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THE DIRAC OPERATOR AND GRAVITATION

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Abstract

We give a brute-force proof of the fact, announced by Alain Connes, that the Wodzicki residue of the inverse square of the Dirac operator is proportional to the Einstein-Hilbert action of general relativity. We show that this also holds for twisted (e.g. by electrodynamics) Dirac operators, and more generally, for Dirac operators pertaining to Clifford connections of general Clifford bundles.

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ERRATUM

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Page. 2 L. 3 read:

$$I=4\mathrm{Tr}_w\{s_{-4}(x,\xi)\}=(2p)^{-4}\int_{\xi\in\Sigma^3}\mathrm{tr}\{s_{-4}(x,\xi)\}d^3\xi\ dv, \tag{3}$$

L.9 read:

$$\mathcal{L}_g=R_{ikmn}(dx^m\wedge dx^n,dx^i\wedge dx^k)=2(g^{im}g^{nk}-g^{in}g^{mk})R_{ikmn}=2\kappa, \tag{5a}$$

L. 10 read $I=\frac{1}{3\cdot 2^5p^2}\kappa=\frac{1}{3\cdot 2^6p^2}L_g$

Page. 3 L. -3 instead of $-\frac{1}{4}\|x\|^{-2}\kappa(x)$ read $-\frac{1}{4}\|x\|^{-4}\kappa(x)$

Page. 11 L. -2 instead of $\frac{1}{4}\kappa$ read $-\frac{1}{4}\kappa$

Page. 12 L. 9 read:

$$=\Delta-\frac{1}{2}\gamma(dx^m)\gamma(dx^n)\otimes R^V(\partial_m,\partial_n)+\frac{1}{4}\kappa\otimes id_W.$$

The Dirac operator and gravitation

Daniel Kastler

In what follows **M** is a 4-dimensional oriented riemannian spin manifold with riemannian metric g (yielding the norm $\| \|$ and the volume-element dv). We recall that the Dirac operator D is locally given as follows in terms of an orthonormal section e_i (with dual section θ^k) of the frame bundle of **M**: one has

$$(1)\qquad \begin{cases} D=i\gamma^i\tilde{\nabla}_i=i\gamma^i(e_i+\sigma_i) \\ \text{with } \sigma_i(x)=\frac{1}{4}\gamma_{ij,k}(x)\gamma^j\gamma^k=\frac{1}{8}\gamma_{ij,k}(x)[\gamma^j\gamma^k-\gamma^k\gamma^j] \end{cases},$$

where the $\gamma_{ij,k}$ represent the Levi-Civita connection ∇ with spin connection $\tilde{\nabla}$, specifically:

$$(2)\qquad \begin{cases} \gamma_{ij,k}=-\gamma_{ik,j}=\frac{1}{2}[c_{ij,k}+c_{ki,j}+c_{kj,i}] \\ \text{with } c^k_{ij}=\theta^k([e_i,e_j]) \end{cases}, \quad i,j,k=1,\dots,4.$$

Here the γ^i are constant self-adjoint Dirac matrices s.t. $\gamma^i\gamma^j+\gamma^j\gamma^i=\delta^{ij}$. In terms of local coordinates x^μ inducing the alternative vierbein $\partial_\mu=S^i{}_\mu(x)e_i$ (with dual vierbein dx^μ) we have $\gamma^i e_i=\gamma^\mu\partial_\mu$, the γ^μ being now x-dependant Dirac matrices s.t. $\gamma^\mu\gamma^\nu+\gamma^\nu\gamma^\mu=g^{\mu\nu}$ (we use latin sub-(super-)scripts for the basis e_i and greek sub-(super-)scripts for the basis ∂_μ, the type of sub-(super-)script specifying the type of Dirac matrices). The specification of the Dirac operator in the greek basis is as follows: one has

$$(1a)\qquad \begin{cases} D=i\gamma^\mu\tilde{\nabla}_\mu=i\gamma^\mu(\partial_\mu+\sigma_\mu) \\ \text{with } \sigma_\mu(x)=S^i{}_\mu(x)\sigma_i(x) \end{cases}.$$

In what follows the notation D^{-1} refers to an inverse modulo smoothing operators.

In this Chapter we establish the following astonishing fact due to Alain Connes[1]:

[1] *Theorem*. *The value of the Wodzicki residue* [1] *on the inverse square of the Dirac operator, namely:*

$$(3)\qquad I=\mathrm{Tr}_\omega\{\sigma_{-4}(x,\xi)\}=(2\pi)^{-4}\int_{\xi\in S^3}\mathrm{tr}\{\sigma_{-4}(x,\xi)\}d^3\xi\, dv,$$

where:

$$(4)\qquad \sigma_{-4}(x,\xi)=\text{part of order -4 of the total symbol } \sigma(x,\xi) \text{ of } D^{-2},$$

coincides up to a constant with the Hilbert-Einstein action $\int \mathcal{L}_g dv$ *of general relativity, where:*

$$(5)\qquad \mathcal{L}_g=R_{\mu\nu}\wedge *(dx^\mu\wedge dx^\nu)$$

(specifically

$$(5a)\qquad \mathcal{L}_g=R_{ikmn}(dx^m\wedge dx^n,dx^i\wedge dx^k)=2(g^{im}g^{nk}-g^{in}g^{mk})R_{ikmn}=-4\kappa,$$

κ the scalar curvature). *One has* $I=\frac{5}{3\cdot 2^5\pi^2}\kappa=-\frac{5}{24\pi^2}L_g$.

We recall the Lichnérowicz formula for the square of the Dirac operator:

$$(6)\qquad D^2=-g^{\mu\nu}(\tilde{\nabla}_\mu\tilde{\nabla}_\nu-\Gamma^\alpha_{\mu\nu}\tilde{\nabla}_\alpha)+\frac{1}{4}\kappa$$

$$=-g^{\mu\nu}[\partial^x_\mu\partial^x_\nu+2\sigma_\mu\cdot\partial^x_\nu-\Gamma^\alpha_{\mu\nu}\partial_\alpha+\partial^x_\mu\,\sigma_\nu+\sigma_\mu\sigma_\nu-\Gamma^\alpha_{\mu\nu}\sigma_\alpha]+\frac{1}{4}\kappa.$$

Our computations are based on the algorithm yielding the principal symbol of a product of pseudo-differential operators in terms of the principal symbols of the factors, namely, with the shorthands $\partial^\alpha_\xi=\partial^\alpha/\partial\xi_\alpha$, $\partial^x_\alpha=\partial_\alpha/\partial x^\alpha$:

$$(7)\qquad \sigma^{PQ}(x,\xi)=\Sigma_\alpha\frac{(-i)^{|\alpha|}}{\alpha!}\partial^\alpha_\xi\sigma^P(x,\xi)\cdot\partial^x_\alpha\sigma^Q(x,\xi).$$

We need to compute the total symbol $\sigma(x,\xi)$ of D^{-2} up to order -4, with D^2 the following sum of terms $(D^2)_k$ of order k:

[1] The Wodzicki residue is a (in fact the unique) trace on the pseudo-differential operators (concentrated on pseudo-differential operators of order -4=-dimM).

