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Cyclic Renormalization and Automorphism Groups of Rooted Trees

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I am the Lorax, I speak for the trees...[‡]

For the ones we love...

[‡]From *The Lorax*, Dr. Seuss

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Internal References.

References within the text are made in a hierarchical fashion. For example, equation (4) in Section 2.6 of Chapter 1 would be referred to as (I, (2.6)(4)) outside of Chapter 1, as (2.6)(4) within Chapter I and as (4) within Section (2.6) of Chapter I. Similarly, Section (2.6) of Chapter I is referred to as (I, (2.6)) outside of Chapter I and (2.6) within Chapter I.

Chapter 0

Introduction

The motivation behind this monograph derives from a relation between renormalizability of certain dynamical systems on the unit interval and group actions on rooted trees. Certain classes of maps of the unit interval, when restricted to invariant Cantor sets, have the form of such automorphisms and better understanding the structure of such automorphism groups contributes to a fuller understanding of the types of maps which have these sorts of restrictions. While a priori these two subjects have quite different audiences, it is our hope that the link we draw between the two may persuade others to investigate similar potential bridges between algebra and dynamics.

Some of renormalization group theory can be traced to connections made between the classes of maps $f(x) = Rx(1-x)$ and $g(x) = S\sin(\pi x)$, for varying parameters R and S. For particular parameter values (at the so-called "accumulation of period doubling" or "boundary of chaos") these maps have infinitely many periodic orbits and the periods of these orbits are all of the form 2^n. It is well known that these two maps share many "smooth characteristics". For example, both maps have invariant Cantor sets of the same Hausdorff dimension; the bifurcation structures of the two parametrized families also have similar metric properties near the two maps $f(x)$ and $g(x)$, properties which are shared by all maps and families in the same so-called universality class.

These characteristics were first observed numerically in the physics community [CT, Fe1, TC1]. It was conjectured that an analogous phenomenon played a role in the transition to chaos in certain experiments in fluid dynamics [CT]. Physicists described this phenomenon in terms of a class of techniques which in the areas of quantum field theory and statistical mechanics has come to be called called renormalization group theory.

Much of this monograph can be viewed as part of the ongoing effort directed towards the mathematical development of renormalization group theory. Recent work (cf. [Su, Mc1, Mc2]) has made great strides in this direction, but many open questions remain.

As stated at the opening, the connection with group theory comes from considering the dynamical systems at the accumulation of period doubling when restricted to their invariant Cantor sets. These Cantor sets are naturally viewed as the ends of a rooted tree with corresponding action given by a particular element of the automorphism group of the tree. The structure of the full automorphism group sheds light on the possible dynamical systems obtained in this fashion and conversely, dynamical considerations indicate possible directions for better understanding the group structure. To say things a bit more precisely, recall that dynamical systems are often studied in terms of periodic structure. The basic periodic systems are $\mathbb{Z}_n := (\mathbb{Z}/n\mathbb{Z}, +1)$, where $+1$ sends r to $r+1$ for $r \in \mathbb{Z}/n\mathbb{Z}$. For a dynamical system (K, f) (K a topological space and $f : K \longrightarrow K$ a continuous map), an orbit of period n is an embedding $\mathbb{Z}_n \longrightarrow (K, f)$. Here we study the dual notion of a (necessarily surjective) morphism $R : (K, f) \longrightarrow \mathbb{Z}_n$, which we call an *n-renormalization* of (K, f). Thus K is the disjoint union of the (open and closed) fibers $K_r = R^{-1}(r)$ and f sends K_r to K_{r+1} for $r \in \mathbb{Z}/n\mathbb{Z}$. Let $Per(K, f)$ (resp., $Ren(K, f)$) denote the set of integers n for which (K, f) admits an orbit of period n (resp. an n-renormalization).

