Algebraic independence of certain Mahler functions

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Abstract We prove algebraic independence of functions satisfying a simple form of algebraic Mahler functional equations. The main result (Theorem 1) partly generalizes a result obtained by Kubota. This result is deduced from a quantitative version of it (Theorem 3), which is proved by using an inductive method originated by Duverney. As an application we can also generalize a recent result by Bundschuh and the second named author (Theorem 2 and its corollary).

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1 Introduction

Throughout this paper let $d \geq 2$ and t denote positive integers. Let K be an arbitrary field of characteristic zero, and $K[[z]]$ be the ring of formal power series in z with coefficients from K. Set

$$\mathcal{F} := 1 + zK[[z]] \quad \text{and} \quad \mathcal{R} := \mathcal{F} \cap K(z).$$

Note that $\mathcal{F}$ is a group under the usual multiplication on $K[[z]]$ and $\mathcal{R}$ is a subgroup of $\mathcal{F}$. We are interested in an arbitrary element $f(z) \in \mathcal{F}$ of the form

$$f(z) = \prod_{\nu=0}^{\infty} R(z^{d^\nu})^{t^\nu}, \quad R(z) \in \mathcal{R}. \tag{1}$$

It is seen that $f(z)$ satisfies the following Mahler functional equation

$$f(z) = R(z) f(z^d)^t. \tag{2}$$

Conversely, the functional equation (2) has a unique solution in $\mathcal{F}$ which is given by the infinite product (1).

There is a way to see when an element $f(z)$ given by (1) belongs to $\mathcal{R}$. That is, if $f(z) \in \mathcal{R}$, then $R(z) = f(z)/f(z^d)^t$. Conversely, if $R(z) = h(z)/h(z^d)^t$ with $h(z) \in \mathcal{R}$, then $f(z) = h(z) \in \mathcal{R}$. Thus, defining a subgroup $\mathcal{H}_t$ of $\mathcal{R}$ by

$$\mathcal{H}_t := \{h(z)/h(z^d)^t \mid h(z) \in \mathcal{R}\},$$

we have shown that $f(z)$ given by (1) belongs to $\mathcal{R}$ if and only if $R(z)$ belongs to $\mathcal{H}_t$. This together with a result by Keiji Nishioka [10] (see also [11, Theorem 1.3]) gives a transcendence criterion such as

$$f(z) \notin \overline{K(z)} \quad \Longleftrightarrow \quad R(z) \notin \mathcal{H}_t. \tag{3}$$

The purpose of this paper is, as an extension of this transcendence criterion, to give an algebraic independence criterion for infinite products of the form (1). To state the main result let us take, for $j = 1, ..., m$,

$$f_j(z) = \prod_{\nu=0}^{\infty} R_j(z^{d^\nu})^{t^\nu}, \quad R_j(z) \in \mathcal{R}. \tag{4}$$

Then we have the following result, where the statement for $t = 1$ is contained in a result proved by Kubota [9, Corollary 8] (see also [11, Theorem 3.5]).

Theorem 1. *Let $f_1(z), ..., f_m(z)$ be elements of $\mathcal{F}$ given by (4). Then they are algebraically independent over $K(z)$ if and only if $R_1(z), ..., R_m(z)$ are multiplicatively independent modulo $\mathcal{H}_t$, that is, $R_1(z)^{i_1} \cdots R_m(z)^{i_m} \notin \mathcal{H}_t$ for any nonzero tuple $(i_1, ..., i_m) \in \mathbb{Z}^m$.*

Let us take $K = \mathbb{C}$, and denote by $\mathcal{P}$ the set of infinite products of the form given by (1). Under this setting we next state Theorem 2 and its corollary, generalizing [4, Theorem 1.11], which enable us to give explicit examples of algebraically independent elements of $\mathcal{P}$. To this end, for $R(z) \in \mathcal{R}$, we denote by $\mathcal{S}(R)$ the set of zeros or poles of $R(z)$ whose absolute values are not equal to 1. Then we denote by $\mathcal{P}_0$ the subset of $\mathcal{P}$ consisting $f(z)$ given by (1) such that $\mathcal{S}(R)$ is empty.

