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Mean-field critical behaviour for correlation length for percolation in high dimensions

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Summary. Extending the method of [27], we prove that the correlation length ξ of independent bond percolation models exhibits mean-field type critical behaviour (i.e. $\xi(p) \sim (p_c - p)^{-1/2}$ as $p \nearrow p_c$) in two situations: i) for nearest-neighbour independent bond percolation models on a d-dimensional hypercubic lattice $\mathbb{Z}^d$, with d sufficiently large, and ii) for a class of "spread-out" independent bond percolation models, which are believed to belong to the same universality class as the nearest-neighbour model, in more than six dimensions. The proof is based on, and extends, a method developed in [27], where it was used to prove the triangle condition and hence mean-field behaviour of the critical exponents γ, β, δ, Δ and ν_2 for the above two cases.

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1. Introduction

In this paper, we continue our analysis of the mean-field critical behaviour for percolation in high dimensions (started in [27]), and prove that the critical exponent ν controlling the divergence of the correlation length exists and takes its mean-field value ($\nu=1/2$; see Sect. 1.1 for definitions of ν and other critical exponents).

The first rigorous proof of the mean-field critical behaviour is by Sokal [39], who proved the critical exponent equality $\alpha=0$ for Ising and φ^4 models in dimensions greater than four. In the analysis, a combination of the infrared bound [20] and correlation inequalities played an essential rôle. Works in this spirit followed, and led to the proof of exponent equalities $\gamma=1$, $\beta=1/2$, $\delta=3$ in dimensions greater than four [1, 4, 19]. However, the rigorous proof of the mean-field behaviour of the exponent ν of correlation length ($\nu=1/2$) turned out to be much more difficult, and we had to wait for the development of a rigorous renormalization group method [21] to obtain a partial result [26, 29].

On the other hand, for related stochastic geometric models (self-avoiding walks, percolation, branched polymers ...), there has been no general proof of the infrared bound. There are even some indications that it is explicitly violated in low dimensions [42, 31, 10]. The important step in proving mean-field properties for these models was taken in [14], where the mean-field property for the exponent ν (i.e. $\nu=1/2$) together with the infrared bound was proved for weakly self-avoiding walk in more than four dimensions. This method was further studied and simplified in [37, 38], and yielded $\gamma=1$, $\nu=1/2$ for strictly self-avoiding walk in sufficiently high dimensions. In the above analysis, "complex activity" of random walks was introduced, which, through Cauchy integral formula, provided considerably detailed information on the critical behaviour, especially those of the susceptibility and the correlation length. (However, it has been shown in [37] that the infrared bound can be proved without using the "complex activity.")

For percolation models, Slade and the present author [27] proved first the infrared bound in high dimensions along the line of argument of [37]. Then, according to [5, 8, 35], mean-field properties ($\gamma=\beta=1$, $\delta=\Delta=2$) followed. Also the analysis provided the proof of the mean-field critical behaviour of the average radius of gyration (also called the correlation length of order two), in particular the exponent equality $\nu_2=1/2$. However, the problem of mean-field behaviour of the correlation length ξ, defined as the inverse of the exponential decay rate of the two point function, and its exponent ν still remained open.

In this paper, we extend the analysis of [27] to prove $\nu=1/2$. Instead of introducing "complex probability" which would correspond to the "complex activity" of self avoiding walk, we carry out our analysis based on the expansion (identity for the two point function) derived in [27], performing a kind of Fourier-Laplace transform on it. A related idea was used in [13] to control massive decay of two-point functions of φ_3^4 theory. The method of this paper, like the one of [27], can also be applied to site percolation, and yields the same results. Also the method can be applied to other systems, once one has an expansion similar to the one used in this paper. (An example is a system of lattice trees and lattice animals [28].)

1.1. The models and their basic properties

As in [27], we consider independent Bernoulli (bond) percolation models on the infinite d-dimensional hypercubic lattice $\mathbb{Z}^d$. (See [22] for a review.) An element of $\mathbb{Z}^d$ is called a *site*, and a pair of *distinct* sites is called a *bond*. To each bond $b=\{x,y\}$ $(x, y\in\mathbb{Z}^d)$, a random variable n_b is associated, which takes the value 0 and 1. The set of random variables $\{n_b\}$ is independent, and the distribution of n_b is given by

$$\mathrm{Prob}(n_b=1)=p_b\equiv p\cdot J_b,$$
$$\mathrm{Prob}(n_b=0)=1-p_b\equiv 1-p\cdot J_b.$$

We require $\mathbb{Z}^d$-invariance (i.e. invariance under translation, reflection and rotation by $\pi/2$) for the $J_{\{xy\}}=J_{\{0,x-y\}}$. We write $\|x\|_\phi\equiv\left(\sum_{\mu=1}^{d}|x_\mu|^\phi\right)^{1/\phi}$, $|x|\equiv\|x\|_2$.

We consider the following possibilities for J_b:

(i) the nearest-neighbour model:

$$J_{\{0,x\}}=\begin{cases}1 & \text{if } x \text{ is a nearest-neighbour of 0 (i.e. if } \|x\|_2=1)\\ 0 & \text{otherwise}\end{cases}.$$

(ii) The spread-out model:

$$J_{\{0,x\}}=L^{-d}g(x/L)$$

where $g\colon\mathbb{R}^d\to[0,\infty)$ is a given function which is normalized so that $\int g(x)\,d^dx=1$ and $\int|x|^2g(x)\,d^dx=1$ (see the remark after Theorem 1.2), and is invariant under rotations by $\pi/2$ and reflections in the coordinate hyperplanes. The parameter L will be taken to be large. A basic example is

$$g(x)=\begin{cases}C & \text{if } \|x\|_\infty\equiv\max_{1\le\mu\le d}|x_\mu|\le l\\ 0 & \text{otherwise}\end{cases}$$

with $l=(3/d)^{1/2}$, $C=(2l)^{-d}$. We require that g decay exponentially at infinity (i.e. there exist positive C and δ such that $g(x)\le Ce^{-\delta\|x\|_1}$).