Suppose that (I, f) is a dynamical system on a compact real interval I. The linear order of I influences the above notions as follows. An orbit of period n is ordered, $x_1 < x_2 < \ldots < x_n$, and the action of f defines a permutation σ of the indices; σ belongs to the set C_n of n-cycles in the symmetric group S_n. Let $C_n(f)$ denote the set of n-cycles that so occur, and $C(f)$ the union of the $C_n(f)$, contained in the union C of the C_n. Define a "forcing" relation, $\Rightarrow$, on the natural numbers, and on the set C, as follows: Let $n, m \in \mathbb{N}$, and $\sigma, \tau \in C$. Then $n \Rightarrow m$, (resp., $\sigma \Rightarrow \tau$), means that, for all f as above, $n \in Per(f)$ implies that $m \in Per(f)$, (resp., $\sigma \in C(f)$ implies $\tau \in C(f)$). A remarkable theorem of Sharkowskii says that $\Rightarrow$ is an (explicit) total order on the natural numbers. (See II, (11.2) below.) Thus $Per(f)$ is always a terminal segment for the Sharkowskii order; f has entropy zero iff $Per(f)$ consists of a sequence (finite or infinite) $1, 2, 4, 8, \ldots$ of consecutive powers of 2. On the set C of cycles, $\Rightarrow$ can sometimes go opposite to the Sharkowskii order.

For renormalizations, suppose now that K is a minimal closed invariant subset of I, and that $R : (K, f) \longrightarrow \mathbb{Z}_n$ is an n-renormalization, as above. We call R an *interval n-renormalization* if each of the fibers K_r is an interval of K, and write $IRen(K, f)$ for the set of integers n for which (K, f) admits an interval n-renormalization. A simple, but fundamental, result (I, (2.6)) is that $IRen(K, f)$ is totally ordered by divisibility. This permits us to coherently organize the interval renormalizations of (K, f) into an inverse sequence

$$(*) \ldots \longrightarrow (\mathbb{Z}/n_i\mathbb{Z}, +1) \longrightarrow (\mathbb{Z}/n_{i-1}\mathbb{Z}, +1) \longrightarrow \ldots \longrightarrow (\mathbb{Z}/n_0\mathbb{Z}, +1)$$

where $\{n_0 = 1 < n_1 < n_2 < \ldots\} = IRen(K, f)$. In turn this defines a (surjective) morphism ϕ from (K, f) to the inverse limit $(\widehat{\mathbb{Z}}, +1)$ of (*), sometimes called a "q-adic adding machine." When $IRen(K, f)$ is infinite (f is "infinitely interval renormalizable") then we show that both ϕ and f are injective, except perhaps for countably many 2-point fibers (which actually occur in given examples). Moreover each n in $Ren(K, f)$ divides some m in $IRen(K, f)$. (See I, (4.1).) We also show that every set of natural numbers totally ordered by divisibility can be realized as some $IRen(K,f)$. It can happen that $IRen(K,f)$ is finite even when $Ren(K,f)$ is infinite.

Interval renormalization has some relation to periodic structure. For example $IRen(K,f)$ is contained in $Per(I,f)$; in fact the indicated periodic orbits occur in the convex hull of K in I, and K is contained in the closure of their union. Further, the complement of the union of the periodic orbits in its closure is sometimes the place to find a minimal closed invariant set K as above.

Chapter II interprets interval renormalizations for unimodal maps in terms of a $*$-product on "itineraries", in the sense of Milnor-Thurston. This permits us to invoke theorems about the itinerary behavior of quadratic maps to deduce analogous results about the interval renormalization structure of such maps.

In Chapter III we take the point of view that the inverse sequence (*) can be interpreted as a rooted tree X, which is "spherically homogeneous," on which $+1$ acts as a "spherically transitive" automorphism. We show that, in $G = Aut(X)$, the spherically transitive automorphisms form a single conjugacy class. Given an interval n-renormalization $R : (K,f) \longrightarrow \mathbb{Z}_n$, we obtain new dynamical systems from f^n restricted to the fibers K_r of R. The latter may or may not be interval equivalent to the original (K,f). In Chapter III, Section 6, we study a group theoretic analogue of this problem.