Theorem 2. *Let $f_j(z)$, $j = 1, ..., m$, be elements of $\mathcal{P} \backslash \mathcal{P}_0$ given by (4) such that $\mathcal{S}(R_j)$ are disjoint. Assume that, for any $\alpha \in \cup \mathcal{S}(R_j)$, there exist infinitely many positive integers n for each of which not all solutions of $z^{d^n} = \alpha$ are zeros of the polynomial*

$$\prod_{\beta \in \cup \mathcal{S}(R_j)} \prod_{\nu=0}^{n-1} (z^{d^\nu} - \beta).$$

Let $g_\ell(z)$, $\ell = 1, ..., s$, be elements of $\mathcal{P}_0$ which are algebraically independent over $\mathbb{C}(z)$. Then the $m+s$ infinite products $f_j(z)$ and $g_\ell(z)$ are algebraically independent over $\mathbb{C}(z)$.

To give a corollary to Theorem 2 we introduce infinite products of cyclotomic polynomials, generalizing those given in [3, 4]. For any positive integer ℓ we denote by $\Phi_\ell(z)$ the ℓth cyclotomic polynomial, where for $\ell = 1$ we denote $\Phi_1(z) := 1 - z \in \mathcal{R}$. Then we define, for any positive integers k and ℓ,

$$g_{k,\ell}(z) = \prod_{\nu=0}^{\infty} \Phi_\ell(z^{kd^\nu})^{t^\nu} \in \mathcal{P}_0. \tag{5}$$

Corollary. *Let $f_j(z)$, $j = 1, ..., m$, be elements of $\mathcal{P} \backslash \mathcal{P}_0$ given by (4) such that $\mathcal{S}(R_j)$ are disjoint and subsets of $\mathbb{R}$. In the case where $d = 2$, assume further that if $\alpha \in \cup \mathcal{S}(R_j)$, then $-\alpha \notin \cup \mathcal{S}(R_j)$. Let $g_{k,\ell}(z)$ with $k \in \mathcal{K}$, $\ell \in \mathcal{L}$ be elements of $\mathcal{P}_0$ given by (5), where $\mathcal{K}$ and $\mathcal{L}$ are finite sets of positive integers relatively prime to d such that $k\ell$ are all distinct for $k \in \mathcal{K}$, $\ell \in \mathcal{L}$. Then the infinite products $f_j(z)$ and*

$g_{k,\ell}(z)$ *are algebraically independent over* $\mathbb{C}(z)$.

The result [4, Theorem 1.11] is a special case of a combination of Theorem 2 and its corollary. More precisely, it is Theorem 2 under the restrictions that $t = 1$, $R_j(z) \in \mathcal{F} \cap K[z]$, and $g_\ell(z) = g_{1,\ell}(z)$ with $\ell \in \mathcal{L}$, a finite set of positive integers relatively prime to d.

The plan of this paper is as follows. In Section 2 we will state and prove Theorem 3, a quantitative version of Theorem 1. The proof of Theorem 3 is carried out by using an inductive method originated by Duverney [5], which has been further developped by [6, 12, 1, 2]. In particular, it closely follows the argument used in the proof of [2, Theorem 3]. In Section 3 we prove Theorem 2 and its corollary. Finally, in Section 4, we briefly mention certain arithmetical results related to the values of $g_{k,\ell}(z)$, which follow from Greuel's result [7, Theorem 1].

2 A quantitative version of Theorem 1

In this section we prove Theorem 3 below, a quantitative version of Theorem 1, from which Theorem 1 directly follows. In what follows, for $g_1(z), ..., g_m(z)$ in $\mathcal{F}$ and for $I = (i_1, ..., i_m) \in \mathbb{Z}^m$, we denote

$$\underline{g}(z) = (g_1(z), ..., g_m(z)) \in \mathcal{F}^m \quad \text{and} \quad \underline{g}^I(z) = g_1(z)^{i_1} \cdots g_m(z)^{i_m} \in \mathcal{F}.$$

For $g(z) \in K[[z]]\backslash\{0\}$ we denote by $\operatorname{ord} g$ the zero order of $g(z)$ at $z = 0$, and for $g(z) = 0$ we denote $\operatorname{ord} g = \infty$.