The bond density p is the only parameter in these models. (Note that in model (ii), p can take values in the interval $[0, L^d/\sup_x g(x)]$.)

If $n_b=1$ we say that b is *occupied*, while if $n_b=0$ we say b is *vacant*. We use $\mathrm{Prob}_p(E)$ to denote the probability of an event E with respect to the joint distribution of the n_b, and denote expectation with respect to this distribution by $\langle\cdot\rangle_p$. (However, we occasionally omit the subscript p.)

Given a bond configuration $\{n_b\}$, two sites x and y in the lattice are said to be *connected* if there exists a path from x to y which consists of occupied bonds. The *connected cluster* $C(x)$ of x is the random set of *sites* defined by

$$C(x)=\{y\in\mathbb{Z}^d\colon y \text{ is connected to } x\}.$$

The number of sites in $C(x)$ is denoted by $|C(x)|$.

We define the *two point function*

$$\tau_p(x,y)=\mathrm{Prob}_p(y \text{ is connected to } x),$$

the susceptibility

$$\chi_p = \sum_x \tau_p(0, x) = \langle |C(0)| \rangle_p,$$

and the mass m_p (inverse of the *correlation length* $\xi(p)$)

$$\xi(p)^{-1} = m_p = -\lim_{n \to \infty} \frac{\ln \tau_p(0, n e_1)}{n} \tag{1.1}$$

where $e_1 \equiv (1, 0, 0, \ldots, 0)$ is the unit vector in 1-direction. The existence of the above limit (which might be zero) is proven by the Harris-FKG inequality [30, 18] through a subadditivity argument [24, 16, 22] (see also Appendix A). We write $\xi(p)$ rather than ξ_p in order to distinguish this from "correlation length of order ϕ" defined in (1.4).

Remark. In the above, we defined the mass by long-distance behaviour of τ_p along a coordinate axis. It is natural to ask about behaviour of $\tau_p(0, x)$ in off-axis directions, i.e. behaviour of $\tau_p(0, nx)$ (as $n \nearrow \infty$) for general $x \in \mathbb{Z}^d$. It can be shown [7] by a subadditive argument that $l(x)$ defined by

$$m_p\, l(x) \equiv -\lim_{n \to \infty} \frac{\ln \tau_p(0, nx)}{n}$$

exists, and is a norm on $\mathbb{Z}^d$ which satisfies $\|x\|_\infty \leqq l(x) \leqq \|x\|_1$. In this sense, $\xi(p) = m_p^{-1}$ characterizes long-distance behaviour of $\tau_p(0, x)$ not only for x on coordinate axes but also for x off the axes.

Some of the basic properties of the models which will be relevant for us are the following. See, e.g., [22] for clear and self-contained derivation of these properties for the nearest-neighbour model. Most of the proofs extend to the spread-out model trivially. More account can be found in Appendix A.

i) For $d \geqq 2$, there exists $p_c \in (0, 1)$ such that $\chi_p < \infty$ for $p < p_c$, and $\chi_p = \infty$ for $p > p_c$ [12, 25]. p_c is called the *critical point.* Usually, p_c is defined as the percolation threshold (i.e., $p_c \equiv \sup\{p \mid \mathrm{Prob}_p(|C(0)| = \infty) = 0\}$), but it has been recently established [32, 3, 33] for a quite wide range of models (including the ones considered here) that these two are identical.

ii) $\chi_p \nearrow \infty$ as $p \nearrow p_c$ [5].

iii) For models with exponentially decaying interactions (i.e. $J_{\{0,x\}} \leqq Ce^{-\delta \|x\|_1}$ with some $0 < \delta$, $C < \infty$), $\xi(p) < \infty$ as long as $\chi_p < \infty$. Moreover, $\xi(p) \nearrow \infty$ as $p \nearrow p_c$. For finite range models (i.e. $J_{\{0x\}} = 0$ for $|x| > R$ with some $R > 0$), these have been proven explicitly in [24, 5]. For infinite range models (but still obeying the bound $J_{\{0,x\}} \leqq Ce^{-\delta \|x\|_1}$), we can argue as in [6], or as in the Appendix I of [2].

iv) The two point function obeys the bound [23, 16]

$$\tau_p(0, x) \leqq e^{-m_p \|x\|_\infty} \tag{1.2}$$

where m_p is defined by (1.1).

In this paper, we are concerned with the *critical behaviour* of the correlation length $\xi(p)$, i.e. its behaviour near the critical point p_c as p approaches p_c. In general, this behaviour (as well as that of χ_p) is expected to be in the form of power laws, and we introduce the critical exponents γ and ν as follows

(1.3)
$$\begin{aligned} \chi_p &\sim (p_c - p)^{-\gamma} \quad \text{as } p \nearrow p_c, \\ \xi(p) &\sim (p_c - p)^{-\nu} \quad \text{as } p \nearrow p_c. \end{aligned}$$

Here $f(p) \sim |p_c - p|^{-\lambda}$ is defined to mean that there are positive constants C_1 and C_2 (which are independent of p) such that

$$C_1 |p_c - p|^{-\lambda} \leqq f(p) \leqq C_2 |p_c - p|^{-\lambda} \quad \text{for } p \text{ close to } p_c.$$

