Lecture Notes in Mathematics

Edited by J.-M. Morel, F. Takens and B. Teissier

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- a subject index: as a rule this is genuinely helpful for the reader.

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Preface

This book grew out of my attempt in August 1998 to compare Carleson's and Fefferman's proofs of the pointwise convergence of Fourier series with Lacey and Thiele's proof of the boundedness of the bilinear Hilbert transform. I started with Carleson's paper and soon realized that my summer vacation would not suffice to understand Carleson's proof.

Bit by bit I began to understand it. I was impressed by the breathtaking proof and started to give a detailed exposition that could be understandable by someone who, like me, was not a specialist in harmonic analysis. I've been working on this project for almost two years and lectured on it at the University of Seville from February to June 2000. Thus, this book is meant for graduate students who want to understand one of the great achievements of the twentieth century.

This is the first exposition of Carleson's theorem about the convergence of Fourier series in book form. It differs from the previous lecture notes, one by Mozzochi [38], and the other by Jørsboe and Mejlbro [26], in that our exposition points out the motivation of every step in the proof. Since its publication in 1966, the theorem has acquired a reputation of being an isolated result, very technical, and not profitable to study. There have also been many attempts to obtain the results by simpler methods. To this day it is the proof that gives the finest results about the maximal operator of Fourier series.

The Carleson analysis of the function, one of the fundamental steps of the proof, has an interesting musical interpretation. A sound wave consists of a periodic variation of pressure occurring around the equilibrium pressure prevailing at a particular time and place. The sound signal f is the variation of the pressure as a function of time. The Carleson analysis gives the score of a musical composition given the sound signal f. The Carleson analysis can be carried out at different levels. Obviously the above assertion is true only if we consider an adequate level.

Carleson's proof has something that reminds me of living organisms. The proof is based on many choices that seem arbitrary. This happens also in living organisms. An example is the **error** in the design of the eyes of the vertebrates. The photoreceptors are situated in the retina, but their outputs emerge on the **wrong** side: inside the eyes. Therefore the axons must finally be packed in the optic nerve that exit the eyes by the so called **blind spot**. But so many fibers (125 million light-sensitive cells) will not pass by a small spot. Hence evolution has solved the problem packing another layer of neurons inside the eyes that have rich interconections with the photoreceptors and with each other. These neurons process the information before it is send to the brain, hence the number of axons that must leave the eye is sustantially reduced (one million axons in each optic nerve). The incoming light must traverse these neurons to reach the photoreceptors, hence evolution has the added problem of making them transparent.

We have tried to arrange the proof so that these things do not happen, so that these arbitrary selections do not shade the idea of the proof. We have had the advantage of the text processor TEX, which has allowed us to rewrite without much pain. (We hope that no signs of these rewritings remain).

By the way, the eyes and the ears process the information in totally different ways. The proof of Carleson follows more the ear than the eyes. But what these neurons are doing in the inside of the eyes is just to solve the problem: How must I compress the information to send images using the least possible number of bits? A problem for which the wavelets are being used today.

I would like this book to be a commentary to the Carleson paper. Therefore we give the Carleson-Hunt theorem following more Carleson's than Hunt's paper.

The chapter on the maximal operator of Fourier series S^*f, gives the first exposition of the consequences of the Carleson-Hunt theorem. Some of the results appear here for the first time.

I wish to express my thanks to Fernando Soria and to N. Yu Antonov for sending me their papers and their comments about the consequences of the Carleson-Hunt theorem. Also to some members of the department of Mathematical Analysis of the University of Seville, especially to Luis Rodríguez-Piazza who showed me the example contained in chapter XIII.

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Part II. The Carleson-Hunt Theorem

Introduction

The origin of Fourier series is the 18th century study by Euler and Daniel Bernoulli of the vibrating string. Bernoulli took the point of view, suggested by physical considerations, that every function can be expanded in a trigonometric series. At this time the prevalent idea was that such an expression implied differentiability properties, and that such an expansion was not possible in general.

Such a question was not one for that time. A response depended on what is understood by a function, a concept that was not clear until the 20th century. The first positive results were given in 1829 by Dirichlet, who proved that the expansion is valid for every continuous function with a finite number of maxima and minima.

A great portion of the mathematics of the first part of the 20th century was motivated by the convergence of Fourier series. For example, Cantor's set theory has its origin in the study of this convergence. Also Lebesgue's measure theory owes its success to its application to Fourier series.

Luzin, in 1913, while considering the properties of Hilbert's transform, conjectured that every function in $\mathcal{L}^2[-\pi,\pi]$ has an a. e. convergent Fourier series. Kolmogorov, in 1923 gave an example of a function in $\mathcal{L}^1[-\pi,\pi]$ with an a. e. divergent Fourier series.