Theorem 3. *Let $f_1(z), ..., f_m(z)$ be elements of $\mathcal{F}$ given by (4), and assume that $R_1(z), ..., R_m(z)$ are multiplicatively independent modulo $\mathcal{H}_t$. Then, for any finite subset Λ of $\mathbb{Z}^m \setminus \{\underline{0}\}$, there exist positive constants $c(\Lambda)$ and M_0 depending on $\underline{f}(z)$ and Λ such that*

$$\operatorname{ord}\left(A_{\underline{0}}(z) + \sum_{I \in \Lambda} A_I(z)\underline{f}^I(z)\right) \leq c(\Lambda)M^{\kappa^{|\Lambda|-1}}, \quad \kappa = 1 + \frac{\log t}{\log d},$$

holds for all $A_I(z) \in K[z], I \in \Lambda \cup \{\underline{0}\}$, not all zero, of degrees not greater than M provided that $M \geq M_0$.

We shall prove the assertion by using induction on $l = |\Lambda|$. Under the assumption on $R_j(z)$, as is mentioned in the introduction, $\underline{f}^I(z)$ is irrational. Hence the assertion of the theorem for $l = 1$ is a consequence of the following result.

Lemma 2.1. *Let $f(z)$ be an infinite product given by (1) having $R(z) = E(z)/F(z)$ with $E(z)$, $F(z) \in \mathcal{F} \cap K[z]$. Assume that $f(z)$ is irrational. Let $Q(z)$ and $P(z)$ be polynomials in $K[z]$, not both zero, of degrees at most N, and set*

$$G(z) = Q(z)f(z) - P(z).$$

Then, denoting by L the maximum of the degree of $E(z)$ and that of $F(z)$, we have

$$\operatorname{ord} G(z) \leq (dt+1)N + L.$$

Proof. We may assume that $\tau := \operatorname{ord} G(z) > N$. Then $\operatorname{ord} Q(z) = \operatorname{ord} P(z)$, and we may assume further that this quantity is zero because the assertion for the general case follows from this particular case. Since $(Q(z^d)f(z^d))^t = (P(z^d) + G(z^d))^t$, we have by the functional equation (2) that

$$F(z)Q(z^d)^t f(z) = E(z)(P(z^d) + G(z^d))^t.$$

Hence we obtain

$$\widehat{G}(z) = \widehat{Q}(z)f(z) - \widehat{P}(z),$$

where

$$\widehat{Q}(z) = F(z)Q(z^d)^t, \quad \widehat{P}(z) = E(z)P(z^d)^t,$$

$$\widehat{G}(z) = E(z)\sum_{i=1}^{t}\binom{t}{i}P(z^d)^{t-i}G(z^d)^i.$$

Let us set

$$\Delta(z) = \begin{vmatrix} Q(z) & -P(z) \\ \widehat{Q}(z) & -\widehat{P}(z) \end{vmatrix} = \begin{vmatrix} Q(z) & G(z) \\ \widehat{Q}(z) & \widehat{G}(z) \end{vmatrix}.$$

Since $\operatorname{ord} Q(z)\widehat{G}(z) = d\tau$ and $\operatorname{ord} \widehat{Q}(z)G(z) = \tau$, we have $\operatorname{ord} \Delta(z) = \tau$, and hence

$$\operatorname{ord} \Delta(z) \leq \deg \Delta(z) \leq (dt+1)N + L.$$

The lemma is proved.

For $\underline{R}(z) = (R_1(z), ..., R_m(z))$ and for $I = (i_1, ..., i_m) \in \mathbb{Z}^m$, we denote

$$\underline{R}^I(z) = R_1(z)^{i_1} \cdots R_m(z)^{i_m} = \frac{E_I(z)}{F_I(z)}, \quad E_I(z),\ F_I(z) \in \mathcal{F} \cap K[z]. \tag{6}$$

Lemma 2.2. *Let the notations and the assumptions be as given in Theorem 3. Let l be the number of the elements of Λ, and $L(\Lambda)$ be a positive integer such that for all $I \in \Lambda$ the degrees of $E_I(z)$ and $F_I(z)$ are at most $L(\Lambda)$. Then, for every positive integer n, there exist nonzero polynomials $Q(n; z)$, $P_I(n; z)$, $I \in \Lambda$, in $K[z]$ of degrees at most $l^2L(\Lambda)t^n$ such that, by denoting*

$$G_I(n; z) = Q(n; z)\underline{f}^I(z)^{t^n} - P_I(n; z), \tag{7}$$

we have

$$(l^2 + l)L(\Lambda)t^n + 1 \leq \operatorname{ord} G_I(n; z) \leq (dt+1)l^2L(\Lambda)t^n + L(\Lambda)t^n. \tag{8}$$

