delta_{0}\right)
\end{aligned}
$$

and hence $\eta_{0}<\frac{9}{2^{1+\frac{\omega}{2}}\left(1+\delta_{0}\right)^{2}} \delta_{0}^{1-\frac{\omega}{2}}+g_{\omega}\left(\delta_{0}\right)$, giving us the desired contradiction.

## 4. Appendix

4.1. Trapped Surface Formation for the Einstein Scalar Field. Here we provide a proof of Christodolou's sharp trapped surface formation criterion as in [9. In the case for the real scalar field, the system of equations (2.4) to (2.12) is reduced to

$$
\begin{gather*}
r \partial_{v} \partial_{u} r+\partial_{v} r \partial_{u} r=-\frac{\Omega^{2}}{4},  \tag{4.1}\\
\partial_{u}\left(\Omega^{-2} \partial_{u} r\right)=-4 \pi r \Omega^{-2}\left|\partial_{u} \phi\right|^{2},  \tag{4.2}\\
\partial_{v}\left(\Omega^{-2} \partial_{v} r\right)=-4 \pi r \Omega^{-2}\left|\partial_{v} \phi\right|^{2},  \tag{4.3}\\
r \partial_{u} \partial_{v} \phi+\partial_{u} r \partial_{v} \phi+\partial_{v} r \partial_{u} \phi=0 . \tag{4.4}
\end{gather*}
$$

Also, the derivatives of the Hawking mass become:

$$
\begin{align*}
\partial_{u} m & =-8 \pi r^{2} \Omega^{-2} \partial_{v} r\left|\partial_{u} \phi\right|^{2},  \tag{4.5}\\
\partial_{v} m & =-8 \pi r^{2} \Omega^{-2} \partial_{u} r\left|\partial_{v} \phi\right|^{2} . \tag{4.6}
\end{align*}
$$

For convenience, we restate theorem 1.1 here.
Theorem 1.1. Define the function

$$
E(x):=\frac{x}{(1+x)^{2}}\left[\ln \left(\frac{1}{2 x}\right)+5-x\right] .
$$

Consider the system (1.1) with characteristic initial data along $u=u_{0}$ and $v=v_{1}$. For initial mass input $\eta_{0}$ along $u=u_{0}$, if the following lower bound holds:

$$
\eta_{0}>E\left(\delta_{0}\right)
$$

then a trapped surface $S_{u, v}$, with properties $\partial_{v} r(u, v)<0$ and $\partial_{u} r(u, v)<0$, forms in the region $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right] \subset \mathcal{R}$.

In this section, we will first give a few technical estimates to the dynamical quantities in the strip $\left[u_{0}, 0\right] \times\left[v_{1}, v_{2}\right]$. These will be used in the proof for Theorem 1.1. We start off by showing that $\partial_{u} r$ is negative and bounded away from 0 .

Lemma 4.1. $\partial_{u} r \leq-\frac{1}{2} \Omega^{2}$ everywhere in $\mathcal{D}\left(0, v_{1}\right) \cup\left(\left[u_{0}, 0\right] \times\left[v_{1}, \infty\right)\right)$
Proof. Rewrite (4.1) as

$$
\partial_{v}\left(r \partial_{u} r\right)=-\frac{\Omega^{2}}{4} .
$$

Note that $\Omega^{2}=1$ along $C$. Integrating both sides from 0 to $v$ and dividing by $r$ :

$$
\begin{equation*}
-\frac{v}{4 r\left(u_{0}, v\right)}=\partial_{u} r\left(u_{0}, v\right) \tag{4.7}
\end{equation*}
$$

Setting $v=0$ in the above gives us

$$
\begin{equation*}
-\frac{1}{4 \partial_{v} r\left(u_{0}, 0\right)}=\partial_{u} r\left(u_{0}, 0\right)=-\partial_{v} r\left(u_{0}, 0\right) \Longrightarrow \partial_{v} r\left(u_{0}, 0\right)=\frac{1}{2} \tag{4.8}
\end{equation*}
$$