(6a) $$D^2=(D^2)_2+(D^2)_1+(D^2)_0 \quad , \quad \begin{cases} (D^2)_2=-g^{\mu\nu}\partial^x_\mu\partial^x_\nu \\ (D^2)_1=-g^{\mu\nu}(2\sigma_\mu\cdot\partial^x_\nu-\Gamma^\alpha_{\mu\nu}\partial^x_\alpha) \\ (D^2)_0=-g^{\mu\nu}(\partial^x_\mu\sigma_\nu+\sigma_\mu\sigma_\nu-\Gamma^\alpha_{\mu\nu}\sigma_\alpha)+\frac{1}{4}\kappa \end{cases} ,$$

with the respective symbols:

(8) $$\begin{cases} \sigma_2(x,\xi)=g^{\mu\nu}(x)\xi_\mu\xi_\nu \\ \sigma_1(x,\xi)=ig^{\mu\nu}(x)[\Gamma^\alpha_{\mu\nu}(x)\xi_\alpha-2\sigma_\mu(x)\xi_\nu] \\ \sigma_0(x)=-g^{\mu\nu}(x)(\partial^x_\mu\sigma_\nu+\sigma_\mu\sigma_\nu-\Gamma^\alpha_{\mu\nu}\sigma_\alpha)(x)+\frac{1}{4}\kappa(x) \end{cases} ,$$

abbreviated as follows, using the shorthand $\Gamma^\mu=g^{\alpha\beta}\Gamma^\mu_{\alpha\beta}$:

(8a) $$\begin{cases} \sigma_2(x,\xi)=\|\xi\|^2 \\ \sigma_1(x,\xi)=i(\Gamma^\mu-2\sigma^\mu)(x)\xi_\mu \\ \sigma_0(x)=-(\partial^{x\mu}\sigma_\mu+\sigma^\mu\sigma_\mu-\Gamma^\mu\sigma_\mu)(x)+\frac{1}{4}\kappa(x) \end{cases} ,$$

We want to compute a parametrix D^{-2} of D^2 up to order -4 using the above recipe: this amounts to computing the parts σ_{-k}, k=2,3,4, in the expansion of the full symbol σ of D^{-2} into terms of decreasing order:

(9) $$\sigma\text{-}D^{-2}=\sigma=\sigma_{-2}+\sigma_{-3}+\sigma_{-4}+\text{ terms of order }\leq -5.$$

Application of (VIII1,7) with $P=D^2$ and $Q=D^{-2}$ yields in the respective orders 0,-1,-2 the recurrence relations:

(10) $$\sigma_2\sigma_{-2}=1,$$

(11) $$\sigma_2\sigma_{-3}+\sigma_1\sigma_{-2}-i\partial^\mu_\xi\sigma_2\cdot\partial^x_\mu\sigma_{-2}=0,$$

(12) $$\sigma_2\sigma_{-4}+\sigma_1\sigma_{-3}+\sigma_0\sigma_{-2}-i\partial^\mu_\xi\sigma_2\cdot\partial^x_\mu\sigma_{-3}-i\partial^\mu_\xi\sigma_1\cdot\partial^x_\mu\sigma_{-2}-\frac{1}{2}\partial^{\mu\nu}_\xi\sigma_2\cdot\partial^x_{\mu\nu}\sigma_{-2}=0,$$

where the relevant terms are read off the following tabulation of products $\partial^\alpha_\xi\sigma_p(x,\xi)\cdot\partial^x_\alpha\sigma_q(x,\xi)$:

	$\|\alpha\|=0$			$\|\alpha\|=1$			$\|\alpha\|=2$		
	σ_2	σ_1	σ_0	σ_2	σ_1	σ_0	σ_2	σ_1	σ_0
σ_{-2}	0	-1	-2	-1	-2	-3	-2	-3	-4
σ_{-3}	-1	-2	-3	-2	-3	-4	-3	-4	-5
σ_{-4}	-2	-3	-4	-3	-4	-5	-4	-5	-6.

With the σ_k as in (VIII1,8a), this reads:

(10a) $$\sigma_{-2}=\|\xi\|^{-2},$$

(11a) $$\|\xi\|^2\sigma_{-3}+i\|\xi\|^{-2}(\Gamma^\mu-2\sigma^\mu)\xi_\mu-i\partial^\mu_\xi\|\xi\|^2\cdot\partial^x_\mu\|\xi\|^{-2}=0,$$

(12a) $$\|\xi\|^2\sigma_{-4}+i(\Gamma^\mu-2\sigma^\mu)\xi_\mu\sigma_{-3}-\|\xi\|^{-2}[(\partial^{x\mu}\sigma_\mu+\sigma^\mu\sigma_\mu-\Gamma^\mu\sigma_\mu)+\frac{1}{4}\|\xi\|^{-2}\kappa(x)$$
$$-i\partial^\mu_\xi\|\xi\|^2\cdot\partial^x_\mu\sigma_{-3}+(\Gamma^\nu-2\sigma^\nu)\delta^\mu{}_\nu\cdot\partial^x_\mu\|\xi\|^{-2}-\frac{1}{2}\partial^{\mu\nu}_\xi\|\xi\|^2\cdot\partial^x_{\mu\nu}\|\xi\|^{-2}=0.$$

With

(13) $$\partial^\mu_\xi\|\xi\|^2=2\xi^\mu,\ \partial^\mu_\xi\|\xi\|^{-2}=-2\|\xi\|^{-4}\xi^\mu,\ \partial^\mu_\xi\|\xi\|^{-4}=-4\|\xi\|^{-6}\xi^\mu,\ \partial^\mu_\xi\|\xi\|^{-6}=-6\|\xi\|^{-8}\xi^\mu,$$
$$\partial^x_\mu\|\xi\|^2=\xi^\alpha\xi^\beta\partial^x_\mu g^{\alpha\beta},\ \partial^x_\mu\|\xi\|^{-2}=-\|\xi\|^{-4}\xi_\alpha\xi_\beta\partial^x_\mu g^{\alpha\beta},\ \partial^x_\mu\|\xi\|^{-6}=-3\|\xi\|^{-8}\xi_\alpha\xi_\beta\partial^x_\mu g^{\alpha\beta},$$
$$\partial^{\mu\nu}_\xi\|\xi\|^2=2g^{\mu\nu},\quad \partial^x_{\mu\nu}\|\xi\|^{-2}=-\|\xi\|^{-4}\xi_\alpha\xi_\beta\partial^x_{\mu\nu}g^{\alpha\beta}+2\|\xi\|^{-6}\xi_\alpha\xi_\beta\partial^x_\mu g^{\alpha\beta}\xi_\gamma\xi_\delta\partial^x_\nu g^{\gamma\delta}$$

we get

(11b) $$i\sigma_{-3}=\|\xi\|^{-4}\xi_\mu(\Gamma^\mu-2\sigma^\mu)+2\|\xi\|^{-6}\xi^\mu\xi_\alpha\xi_\beta\partial^x_\mu g^{\alpha\beta}$$

and

(12b) $$\sigma_{-4}=-i\|\xi\|^{-2}(\Gamma^\mu-2\sigma^\mu)\xi_\mu\sigma_{-3}+\|\xi\|^{-4}(\partial^{x\mu}\sigma_\mu+\sigma^\mu\sigma_\mu-\Gamma^\mu\sigma_\mu)-\frac{1}{4}\|\xi\|^{-4}\kappa(x)$$
$$+2i\|\xi\|^{-2}\xi^\mu\cdot\partial^x_\mu\sigma_{-3}+\|\xi\|^{-6}\xi_\alpha\xi_\beta(\Gamma^\mu-2\sigma^\mu)\partial^x_\mu g^{\alpha\beta}+\frac{1}{2}\|\xi\|^{-2}\partial^{\mu\nu}_\xi\|\xi\|^2\cdot\partial^x_{\mu\nu}\|\xi\|^{-2}$$
$$=-\|\xi\|^{-6}\xi_\mu\xi_\nu(\Gamma^\mu-2\sigma^\mu)(\Gamma^\nu-2\sigma^\nu)-2\|\xi\|^{-8}\xi^\mu\xi_\nu\xi_\alpha\xi_\beta(\Gamma^\nu-2\sigma^\nu)\partial^x_\mu g^{\alpha\beta}$$
$$+\|\xi\|^{-4}[(\partial^{x\mu}\sigma_\mu+\sigma^\mu\sigma_\mu-\Gamma^\mu\sigma_\mu)-\frac{1}{4}\|\xi\|^{-2}\kappa(x)-2i\|\xi\|^{-2}\xi^\mu\cdot\partial^x_\mu\sigma_{-3}$$
$$+\|\xi\|^{-6}\xi_\alpha\xi_\beta(\Gamma^\mu-2\sigma^\mu)\partial^x_\mu g^{\alpha\beta}-\|\xi\|^{-6}\xi_\alpha\xi_\beta g^{\mu\nu}\partial^x_{\mu\nu}g^{\alpha\beta}$$
$$+2\|\xi\|^{-8}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta g^{\mu\nu}\partial^x_\mu g^{\alpha\beta}\partial^x_\nu g^{\gamma\delta}$$