Proof. We take $Q(n; z)$ such that, in the power series expansion of $Q(n; z)\underline{f}^I(z)^{t^n}$, $I \in \Lambda$, the coefficients of z^ν with $\nu = l^2L(\Lambda)t^n + 1, ..., (l^2 + l)L(\Lambda)t^n$ vanish. To this end we consider a system of $l^2L(\Lambda)t^n$ linear homogeneous equations with coefficients from K having $l^2L(\Lambda)t^n+1$ unknowns, which has a nontrivial solution in K. Then we denote by $P_I(n;z)$, $I\in\Lambda$, the first $l^2L(\Lambda)t^n+1$ terms of $Q(n;z)\underline{f}^I(z)^{t^n}$, which clearly give the lower estimate in (8). The upper estimate in (8) is a consequence of Lemma 2.1. The lemma is proved.

Under the setting given in (6), for every positive integer n, we denote

$$E_I(n;z)=\prod_{k=0}^{n-1}E_I(z^{d^k})^{t^k},\quad F_I(n;z)=\prod_{k=0}^{n-1}F_I(z^{d^k})^{t^k}.$$

Then, by the functional equation (2) for each $f=f_j$ with $R=R_j$, we obtain

$$\underline{f}^I(z)=R_I(n;z)\underline{f}^I(z^{d^n})^{t^n},\quad R_I(n;z)=\frac{E_I(n;z)}{F_I(n;z)}. \tag{9}$$

For the inductive procedure given below it is crucial that $R_I(n;z)$ approximates $\underline{f}^I(z)$ very well in the sense

$$\underline{f}^I(z)-R_I(n;z)\in z^{d^n}K[[z]]. \tag{10}$$

Completion of the proof of Theorem 3. Assume that $l\geq 2$. Then, under the induction hypothesis that the assertion holds for any subset of $\mathbb{Z}^m\setminus\{\underline{0}\}$ with elements less than l, we shall prove the assertion for an arbitrarily given subset Λ of $\mathbb{Z}^m\setminus\{\underline{0}\}$ with l elements. Let us denote

$$\Omega(z)=A_{\underline{0}}(z)+\sum_{I\in\Lambda}A_I(z)\underline{f}^I(z).$$

To estimate $\operatorname{ord}\Omega(z)$ from above we take $Q(n;z)$ and $P_I(n;z)$, $G_I(n;z)$, $I\in\Lambda$, given in Lemma 2.2, where we will fix n later. Replacing z by z^{d^n} in (7) and multiplying $R_I(n;z)$ in both sides, we deduce from (9) that

$$Q(n;z^{d^n})\underline{f}^I(z)-R_I(n;z)P_I(n;z^{d^n})=R_I(n;z)G_I(n;z^{d^n}).$$

We then obtain an equality

$$Q(n;z^{d^n})\Omega(z)-P(n;z)=G(n;z), \tag{11}$$

where

$$P(n;z)=Q(n;z^{d^n})A_{\underline{0}}(z)+\sum_{I\in\Lambda}A_I(z)R_I(n;z)P_I(n;z^{d^n}),$$

$$G(n;z)=\sum_{I\in\Lambda}A_I(z)R_I(n;z)G_I(n;z^{d^n}).$$

Let $\tau(n)$ be the minimum of $\operatorname{ord}G_I(n;z)$ for $I\in\Lambda$, and take a $J\in\Lambda$ such that $\tau(n)=\operatorname{ord}G_J(n;z)$. Denoting by $g_I(n)$, $I\in\Lambda$, the coefficient of $z^{\tau(n)}$ in $G_I(n;z)$, we deduce from (10) an important relation

$$G(n;z)-z^{\tau(n)d^n}\sum_{I\in\Lambda}g_I(n)A_I(z)\underline{f}^I(z)\in z^{(\tau(n)+1)d^n}K[[z]]. \tag{12}$$

Let us set

$$\widehat{\Omega}(z) = g_J(n)A_J(z) + \sum_{I\in\Lambda\setminus\{J\}} g_I(n)A_I(z)\underline{f}^{I-J}(z),$$

for which we have

$$\sum_{I\in\Lambda} g_I(n)A_I(z)\underline{f}^I(z) = \underline{f}^J(z)\widehat{\Omega}(z).$$