A. P. Calderon (in 1959) proved that if the Fourier series of every function in $\mathcal{L}^2[-\pi,\pi]$ converges a. e., then

$$\mathfrak{m}\{x : \sup_n |S_n f(x)| > y\} \le C\frac{\|f\|_2}{y^2}.$$

For many people the belief in Luzin's conjecture was destroyed; it seemed too good to be true.

So, it was a surprise when Carleson, in 1966, proved Luzin's conjecture. The next year Hunt proved the a. e. convergence of the Fourier series of every $f \in \mathcal{L}^p[-\pi,\pi]$ for $1 < p \le \infty$.

Kolmogorov's example is in fact a function in $L\log\log L$ with a. e. divergent Fourier series. Hunt proved that every function in $L(\log L)^2$ has an a. e. convergent Fourier series. Sjölin, in 1969, sharpened this result: every function in the space $L\log L\log\log L$ has an a. e. convergent Fourier series. The last result in this direction is that of Antonov (in 1996) who proved the same for functions in $L\log L(\log\log\log L)$. Also, there are some quasi-Banach spaces of functions with a. e. convergent Fourier series given by Soria in 1985.

In the other direction, the results of Kolmogorov were sharpened by Chen in 1969 giving functions in $L(\log\log L)^{1-\varepsilon}$ with a. e. divergent Fourier series. Recently, Konyagin (1999) has obtained the same result for the space $L\varphi(L)$, whenever φ satisfies $\varphi(t) = o(\sqrt{(\log t/\log\log t)})$.

Apart from the proof of Carleson, there have been two others. First, the one by Fefferman in 1973. He says: *However, our proof is very inefficient near* L^1*. Carleson's construction can be pushed down as far as* $L\log L(\log\log L)$*, but our proof seems unavoidably restricted to* $L(\log L)^M$ *for some large* M*.* Then we have the recent proof of Lacey and Thiele; they are not so explicit as Fefferman, but what they prove (as far as I know) is limited to the case $p = 2$.

The proof of Lacey and Thiele is based on ideas from Fefferman proof, also the proof of Fefferman has been very important since it has inspired these two authors in his magnificent proof of the boundedness of the bilinear Hilbert transform. The trees and forests that appears in these proofs has some resemblance to the notes of the function and the allowed pairs of the proof of Carleson that are introduced in chapter eight and nine of this book, but to understand better these relationships will be matter for other book.

The aim of this book is the exposition of the principal result about the convergence of Fourier series, that is, the Carleson-Hunt Theorem.

The book has three parts. The first part gives a review of some results needed in the proof and consists of three chapters.

In the first chapter we give a review of the Hardy-Littlewod maximal function. We prove that this operator transforms $\mathcal{L}^p$ into $\mathcal{L}^p$ for $1 < p \le \infty$. The differentiation theorem allows one to see the great success we get with a pointwise convergence problem by applying the idea of the maximal function. This makes it reasonable to consider the maximal operator $S^*f(x) = \sup_n |S_n(f,x)|$, in the problem of convergence of Fourier series.

In chapter two we give elementary results about Fourier series. We see the relevance of the conjugate function and explain the elements on which Luzin, in 1913, founded his conjecture about the convergence of Fourier series of $\mathcal{L}^2$ functions. We also present Dini's and Jordan's tests of convergence in conformity with the law: s/he who does not know these criteria must not read Carleson's proof.

The properties of Hilbert's transform, needed in the proof of Carleson's Theorem are treated in chapter three.

In the second part we give the exposition of the Carleson-Hunt Theorem. The basic idea of the proof is the following. Our aim is to bound what we call Carleson integrals

$$\text{p.v.}\int_{-\pi}^{\pi} \frac{e^{in(x-t)}}{x-t} f(t)\,dt.$$

To this end we consider a partition Π of the interval $[-\pi,\pi]$ into subintervals, one of them $I(x)$ containing the point x, and write the integral as

$$\text{p.v.}\int_{-\pi}^{\pi}\frac{e^{in(x-t)}}{x-t}f(t)\,dt = \text{p.v.}\int_{I(x)}\frac{e^{in(x-t)}}{x-t}f(t)\,dt \\ + \sum_{J\in\Pi, J\neq I(x)}\int_J \frac{e^{in(x-t)}f(t)-M_J}{x-t}\,dt + \int_J \frac{M_J}{x-t}\,dt.$$

where M_J is the mean value of $e^{in(x-t)}f(t)$ on the interval J.

The last sum can be conveniently bounded so that, in fact, we have changed the problem of bounding the first integral to the analogous problem for the integral on $I(x)$. After a change of scale we see that we have a similar integral, but the number of cycles in the exponent n has decreased in number.