Regrouping the terms and inserting

(14) $$i\partial^x_\mu\sigma_{-3}=-2\|\xi\|^{-6}\xi_\nu\xi_\alpha\xi_\beta(\Gamma^\nu-2\sigma^\nu)\partial^x_\mu g^{\alpha\beta}+\|\xi\|^{-4}\xi_\nu\partial^x_\mu(\Gamma^\nu-2\sigma^\nu)$$
$$-6\|\xi\|^{-8}\xi^\nu\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\partial^x_\mu g^{\alpha\beta}\partial^x_\nu g^{\gamma\delta}+2\|\xi\|^{-6}\xi_\alpha\xi_\gamma\xi_\delta\partial^x_\mu g^{\nu\alpha}\partial^x_\nu g^{\gamma\delta}$$
$$+2\|\xi\|^{-6}\xi^\nu\xi_\gamma\xi_\delta\partial^x_{\mu\nu}g^{\gamma\delta},$$

hence

(14a) $$2i\|\xi\|^{-2}\xi^\mu\cdot\partial^x_\mu\sigma_{-3}=-4\|\xi\|^{-8}\xi^\mu\xi_\nu\xi_\alpha\xi_\beta(\Gamma^\nu-2\sigma^\nu)\partial^x_\mu g^{\alpha\beta}+2\|\xi\|^{-6}\xi^\mu\xi_\nu\partial^x_\mu(\Gamma^\nu-2\sigma^\nu)$$
$$-12\|\xi\|^{-10}\xi^\mu\xi^\nu\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\partial^x_\mu g^{\alpha\beta}\partial^x_\nu g^{\gamma\delta}+4\|\xi\|^{-8}\xi^\mu\xi_\alpha\xi_\gamma\xi_\delta\partial^x_\mu g^{\nu\alpha}\partial^x_\nu g^{\gamma\delta}$$
$$+4\|\xi\|^{-8}\xi^\mu\xi^\nu\xi_\gamma\xi_\delta\partial^x_{\mu\nu}g^{\gamma\delta},$$

we get for σ_{-4} the sum of terms:

(15)
$$A=-\|\xi\|^{-6}\xi_\mu\xi_\nu\Gamma^\mu\Gamma^\nu+\|\xi\|^{-4}[g_{\mu\nu}-4\|\xi\|^{-2}\xi_\mu\xi_\nu][\sigma^\mu\sigma^\nu-\Gamma^\mu\sigma^\nu]$$
$$B=\|\xi\|^{-4}\partial^{x\mu}\sigma_\mu-\frac{1}{4}\|\xi\|^{-4}\kappa(x)$$
$$C=-6\|\xi\|^{-8}\xi^\mu\xi_\nu\xi_\alpha\xi_\beta(\Gamma^\nu-2\sigma^\nu)\partial^x_\mu g^{\alpha\beta}$$
$$D=2\|\xi\|^{-6}\xi^\mu\xi_\nu\partial^x_\mu(\Gamma^\nu-2\sigma^\nu)$$
$$E=-12\|\xi\|^{-10}\xi^\mu\xi^\nu\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta\partial^x_\mu g^{\alpha\beta}\partial^x_\nu g^{\gamma\delta}$$
$$F=4\|\xi\|^{-8}\xi^\mu\xi_\alpha\xi_\gamma\xi_\delta\partial^x_\mu g^{\nu\alpha}\partial^x_\nu g^{\gamma\delta}$$
$$G=\|\xi\|^{-6}\xi_\alpha\xi_\beta(\Gamma^\mu-2\sigma^\mu)\partial^x_\mu g^{\alpha\beta}$$
$$H=4\|\xi\|^{-8}\xi^\mu\xi^\nu\xi_\gamma\xi_\delta\partial^x_{\mu\nu}g^{\gamma\delta}$$
$$K=-\|\xi\|^{-6}\xi_\alpha\xi_\beta g^{\mu\nu}\partial^x_{\mu\nu}g^{\alpha\beta}$$
$$L=2\|\xi\|^{-8}\xi_\alpha\xi_\beta\xi_\gamma\xi_\delta g^{\mu\nu}\partial^x_\mu g^{\alpha\beta}\partial^x_\nu g^{\gamma\delta}$$

We have to take the Clifford trace and integrate over the sphere S^3 (commuting procedures). Owing to (VIII1,1,1a), all the terms linear in σ_μ vanish under the Clifford trace. We proceed to the integration over S^3, using the following facts, for which we refer to Appendix **[VIIID]**. We have, using the shorthand $\int=\frac{1}{2\pi^2}\int_{\xi\in S^3}dv$:

(16) $$\begin{cases}\int\xi^\mu\xi^\nu=\frac{1}{4}[\mu\nu]\\ \xi^\mu\xi^\nu\xi^\alpha\xi^\beta=\frac{1}{3\cdot 2^3}[\mu\nu\alpha\beta]\\ \xi^\mu\xi^\nu\xi^\alpha\xi^\beta\xi^\gamma\xi^\delta=\frac{1}{3\cdot 2^6}[\mu\nu\alpha\beta\gamma\delta]\end{cases}$$

where $[\mu\nu.....\gamma\delta]$ stands for the sum of products of $g^{\alpha\beta}$ determined by all "pairings" of $\mu\nu.....\gamma\delta$. We now average over S^3 the terms surviving in (VIII1,15) yields (we now write ∂_μ instead of ∂^x_μ without risking confusion, and use the fact that one has, $\cong$ indicating equivalence when multiplied with an expression symmetric in $\alpha\beta$, and γ,δ:

(17) $$[{}^{\mu\nu}{}_{\alpha\beta\gamma\delta}]\cong g^{\mu\nu}[\alpha\beta\gamma\delta]+2\delta^\mu{}_\alpha[{}^\nu{}_{\beta\gamma\delta}]+2\delta^\mu{}_\gamma[{}^\nu{}_{\alpha\beta\delta}].$$

We get the terms:

(18)
$$A\ \rightarrow\ -\frac{1}{4}[\mu\nu]\Gamma^\mu\Gamma^\nu=-\frac{1}{4}g_{\mu\nu}\Gamma^\mu\Gamma^\nu$$
$$B\ \rightarrow\ -\frac{1}{4}\kappa(x)$$
$$C\ \rightarrow\ -6\frac{1}{3\cdot 2^3}[{}^\mu{}_{\nu\alpha\beta}]\Gamma^\nu\partial_\mu g^{\alpha\beta}=-\frac{1}{4}[\delta^\mu{}_\nu g_{\alpha\beta}+2\delta^\mu{}_\alpha g_{\nu\beta}]\Gamma^\nu\partial_\mu g^{\alpha\beta}$$
$$=-\frac{1}{4}\Gamma^\mu g_{\alpha\beta}\partial_\mu g^{\alpha\beta}-\frac{1}{2}\Gamma^\nu g_{\nu\beta}\partial_\mu g^{\mu\beta}$$
$$D\ \rightarrow\ 2\frac{1}{4}[{}^\mu{}_\nu]\partial_\mu\Gamma^\nu=\frac{1}{2}\delta^\mu{}_\nu\partial_\mu\Gamma^\nu=\frac{1}{2}\partial_\mu\Gamma^\mu$$
$$E\ \rightarrow\ -12\frac{1}{3\cdot 2^6}[{}^{\mu\nu}{}_{\alpha\beta\gamma\delta}]\partial_\mu g^{\alpha\beta}\partial_\nu g^{\gamma\delta}$$
$$=-\frac{1}{16}\{g^{\mu\nu}g_{\alpha\beta}g_{\gamma\delta}+2g^{\mu\nu}g_{\alpha\gamma}g_{\beta\delta}+2\delta^\mu{}_\alpha\delta^\nu{}_\beta g_{\gamma\delta}$$
$$+4\delta^\mu{}_\alpha\delta^\nu{}_\gamma g_{\beta\delta}+4\delta^\mu{}_\gamma\delta^\nu{}_\alpha g_{\beta\delta}+2\delta^\mu{}_\gamma\delta^\nu{}_\delta g_{\alpha\beta}\}\partial_\mu g^{\alpha\beta}\partial_\nu g^{\gamma\delta}$$
$$=-\frac{1}{16}g^{\mu\nu}g_{\alpha\beta}g_{\gamma\delta}\partial_\mu g^{\alpha\beta}\partial_\nu g^{\gamma\delta}-\frac{1}{8}g^{\mu\nu}g_{\alpha\gamma}g_{\beta\delta}\partial_\nu g^{\alpha\beta}\partial_\mu g^{\gamma\delta}$$
$$-\frac{1}{8}g_{\gamma\delta}\partial_\mu g^{\mu\nu}\partial_\nu g^{\gamma\delta}-\frac{1}{4}g_{\beta\delta}\partial_\mu g^{\mu\beta}\partial_\nu g^{\nu\delta}$$
$$-\frac{1}{4}g_{\beta\delta}\partial_\mu g^{\nu\beta}\partial_\nu g^{\mu\delta}-\frac{1}{8}g_{\alpha\beta}\partial_\mu g^{\alpha\beta}\partial_\nu g^{\mu\nu}$$
$$F\ \rightarrow\ 4\frac{1}{3\cdot 2^3}[{}^\mu{}_{\alpha\gamma\delta}]\partial_\mu g^{\nu\alpha}\partial_\nu g^{\gamma\delta}$$
$$=\frac{1}{6}\delta^\mu{}_\alpha g_{\gamma\delta}\partial_\mu g^{\nu\alpha}\partial_\nu g^{\gamma\delta}+\frac{1}{3}\delta^\mu{}_\gamma g_{\alpha\delta}\partial_\mu g^{\nu\alpha}\partial_\nu g^{\gamma\delta}$$
$$=\frac{1}{6}g_{\gamma\delta}\partial_\mu g^{\mu\nu}\partial_\nu g^{\gamma\delta}+\frac{1}{3}g_{\alpha\delta}\partial_\mu g^{\nu\alpha}\partial_\nu g^{\mu\delta}$$
$$G\ \rightarrow\ \frac{1}{4}[\alpha\beta]\Gamma^\mu\partial^x_\mu g^{\alpha\beta}=\frac{1}{4}\Gamma^\mu g_{\alpha\beta}\partial^x_\mu g^{\alpha\beta}$$

$$H \rightarrow 4\frac{1}{3\cdot 2^3}[{}^{\mu\nu}{}_{\gamma\delta}]\partial_{\mu\nu}g^{\gamma\delta}=\frac{1}{6}[g^{\mu\nu}g_{\gamma\delta}+2\delta^{\mu}{}_{\gamma}\delta^{\nu}{}_{\delta}]\partial_{\mu\nu}g^{\gamma\delta}$$

$$K \rightarrow -\frac{1}{4}[\alpha\beta]g^{\mu\nu}\partial_{\mu\nu}g^{\alpha\beta}=-\frac{1}{4}g_{\alpha\beta}g^{\mu\nu}\partial_{\mu\nu})g^{\alpha\beta}=\frac{1}{4}g^{\alpha\beta}g^{\mu\nu}\partial_{\mu\nu}g_{\alpha\beta}$$

$$\begin{aligned} L \rightarrow\ & 2\frac{1}{3\cdot 2^3}g^{\mu\nu}[\alpha\beta\gamma\delta]\partial^{x}_{\mu}g^{\alpha\beta}\partial_{\nu}g^{\gamma\delta} \\ & =\frac{1}{12}g^{\mu\nu}[g_{\alpha\beta}g_{\gamma\delta}+2g_{\alpha\gamma}g_{\beta\delta}]\partial_{\mu}g^{\alpha\beta}\partial_{\nu}g^{\gamma\delta} \end{aligned}$$

We now evaluate the terms containing the Γ^{μ} and convert into expressions in which the partial derivatives act on the $g^{\alpha\beta}$ with upper indices: we recall that:

(19) $$\Gamma^{\mu}=g^{\alpha\beta}\Gamma^{\mu}_{\alpha\beta}=g^{\alpha\beta}g^{\mu\sigma}\Gamma_{\alpha\beta,\sigma}=g^{\alpha\beta}g^{\mu\sigma}[\partial_{\alpha}g_{\beta\sigma}-\frac{1}{2}\partial_{\sigma}g_{\alpha\beta}],$$

and:

(20) $$\partial_{\mu}g^{\alpha\beta}=-g^{\alpha\sigma}g^{\beta\tau}\partial_{\mu}g_{\sigma\tau}\ ,\qquad g_{\alpha\gamma}\partial_{\mu}g^{\alpha\beta}=-g^{\alpha\beta}\partial_{\mu}g_{\alpha\gamma},$$

In what follows the following shorthands make the corresponding expressions easily recognizable:

(21) $$\begin{cases} U=g^{\alpha\beta}g^{\gamma\delta}\partial_{\alpha\beta}g_{\gamma\delta} \\ X=g^{\alpha\beta}g^{\gamma\delta}\partial_{\alpha\gamma}g_{\beta\delta} \\ hhG=g^{\alpha\beta}g^{\gamma\delta}g^{\sigma\tau}\partial_{\alpha}g_{\beta\sigma}\partial_{\gamma}g_{\delta\tau} \\ ghH=g^{\alpha\beta}g^{\gamma\delta}g^{\sigma\tau}\partial_{\alpha}g_{\beta\sigma}\partial_{\tau}g_{\gamma\delta} \\ gg\Delta=g^{\alpha\beta}g^{\gamma\delta}g^{\sigma\tau}\partial_{\sigma}g_{\alpha\beta}\partial_{\tau}g_{\gamma\delta} \\ GHH=g^{\mu\sigma}g^{\alpha\eta}g^{\xi\beta}\partial_{\mu}g_{\eta\xi}\partial_{\alpha}g_{\beta\sigma} \\ \Delta GG=g^{\mu\sigma}g^{\alpha\eta}g^{\xi\beta}\partial_{\mu}g_{\eta\xi}\partial_{\sigma}g_{\alpha\beta} \end{cases}$$

where g, G, h, H, Δ, stand for contractions between the following kinds of pairs of indices: indices of letters g (same or different), indices of letters g and ∂ (nearby or remote), indices of letters ∂ and ∂.