It follows from the induction hypothesis that

$$\operatorname{ord}\widehat{\Omega}(z) \le c(\widehat{\Lambda})M^{\kappa^{l-2}}, \quad \widehat{\Lambda} = \{I - J : I \in \Lambda \setminus \{J\}\},$$

where $c(\widehat{\Lambda}) \ge 1$ is a constant depending on $\underline{f}(z)$ and $\widehat{\Lambda}$. We now fix n so that

$$d^{n-1} \le c(\widehat{\Lambda})M^{\kappa^{l-2}} < d^n, \tag{13}$$

which together with (12) allows us to have $G(n;z) \ne 0$, and hence

$$\operatorname{ord} G(n;z) \le (\tau(n) + 1)d^n. \tag{14}$$

Thus, if $P(n;z) = 0$, then $\operatorname{ord}\Omega(z) \le \operatorname{ord} G(n;z)$ by (11). Since $dt = d^\kappa$ by the definition, combining (13), (14) and an upper bound of $\tau(n)$ given in (8), we obtain the desired assertion. On the other hand, if $P(n;z) \ne 0$, then, on noting that

$$P(n;z)\prod_{I\in\Lambda} F_I(n;z)$$

is a polynomial in $K[z]$ of degree at most $(l^2 + l)L(\Lambda)(dt)^n + M$, we obtain

$$\operatorname{ord} P(n;z) \le (l^2 + l)L(\Lambda)(dt)^n + M.$$

Since we deduce from a lower bound of $\tau(n)$ given in (8) that

$$\operatorname{ord} G(n;z) \ge (l^2 + l)L(\Lambda)(dt)^n + d^n,$$

comparing these bounds, we have $\operatorname{ord} P(n;z) < \operatorname{ord} G(n;z)$ under (13), and therefore again $\operatorname{ord}\Omega(z) \le \operatorname{ord} G(n;z)$. Thus the assertion holds also in the case where $P(n;z) \ne 0$. This completes the proof of the theorem.

3 Proof of Theorem 2 and its corollary

We first prove Theorem 2. To this end we need the following lemma, which is essentially [4, Lemma 3.2] when $t = 1$ and gives a sufficient condition by which $R(z) \in \mathcal{R}$ does not belong to $\mathcal{H}_t$.

Lemma 3.1. *Let $R(z)$ be an element of $\mathcal{R}$ such that $\mathcal{S}(R)$ is nonempty , where we denote by ω (resp. Ω), if there exists, an element of $\mathcal{S}(R)$ of minimal (resp. maximal) absolute value less than 1 (resp. greater than 1). Assume that there exist* *infinitely many positive integers n for each of which not all solutions of $z^{d^n} = \omega$ (or of $z^{d^n} = \Omega$) are zeros or poles of the product*

$$R^{\langle n \rangle}(z) := R(z)R(z^d)^t \cdots R(z^{d^{n-1}})^{t^{n-1}}.$$

Then $R(z)$ does not belong to $\mathcal{H}_t$.

Proof. We prove the assertion for the case where ω exists and satisfies the condition given in the lemma. The proof of the other case is similar. Assume, on the contrary, that $R(z) = h(z)/h(z^d)^t$ with some $h(z) \in \mathcal{R}$. Note first that $h(z)$ does not have any zero or pole whose absolute value is less than $|\omega|$. In fact, if $\hat{\omega}$ is such a zero or a pole of minimal absolute value, then $\hat{\omega} \in \mathcal{S}(R)$ because $\hat{\omega}^d \notin \mathcal{S}(h)$, which is a contradiction. Hence $\omega \in \mathcal{S}(h)$ because $\omega \in \mathcal{S}(R)$ and $\omega^d \notin \mathcal{S}(h)$.

Let n be a positive integer for which a d^n th root w^{1/d^n} of ω does not belong to $\mathcal{S}(R^{\langle n \rangle})$. Since we have $R^{\langle n \rangle}(z) = h(z)/h(z^{d^n})^{t^n}$, and since $\omega \in \mathcal{S}(h)$, we should have $\omega^{1/d^n} \in \mathcal{S}(h)$. Hence, under the assumption of the lemma, $h(z)$ has infinitely many zeros or poles, contradicting the rationality of $h(z)$. Thus the lemma is proved.