To compare the result of this paper with that of [27], we also introduce the *correlation length of order* ϕ ($\phi = 2$ case is sometimes called *average radius of gyration*) $\xi_\phi(p)$ as follows:

(1.4)
$$(\xi_\phi(p))^\phi \equiv \frac{\sum_x |x|^\phi \tau_p(0, x)}{\sum_x \tau_p(0, x)}$$

for $\phi > 0$. By Hölder's inequality, this $\xi_\phi(p)$ is nondecreasing in ϕ. We define its critical exponent ν_ϕ by the relation:

$$\xi_\phi(p) \sim (p_c - p)^{-\nu_\phi} \quad \text{as } p \nearrow p_c.$$

Remark. The above $\xi(p)$ and $\xi_\phi(p)$ are formally equivalent. E.g. if we *assume* a simple scaling form for the two point function

(1.5)
$$\tau_p(0, x) \approx f(l(x))\, e^{-l(x)/\xi(p)} \quad \text{as } |x| \nearrow \infty$$

where $f(z)$ is a slowly varying real function (e.g. $f(z) \approx z^{-(d-2+\eta)}$), then $\xi(p) \approx \xi_\phi(p)$ for p near p_c, and thus $\nu = \nu_\phi$. However, at present, there is no rigorous proof of the scaling form (1.5), nor the proof that $\xi(p)$ and $\xi_\phi(p)$ exhibit the same critical behaviour. Incidentally it has been proven [34] that

$$0 \leqq \nu - \nu_\phi \leqq \frac{\gamma - \nu}{\phi}$$

and thus

$$\lim_{\phi \to \infty} \nu_\phi = \nu$$

assuming the existence of the exponents.

On the Bethe lattice, the susceptibility obeys the simple power law, i.e. the above exponent γ exists and takes the value $\gamma = 1$. Also with a suitable (slightly artificial?) introduction of distance on the Bethe lattice, $\xi_2(p)$ also obeys the power law, with $\nu_2 = 1/2$. See [22] for more details on the Bethe lattice calculation. The Bethe lattice critical exponents are known as the *mean-field* values, and it is expected for the above models (i) and (ii) in more than six dimensions all critical exponents take their mean-field values (including $\nu = \nu_2 = 1/2$).

1.2. Main results

In [27] it was proved for the models (i) and (ii) satisfying the conditions of Theorems 1.1, 1.2 below that the susceptibility χ_p and the average radius of gyration $\xi_2(p)$ exhibit mean-field critical behaviour, i.e.

$$\chi_p \sim |p_c - p|^{-1}, \qquad \xi_2(p) \sim |p_c - p|^{-1/2} \tag{1.6}$$

together with other quantities (in particular, it was proven $\beta = 1$, and $\delta = \Delta = 2$). In this paper, we prove similar results for the correlation length $\xi(p)$. I.e.,

Theorem 1.1. *For the nearest-neighbour independent bond percolation model* (i) *on* $\mathbb{Z}^d$, *there exists* $d_0 > 6$ *such that for* $d \geqq d_0$ *the correlation length* $\xi(p)$ *exhibits mean-field type critical behaviour.* I.e., *there are positive and finite constants* (*independent of* p) C_1, C_2, *such that*

$$C_1 |p_c - p|^{-1/2} \leqq \xi(p) \leqq C_2 |p_c - p|^{-1/2} \tag{1.7}$$

for $p \in [p_c/2, p_c)$.

Theorem 1.2. *The correlation length* $\xi(p)$ *exhibits mean-field type critical behaviour* (i.e. (1.7) *holds for* $p \in [p_c/2, p_c)$) *for* $d > 6$, *for the spread-out models* (ii), *if* L *is sufficiently large* (*depending on* d *and* g) *and if* g *is* $\mathbb{Z}^d$-*invariant*, $\dfrac{\partial^d g}{\partial x_1 \partial x_2 \ldots \partial x_d}$ *is piecewise continuous and satisfies the following conditions*:

$$\int d^d x\, g(x) = 1, \qquad \int d^d x\, g(x)\, |x|^2 = 1, \tag{1.8}$$

$$\left.\begin{array}{l} g(x) \cdot e^{\delta \|x\|_1} \in L_\infty(\mathbb{R}^d) \\ \int |\partial^I g(x)|\, e^{\delta \|x\|_1} d^d x < \infty \end{array}\right\} \quad \text{for some } \delta > 0 \tag{1.9}$$

where the derivative is interpreted as a distribution, and $\partial^I \equiv \prod_{\mu \in I} \dfrac{\partial}{\partial x_\mu}$ *with* $I \subset \{1, 2, \ldots, d\}$.

Remarks. 1. Without loss of generality, we can normalize $g(x)$ as in the first condition of (1.8), and can fix the scale of x (how far $g(x)$ is spread-out) as in the second condition of (1.8).

2. In [41, Sect. 6], it has been shown that exponents γ, Δ and ν (when exist) satisfy the hyperscaling inequality

$$d\nu \geqq 2\Delta - \gamma,$$

and thus that γ, Δ and ν cannot simultaneously take their mean-field values in dimensions less than six. (Similar conclusion was derived for different exponents in [17].) Taking the result of this paper ($\nu = 1/2$ in high dimensions) and that of [27] ($\gamma = 1$, $\Delta = 2$ in high dimensions) into account, the above inequality of [41] now implies that at least one of the exponents does take on different values depending on whether the dimension is high or low (or some of the exponents does not exist in low dimensions).