– Computation of A: we have:

$$\begin{aligned} -\frac{1}{4}g_{\mu\nu}\Gamma^{\mu}\Gamma^{\nu} &=-\frac{1}{4}g_{\mu\nu}g^{\alpha\beta}g^{\mu\sigma}g^{\gamma\delta}g^{\nu\tau}[\partial_{\alpha}g_{\beta\sigma}-\frac{1}{2}\partial_{\sigma}g_{\alpha\beta}]\cdot[\partial_{\gamma}g_{\delta\tau}-\frac{1}{2}\partial_{\tau}g_{\gamma\delta}], \\ &=-\frac{1}{4}g^{\alpha\beta}g^{\gamma\delta}g^{\sigma\tau}[\partial_{\alpha}g_{\beta\sigma}-\frac{1}{2}\partial_{\sigma}g_{\alpha\beta}]\cdot[\partial_{\gamma}g_{\delta\tau}-\frac{1}{2}\partial_{\tau}g_{\gamma\delta}] \\ &=-\frac{1}{4}hhG+\frac{1}{8}ghH+\frac{1}{8}ghH-\frac{1}{16}gg\Delta \end{aligned}$$

hence

(22) $$A \rightarrow -\frac{1}{4}hhG+\frac{1}{4}ghH-\frac{1}{16}gg\Delta$$

– Computation of C: we have:

$$\begin{aligned} -\frac{1}{4}\Gamma^{\mu}g_{\alpha\beta}\partial_{\mu}g^{\alpha\beta}-\frac{1}{2}\Gamma^{\nu}g_{\nu\beta}\partial_{\mu}g^{\mu\beta} &=\frac{1}{4}\Gamma^{\mu}g^{\alpha\beta}\partial_{\mu}g_{\alpha\beta}+\frac{1}{2}\Gamma^{\nu}g^{\mu\beta}\partial_{\mu}g_{\nu\beta} \\ &=\frac{1}{4}g^{\sigma\tau}g^{\mu\rho}g^{\alpha\beta}[\partial_{\sigma}g_{\tau\rho}-\frac{1}{2}\partial_{\rho}g_{\sigma\tau}]\partial_{\mu}g_{\alpha\beta}+\frac{1}{2}g^{\sigma\tau}g^{\nu\rho}g^{\mu\beta}\partial_{\mu}g_{\nu\beta}[\partial_{\sigma}g_{\tau\rho}-\frac{1}{2}\partial_{\rho}g_{\sigma\tau}] \\ &=\frac{1}{4}ghH-\frac{1}{8}gg\Delta+\frac{1}{2}hhG-\frac{1}{4}ghH, \end{aligned}$$

hence

(23) $$C \rightarrow \frac{1}{2}hhG-\frac{1}{8}gg\Delta$$

– Computation of D: we have:

$$\begin{aligned} -\frac{1}{2}\partial_{\mu}\Gamma^{\mu} &=\frac{1}{2}\partial_{\mu}\{g^{\sigma\tau}g^{\mu\rho}[\partial_{\sigma}g_{\tau\rho}-\frac{1}{2}\partial_{\rho}g_{\sigma\tau}]\} \\ &=\frac{1}{2}g^{\sigma\tau}g^{\mu\rho}[\partial_{\mu\sigma}g_{\tau\rho}-\frac{1}{2}\partial_{\mu\rho}g_{\sigma\tau}]+\frac{1}{2}\partial_{\mu}[g^{\sigma\tau}g^{\mu\rho}]\cdot[\partial_{\sigma}g_{\tau\rho}-\frac{1}{2}\partial_{\rho}g_{\sigma\tau}] \\ &=\frac{1}{2}X-\frac{1}{4}U+\frac{1}{2}[g^{\sigma\tau}\partial_{\mu}g^{\mu\rho}+g^{\mu\rho}\partial_{\mu}g^{\sigma\tau}]\cdot[\partial_{\sigma}g_{\tau\rho}-\frac{1}{2}\partial_{\rho}g_{\sigma\tau}] \\ &=\frac{1}{2}X-\frac{1}{4}U-\frac{1}{2}[g^{\sigma\tau}g^{\mu\alpha}g^{\rho\beta}\partial_{\mu}g_{\alpha\beta}+g^{\mu\rho}g^{\sigma\alpha}g^{\tau\beta}\partial_{\mu}g_{\alpha\beta}]\cdot[\partial_{\sigma}g_{\tau\rho}-\frac{1}{2}\partial_{\rho}g_{\sigma\tau}] \end{aligned}$$

hence

(24) $$D \rightarrow \frac{1}{2}X-\frac{1}{4}U-\frac{1}{2}hhG+\frac{1}{4}ghH-\frac{1}{2}GHH+\frac{1}{4}\Delta GG$$

– Computation of E: we have:

$$\begin{aligned} &-\frac{1}{16}g^{\mu\nu}g_{\alpha\beta}g_{\gamma\delta}\partial_{\mu}g^{\alpha\beta}\partial_{\nu}g^{\gamma\delta}-\frac{1}{8}g^{\mu\nu}g_{\alpha\gamma}g_{\beta\delta}\partial_{\nu}g^{\alpha\beta}\partial_{\mu}g^{\gamma\delta} \\ &\qquad -\frac{1}{8}g_{\gamma\delta}\partial_{\mu}g^{\mu\nu}\partial_{\nu}g^{\gamma\delta}-\frac{1}{4}g_{\beta\delta}\partial_{\mu}g^{\mu\beta}\partial_{\nu}g^{\nu\delta} \\ &\qquad -\frac{1}{4}g_{\beta\delta}\partial_{\mu}g^{\nu\beta}\partial_{\nu}g^{\mu\delta}-\frac{1}{8}g_{\alpha\beta}\partial_{\mu}g^{\alpha\beta}\partial_{\nu}g^{\mu\nu} \\ &=-\frac{1}{16}g^{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\partial_{\mu}g_{\alpha\beta}\partial_{\nu}g_{\gamma\delta}-\frac{1}{8}g^{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\partial_{\nu}g_{\alpha\gamma}\partial_{\mu}g_{\beta\delta} \\ &\qquad -\frac{1}{8}g^{\gamma\delta}g^{\mu\sigma}g^{\nu\tau}\partial_{\mu}g_{\sigma\tau}\partial_{\nu}g_{\gamma\delta}-\frac{1}{4}g^{\nu\delta}g^{\mu\sigma}g^{\beta\tau}\partial_{\mu}g_{\sigma\tau}\partial_{\nu}g_{\beta\delta} \\ &\qquad -\frac{1}{4}g^{\mu\delta}g^{\nu\sigma}g^{\beta\tau}\partial_{\mu}g_{\sigma\tau}\partial_{\nu}g_{\beta\delta}-\frac{1}{8}g^{\alpha\beta}g^{\mu\sigma}g^{\nu\tau}\partial_{\mu}g_{\alpha\beta}\partial_{\nu}g_{\sigma\tau} \\ &=-\frac{1}{16}gg\Delta-\frac{1}{8}\Delta GG-\frac{1}{8}ghH-\frac{1}{4}hhG-\frac{1}{4}GHH-\frac{1}{8}ghH \end{aligned}$$

hence

$$(25)\qquad E \rightarrow -\tfrac{1}{4}ghH-\tfrac{1}{4}hhG-\tfrac{1}{16}gg\Delta-\tfrac{1}{8}\Delta GG-\tfrac{1}{4}GHH$$