Since $\Omega^{2}=1$ on $C$ as well, (4.3) gives us that $\partial_{v} \partial_{v} r \leq 0$, i.e. $r$ is concave with respect to $v$. Combining this with the fact that $r\left(u_{0}, 0\right)=v\left(u_{0}, 0\right)=0$, we have:

$$
\frac{r}{v}\left(u_{0}, v\right) \leq \partial_{v} r\left(u_{0}, 0\right)
$$

Hence,

$$
\frac{r}{v}\left(u_{0}, v\right) \leq \frac{1}{2}
$$

Substitute this into (4.7), we get

$$
\partial_{u} r\left(u_{0}, v\right) \leq-\frac{1}{2}
$$

By (4.3), $\Omega^{-2} \partial_{u} r$ is decreasing along incoming null geodesics. Hence for a general point in $\mathcal{D}\left(0, v_{1}\right) \cup\left(\left[u_{0}, 0\right] \times\left[v_{1}, \infty\right)\right)$, we have $\Omega^{-2} \partial_{u} r \leq-\frac{1}{2}$.

Remark 8. $m(u, v) \geq 0$ for all $(u, v) \in \mathcal{D}\left(0, v_{1}\right) \cup\left(\left[u_{0}, 0\right] \times\left[v_{1}, \infty\right)\right)$

Proof. Given any point $(u, v) \in \mathcal{R}$, we can extend the outgoing null geodesic backwards until it intersects $\Gamma$ at some coordinate $\left(u, v_{c}\right)$, so that $r\left(u, v_{c}\right)=0$. Using (4.6), we have

$$
\partial_{v} m=-8 \pi r^{2} \Omega^{-2} \partial_{u} r\left|\partial_{v} \phi\right|^{2},
$$

and since $\partial_{u} r \leq 0$ by Lemma 4.1, we get that $\partial_{v} m \geq 0$. Combining with the fact that $m\left(u, v_{c}\right)=0$, we obtain the desired result.

Next, we show that the mixed derivative of $r$ is always negative. This places a upper bound on the growth on the ratio $\frac{r_{2}}{r_{1}}$.

Proposition 4.2. Assume that $\mathcal{D}\left(0, v_{1}\right) \cup \mathcal{R}$ is free of trapped surfaces. Then i) $\partial_{u} \partial_{v} r \leq 0$ in $\mathcal{R}$ and ii) $\delta(x):=\frac{r_{2}}{r_{1}}-1 \leq \frac{1}{2}$ for $u \in\left[u_{0}, u_{*}\right]$.

Proof. We rewrite (4.1) into the following equivalent form:

$$
\begin{equation*}
\partial_{u} \partial_{v} r=-\frac{\Omega^{2}}{2} \frac{m}{r^{2}} \tag{4.9}
\end{equation*}
$$

Since $m \geq 0$, the right side of the above equation is non-positive. This proves the first part of the lemma.

Integrating with respect to $u$, we get:

$$
\partial_{v} r(u)-\partial_{v} r\left(u_{0}\right) \leq 0 \Longrightarrow \partial_{v} r(u) \leq \partial_{v} r\left(u_{0}\right)
$$

Integrating the above inequality with respect to $v$,

$$
r_{2}(u)-r_{1}(u) \leq r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right), \text { for all } u \in\left[u_{0}, u_{*}\right] .
$$

Hence, we can use the above inequality to compute a bound for $\delta(u)$ :

$$
\begin{align*}
\delta(u)=\frac{r_{2}}{r_{1}}-1 & =\frac{r_{2}-r_{1}}{r_{2}-\left(r_{2}-r_{1}\right)} \leq \frac{r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right)}{r_{2}(u)-\left(r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right)\right)} \\
& \leq \frac{\delta_{0}}{\frac{r_{2}(u)}{r_{1}\left(u_{0}\right)}-\delta_{0}}=\frac{\delta_{0}}{x(u)\left(1+\delta_{0}\right)-\delta_{0}}, \quad \text { for all } u \in\left[u_{0}, u_{*}\right] \tag{4.10}
\end{align*}
$$

where $x(u):=r_{2}(u) / r_{2}\left(u_{0}\right)$. Recall $r_{2}\left(u_{*}\right):=\frac{3 \delta_{0}}{1+\delta_{0}} \cdot r_{2}\left(u_{0}\right)$ and since $x(u)$ is monotonically decreasing, we have

$$
x(u) \geq x\left(u_{*}\right)=\frac{3 \delta_{0}}{1+\delta_{0}} \text { for } u \in\left[u_{0}, u_{*}\right]
$$

and hence

$$
\delta(u) \leq \frac{\delta_{0}}{2 \delta_{0}}=\frac{1}{2} \text { for all } u \in\left[u_{0}, u_{*}\right]
$$

Next we prove two key lemmas. In the first one we bound the difference in $r \partial_{u} \phi$ between $v=v_{1}$ and $v=v_{2}$. Then in the second we bound the ratio of $\partial_{v} r$ between $v=v_{1}$ and $v=v_{2}$.