3. The above Theorems 1.1, 1.2 and Theorems 1.1, 1.2 of [27] imply that the scaling (continuum) limit of percolation is trivial (gaussian) in high-dimensions in the following sense: We define n-point connectivity function as

$$\tau_{n,p}(x_1, x_2, \ldots, x_n) \equiv \mathrm{Prob}_p(C(x_1) \ni x_2, \ldots, x_n)$$

and define the renormalized coupling (for $n \geqq 3$)

$$g_{\mathrm{ren},n}(p) \equiv \frac{\sum_{x_2, \ldots, x_n} \tau_{n,p}(x_1, x_2, \ldots, x_n)}{(\chi_p)^{n/2} (\xi(p))^{d\left(\frac{n}{2}-1\right)}}.$$

Then for $n \geqq 3$, $g_{\mathrm{ren},n}(p) \searrow 0$ as $p \nearrow p_c$ for models (i) and (ii) which satisfy the assumptions of the above Theorem 1.1 or Theorem 1.2. The proof follows immediately, if one uses the tree graph bound [5] to bound

$$\sum_{x_2, \ldots, x_n} \tau_{n,p}(x_1, x_2, \ldots, x_n) \leqq C_n \chi_p^{2n-3}$$

and combine it with the above results of the mean-field property of χ_p and $\xi(p)$.

We are planning to come back to this and related problems (slightly artificial "nontrivial" continuum limits in $d<6$) in future publications.

1.3. Framework of the proof

In this section, we present the general framework of the proof of our main results. The proof is based on the following two properties (Prop. 1.3 and Prop. 1.4). The former holds quite generally, and can be proven along the line of argument of [6, 36]. The latter is the key of the proof, and is proven using the identity for the two point function derived in [27]. Actually, the proof follows steps quite similar to the proof of Theorems 1.1 and 1.2 of [27]. But here, the parameter m plays the rôle of p of [27].

We start from the expression for the two point function $\tau_p(0,x)$ which was derived in [27, Sect. 2]:

$$\tau_p(0,x) = G_N(0,x) + \sum_y p_{0y}\, \tau_p(y,x) + \sum_y \Pi_{\leqq N}(0,y)\, \tau_p(y,x) \tag{1.10}$$

where G_N and $\Pi_{\leqq N}$ obey the bounds described in Prop. 2.6 of [27]. (The bounds are explicitly stated again in the following Sect. 2.) Multiplying (1.10) by e^{mx_1}, we have

$$\tau_p^{(m)}(0,x) = G_N^{(m)}(0,x) + \sum_y p_{0y}^{(m)}\, \tau_p^{(m)}(y,x) + \sum_y \Pi_{\leqq N}^{(m)}(0,y)\, \tau_p^{(m)}(y,x) \tag{1.11}$$

where, and throughout the paper, for arbitrary $f(x)$, $g(x,y)$ ($x, y \in \mathbb{Z}^d$), we write

$$f^{(m)}(x) \equiv f(x)\, e^{mx_1}, \qquad g^{(m)}(x,y) \equiv g(x,y)\, e^{m(y_1 - x_1)}. \tag{1.12}$$

Throughout the paper, the Fourier transform $\hat{f}(k)$ of a function $f(x)$ and $\hat{g}(k)$ of a translation-invariant function $g(x,y)=g(0,y-x)$ are defined as ($k\cdot x \equiv \Sigma_{\mu=1}^{d} k_\mu x_\mu$)

$$(1.13)\qquad \hat{f}(k) \equiv \sum_{x\in\mathbb{Z}^d} f(x)\, e^{ik\cdot x}, \qquad \hat{g}(k) \equiv \sum_{y\in\mathbb{Z}^d} g(x,y)\, e^{ik\cdot(y-x)}$$

and the momentum k is an element of the Brillouin zone: $k\in[-\pi,\pi]^d$. We write $|k|^2 \equiv \Sigma_{\mu=1}^{d} k_\mu^2$. We abbreviate the integral over the Brillouin zone as

$$\int \frac{d^d k}{(2\pi)^d} \equiv \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d}.$$

We now take the Fourier transform of both sides of (1.10) and (1.11) [for $m<m_p$ the Fourier transform of $\tau_p^{(m)}$ exists, because from (1.2), $\tau_p^{(m)}(0,x) \leqq e^{-(m_p-m)\|x\|_\infty}$]. We get

$$(1.14)\qquad \hat{\tau}_p^{(m)}(k) = \frac{\hat{G}_N^{(m)}(k)}{1-(p/p_G)\,\hat{D}^{(m)}(k) - \hat{\Pi}_{\leqq N}^{(m)}(k)}$$

and the corresponding equation for $m=0$. Here

$$(1.15)\qquad \hat{D}^{(m)}(k) \equiv \frac{\sum_y J_{0y}\, e^{my_1}\, e^{ik\cdot y}}{\sum_y J_{0y}}$$

and p_G is defined so that

$$(1.16)\qquad p_G \cdot \sum_y J_{0y} = 1.$$

Taking the inverse of (1.14) and the corresponding expression for $m=0$ (both for $k=0$) and subtracting, we get $[\chi_p^{(m)} \equiv \hat{\tau}_p^{(m)}(0) = \Sigma_x \tau_p^{(m)}(0,x)]$

$$(1.17)\qquad \chi_p^{-1} - (\chi_p^{(m)})^{-1} = \frac{(p/p_G)\,\{\hat{D}^{(m)}(0) - \hat{D}(0)\} + \hat{\Pi}_{\leqq N}^{(m)}(0) - \hat{\Pi}_{\leqq N}(0) + (\hat{G}_N^{(m)}(0) - \hat{G}_N(0))\,\chi_p^{-1}}{\hat{G}_N^{(m)}(0)}.$$

As for the left hand side, we have, using the Aizenman-Simon inequality,

Proposition 1.3. *For model* (i) (*respectively for model* (ii) *with* $g(x)$ *satisfying* (1.9)),

$$(1.18)\qquad \chi^{(m)} \nearrow \infty \qquad \textit{as } m \nearrow m_p$$

for all $p<p_c$ (*resp. for all* $p<p_c$ *for which* $m_p<\delta$).