– Computation of F: we have:

$$\tfrac{1}{6}g_{\gamma\delta}\partial_\mu g^{\mu\nu}\partial_\nu g^{\gamma\delta}+\tfrac{1}{3}g_{\alpha\delta}\partial_\mu g^{\nu\alpha}\partial_\nu g^{\mu\delta}$$
$$=\tfrac{1}{6}g^{\gamma\delta}g^{\mu\sigma}g^{\nu\tau}\partial_\mu g_{\sigma\tau}\partial_\nu g_{\gamma\delta}+\tfrac{1}{3}g^{\nu\alpha}g^{\mu\sigma}g^{\delta\tau}\partial_\mu g_{\alpha\delta}\partial_\nu g_{\sigma\tau}$$

hence

$$(26)\qquad F \rightarrow \tfrac{1}{6}ghH+\tfrac{1}{3}GHH$$

– Computation of G: we have:

$$\tfrac{1}{4}\Gamma^\mu g_{\alpha\beta}\partial_\mu g^{\alpha\beta}=-\tfrac{1}{4}\Gamma^\mu g^{\alpha\beta}\partial_\mu g_{\alpha\beta}=-\tfrac{1}{4}g^{\alpha\beta}g^{\sigma\tau}g^{\mu\rho}[\partial_\sigma g_{\tau\rho}-\tfrac{1}{2}\partial_\rho g_{\sigma\tau}]\partial_\mu g_{\alpha\beta}$$

hence $g^{\alpha\beta}g^{\sigma\tau}g^{\mu\rho}\partial_\rho g_{\sigma\tau}\partial_\mu g_{\alpha\beta}$

$$(27)\qquad G \rightarrow -\tfrac{1}{4}ghH+\tfrac{1}{8}gg\Delta$$

– Computation of H: we have:

$$\partial_{\mu\nu}g^{\gamma\delta}=\partial_\mu\partial_\nu g^{\gamma\delta}=-\partial_\mu[g^{\gamma\sigma}g^{\delta\tau}\partial_\nu g_{\sigma\tau}]$$
$$=-g^{\gamma\sigma}g^{\delta\tau}\partial_{\mu\nu}g_{\sigma\tau}-g^{\gamma\sigma}\partial_\nu g_{\sigma\tau}\partial_\mu g^{\delta\tau}-g^{\delta\tau}\partial_\nu g_{\sigma\tau}\partial_\mu g^{\gamma\sigma}$$
$$=-g^{\gamma\sigma}g^{\delta\tau}\partial_{\mu\nu}g_{\sigma\tau}+g^{\gamma\sigma}g^{\delta\alpha}g^{\tau\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}+g^{\delta\tau}g^{\gamma\alpha}g^{\sigma\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}$$

hence

$$\tfrac{1}{6}g^{\mu\nu}g_{\gamma\delta}\partial_{\mu\nu}g^{\gamma\delta}=-\tfrac{1}{6}g^{\mu\nu}\delta^\sigma{}_\delta g^{\delta\tau}\partial_{\mu\nu}g_{\sigma\tau}+\tfrac{1}{6}g^{\mu\nu}\delta^\sigma{}_\delta g^{\delta\alpha}g^{\tau\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}$$
$$+\tfrac{1}{6}g^{\mu\nu}\delta^\tau{}_\gamma g^{\gamma\alpha}g^{\sigma\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}$$
$$=-\tfrac{1}{6}g^{\mu\nu}g^{\delta\tau}\partial_{\mu\nu}g_{\delta\tau}+\tfrac{1}{6}g^{\mu\nu}g^{\delta\alpha}g^{\tau\beta}\partial_\nu g_{\delta\tau}\partial_\mu g_{\alpha\beta}$$
$$+\tfrac{1}{6}g^{\mu\nu}g^{\gamma\alpha}g^{\sigma\beta}\partial_\nu g_{\sigma\gamma}\partial_\mu g_{\alpha\beta}$$
$$=-\tfrac{1}{6}U+\tfrac{1}{6}\Delta GG+\tfrac{1}{6}\Delta GG=-\tfrac{1}{6}U+\tfrac{1}{3}\Delta GG$$

whilst

$$-\tfrac{1}{6}2\delta^\mu{}_\gamma\delta^\nu{}_\delta\partial_{\mu\nu}g^{\gamma\delta}=-\tfrac{1}{3}\partial_{\mu\nu}g^{\mu\nu}=\tfrac{1}{3}\partial_\mu[g^{\mu\sigma}g^{\nu\tau}\partial_\nu g_{\sigma\tau}]$$
$$=\tfrac{1}{3}g^{\mu\sigma}g^{\nu\tau}\partial_{\mu\nu}g_{\sigma\tau}-\tfrac{1}{3}g^{\mu\sigma}g^{\nu\alpha}g^{\tau\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}$$
$$-\tfrac{1}{3}g^{\nu\tau}g^{\mu\alpha}g^{\sigma\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}\ =\tfrac{1}{3}X-\tfrac{1}{3}GHH-\tfrac{1}{3}hhG$$

hence

$$(28)\qquad H \rightarrow -\tfrac{1}{6}U-\tfrac{1}{3}X+\tfrac{1}{3}\Delta GG+\tfrac{1}{3}GHH+\tfrac{1}{3}hhG.$$

– Computation of K: we have:

$$-\tfrac{1}{4}g_{\gamma\delta}g^{\mu\nu}\partial_{\mu\nu}g^{\gamma\delta}$$
$$=-\tfrac{1}{4}g_{\gamma\delta}g^{\mu\nu}[-g^{\gamma\sigma}g^{\delta\tau}\partial_{\mu\nu}g_{\sigma\tau}+g^{\gamma\sigma}g^{\delta\alpha}g^{\tau\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}+g^{\delta\tau}g^{\gamma\alpha}g^{\sigma\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}]$$
$$=\tfrac{1}{4}\delta^\sigma{}_\delta g^{\mu\nu}g^{\delta\tau}\partial_{\mu\nu}g_{\sigma\tau}-\tfrac{1}{4}\delta^\sigma{}_\delta g^{\mu\nu}g^{\delta\alpha}g^{\tau\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}-\tfrac{1}{4}\delta^\tau{}_\gamma g^{\mu\nu}g^{\gamma\alpha}g^{\sigma\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}$$
$$=\tfrac{1}{4}g^{\mu\nu}g^{\sigma\tau}\partial_{\mu\nu}g_{\sigma\tau}-\tfrac{1}{4}g^{\mu\nu}g^{\sigma\alpha}g^{\tau\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}-\tfrac{1}{4}g^{\mu\nu}g^{\tau\alpha}g^{\sigma\beta}\partial_\nu g_{\sigma\tau}\partial_\mu g_{\alpha\beta}$$
$$=\tfrac{1}{4}U-\tfrac{1}{4}\Delta GG-\tfrac{1}{4}\Delta GG$$

hence

$$(29)\qquad K \rightarrow \tfrac{1}{4}U-\tfrac{1}{2}\Delta GG$$

– Computation of L: we have:

$$\tfrac{1}{12}g^{\mu\nu}[g_{\alpha\beta}g_{\gamma\delta}+2g_{\alpha\gamma}g_{\beta\delta}]\partial_\mu g^{\alpha\beta}\partial_\nu g^{\gamma\delta}$$
$$=\tfrac{1}{12}g^{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\partial_\mu g_{\alpha\beta}\partial_\nu g_{\gamma\delta}+\tfrac{1}{6}g^{\mu\nu}g^{\alpha\beta}g^{\gamma\delta}\partial_\mu g_{\alpha\gamma}\partial_\nu g_{\beta\delta}$$

hence

$$(30)\qquad L \rightarrow \tfrac{1}{12}gg\Delta+\tfrac{1}{6}\Delta GG$$

We finally add up the contributions (VIII1,22) through (VIII1,30), which we recapitulate for the convenience of the reader:

$$(22)\qquad A \rightarrow -\tfrac{1}{4}hhG+\tfrac{1}{4}ghH-\tfrac{1}{16}gg\Delta$$