Lemma 4.3. Define $\Theta:=r_{2} \partial_{u} \phi_{2}-r_{1} \partial_{u} \phi_{1}$. Suppose that $\mathcal{D}\left(0, v_{1}\right) \cup \mathcal{R}$ is free of trapped surfaces. Then

$$
\Theta(u)^{2} \leq \frac{\partial_{u} r_{2}}{8 \pi \Omega_{2}^{-2} \partial_{v} r_{2}}\left(m_{2}-m_{1}\right)\left(\frac{1}{r_{2}}-\frac{1}{r_{1}}\right)(u)
$$

for all $u \in\left[u_{0}, u_{*}\right]$.
Proof. We can write the wave equation (4.4) as

$$
\partial_{v}\left(r \partial_{u} \phi\right)=-\partial_{u} r \partial_{v} \phi .
$$

By integrating the above equation, we get

$$
\begin{align*}
\Theta^{2} & =\left(r_{2} \partial_{u} \phi_{2}-r_{1} \partial_{u} \phi_{1}\right)^{2} \\
& =\left|\int_{v_{1}}^{v_{2}}-\partial_{u} r \partial_{v} \phi d v\right|^{2} \leq\left(\int_{v_{1}}^{v_{2}}-\partial_{u} r\left|\partial_{v} \phi\right| d v\right)^{2} \\
& \leq \frac{1}{8 \pi} \int_{v_{1}}^{v_{2}}-8 \pi r^{2} \partial_{u} r \Omega^{-2}\left|\partial_{v} \phi\right|^{2} d v \cdot \int_{v_{1}}^{v_{2}}-\frac{\partial_{u} r}{r^{2} \Omega^{-2}} d v \tag{4.11}
\end{align*}
$$

where we have applied Holder's inequality for the last inequality.
The first integral can be written in terms of the hawking mass:

$$
\begin{align*}
\int_{v_{1}}^{v_{2}}-8 \pi r^{2} \partial_{u} r \Omega^{-2}\left|\partial_{v} \phi\right|^{2} d v & =\int_{v_{1}}^{v_{2}} \partial_{v} m d v \\
& =m_{2}-m_{1} \tag{4.12}
\end{align*}
$$

To bound the second integral, we apply Proposition 4.2 to get $\partial_{v} \partial_{u} r \leq 0$, and hence $\partial_{u} r \geq \partial_{u} r_{2}$. Also, equation (4.3) implies that $\Omega_{2}^{-2} \partial_{v} r_{2} \leq \Omega^{-2} \partial_{v} r$. Combining these two pieces of information, we have

$$
\begin{align*}
\int_{v_{1}}^{v_{2}}-\frac{\partial_{u} r}{r^{2} \Omega^{-2}} d v & =\int_{r_{1}}^{r_{2}}-\frac{\partial_{u} r}{r^{2} \Omega^{-2} \partial_{v} r} d r \leq-\partial_{u} r_{2} \int_{r_{1}}^{r_{2}} \frac{1}{r^{2} \Omega^{-2} \partial_{v} r} d r \\
& \leq \frac{-\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}} \int_{r_{1}}^{r_{2}} \frac{1}{r^{2}} d r=\frac{\partial_{u} r_{2}}{\Omega_{2}^{-2} \partial_{v} r_{2}}\left(\frac{1}{r_{2}}-\frac{1}{r_{1}}\right) \tag{4.13}
\end{align*}
$$