Proof of Theorem 2. Assume on the contrary that $f_j(z)$ and $g_\ell(z)$ are algebraically dependent over $\mathbb{C}(z)$. Denoting

$$g_\ell(z) = \prod_{\nu=0}^{\infty} S_\ell(z^{d^\nu})^{t^\nu}, \quad S_\ell(z) \in \mathcal{R},$$

we deduce from Theorem 1 that there exists a nontrivial monomial of $R_j(z)$ and $S_\ell(z)$ which belongs to $\mathcal{H}_t$. If some $R_j(z)$ appears in the monomial, then under the assumptions of Theorem 1, it follows from Lemma 3.1 that this monomial does not belong to $\mathcal{H}_t$, which is a contradiction. Therefore, none of $R_j(z)$ appears in this monomial. However, this is again a contradiction by Theorem 1 because $g_\ell(z)$ are assumed to be algebraically independent over $\mathbb{C}(z)$. Thus the theorem is proved.

We next prove the corollary to Theorem 2 by using the following two results. The statement of the first one with its proof is contained in the proof of [4, Corollary 1.3]. For the convenience of the reader we recall a short proof below.

Lemma 3.2. *Let $\{\alpha_1, ..., \alpha_s\}$ be a set of real numbers, and α be its element. For any positive integer n we define $\mathcal{T}_n := \{\beta \in \mathbb{C} \,|\, \beta^{d^\mu} = \alpha_\nu \ (0 \le \mu < n,\ 1 \le \nu \le s)\}$. If $d \ge 3$, or, if $d = 2$ and $-\alpha_\nu \notin \{\alpha_1, ..., \alpha_s\}$ for each ν, there exists a d^n th root of α which does not belong to $\mathcal{T}_n$.*

Proof. Asume first that $d \ge 3$. Then a d^n th root of α with an argument $2\pi/d^n$ or π/d^n corressponding to $\alpha > 0$ or $\alpha < 0$, respectively, does not belong to $\mathcal{T}_n$.

Assume next that $d = 2$ and $-\alpha_\nu \notin \{\alpha_1, ..., \alpha_s\}$ for each ν. By the latter assumption the absolute values of α_ν are all distinct. Since all 2^n th roots of α have the same absolute values, for each μ with $1 \le \mu < n$ there exists at most one ν with $1 \le \nu \le s$ such that the equation $z^{2^\mu} = \alpha_\nu$ have 2^n th roots of α as solutions. Therefore, there exist at most $1 + 2 + 2^2 + \cdots + 2^{n-1}$ $(= 2^n - 1)$ elements of

$\mathcal{T}_n$ which are 2^n th roots of α. Thus the assertion of the lemma also holds in this case.

Lemma 3.3. *Let $\mathcal{K}$ and $\mathcal{L}$ be finite sets of positive integers relatively prime to d such that $k\ell$ are all distinct for $k \in \mathcal{K}$, $\ell \in \mathcal{L}$. Then the polynomials $\Phi_\ell(z^k)$ for $k \in \mathcal{K}$, $\ell \in \mathcal{L}$ are multiplicatively independent modulo $\mathcal{H}_t$.*

Proof. Assume, on the contrary, that there exists a tuple $I = (i(k,\ell))_{k\in\mathcal{K},\ell\in\mathcal{L}}$ of integers $i(k,\ell)$, not all zero, such that

$$R(z) := \prod_{k\in\mathcal{K}} \prod_{\ell\in\mathcal{L}} \Phi_\ell(z^k)^{i(k,\ell)} = \frac{h(z)}{h(z^d)^t} \in \mathcal{H}_t, \quad h(z) \in \mathcal{R}.$$

Denoting $h(z) = a(z)/b(z)$ with relatively prime polynomials $a(z)$ and $b(z)$, we have

$$\prod_{k\in\mathcal{K}} \prod_{\ell\in\mathcal{L}} \Phi_\ell(z^k)^{i(k,\ell)} = \frac{a(z)}{b(z)} \frac{b(z^d)^t}{a(z^d)^t}.$$

To reach a contradiction we compare the left-hand side and the right-hand side each of which expresses $R(z)$. Let $i(k,\ell)$ be the component of I such that $k\ell$ is maximal among the all nonzero components of I. We see from the left-hand side that a primitive $k\ell$ th root ζ of unity is a zero or a pole of $R(z)$, where we may assume that it is a zero. Then we see from the right-hand side that $a(\zeta) = 0$ or $b(\zeta^d) = 0$.

Assume first that $a(\zeta) = 0$. Then $\zeta^{1/d}$ is a zero of $a(z^d)$. Since $\zeta^{1/d}$ is a primitive $dk\ell$ th root of unity, and since the left-hand side does not have any primitive $dk\ell$ th root of unit as a pole, we have $a(\zeta^{1/d}) = 0$. Inductively, we have $a(\zeta^{1/d^n}) = 0$ for all $n \geq 0$, which is impossible.