Finally, in Chapter IV we investigate the normal subgroup structure of $G = Aut(X)$, using a description of G as an infinite iterated wreath product of symmetric groups. In the course of this we construct certain abelian characters (multi-signatures in the case of the dyadic tree) in terms of which one can characterize the spherically transitive automorphisms. The kernels of restrictions of G to finite radius balls centered at the root of X define natural normal subgroups of G, which are somewhat analogous to principal congruence subgroups in p-adic algebraic groups. These and certain abelian characters defined on them afford a general description of the normal subgroups of G. This analysis applies as well to certain subgroups of G also constructed as iterated wreath products.

It is natural to ask if the ideas in this paper extend to higher dimensions. It appears that one of the fundamental facts which allow group theory to play a role in the combinatorial discussion of interval maps is that periodic orbits of such maps are naturally described by permutations, as determined by the linear order of the orbit in the real line. Furthermore, continuity of the map implies that the permutation representing a periodic orbit in one dimension yields some information about the map as it provides some information about the way in which the intervals between points are mapped. In general, in higher dimensions we appear to lose the natural ordering as well as strong influence of finite orbits on the large scale structure of the map. In two dimensions though, the latter aspect does seem to have a natural counterpart in the form of the mapping class of the map restricted to the punctured manifold obtained by removing the periodic orbit. After selecting a suspension on the map, there is an associated braid, and the braid group in dimension two in some sense replaces the symmetric group used in the one-dimensional dynamics discussed here. These sorts of ideas may be found in [GST] and the references therein where analogous considerations permit the definition and investigation of two-dimensional infinitely renormalizable dynamical systems. Despite this remark it remains unclear as to what sorts of objects would assume that role that trees play in the one-dimensional case.

Remark. Our bibliography is far from exhaustive and we apologize for any instances in which we may not have given proper credit. For a fairly comprehensive bibliography for one-dimensional dynamics see [MS]. Those interested in the more group theoretic aspects of actions on trees might start with [Se]. The book [Rot] is an excellent group theory resource. A classic text for permutation groups is [Wie] while the paper [We] serves as a nice introduction to wreath products and contains many early references to the origins of the subject.

Overview of Chapter I

Let (K, f) be a dynamical system, consisting of a topological space K and a continuous map $f : K \longrightarrow K$. **Renormalization** procedures generally involve a choice of some subspace $H \subset K$ and an appropriate "first return" map of f-orbits, starting in H, back to H. The resulting dynamical system on H is then called a **renormalization of** (K, f). In general, the time of first return varies with the point of departure, and the f-transforms of H need not cover K. (See, for example, [MS] for interval dynamics, [LyMil] for interval dynamics with non-uniform return times, and [BRTT] for renormalization on n-tori.)

The "cyclic renormalizations" that we study here correspond to a fixed time of return, given by some integral power f^n of f. More precisely, an n-**renormalization** of (K, f) is, for us, a morphism of dynamical systems

$$\phi_n : (K, f) \longrightarrow (\mathbb{Z}/n\mathbb{Z}, +1). \tag{1}$$

Thus the fibers $K_r = \phi_n^{-1}(r)$ $(r \in \mathbb{Z}/n\mathbb{Z})$ are open-closed sets partitioning K, and $f(K_r) \subset K_{r+1}$ for all r. Then each (K_r, f_r) where $f_r = f^n|_{K_r}$ is a renormalization of (K, f) in the sense described above. The fiber K_0 here corresponds to the H above, and its f-transforms cover K.

If (K, f) is **minimal**, i.e. if each f-orbit is dense in K, then the K_r are just the f^n-orbit closures, and each (K_r, f_r) is again minimal. Moreover ϕ_n in (1) above is determined by n, up to a translation of $\mathbb{Z}/n\mathbb{Z}$. We put

$$Ren(K, f) = \{n \geq 1 \mid (K, f) \textit{ admits an } n\textit{-renormalization}\}. \tag{2}$$