$$(23)\qquad C \rightarrow \tfrac{1}{2}hhG-\tfrac{1}{8}gg\Delta$$

$$(24)\qquad D \rightarrow \tfrac{1}{2}X-\tfrac{1}{4}U-\tfrac{1}{2}hhG+\tfrac{1}{4}ghH-\tfrac{1}{2}GHH+\tfrac{1}{4}\Delta GG$$

$$(25)\qquad E \rightarrow -\tfrac{1}{4}ghH-\tfrac{1}{4}hhG-\tfrac{1}{16}gg\Delta-\tfrac{1}{8}\Delta GG-\tfrac{1}{4}GHH$$

$$(26)\qquad F \rightarrow \tfrac{1}{6}ghH+\tfrac{1}{3}GHH$$

$$(27)\qquad G \to -\frac{1}{4}ghH+\frac{1}{8}gg\Delta$$

$$(28)\qquad H \to -\frac{1}{6}U-\frac{1}{3}X+\frac{1}{3}\Delta GG+\frac{1}{3}GHH+\frac{1}{3}hhG$$

$$(29)\qquad K \to \frac{1}{4}U-\frac{1}{2}\Delta GG$$

$$(30)\qquad L \to \frac{1}{12}gg\Delta+\frac{1}{6}\Delta GG$$

The sum of those terms amounts to:

$$(31)\qquad \left(-\frac{1}{4}-\frac{1}{6}+\frac{1}{4}\right)U+\left(\frac{1}{2}-\frac{1}{3}\right)X+\left(-\frac{1}{4}+\frac{1}{2}-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}\right)hhG+\left(\frac{1}{4}+\frac{1}{4}-\frac{1}{4}+\frac{1}{6}-\frac{1}{4}\right)ghH$$
$$+\left(\frac{1}{4}+\frac{1}{4}-\frac{1}{4}+\frac{1}{6}-\frac{1}{4}\right)ghH+\left(-\frac{1}{16}-\frac{1}{8}-\frac{1}{16}+\frac{1}{8}+\frac{1}{12}\right)gg\Delta$$
$$+\left(-\frac{1}{2}-\frac{1}{4}+\frac{1}{3}+\frac{1}{3}\right)GHH+\left(\frac{1}{4}-\frac{1}{8}+\frac{1}{3}-\frac{1}{2}+\frac{1}{6}\right)\Delta GG$$
$$=-\frac{1}{6}U+\frac{1}{6}X-\frac{1}{6}hhG+\frac{1}{6}ghH-\frac{1}{24}gg\Delta-\frac{1}{12}GHH-\frac{1}{8}\Delta GG]$$
$$=-\frac{1}{6}[U-X+hhG-ghH+\frac{1}{4}gg\Delta+\frac{1}{2}GHH+\frac{3}{4}\Delta GG]=-\frac{1}{6}\kappa$$

which, added to the contribution $-\frac{1}{4}\kappa(x)$ of the term B, yields **Theorem [1]**.

With $\mathbf{M}$ a compact riemannian manifold, denoting by $\mathbb{C}l_{\mathbf{M}}$ the set of smooth sections of the vector bundle with fibre over $x\in\mathbf{M}$ the Clifford algebra over the cotangent space of x (a $\mathbb{Z}/2$-graded complex algebra and $C^\infty(\mathbf{M})$-module), we now consider an additional smooth vector bundle $\mathcal{V}$ over $\mathbf{M}$ (with $C^\infty(\mathbf{M})$-module of smooth sections W), equipped with a connexion $\nabla^{\mathcal{V}}$, with corresponding curvature-tensor $R^{\mathcal{V}}$. We consider the tensor-product vector bundle $\mathcal{S}\otimes\mathcal{V}$ (with $C^\infty(\mathbf{M})$-module of smooth sections $\mathbb{S}_{\mathbf{M}}\otimes W$) which becomes a *Clifford bundle* via the definition:

$$(32)\qquad c(a)=\gamma(a)\otimes id_W \qquad ,\ a\in\mathbb{C}l_{\mathbf{M}},$$

and which we equip with the *compound connexion*:

$$(33)\qquad \boldsymbol{\nabla}_\xi=\tilde{\nabla}_\xi\otimes id_W+id_{\mathbb{S}_M}\otimes\nabla^{\mathcal{V}}_\xi \qquad ,\ \xi,\eta\in\mathcal{X}(\mathbf{M}),$$

the latter becoming a *Clifford connection* in the sense that:

$$(34)\qquad [\boldsymbol{\nabla}_\xi, c(a)]= c(\nabla_\xi a)] \qquad ,\ a\in\mathbb{C}l_{\mathbf{M}},\xi\in\mathcal{X}(\mathbf{M}),$$

where ∇ is the connexion of $\mathbb{C}l_{\mathbf{M}}$ induced by the Levi-Civita connexion of $\mathbf{M}$.The corresponding *twisted Dirac operator* $\mathbb{D}$, and *twisted connexion-laplacian* $\mathbb{\Delta}$ are then respectively locally specified as follows:

$$(35)\qquad \mathbb{D}=ic(dx^\mu)\boldsymbol{\nabla}_\mu,$$

and

$$(36)\qquad \mathbb{\Delta}=-g^{\mu\nu}(\boldsymbol{\nabla}_\mu\boldsymbol{\nabla}_\nu-\Gamma^{\alpha}_{\mu\nu}\boldsymbol{\nabla}_\alpha).$$

We have the following Lichnérowicz formulae for the square of the twisted Dirac operator:

$$(37)\qquad \mathbb{D}^2=\mathbb{\Delta}-\frac{1}{2}R^{\mathcal{V}}(\partial_\mu,\partial_\nu)c(dx^\mu)c(dx^\nu)+\frac{1}{4}\kappa$$
$$=\mathbb{\Delta}-\frac{1}{2}R^{\mathcal{V}}(\partial_\mu,\partial_\nu)\gamma(dx^\mu)\gamma(dx^\nu)\otimes id_W+\frac{1}{4}\kappa\otimes id_W.$$

[2] ***Proposition***. *The Wodzicki residue of* $\mathbb{D}^{-2}$ *coincides with that of of* D^{-2}, *thus also yields a multiple of the Einstein-Hilbert action.*

<u>Proof</u>: Let $a_\mu dx^\mu$ be the connexion one-form of the connexion $\nabla^{\mathcal{V}}$: the connexion one-form of the compound connexion (33) then reads:

$$(33a)\qquad \boldsymbol{\sigma}_\mu=\sigma_\mu\otimes id_W+id_{\mathbb{S}_M}\otimes a_\mu=\text{"}\sigma_\mu+a_\mu\text{"},$$

the computation of the Wodzicki residue of $\mathbb{D}^{-2}$ then results of that of of D^{-2} trough the changes:

$$(38)\qquad \begin{cases}\sigma_\mu \to \boldsymbol{\sigma}_\mu=\sigma_\mu+a_\mu\\ \kappa \to \boldsymbol{\kappa}=\kappa-2R^{\mathcal{V}}_{\mu\nu}\gamma^\mu\gamma^\nu\end{cases},$$

with corresponding replacements $\mathbb{A}\to A$ through $\mathbb{L}\to L$ which we now compute. The term $\mathbb{A}$, obtained from A through the change $\sigma_\mu \to \boldsymbol{\sigma}_\mu$, vanishes as the latter in the integration over S^3. As for the other terms, we have the following changes, leading to the indicated results after taking the Clifford trace and integrating over S^3:

$$(39)\qquad \mathbb{B}-B=\|\xi\|^{-4}\partial^x{}^\mu a_\mu+\frac{1}{2}\|\xi\|^{-4}R^{\mathcal{V}}_{\mu\nu}\gamma^\mu\gamma^\nu \quad\to\quad \partial^\mu a_\mu\ ,$$
$$\mathbb{C}-C=12\|\xi\|^{-8}\xi^\mu\xi_\nu\xi_\alpha\xi_\beta a^\nu\partial^x_\mu g^{\alpha\beta} \quad\to\quad 12\,\frac{1}{3\cdot 2^3}[^\mu{}_{\nu\alpha\beta}]a^\nu\partial_\mu g^{\alpha\beta}$$
$$=\frac{1}{2}[\delta^\mu{}_\nu g_{\alpha\beta}+2\delta^\mu{}_\alpha g_{\nu\beta}]a^\nu\partial_\mu g^{\alpha\beta}$$
$$=\frac{1}{2}a^\mu g_{\alpha\beta}\partial_\mu g^{\alpha\beta}+a^\nu g_{\nu\beta}\partial_\alpha g^{\alpha\beta}\ ,$$

$$\mathbb{D}\text{-}D=-4\|\xi\|^{-6}\xi^\mu\xi_\nu\partial^x_\mu a^\nu \to -\delta^\mu{}_\nu\partial_\mu a^\nu=-\partial_\mu a^\mu=-\partial_\mu(g^{\mu\nu}a_\nu)$$

$$=-g^{\mu\nu}\partial_\mu a_\nu-a_\nu\partial_\mu g^{\mu\nu}=-\partial^\mu a_\mu-a_\nu\partial_\mu g^{\mu\nu},$$

$$\mathbb{E}\text{-}E=0,$$

$$\mathbb{F}\text{-}F=0,$$

$$\mathbb{G}\text{-}G=-2\|\xi\|^{-6}\xi_\alpha\xi_\beta a^\mu\partial^x_\mu g^{\alpha\beta} \to -\frac{1}{2}a^\mu g_{\alpha\beta}\partial_\mu g^{\alpha\beta},$$

$$\mathbb{H}\text{-}H=0,$$

$$\mathbb{K}\text{-}K=0,$$

$$\mathbb{L}\text{-}L=0,$$

adding up to zero.

In fact the above result can be generalized further to the (generalized twisted) Dirac operators pertaining to Clifford connections of general Clifford bundles [2]. Let $\mathcal{E}$ be a $\mathbb{Z}/2$-graded vector bundle over **M**, with $C^\infty(\mathbf{M})$-module of smooth sections $\mathbf{E}$: $\mathcal{E}$ is called a *Clifford bundle* whenever there is a homomorphism of $\mathbb{Z}/2$-graded complex algebras c: $\mathbb{C}l_M\to End_{C^\infty(M)}\mathbf{E}$. Furthermore a connection ∇ of $\mathcal{E}$ is called a *Clifford connection* whenever all ∇_ξ, $\xi\in\mathcal{X}(\mathbf{M})$, are even, and one has

(34a) $$[\nabla_\xi, c(a)]=c(\nabla_\xi a)] \quad , a\in\mathbb{C}l_M, \xi\in\mathcal{X}(\mathbf{M})$$

(generalizing(34)). Those elements then specify as follows a generalized Dirac operator $\mathbb{D}_\nabla$:

(35a) $$\mathbb{D}_\nabla=ic(dx^\mu)\nabla_\mu$$

(generalizing(35)) giving rise to the generalized Lichnérowicz formula:

(37a) $$\mathbb{D}^2=\Delta-\frac{1}{2}F^{\mathcal{E}/S}(\partial_\mu,\partial_\nu)c(dx^\mu)c(dx^\nu)+\frac{1}{4}\kappa,$$

where $F^{\mathcal{E}/S}$ is the so-called *twisting curvature* of the bundle $\mathcal{E}$ ([2], Prop. 3.43). The replacement $R^\nu\to F^{\mathcal{E}/S}$ then leaves the above calculation unchanged, to the effect that one has:

[3] *Proposition.* *The Wodzicki residue of* $\mathbb{D}_\nabla^{-2}$ *also yields a multiple of the Einstein-Hilbert action.*

Appendix A. *The Einstein-Hilbert action. Scalar curvature.*

With **M** a 4-dimensional riemannian manifold with metric g (inducing the volume-element dv and the scalar product $(\cdot,\cdot)$ on tensors), the Levi-Civita connection ∇ is defined as follows in terms of local coordinates:

(A,1) $$\nabla\partial_i=\Gamma^k_{ij}\partial_k dx^j \quad \text{with} \quad \begin{cases}\Gamma^k_{ij}=g^{km}\Gamma_{ij,m}\\ \Gamma_{ij,m}=\frac{1}{2}[\partial_i g_{jm}+\partial_j g_{im}-\partial_m g_{ij}]\end{cases}.$$

The corresponding curvature ∇^2 is the two-form with values endomorphisms of the tangent bundle of **M** locally given by the matrix:

(A,2) $$R^j{}_k=d\omega^j{}_k+\omega^j{}_s\wedge\omega^s{}_k=\frac{1}{2}R^j{}_{kmn}dx^m\wedge dx^n,$$

explicitely given by:

(A,3) $$R^i{}_{kmn}=\partial_m\Gamma^i_{nk}-\partial_n\Gamma^i_{mk}+\Gamma^i_{ms}\Gamma^s_{nk}-\Gamma^i_{ns}\Gamma^s_{mk},$$

alternatively:

(A,4) $$R_{jkmn}=g_{ij}R^i{}_{kmn}=\frac{1}{2}\partial_m(\partial_k g_{nj}-\partial_j g_{nk})-\frac{1}{2}\partial_n(\partial_k g_{mj}-\partial_j g_{mk})+g_{st}\Gamma^s_{nj}\Gamma_{mk}-g_{st}\Gamma^s_{mj}\Gamma^t_{nk}$$

$$=\frac{1}{2}\partial_m(\partial_k g_{nj}-\partial_j g_{nk})-g_{st}\Gamma^s_{mj}\Gamma^t_{nk}-(m\leftrightarrow n)$$

The corresponding scalar curvature is

(A,5) $$\kappa=-R^{mn}{}_{mn}=-g^{mj}g^{nk}R_{jkmn}=(g^{mi}g^{nk}-g^{ni}g^{mk})[\partial_{mi}g_{nk}+g_{st}\Gamma^s_{mi}\Gamma^t_{nk}].$$

In terms of the shorthands (21) one has:

(A,6) $$\kappa=U-X+hhG-ghH+\frac{1}{4}gg\Delta+\frac{1}{2}GHH-\frac{3}{4}\Delta GG.$$

The *Einstein-Hilbert action* is by definition:

(A,7) $$L_g=R_{ik}\wedge*(dx^i\wedge dx^k)=\frac{1}{2}R_{ikmn}dx^m\wedge dx^n*(dx^i\wedge dx^k),$$

alternatively:

(A,8) $$L_g=\mathcal{L}_g dv,$$

with the lagrangian density:

$$\mathcal{L}_g=\tfrac{1}{2}R_{ikmn}\, g(dx^m\wedge dx^n, dx^i\wedge dx^k)=(g^{im}g^{nk}-g^{in}g^{mk})R_{ikmn} \tag{A.9}$$

$$=2g^{im}g^{nk}R_{ikmn}=2R^{mn}{}_{mn}=-2\kappa.$$

Note: After completion of this work, we had the visit in Marseille of Markus Walze and Wolfgang Kalau from Mainz (R.F.A.), who reported about an analogous calculation (using normal coordinates) leading to the same results.

Bibliography

[1] Alain Connes. Private communication. See also Alain Connes, Non-commutative geometry and physics, Proceedings of the Les Houches School, 1992.

[2] N.Berline, E.Getzler, M.Vergne. Heat kernels and Dirac Operators. Grundlehren der mathematischen Wissentschaften 298, Springer-Verlag (1991).