Therefore we can repeat the reasoning. With this procedure we obtain the theorem that $S_n(f,x) = o(\log\log n)$ a. e., for every $f \in \mathcal{L}^2[-\pi,\pi]$. We think that to understand the proof of Carleson's theorem it is important to start with this theorem, because this is how the proof was generated. Only after we have understood this proof can we understand the very clever modifications that Carleson devised to obtain his theorem.

The next three chapters are dedicated to this end. The first deals with the actual bound of the second and third terms, and the problem of how we must choose the partition to optimize these bounds.

In Chapter five we prove that the bounds are good, with the exception of sets of controlled measure. Then, in Chapter six, given $f \in \mathcal{L}^2[-\pi,\pi]$, $y>0$, $N\in N$, and $\varepsilon>0$, we define a measurable set E with $\mathfrak{m}(E) < A\varepsilon$ and such that

$$\sup_{0\le n\le N/4}\left|\text{p.v.}\int_{-\pi}^{\pi}\frac{e^{in(x-t)}}{x-t}f(t)\,dt\right| \le C\frac{\|f\|_2}{\sqrt{\varepsilon}}(\log N).$$

From this estimate we obtain the desired conclusion that

$$S_n(f,x) = o(\log\log n) \quad a.\ e.$$

These three chapters follow Carleson's paper, where instead of $f\in\mathcal{L}^2$ he assumed that $|f|(\log^+|f|)^{1+\delta}\in\mathcal{L}^1$, reaching the same conclusion. Since we shall obtain further results, we have taken the simpler hypothesis that $f\in\mathcal{L}^2$. In fact, our motivation to include the proof is to allow the reader to understand the modifications contained in the next five chapters.

The logarithmic term appears in the above proof because every time we apply the basic procedure, we must put apart in the set E a small subset where the bound is not good. We have to put in a term $\log N$ in order to obtain a controlled measure. In fact, we are considering all pairs (n,J) formed by a dyadic subinterval J of $[-\pi,\pi]$, and the number of cycles of the Carleson integral. If we consider the procedure of chapters four to six, then we suspect that we do not need all of these pairs. This is the basic observation on which all the clever reasoning of Carleson is founded.

In chapter seven we determine which pairs are needed. Carleson made an analysis of the function to detect which pairs these are. If we think of f as the sound signal of a piece of music, then this analysis can be seen as a process to derive from f the score of this piece of music. In this chapter we define the set $\mathcal{Q}^j$ of notes of f to the level j.

In chapter eight we define the set $\mathcal{R}^j$ of allowed pairs. This is an enlargement of the set of notes of f, so that we can achieve two objectives. The principal objective is that if $\alpha = (n, J)$ is a pair such that $\alpha \notin \mathcal{R}^j$, then the sounds of the notes of f (at level j) that have a duration containing J, is essentially a single note or a rest. This is very important because if we consider a Carleson integral $\mathcal{C}_\alpha f(x)$ with this pair, then we have a candidate note, the sound of f, that is an allowed pair and therefore can be used in the basic procedure of chapter four.

Chapter nine is the most difficult part of the proof. In it we see how, given an arbitrary Carleson integral $\mathcal{C}_\alpha f(x)$, we can obtain an allowed pair ξ such that we can apply the procedure of chapter four, and a change of frequency to bound this integral.

In chapter ten we apply all this machinery to prove the basic inequality of Theorem 10.2.

The last part of the book is dedicated to deriving some consequences of the proof of Carleson-Hunt.

First, in chapter eleven we prove a version of the Marcinkiewicz interpolation theorem and give the definition and first properties of the spaces that we shall need in chapter twelve. In particular, we study a class of spaces near $\mathcal{L}^1(\mu)$ that play a prominent role. We prove that they are atomic spaces, a fact that allows very neat proofs in the following chapter.

In Chapter twelve we study the maximal operator S^*f of Fourier series. In it we give detailed and explicit versions of Hunt's theorem, with improved constants. We end the chapter by defining two quasi-Banach spaces, Q and QA, of functions with almost everywhere convergent Fourier series. These spaces improve the known results of Sjölin, Soria and Antonov, and the proofs are simpler.

In the last chapter we consider the Fourier transform on $\mathbf{R}$. We consider the problem of when we can obtain the Fourier transform of a function $f \in \mathcal{L}^p(\mathbf{R})$ by the formula

$$\widehat{f}(x) = \lim_{a\to+\infty} \int_{-a}^{a} f(t)e^{-2\pi ixt}\,dt.$$

We prove by an example (Example 13.2) that our results are optimal.