Then (cf. (1.5)) the set $Ren(K, f)$ is stable under divisors and LCM's (least common multiples). It is convenient to introduce (cf. (1.6)) the **supernatural number**‡

$$Q = Q(K, f) = LCM(Ren(K, f)). \tag{3}$$

Then

$$Ren(K, f) = Div(Q) = \{\textit{integral divisors of } Q\}. \tag{4}$$

Assume that some $x_0 \in K$ has a dense f-orbit. Choosing our n-renormalization ϕ_n $(n \in Ren(K, f))$ so that $\phi_n(x_0) = 0$, they form an inverse system (with respect to divisibility) and we obtain a morphism

$$\widehat{\phi}_Q : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_Q, +1), \tag{5}$$

where $\widehat{\mathbb{Z}}_Q = \varprojlim_{n|Q} \mathbb{Z}/n\mathbb{Z}$, and $(\widehat{\mathbb{Z}}_Q, +1)$ is called the ***Q*-adic adding machine.**

Now suppose that (K, f) arises from a dynamical system $g : I \longrightarrow I$ on a closed real interval $I = [a, b]$, $a < b$, with K a minimal closed g-invariant

‡A **supernatural number** Q is a formal product, $Q = \prod_p p^{e_p}$, p varying over all primes, and $0 \leq e_p \leq \infty$ for each p. It is clear what it means for one such number to divide another.

subset and $f = g|_K$. Then K inherits a (linear) order structure from I, so we may speak of K-intervals. Moreover the topology on K is the order topology. An n-renormalization ϕ_n of (K, f) as in (1) above is called an **interval n-renormalization** if its fibers K_r are all K-intervals. These form a partition of K by intervals, so they occur in a definite order in K. This defines a unique linear order on $\mathbb{Z}/n\mathbb{Z}$ so that ϕ_n is weak order preserving (i.e. ϕ_n preserves $\leq$). We put

$$\begin{aligned} IRen(K,f) &= \{n \geq 1 \mid (K,f) \textit{ admits an interval } n\textit{-renormalization}\} \\ &\subset Ren(K,f). \end{aligned} \tag{6}$$

The fundamental observation about this (Theorem (2.6)) is:

$$IRen(K,f) \textit{ is totally ordered by divisibility.} \tag{7}$$

Thus we can write:

$$IRen(K,f) = \{n_0(=1) < n_1 < n_2 < \cdots\}, \textit{ with } n_i \mid n_{i+1}. \tag{8}$$

Put

$$q_i = \frac{n_i}{n_{i-1}}, \tag{9}$$

so that

$$n_h = q_1 \cdot q_2 \cdots q_h. \tag{10}$$

Of course $IRen(K, f)$ may be finite or infinite. We put

$$\mathbf{q} = \mathbf{q}(K,f) = (q_1, q_2, q_3, \ldots) \tag{11}$$

and call this the **interval renormalization index** of (K, f). As above, we obtain a natural morphism

$$\widehat{\phi}_{\mathbf{q}} : (K,f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1), \tag{12}$$

where $\widehat{\mathbb{Z}}_{\mathbf{q}} = \varprojlim_h \mathbb{Z}/n_h\mathbb{Z}$, and $(\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$ is called the $\mathbf{q}$-adic adding machine. In terms of supernatural numbers, we have

$$\widehat{\mathbb{Z}}_{\mathbf{q}} = \widehat{\mathbb{Z}}_{Q(\mathbf{q})}, \tag{13}$$

where

$$Q(\mathbf{q}) = LCM(IRen(K,f)) = \prod_h q_h. \tag{14}$$

The K-induced order on each $\mathbb{Z}/n_h\mathbb{Z}$ gives, in the limit, a linear order on $\widehat{\mathbb{Z}}_{\mathbf{q}}$ so that $\widehat{\phi}_{\mathbf{q}}$ is weak order preserving.

If $K = \{1, 2, \ldots, N\}$, with its natural order, then f is just a transitive permutation of K, and $IRen(K, f) = \{n_0 = 1 < n_1 < n_2 < \cdots < n_m = N\}$ with $n_{i-1} \mid n_i \quad (i = 1, \ldots, m)$. In the special case that $N = 2^M$ we have $IRen(K, f) = \{1, 2, 4, 8, \ldots, 2^M\}$ if and only if f is a simple permutation in the sense of [Bl].