Substituting (4.12) and (4.13) back into (4.11) gives us the desired result.
Lemma 4.4. Assume that $\mathcal{D}\left(0, v_{1}\right) \cup \mathcal{R}$ is free of trapped surfaces. Then

$$
\frac{\Omega_{2}^{-2} \partial_{v} r_{2}}{\Omega_{1}^{-2} \partial_{v} r_{1}}(u) \leq e^{-\eta(u)}
$$

for all $u \in\left[u_{0}, u_{*}\right]$.
Proof. Dividing both sides of equation (4.3) by $\Omega^{-2} \partial_{v} r$ and integrating from $v_{1}$ to $v_{2}$, we get

$$
\ln \left|\Omega_{2}^{-2} \partial_{v} r_{2}\right|-\ln \left|\Omega_{1}^{-2} \partial_{v} r_{1}\right|=\ln \left(\frac{\Omega_{2}^{-2} \partial_{v} r_{2}}{\Omega_{1}^{-2} \partial_{v} r_{1}}\right)=-4 \pi \int_{v_{1}}^{v_{2}} \frac{r\left|\partial_{v} \phi\right|^{2}}{\partial_{v} r} d v
$$

By equation (4.6) and the definition of the Hawking mass, we have

$$
\frac{\partial_{v} m}{r-2 m}=\frac{-8 \pi r^{2} \Omega^{-2} \partial_{u} r\left|\partial_{v} \phi\right|^{2}}{-4 r \Omega^{-2} \partial_{u} r \partial_{v} r}=\frac{2 \pi r\left|\partial_{v} \phi\right|^{2}}{\partial_{v} r}
$$



Figure 4. Idea of Proof of Theorem 1.1

Hence for any $u \in\left[u_{0}, u_{*}\right]$

$$
\begin{aligned}
\ln \left(\frac{\Omega_{2}^{-2} \partial_{v} r_{2}}{\Omega_{1}^{-2} \partial_{v} r_{1}}\right) & =-2 \int_{v_{1}}^{v_{2}} \frac{\partial_{v} m}{r-2 m} d v \leq-2 \int_{v_{1}}^{v_{2}} \frac{1}{r} \partial_{v} m d v \\
& \leq \frac{-2}{r_{2}} \int_{v_{1}}^{v_{2}} \partial_{v} m d v=-\frac{2\left(m_{2}-m_{1}\right)}{r_{2}}=-\eta
\end{aligned}
$$

Exponentiating both sides of the above inequality gives us the desired result.

Now we are ready to prove Theorem 1.1.
Proof. (Theorem 1.1) We consider the dimensionless length scale $x(u):=\frac{r_{2}(u)}{r_{2}\left(u_{0}\right)}$. Note that $x$ decreases as $u$ increases and $x\left(u_{0}\right)=1$. We will show that $\frac{d \eta}{d x}$ is bounded from above, i.e. $\frac{d \eta}{d u}$ is bounded from below, and hence obtain a lower bound for $\eta\left(u_{*}\right)$. If this lower bound is greater than 1 , this implies $S\left(u_{*}, v_{2}\right)$ is a trapped surface, for

$$
\begin{aligned}
\eta\left(u_{*}\right)=\frac{2\left(m_{2}-m_{1}\right)}{r_{2}}\left(u_{*}\right)>1 & \Longrightarrow \frac{2 m_{2}}{r_{2}}\left(u_{*}\right)>1 \\
& \Longrightarrow S\left(u_{*}, v\right) \text { is a trapped surface. }
\end{aligned}
$$

See Figure 4 for an illustration.
To be precise, we prove a Gronwall-like inequality under the assumption that there is no trapped surface formed before $u_{*}$. In particular, we assume that $\partial_{v} r_{2}(u)>0$ for all $u \in\left[u_{0}, u_{*}\right]$. We show that this assumption will lead
to a contradiction.
Assuming that $\partial_{v} r_{2}(u)>0$ for all $u \in\left[u_{0}, u_{*}\right]$, the following chain of identities hold in the region $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right]$ :