Proof. This can be proven along the line of argument of [36, 6]. We include its proof in Appendix A for the convenience of the reader. □

Now for the quantities on the right hand side, we have the following proposition which is the main technical result of the paper. This proposition, in essence, states that in (1.17) $\hat{D}^{(m)}(0) - \hat{D}(0)$ is the main term, and that the rest (i.e. $\hat{\Pi}_{\leqq N}^{(m)}(0) - \hat{\Pi}_{\leqq N}(0)$, $\hat{G}_N^{(m)}(0) - \hat{G}_N(0)$) is a "correction" of higher order in d^{-1} or L^{-1}.

Proposition 1.4. *Consider model* (i) (*respectively model* (ii), *which satisfies the conditions of Theorem* 1.2). *Uniformly in* $p \in [p_G, p_c)$, *we have the following: For any* $\varepsilon > 0$ *there exists* $d_0 > 6$ (*resp.* $L_0 \geqq 1$) [d_0 *and* L_0 *are independent of* p] *such that for* $d \geqq d_0$ (*resp.* $L \geqq L_0$) *and for* $N \geqq N_0(\varepsilon)$

$$|\hat{\Pi}^{(m)}_{\leqq N}(0) - \hat{\Pi}_{\leqq N}(0)| \leqq \varepsilon m^2/d, \tag{1.19}$$

$$|\hat{G}^{(m)}_N(0) - \hat{G}_N(0)| \leqq \varepsilon m^2/d, \tag{1.20}$$

and

$$|\hat{G}^{(m)}_N(0) - 1| \leqq \varepsilon \tag{1.21}$$

hold for $0 < m < \min\{m_p, d^{-1/2}\}$ (resp. $0 < m < \min\{\delta L^{-2}, L^{-1}, m_p\}$). *Also for the model* (i), *for all* $m \geqq 0$

$$\hat{D}^{(m)}(0) - \hat{D}(0) = \frac{\cosh m - 1}{d} \geqq \frac{m^2}{2d} \tag{1.22}$$

and for the model (ii), *for sufficiently large* L,

$$\hat{D}^{(m)}(0) - \hat{D}(0) \begin{cases} \geqq m^2 L^2/3d & \text{for all } m \geqq 0 \\ \leqq m^2 L^2/d & \text{for } 0 \leqq m \leqq \delta L^{-2} \end{cases}. \tag{1.23}$$

Proof of Theorems 1.1 and 1.2, given Propositions 1.3 and 1.4. We choose various constants in the following way: (0) Fix $\varepsilon < 1/10$. (1) Fix $d \geqq d_0$ (model (i)) or $L \geqq L_0$ (model (ii)) and $N \geqq N_0(\varepsilon)$ so that (1.19) to (1.22) or (1.23) of Prop. 1.4 should hold. (2) Take $p (p_G \leqq p < p_c)$ sufficiently close to p_c so that $m_p < d^{-1/2}$ (model (i)) or $m_p < \min\{L^{-1}, \delta L^{-2}\}$ (model (ii)). Existence of such p is guaranteed, because $\xi(p) \nearrow \infty$ as $p \nearrow p_c$. (3) Now let $m \nearrow m_p$ for any fixed p chosen in (2).

As $m \nearrow m_p$, by Prop. 1.3, $\chi_p^{(m)} \nearrow \infty$, and thus by Prop. 1.4 and by (1.17) (we know, from [27], that $p/p_G \leqq p_c/p_G \leqq 3$),

$$\begin{aligned} &m_p^2/3d \leqq \chi_p^{-1} \leqq m_p^2/d && (\text{model (i), for } 0 < m_p \leqq d^{-1/2}) \\ &L^2 m_p^2/5d \leqq \chi_p^{-1} \leqq 4L^2 m_p^2/d && (\text{model (ii), for } 0 < m_p \leqq \min\{L^{-1}, \delta L^{-2}\}). \end{aligned} \tag{1.24}$$

Because it has been proven $\chi_p \sim (p_c - p)^{-1}$ for these models [27], this immediately implies our main theorems. (More precisely this proves (1.7) for p close to p_c as chosen above. However, this can be easily extended to smaller p down to, say, $p_c/2$, because both χ_p and $\xi(p)$ are uniformly finite and positive.) □

Remark. In [27], it was shown (for p close to p_c)

$$\begin{aligned} &\xi_2(p)^2 \sim \chi_p && \text{for model (i),} \\ &\xi_2(p^2) \sim L^2 \chi_p && \text{for model (ii).} \end{aligned}$$

On the other hand, the above (1.24) can be written as

$$\begin{aligned} &\xi(p)^2 \sim \chi_p/d && \text{for model (i),} \\ &\xi(p)^2 \sim L^2 \chi_p/d && \text{for model (ii).} \end{aligned}$$

The difference between $\xi(p)$ and $\xi_2(p)$, $\xi(p)^2 \sim \xi_2(p)^2/d$, can be explained by the fact that $\xi_2(p)$ measures the distance in $\|x\|_2$-norm, whereas $\xi(p)$ measures in $\|x\|_\infty$-norm.

The above Prop. 1.4 is similar to the result of [27], especially Prop. 4.3 and Lemma 4.5, that in (1.14) (for $m=0$) $\hat{G}_N(k) \approx 1$ and $1-(p/p_G)\,\hat{D}(k)$ give the main contribution, and that the rest is a correction of order ε. So the proof of Prop. 1.4 is carried out in a way parallel to that of Theorem 1.1 and Theorem 1.2 of [27].