We can summarize many of our results in Chapter I as follows. Suppose that we are given

$$Q = \text{ a supernatural number} \tag{15}$$

and

$$\mathbf{q} = (q_1, q_2, q_3, \ldots) \text{ a sequence, finite or infinite, of integers } q_i \geq 2, \text{ such that } Q(\mathbf{q}) := \prod_i q_i \text{ divides } Q. \tag{16}$$

Then the results of Sections 1, 3, and 4 give the following.

Theorem. *(a) There is a compact real ordered minimal dynamical system (K, f) such that $Q(K, f) = Q$ and $\mathbf{q}(K, f) = \mathbf{q}$ iff either $\mathbf{q}$ is finite, or $\mathbf{q}$ is infinite and $Q(\mathbf{q}) = Q$.*

(b) Suppose that (K, f) is a compact real dynamical system with a dense orbit, and that (K, f) is infinitely interval renormalizable, i.e. that $\mathbf{q} = \mathbf{q}(K, f)$ is infinite. Then

$$\widehat{\phi}_{\mathbf{q}} : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1)$$

is surjective, and injective except perhaps for countably many 2-point fibers. Moreover $f : K \longrightarrow K$ is surjective, and injective, except perhaps, for countably many 2-point fibers. Further, K is a Cantor set and (K, f) is minimal.

In (4.6) we construct, using a “Denjoy expansion technique”, examples where the 2-point fibers of $\widehat{\phi}_{\mathbf{q}}$ and f do in fact occur.

In Section 4 we anticipate examples (constructed in Ch. II, Section 3) of (K, f) where K is a minimal closed invariant set for a unimodal dynamical system f on a real interval I, and with $IRen(K, f)$ prescribed in advance. In Section 5 we relate $IRen(K, f)$ to periodic points of (I, f). Self-similarity operators are defined in Section 6.

Overview of Chapter II

Let (J, f) be a unimodal map on a real interval $J = [a, b]$, with maximum $M = f(C)$, increasing on $L = [a, C)$, and decreasing on $R = (C, b]$. Then each $x \in J$ has an “address” $A(x) \in \{L, C, R\}$ such that $x \in A(x)$. The f-orbit $f^*(x) = (x, f(x), f^2(x), \ldots)$ then has an address

$$Af^*(x) = (A(x), Af(x), Af^2(x), \ldots),$$

called the “itinerary” of x. The itinerary

$$K(f) = Af^*(M)$$

of the "postcritical orbit" is called the **kneading sequence** of f [MilTh] (see also [MSS, My]). It symbolically encodes much of the dynamics of (J, f), especially on the f-orbit closure $\overline{O_f(M)}$ of M.

Consider the monoid $G = G_0 \cup G_0 C$, where G_0 is freely generated by $\{L, R\}$, and subject to the relations $CX = C$ for all $X \in G$. We interpret itineraries as either finite words in $G_0 C$, or as infinite words, in $\hat{G}_0$, with letters L and R.

The central aim of Chapter III is to show how an interval n renormalization of (K, f), where $K = \overline{O_f(x)}$, is reflected in a "$\star$-product" factorization, $Af^*(x) = \alpha \star \beta$ where $\alpha \in G_0$ has length $n - 1$ (cf. Theorem (7.1)). In fact $IRen(K, f)$ can be intrinsically recovered from the itinerary $Af^*(x)$ (cf. (9.4)). By iterated star products, we construct in Section 8, elements $\kappa \in \hat{G}_0$ with prescribed initial renormalization. Then in Section 9 we quote results of [MilTh] (see also [CEc]) affirming that all such κ can be realized in the form $Af^*(x)$, $x \in J$, where (J, f) is a quadratic unimodal map.

Sections 10 and 11 relate the previous discussion to periodic orbits, and cyclic permutations.

Overview of Chapter III

Chapters III and IV are essentially group theoretic. They are partly motivated by Chapters I and II, but are mathematically independent of them.

First the motivation. Let (K, f) be a minimal ordered dynamical system that is infinitely interval renormalizable. Put

(1) $$IRen(K, f) = \{n_0 = 1 < n_1 < n_2 \cdots\}$$

and

(2) $$\mathbf{q} = \mathbf{q}(K, f) = (q_1, q_2, a_3, \ldots)$$

with $q_i = n_i / n_{i-1}$.