$$
\begin{align*}
\frac{d \eta}{d x} & =\frac{d \eta}{d u} / \frac{d x}{d u}=\frac{r_{2}\left(u_{0}\right)}{\partial_{u} r_{2}}\left(-\frac{2 \partial_{u} r_{2}}{r_{2}^{2}}\left(m_{2}-m_{1}\right)+\frac{2}{r_{2}} \partial_{u}\left(m_{2}-m_{1}\right)\right) \\
& =-\frac{\eta}{x}+\frac{2}{x \partial_{u} r_{2}}\left(-8 \pi r_{2}^{2} \Omega_{2}^{-2} \partial_{v} r_{2}\left|\partial_{u} \phi_{2}\right|^{2}+8 \pi r_{1}^{2} \Omega_{1}^{-2} \partial_{v} r_{1}\left|\partial_{u} \phi_{1}\right|^{2}\right) \\
& =-\frac{\eta}{x}-\frac{16 \pi \partial_{v} r_{2} \Omega_{2}^{-2}}{x \partial_{u} r_{2}}\left(r_{2}^{2}\left|\partial_{u} \phi_{2}\right|^{2}-\frac{\Omega_{1}^{-2} \partial_{v} r_{1}}{\Omega_{2}^{-2} \partial_{v} r_{2}} r_{1}^{2}\left|\partial_{u} \phi_{1}\right|^{2}\right) . \tag{4.14}
\end{align*}
$$

Using Lemma 4.4, we can bound the factor in the second term:

$$
\begin{aligned}
r_{2}^{2}\left|\partial_{u} \phi_{2}\right|^{2}-\frac{\Omega_{1}^{-2} \partial_{v} r_{1}}{\Omega_{2}^{-2} \partial_{v} r_{2}} r_{1}^{2}\left|\partial_{u} \phi_{1}\right|^{2} & \leq r_{2}^{2}\left|\partial_{u} \phi_{2}\right|^{2}-e^{\eta} r_{1}^{2}\left|\partial_{u} \phi_{1}\right|^{2} \\
& =\Theta^{2}+2 \Theta \partial_{u} \phi_{1} r_{1}+\left(1-e^{\eta}\right) r_{1}^{2}\left|\partial_{u} \phi_{1}\right|^{2}
\end{aligned}
$$

The last expression, being a quadratic in $\Theta$, can be bounded by a monic quadratic polynomial in $\Theta$ :

$$
\Theta^{2}+2 \Theta \partial_{u} \phi_{1} r_{1}+\left(1-e^{\eta}\right) r_{1}^{2}\left|\partial_{u} \phi_{1}\right|^{2} \leq\left(1+\frac{1}{e^{\eta}-1}\right) \Theta^{2} \leq\left(1+\frac{1}{\eta}\right) \Theta^{2}
$$

since $\eta \geq 0$. This last inequality combines with (4.14) to give:

$$
\frac{d \eta}{d x} \leq-\frac{\eta}{x}-\frac{16 \pi \partial_{v} r_{2} \Omega_{2}^{-2}}{x \partial_{u} r_{2}}\left(1+\frac{1}{\eta}\right) \Theta^{2}
$$

Applying Lemma 4.3, we have

$$
\begin{align*}
\frac{d \eta}{d x} & \leq-\frac{\eta}{x}-\frac{2}{x}\left(1+\frac{1}{\eta}\right)\left(\frac{1}{r_{2}}-\frac{1}{r_{1}}\right)\left(m_{2}-m_{1}\right) \\
& =-\frac{\eta}{x}+\frac{\eta}{x}\left(1+\frac{1}{\eta}\right)\left(\frac{r_{2}}{r_{1}}-1\right) . \tag{4.15}
\end{align*}
$$

Using (4.10) we get

$$
\delta=\frac{r_{2}-r_{1}}{r_{2}-\left(r_{2}-r_{1}\right)} \leq \frac{r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right)}{r_{2}(u)-\left(r_{2}\left(u_{0}\right)-r_{1}\left(u_{0}\right)\right)}=\frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}} .
$$

Combining the above with (4.15), we obtain

$$
\frac{d \eta}{d x} \leq \eta\left(\frac{\delta_{0}}{x^{2}\left(1+\delta_{0}\right)-x \delta_{0}}-\frac{1}{x}\right)+\frac{\delta_{0}}{x^{2}\left(1+\delta_{0}\right)-x \delta_{0}}
$$

Defining $g(x):=1-\frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}}$ and $f(x):=\frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}}$, we obtain the following differential inequality:

$$
\frac{d \eta}{d x}+\eta \frac{g(x)}{x}-\frac{f(x)}{x} \leq 0
$$

To solve this differential inequality, we multiply by an integrating factor then integrate with respect to $x$ :

$$
\begin{aligned}
& \frac{d}{d x}\left(e^{-\int_{x}^{1} \frac{g(s)}{s} d s} \eta(x)\right)-e^{-\int_{x}^{1} \frac{g(s)}{s} d s} \frac{f(x)}{x} \leq 0 \\
\Longrightarrow & {\left[e^{-\int_{x^{\prime}}^{1} \frac{g(s)}{s} d s} \eta\left(x^{\prime}\right)\right]_{x^{\prime}=x}^{x^{\prime}=1} \leq \int_{x}^{1} e^{-\int_{x^{\prime}}^{1} \frac{q(s)}{s} d s} \frac{f}{x^{\prime}} d x^{\prime} . }
\end{aligned}
$$