For the convenience of the reader who has some knowledge of the method of [27], we briefly explain the method of proof of Prop. 1.4. In [27], we introduced quantities like T, W, and proved (i) the continuity of T, W in p (ii) T, $W \leqq 4\varepsilon \Rightarrow T$, $W \leqq 3\varepsilon$, for $p<p_c$. Because $W=T=0$ at $p=0$ (more precisely, T, $W \leqq \varepsilon$ for p near zero), it then followed that T, $W \leqq 3\varepsilon$ for all $p<p_c$. In this paper, to prove Prop. 1.4, we follow similar steps for e^{mx_1}-weighted quantities, but now with the parameter m playing the rôle of p as follows. We first introduce $T^{(m)}$, $W^{(m)}$, defined by replacing τ_p by $\tau_p^{(m)}$ (see Sect. 3 for precise definition). We then prove (i) the continuity of $T^{(m)}$ and $W^{(m)}$ in m (instead of p), (ii) $T^{(m)}$, $W^{(m)} \leqq 4\varepsilon \Rightarrow T^{(m)}$, $W^{(m)} \leqq 3\varepsilon$, for $|m|<m_p$. Because $T^{(m)}$ and $W^{(m)}$ coincide with T and W at $m=0$, by the result of [27] (i.e. T, $W \leqq 3\varepsilon$) it follows immediately that $T^{(m)}$, $W^{(m)} \leqq 3\varepsilon$. This in turn implies $\hat{\Pi}^{(m)}_{\leqq N}$ etc. is very small, that is, Prop. 1.4.

1.4. *Organization of the paper*

This work is a natural extension of that reported in [27]. Although considerable effort has been made to make the presentation rather self-contained, readers are advised to read (Sect. 1 of) [27] first to get some understanding of the basic strategy of [27]. The rest of the paper is organized as follows:

In Sect. 2, we recall the key identity for the two point function derived in [27]. We also present the corresponding identity for $\tau_p^{(m)}$. In Sect. 3, as in Sect. 3 of [27], we show how to bound each term of the identity in terms of basic quantities, like $T^{(m)}$, $W^{(m)}$. Based on this, we prove Prop. 1.3 for the spread-out model (model ii) in Sect. 4, and for the nearest-neighbour model (model i) in Sect. 5. In Appendix A, we briefly summarize basic properties of the two point function and the correlation length, and in Appendix B, we prove several properties of gaussian (simple random walk) models used in Sects. 4 and 5.

2. The expansion

In this section, as a preparation for the proof of Prop. 1.4, we recall the identity for the two point function τ_p derived in [27] and study its direct consequences. This section contains the analysis which corresponds to Sect. 2 of [27].

Throughout the paper, we use the following diagrammatic notation. The two point function $\tau_p(x, y)$ is represented by a straight line x——y. (Note that in [27], the two point function was represented by a wavy line.) We use a wavy line to represent e^{mx_1}-weighted two point function: $\tau_p^{(m)}(x, y) = x \sim\!\sim\!\sim y$. A pair of bars $y \,||\, y'$ represents $p_{yy'}$. We follow the convention that *unlabelled vertices are summed over*. For example,

0  $= \sum_{y, y'} \tau_p(0, y)\, p_{yy'}\, \tau_p^{(m)}(y', 0)$

We also use the convention as in [27] that in unshaded loops the summation over their vertices is constrained so that at least two of the vertices must be distinct. Shaded loops have no restrictions.

In [27], an identity for the two point function was derived [27, Prop. 2.3],

$$\tau_p(0,x)=\delta_{0,x}+\sum_{n=0}^{N}(-1)^n g_n(0,x)+(-1)^{N+1}R_N(0,x) + \sum_y p_{0y}\,\tau_p(y,x)+\sum_{n=0}^{N}(-1)^n\sum_{y'}\Pi_n(0,y')\,\tau_p(y',x). \tag{2.1}$$

Here $g_n(0,x)$, $R_N(0,x)$ and $\Pi_n(0,x)$ are even functions in each x_μ, and obey the bound [27, Prop. 2.4]:

$$\begin{aligned} &0\leqq g_n(0,x)\ \leqq h_n(0,x),\\ &0\leqq \Pi_n(0,y')\leqq \sum_y h_n(0,y)\,p_{yy'},\\ &0\leqq R_N(0,x)\leqq \sum_y h_N(0,y)\,p_{yy'}\,\tau_p(y',x) \end{aligned} \tag{2.2}$$

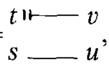
where h_n are defined as

$$h_0(0,x)=\tau_p(0,x)^2\cdot I[x\neq 0]$$

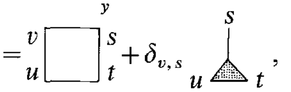
and for $n\geqq 1$

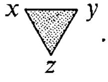
$$h_n(0,x)=\sum_{s_k,t_k,u_k,v_k} A_3(0,s_1,t_1)\prod_{i=1}^{n} B_1(s_i,t_i,u_i,v_i) \cdot\prod_{j=2}^{n} B_2(u_{j-1},v_{j-1},s_j,t_j)\,A_3(u_n,v_n,x) \tag{2.3}$$

with

$$B_1(s,t,u,v)\equiv\sum_y p_{ty}\,\tau_p(y,v)\,\tau_p(s,u)=$$ ,

$$B_2(u,v,s,t)\equiv\tau_p(v,s)\,\tau_p(s,t)\,\tau_p(t,u)\,\tau_p(u,v)\,\{1-I[v=s=t=u]\} +\delta_{v,s}\sum_y\tau_p(s,y)\,\tau_p(y,t)\,\tau_p(t,u)\,\tau_p(u,y)$$

$=$ ,

$$A_3(x,y,z)\equiv\tau_p(x,y)\,\tau_p(y,z)\,\tau_p(z,x)=$$ .