Then we have the inverse sequence of sets

(3) $$X_0 \xleftarrow{p} X_1 \xleftarrow{p} X_2 \xleftarrow{p} \cdots$$

where

(4) $$X_m = \mathbb{Z}/n_m\mathbb{Z}$$

and p is the natural projection. The interval renormalizations

(5) $$(K, f) \longrightarrow (X_m, g_m) = (\mathbb{Z}/n_m\mathbb{Z}, +1)$$

induce

(6) $$\widehat{\phi} : (K, f) \longrightarrow (\widehat{\mathbb{Z}}_{\mathbf{q}}, +1) = \varprojlim_m (X_m, g_m).$$

An inverse sequence of (finite) sets, as in (3), can be interpreted as a (locally finite) rooted tree, X, with vertex set

$$VX = \coprod_{m\geq 0} X_m,$$

its root being the single vertex $x_0 \in X_0$ and with edges joining x to $p(x)$ for all $x \neq x_0$. Then X_m is the sphere of radius m centered at x_0. In our case, each $p : X_m \longrightarrow X_{m-1}$ is a surjective homomorphism with kernel of order q_m. Hence, each fiber of $p : X_m \longrightarrow X_{m-1}$ has q_m elements, so X is what we call a **spherically homogeneous rooted tree of index q**. Moreover, the maps $g_m : X_m \longrightarrow X_m$ assemble to define an automorphism g of the rooted tree X which acts transitively on each of the spheres X_m; i.e. g is what we call a **spherically transitive automorphism of the rooted tree X.**

In Chapter III we study rooted trees X defined by any inverse sequence of finite sets

(7) $$X_0 = \{x_0\} \longleftarrow X_1 \longleftarrow X_2 \longleftarrow \cdots$$

and their automorphism groups

(8) $$G = Aut(X).$$

Then X admits spherically transitive automorphisms iff X is spherically homogeneous, say of index

(9) $$\mathbf{q} = \mathbf{q}(X) = (q_1, q_2, q_3, \ldots),$$

where q_m is the cardinal of each fiber of $p : X_m \longrightarrow X_{m-1}$. In this case, $\mathbf{q}$ determines X up to isomorphism so we can write

(10) $$X = X(\mathbf{q}) \textit{ and } G = G(\mathbf{q}) = Aut(X(\mathbf{q})).$$

In Theorem (4.6) we show that the spherically transitive automorphisms of $X(\mathbf{q})$ are all conjugate in $G(\mathbf{q})$. Moreover, if g is one of them, then $Z_{G(\mathbf{q})}(g) = \overline{\langle g\rangle} \cong \widehat{\mathbb{Z}}_{\mathbf{q}}$, where $\overline{\langle g\rangle}$ denotes the closure of the cyclic group $\langle g\rangle$ in the profinite group $G(\mathbf{q})$, and $\widehat{\mathbb{Z}}_{\mathbf{q}}$ denotes, as above, the q-adic integers.

In the course of this discussion we obtain a description of $G(\mathbf{q})$ as an infinite iterated wreath product. This structure is used in Section 6 to analyze some group theoretic "renormalization operators".

Overview of Chapter IV

This chapter gives a fairly detailed analysis of the normal subgroups of

(1) $$G(\mathbf{q}) = Aut(X(\mathbf{q})),$$

where

(2) $$\mathbf{q} = (q_1, q_2, a_3, \ldots),$$

and of certain of its other subgroups, defined as follows. Let

$$\mathbf{Y} = (Y_1, Y_2, Y_3, \ldots) \tag{3}$$

be a sequence of sets with

$$|Y_m| = q_m. \tag{4}$$

Then we can define $X(\mathbf{q})$ by the inverse sequence

$$X_0 \xleftarrow{p} X_1 \xleftarrow{p} X_2 \xleftarrow{p} \cdots \tag{5}$$

where $X_m = Y_1 \times Y_2 \times \cdots \times Y_m$ and $p : X_m \longrightarrow X_{m-1}$ is a projection away from the last factor. Let