We denote $G(x):=\int_{x}^{1} \frac{g(s)}{s} d s$ and $F(x):=\int_{x}^{1} e^{-G\left(x^{\prime}\right)} \frac{f}{x^{\prime}} d x^{\prime}$. In this notation, we get

$$
\eta_{0}-e^{-G(x)} \eta(x) \leq F(x) \Longrightarrow \eta(x) \geq e^{G(x)}\left(-F(x)+\eta_{0}\right) .
$$

Hence, in the region $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right]$ free of trapped surfaces, we conclude that the following inequality holds:

$$
\begin{equation*}
\eta(x) \geq e^{G(x)}\left(-F(x)+\eta_{0}\right) . \tag{4.16}
\end{equation*}
$$

Now, we compute explicit expressions for $G(x)$ and $F(x)$ :

$$
\begin{gathered}
G(x)=\int_{x}^{1} \frac{1}{s}-\frac{1}{s} \frac{\delta_{0}}{s\left(1+\delta_{0}\right)-\delta_{0}} d s \\
=\int_{x}^{1} \frac{1}{s}+\frac{1}{s}-\frac{1+\delta_{0}}{s\left(1+\delta_{0}\right)-\delta_{0}} d s \\
=\left.\ln \left(\frac{s^{2}}{s\left(1+\delta_{0}\right)-\delta_{0}}\right)\right|_{x} ^{1}=\ln \left(\frac{x\left(1+\delta_{0}\right)-\delta_{0}}{x^{2}}\right) \\
F(x)=\int_{x}^{1} \frac{s^{2}}{s\left(1+\delta_{0}\right)-\delta_{0}} \frac{f}{s} d s=\int_{x}^{1} \frac{\delta_{0} s}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{2}} d s \\
= \\
=\frac{\delta_{0}}{1+\delta_{0}} \int_{x}^{1} \frac{1}{s\left(1+\delta_{0}\right)-\delta_{0}}+\frac{\delta_{0}}{\left(s\left(1+\delta_{0}\right)-\delta_{0}\right)^{2}} d s \\
= \\
\left.\frac{\delta_{0}}{\left(1+\delta_{0}\right)^{2}} \ln \left(s\left(1+\delta_{0}\right)-\delta_{0}\right)\right|_{x} ^{1}-\left.\frac{\delta_{0}^{2}}{\left(1+\delta_{0}\right)^{2}} \frac{1}{s\left(1+\delta_{0}\right)-\delta_{0}}\right|_{x} ^{1} \\
= \\
\frac{\delta_{0}}{\left(1+\delta_{0}\right)^{2}}\left(\ln \left(\frac{1}{x\left(1+\delta_{0}\right)-\delta_{0}}\right)+\delta_{0}\left(\frac{1}{x\left(1+\delta_{0}\right)-\delta_{0}}-1\right)\right) .
\end{gathered}
$$

Using the assumption that there is no trapped surface or MOTS, we have $\eta(x)=\frac{2\left(m_{2}-m_{1}\right)}{r_{2}} \leq \frac{2 m_{2}}{r_{2}}<1$ for $x \in\left[\frac{3 \delta_{0}}{1+\delta_{0}}, 1\right]$. Rearranging (4.16) results in

$$
\eta_{0} \leq e^{-G(x)} \eta(x)+F(x)<e^{-G(x)}+F(x), \text { for all } x \in\left[\frac{3 \delta_{0}}{1+\delta_{0}}, 1\right]
$$

In particular, we can substitute $x=\frac{3 \delta_{0}}{1+\delta_{0}}$ into the above equation and get

$$
\eta_{0}<E\left(\delta_{0}\right)=\frac{\delta_{0}}{\left(1+\delta_{0}\right)^{2}}\left[\log \left(\frac{1}{2 \delta_{0}}\right)+5-\delta_{0}\right] .
$$

This gives us the desired contradiction.