We illustrate $h_0(0,x)$, $h_1(0,x)$ and $h_2(0,x)$ in Fig. 1.

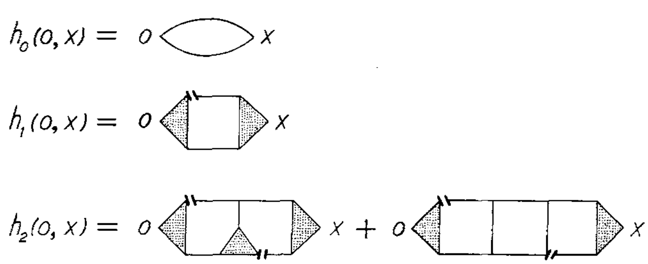


Fig. 1. Diagrammatic representation for $h_N(0,x)$ for $N=0,1,2$.

Multiplying both sides of (2.1) by e^{mx_1}, we can immediately get an identity for $\tau_p^{(m)}(0,x) \equiv \tau_p(0,x)\, e^{mx_1}$:

$$\tau_p^{(m)}(0,x) = \delta_{0,x} + \sum_{n=0}^{N} (-1)^n g_n^{(m)}(0,x) + (-1)^{N+1} R_N^{(m)}(0,x) \tag{2.4}$$
$$+ \sum_y p_{0y}^{(m)} \tau_p^{(m)}(y,x) + \sum_{n=0}^{N} (-1)^n \sum_{y'} \Pi_n^{(m)}(0,y')\, \tau_p^{(m)}(y',x)$$

where we followed the convention of (1.12). Taking the Fourier transform, we get [following the convention of (1.13)]

$$\hat{\tau}_p^{(m)}(k) = \frac{\hat{G}_N^{(m)}(k)}{1 - (p/p_G)\,\hat{D}^{(m)}(k) - \hat{\Pi}_{\leqq N}^{(m)}(k)} \tag{2.5}$$

where

$$\hat{G}_N^{(m)}(k) \equiv 1 + \sum_{n=0}^{N} (-1)^n \hat{g}_n^{(m)}(k) + (-1)^{N+1} \hat{R}_N^{(m)}(k),$$
$$\hat{\Pi}_{\leqq N}^{(m)}(k) \equiv \sum_{n=0}^{N} (-1)^n \hat{\Pi}_n^{(m)}(k),$$

and $\hat{D}^{(m)}(k)$ and p_G are defined by (1.15) and (1.16).

As an immediate consequence of the identity (2.4), we have the following lemma, which corresponds to Prop. 2.6 of [27].

Proposition 2.1. *The above Fourier transforms satisfy the following*:

$$|\hat{g}_n^{(m)}(k)| \leqq \sum_x h_n^{(m)}(0,x), \tag{2.6}$$

$$|\operatorname{Im} \hat{g}_n^{(m)}(k)| \leqq m|k_1| \sum_x h_n^{(m)}(0,x)\, |x_1|^2, \tag{2.7}$$

$$|\hat{\Pi}_n^{(m)}(k)| \leqq \sum_v p_{0v}^{(m)} \sum_x h_n^{(m)}(0,x), \tag{2.8}$$

$$|\operatorname{Im} \hat{\Pi}_n^{(m)}(k)| \leqq 2m|k_1| \{\sum_v p_{0v}^{(m)} \sum_x h_n^{(m)}(0,x)\, |x_1|^2 + \sum_v p_{0v}^{(m)} |v_1|^2 \sum_x h_n^{(m)}(0,x)\}, \tag{2.9}$$

$$0 \leqq \operatorname{Re}(\hat{\Pi}_n^{(m)}(0) - \hat{\Pi}_n^{(m)}(k)) \tag{2.10}$$
$$\leqq \sum_{\mu=1}^{d} k_\mu^2 \{\sum_v p_{0v}^{(m)} \sum_x h_n^{(m)}(0,x)\, |x_\mu|^2 + \sum_v p_{0v}^{(m)} |v_\mu|^2 \sum_x h_n^{(m)}(0,x)\},$$

$$|\hat{R}_N^{(m)}(k)| \leqq \sum_x \tau_p^{(m)}(0,x) \sum_v p_{0v}^{(m)} \sum_y h_N^{(m)}(0,y), \tag{2.11}$$

$$|\operatorname{Im} \hat{R}_N^{(m)}(k)| \leqq 3m|k_1| \Bigg[\sum_x \tau_p^{(m)}(0,x) \sum_v p_{0v}^{(m)} \sum_y h_N^{(m)}(0,y)\, |y_1|^2 \tag{2.12}$$
$$+ \sum_x \tau_p^{(m)}(0,x) \sum_v p_{0v}^{(m)} |v_1|^2 \sum_y h_N^{(m)}(0,y)$$

$$+\sum_x \tau_p^{(m)}(0,x)\,|x_1|^2 \sum_v p_{0v}^{(m)} \sum_y h_N^{(m)}(0,y)\Big],$$

for $s=1,2\,(\partial_\mu \equiv \partial/\partial k_\mu)$

(2.13) $$|\partial_\mu^s \hat{g}_n^{(m)}(k)| \leqq \sum_x h_n^{(m)}(0,x)\,|x_\mu|^2,$$

(2.14) $$|\partial_\mu^s \hat{\Pi}_n^{(m)}(k)| \leqq 2\sum_v p_{0v}^{(m)} \sum_x h_n^{(m)}(0,x)\,|x_\mu|^2 + 2\sum_v p_{0v}^{(m)}|v_\mu|^2 \sum_x h_n^{(m)}(0,x),$$