Assume next that $b(\zeta^d) = 0$. Then, for a primitive d th root of unity η, $\eta\zeta$ is a zero of $b(z^d)$. Note that $\eta\zeta$ is a primitive $dk\ell$ th root of unity because $k\ell$ is relatively prime to d. Since the right-hand side does not have any primitive $dk\ell$ th root of unity as a zero, we have $b(\eta\zeta) = 0$. Hence a primitive $d^2k\ell$ th root of unity $(\eta\zeta)^{1/d}$ is a zero of $b(z^d)$, which implies that it is also a zero of $b(z)$ by the same reason as is given above. Inductively, we have $b((\eta\zeta)^{1/d^n}) = 0$ for all $n \geq 0$, which is impossible. Thus the lemma is proved.

Remark. Assume that $d = 2$. Then the assumption given in Lemma 3.3 does not hold for $\mathcal{K} = \{1\}$ and $\mathcal{L} = \{1, 2\}$. In this case, for any positive integer t, we have

$$\Phi_1(z)^{t-1}\Phi_2(z)^t = \frac{h(z)}{h(z^2)^t} \in \mathcal{H}_t, \quad h(z) = \frac{1}{1-z},$$

and $g_{1,1}(z)^{t-1}g_{1,2}(z)^t = h(z)$. It would be an interestinf question to know a necessary and sufficient condition on $\mathcal{K}$ and $\mathcal{L}$ such that $\Phi_\ell(z^k)$ with $k \in \mathcal{K}$, $\ell \in \mathcal{L}$ are multiplicatively independent modulo $\mathcal{H}_t$.

Proof of the corollary. Under the assumption on $\cup\mathcal{S}(R_j)$ given in the corollary it follows from Lemma 3.2 that the assumption on $\cup\mathcal{S}(R_j)$ given in Theorem 2 is fulfilled. Also it follows from Lemma 3.3 that $g_{k,\ell}(z)$ are algebraically independent over $\mathbb{C}(z)$. Hence the corollary is deduced from Theorem 2.

4 Remarks on arithmetical results

In [7] Greuel proved a general result on algebraic independence of values of Mahler functions satisfying certain implicit functional equations. We here briefly mention arithmetical results on the functions $g_{k,\ell}(z)$ given by (5), which follow from Greuel's general result. For the purpose we denote $g_{k,\ell}(z)$ by $g_{k,\ell,t}(z)$ to show the parameter t explicitly.

We first state Greuel's result [7, Corollary 3]) : Let us take $g_{1,1,t_j}(z)$, $j = 1, ..., m$, where t_j are distinct positive integers. Then these functions are algebraically independent over $\mathbb{C}(z)$ (see also [8] for more general result on algebraic independence of Mahler functions), and hence it follows from [7, Theorem 1] that, for any algebraic number α with $0 < |\alpha| < 1$, the values $g_{1,1,t_1}(\alpha), ..., g_{1,1,t_m}(\alpha)$ are algebraically independent over the rationals provided that

$$\log d > (2m^2 - 1 + (4m^4 - 2m^2 + m)^{1/2}) \log(t_1 \cdots t_m). \tag{15}$$

We next state our result : Let us take $g_{k,\ell,t}(z)$ with $k \in \mathcal{K}$, $\ell \in \mathcal{L}$, where $\mathcal{K}$ and $\mathcal{L}$ are those given in the corollary to Theorem 2. Then these functions are algebraically independent over $\mathbb{C}(z)$ (by the corollary to Theorem 2), and hence it follows from [7, Theorem 1] that, for any algebraic number α with $0 < |\alpha| < 1$, the values $g_{k,\ell,t}(\alpha)$ with $k \in \mathcal{K}$, $\ell \in \mathcal{L}$ are algebraically independent over the rationals provided that (15) holds with $|\mathcal{K}||\mathcal{L}|$ and $|\mathcal{K}||\mathcal{L}| \log t$ on the right-hand side instead of m and $\log(t_1 \cdots t_m)$, respectively.

Note that the arithmetical part of this result for $t = 1$, where the condition on d does not appear, follows also from Kubota's result [9, Theorem 3] (see also [11, Theorem 3.6.1]. It would be interesting to consider common generalization of the above results under a weaker condition than the condition like (15).

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