### 4.2. A Special Case of Minkowskian incoming characteristic initial

 data. Prescribe Minkowskian data along $v=v_{1}$, we can improve the lower bound required on $\eta_{0}$ in Theorem 1.1.Theorem 1.2. Assume that Minkowskian data are prescribed along $v=v_{1}$ and require $\phi\left(u, v_{1}\right)=0$. Suppose that the following lower bound on $\eta_{0}$ holds:

$$
\eta_{0}>\frac{9}{2} \delta_{0},
$$

then there exist a MOTS or a trapped surface in $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right] \subset \mathcal{R}$, i.e. $\partial_{v} r \leq 0$ at some point in $\left[u_{0}, u_{*}\right] \times\left[v_{1}, v_{2}\right]$.

Proof. (Theorem 1.2) In this special case, we have $\phi_{1} \equiv 0$ and $m_{1} \equiv 0$. Equation (4.14) now reads:

$$
\frac{d \eta}{d x}=-\frac{\eta}{x}-\frac{16 \pi \partial_{v} r_{2} \Omega_{2}^{-2}}{x \partial_{u} r_{2}} r_{2}^{2}\left|\partial_{u} \phi_{2}\right|^{2}
$$

and we also have

$$
\Theta^{2}=r_{2}^{2}\left|\partial_{u} \phi_{2}\right|^{2} .
$$

Combining the above equations, followed by applying Lemma 4.3, we get:

$$
\begin{aligned}
\frac{d \eta}{d x}=-\frac{\eta}{x}-\frac{16 \pi \partial_{v} r_{2} \Omega_{2}^{-2}}{x \partial_{u} r_{2}} \Theta^{2} & \leq-\frac{\eta}{x}-\frac{2}{x}\left(\frac{1}{r_{2}}-\frac{1}{r_{1}}\right)\left(m_{2}-m_{1}\right) \\
& =-\frac{\eta}{x}+\frac{\eta}{x}\left(\frac{r_{2}}{r_{1}}-1\right) \\
& \leq-\frac{\eta}{x}+\frac{\eta}{x} \frac{\delta_{0}}{x\left(1+\delta_{0}\right)-\delta_{0}} .
\end{aligned}
$$

Integrating the above inequality:

$$
\begin{aligned}
& \int_{x}^{1} \frac{1}{\eta} d \eta \leq \int_{x}^{1}-\frac{1}{s}+\frac{1}{s} \frac{\delta_{0}}{s\left(1+\delta_{0}\right)-\delta_{0}} d s=\int_{x}^{1}-\frac{2}{s}+\frac{1+\delta_{0}}{s\left(1+\delta_{0}\right)-\delta_{0}} d s \\
& \Longrightarrow \ln \left(\frac{\eta_{0}}{\eta(x)}\right) \leq \ln \left(\frac{x^{2}}{x\left(1+\delta_{0}\right)-\delta_{0}}\right) \\
& \Longrightarrow \eta_{0} \leq \eta(x) \frac{x^{2}}{x\left(1+\delta_{0}\right)-\delta_{0}} .
\end{aligned}
$$

Under the assumption of no trapped surfaces or MOTS, we have $\eta(x)<1$ for all $x \in\left[\frac{3 \delta_{0}}{1+\delta_{0}}, 1\right]$, hence

$$
\eta_{0} \leq \frac{x^{2}}{x\left(1+\delta_{0}\right)-\delta_{0}}, \text { for all } x \in\left[\frac{3 \delta_{0}}{1+\delta_{0}}, 1\right] .
$$

In particular, choosing $x=\frac{3 \delta_{0}}{1+\delta_{0}}$ we have

$$
\eta_{0} \leq \frac{9}{2} \delta_{0} .
$$

This gives us the desired contradiction to the hypothesis.

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[^0]:    ${ }^{1} \mathrm{~A}$ trapped surface is a two-dimensional sphere, with both incoming and outgoing null expansions negative.

[^1]:    ${ }^{2}$ For more discussions about scaling consideration, interested readers are also refereed to [1] and 3] by the first author.

[^2]:    ${ }^{3}$ This definition for $r$ implies that $\left.g\right|_{S}=r^{2}\left(d \theta^{2}+\sin ^{2} \theta d \phi^{2}\right)$