$$\mathbf{Q} = (Q_1, Q_2, Q_3, \ldots) \tag{6}$$

be a sequence of groups with Q_m a group of permutations of Y_m. Then we can inductively construct the wreath products

$$Q(m) = Q_m^{X_{m-1}} \rtimes Q(m-1), \tag{7}$$

with a natural action of $Q(m)$ on X_m, starting with $Q(1) = Q_1$ acting as given on $X_1 = Y_1$. Then

$$G((\mathbf{Q}, \mathbf{Y})) = \varprojlim_m Q(m) \tag{8}$$

is naturally a closed subgroup of $G(\mathbf{q}) = G((\mathbf{S}, \mathbf{Y}))$, where S_m is taken to be the full symmetric group on Y_m. We can write $G((\mathbf{Q}, \mathbf{Y}))$ explicitly as an infinite iterated wreath product,

$$G((\mathbf{Q}, \mathbf{Y})) = \cdots \rtimes Q_m^{X_{m-1}} \rtimes \cdots \rtimes Q_2^{X_1} \rtimes Q_1. \tag{9}$$

There is a canonical homomorphism compatible with (9),

$$\bar{\sigma} : G((\mathbf{Q}, \mathbf{Y})) \longrightarrow \cdots \times Q_m^{\mathrm{ab}} \times \cdots \times Q_2^{\mathrm{ab}} \times Q_1^{\mathrm{ab}}, \quad \bar{\sigma}(g) = (\sigma_m(g))_{m \geq 1}, \tag{10}$$

where Q_m^{ab} denotes the abelianization of Q_m, and

(11) *$Ker(\bar{\sigma})$ contains the closure of the commutator subgroup of $G((\mathbf{Q}, \mathbf{Y}))$.*

If each Q_m is transitive on Y_m then each $Q(m)$ is transitive on X_m, and the inclusion (11) is an equality.

Suppose that each Q_m is cyclic (hence abelian), generated by a q_m-cycle on Y_m. Then we have

$$\bar{\sigma} : G((\mathbf{Q}, \mathbf{Y})) \longrightarrow \prod_{m \geq 1} Q_m.$$

In this case (cf. (4.4) and (4.5)), $G((\mathbf{Q},\mathbf{Y}))$ contains spherically transitive elements; $g \in G((\mathbf{Q},\mathbf{Y}))$ is spherically transitive iff $\sigma_m(g)$ generates Q_m for all $m \geq 1$; and two spherically transitive elements g, g' are conjugate in $G((\mathbf{Q},\mathbf{Y}))$ iff $\bar{\sigma}(g) = \bar{\sigma}(g')$.

Note that, in the case of the dyadic tree, $\mathbf{q} = (2,2,2,\ldots)$, the previous paragraph applies to the full group $G(\mathbf{q})$.

Finally, in Theorem (5.4), under the assumption that Q_m acts primitively on Y_m for each m (e.g. when Q_m is the full symmetric group) we give an analysis of all the normal subgroups of $G((\mathbf{Q},\mathbf{Y}))$. The result is too technical to state here.

If H is a rank 1 simple algebraic group (e.g. $H = PSL_2$) over a p-adic field F, then its Bruhat-Tits building X (cf. [Se]) is a tree on which $H(F)$ acts, with quotient $H(F)\backslash X = \circ\!\!-\!\!\circ$. The maximal compact subgroups of $H(F)$ are vertex stabilizers in X. If $x_0 \in X$ then $H(F)_{x_0} \cong H(A)$ where A is the ring of integral elements of F. Thus we have $H(A) \leq Aut(X, x_0)$, the automorphism group of the spherically homogeneous rooted tree (X, x_0). The congruence subgroups,

$$Ker\Big(H(A) \longrightarrow H(A/M^m)\Big),$$

where M is the maximal ideal of A, coincide with the groups

$$Ker\Big(H(A) \xrightarrow{res} Aut(B_m(x_0))\Big),$$

where $B_m(x_0)$ denotes the ball of radius m about x_0 in X.

In this light, we can think of the description of normal subgroup of $G(\mathbf{q})$ as a combinatorial analogue of the local congruence subgroup problem for the groups $H(A)$ (see e.g. [BMS]).

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