(2.15) $$\begin{aligned}|\partial_\mu^s \hat{R}_N^{(m)}(k)| \leqq\ & 3\sum_x \tau_p^{(m)}(0,x)\,\{\sum_v p_{0v}^{(m)} \sum_y h_N^{(m)}(0,y)\,|y_\mu|^2 \\ & + \sum_v p_{0v}^{(m)}|v_\mu|^2 \sum_y h_N^{(m)}(0,y)\} \\ & + 3\sum_x \tau_p^{(m)}(0,x)\,|x_\mu|^2 \sum_v p_{0v}^{(m)} \sum_y h_N^{(m)}(0,y),\end{aligned}$$

and

(2.16) $$0 \leqq \hat{g}_n^{(m)}(0) - \hat{g}_n(0) \leqq \frac{m^2}{2} \sum_x h_n^{(m)}(0,x)\,|x_1|^2,$$

(2.17) $$\begin{aligned}0 \leqq\ & \hat{R}_N^{(m)}(0) - \hat{R}_N(0) \\ \leqq\ & \frac{3m^2}{2}\Big[\sum_x \tau_p^{(m)}(0,x)\,\{\sum_v p_{0v}^{(m)} \sum_y h_N^{(m)}(0,y)\,|y_1|^2 + \sum_v p_{0v}^{(m)}|v_1|^2 \sum_y h_N^{(m)}(0,y)\} \\ & + \sum_x \tau_p^{(m)}(0,x)\,|x_1|^2 \sum_v p_{0v}^{(m)} \sum_y h_N^{(m)}(0,y)\Big],\end{aligned}$$

(2.18) $$0 \leqq \hat{\Pi}_n^{(m)}(0) - \hat{\Pi}_n(0) \leqq m^2 \{\sum_v p_{0v}^{(m)} \sum_x h_n^{(m)}(0,x)\,|x_1|^2 + \sum_v p_{0v}^{(m)}|v_1|^2 \sum_x h_n^{(m)}(0,x)\}$$

Proof. (2.6), (2.8), (2.11) and (2.13) to (2.15) follow in exactly the same way as in the proof of Prop. 2.6 of [27], once one notices that $\Pi_n^{(m)}$, $g_n^{(m)}$ satisfy the bounds

(2.19) $$0 \leqq g_n^{(m)}(0,x) \leqq h_n(0,x)\,e^{mx_1} \equiv h_n^{(m)}(0,x),$$

(2.20) $$0 \leqq \Pi_n^{(m)}(0,y') \leqq \sum_y h_n^{(m)}(0,y)\,p_{yy'}^{(m)}$$

which follow immediately from (2.2). For example for (2.6),

$$|\hat{g}_n^{(m)}(k)| = |\sum_x e^{ik\cdot x} g_n^{(m)}(0,x)| \leqq \sum_x g_n^{(m)}(0,x) \leqq \sum_x h_n^{(m)}(0,x).$$

(2.10) is proved similarly. We first write

$$\begin{aligned}\mathrm{Re}(\hat{\Pi}_n^{(m)}(0) - \hat{\Pi}_n^{(m)}(k)) &= \sum_x (1-\cos(k\cdot x))\,\Pi_n^{(m)}(0,x) \\ &\leq \sum_{\mu,\nu=1}^d \frac{k_\mu k_\nu}{2} x_\mu x_\nu\, \Pi_n^{(m)}(0,x) = \sum_{\mu=1}^d \frac{k_\mu^2}{2} x_\mu^2\, \Pi_n^{(m)}(0,x)\end{aligned}$$

and then use (2.20) with the triangle inequality. In the above, contribution from $\mu \neq \nu$ terms vanishes because $\Pi_n^{(m)}(0, x)$ is an even function in each $x_2, \ldots, x_d$.

To prove (2.7) we first observe, because $g_n(0, x)$ is even in each $x_1, \ldots, x_d$, that we can write

$$\begin{aligned} \text{(2.21)} \qquad \operatorname{Im} \hat{g}_n^{(m)}(k) &\equiv \operatorname{Im} \sum_x g_n(0, x)\, e^{m x_1} e^{ik \cdot x} \\ &= \sum_x g_n(0, x)\, e^{m x_1} \sin(k_1 x_1) \cdot \prod_{\nu=2}^{d} \cos(k_\nu x_\nu) \\ &= \sum_x g_n(0, x) \sinh(m x_1) \sin(k_1 x_1) \cdot \prod_{\nu=2}^{d} \cos(k_\nu x_\nu). \end{aligned}$$

Taking the absolute value,

$$\begin{aligned} \text{(2.22)} \qquad |\operatorname{Im} \hat{g}_n^{(m)}(k)| &\leqq \sum_x g_n(0, x)\, |\sinh(m x_1)| \cdot |\sin(k_1 x_1)| \\ &= \sum_x g_n(0, x) \cosh(m x_1)\, |\tanh(m x_1)| \cdot |\sin(k_1 x_1)| \\ &\leqq \sum_x g_n(0, x) \cosh(m x_1)\, m|x_1| \cdot |k_1 x_1| \\ &= m|k_1| \sum_x g_n^{(m)}(0, x)\, |x_1|^2 \leqq m|k_1| \sum_x h_n^{(m)}(0, x)\, |x_1|^2. \end{aligned}$$

(2.9) and (2.12) are proved similarly.

(2.16) is proved as follows: we first write

$$\hat{g}_n^{(m)}(0) - \hat{g}_n(0) = \sum_x g_n(0, x)\,(e^{m x_1} - 1) = \sum_x g_n(0, x)\,[\cosh(m x_1) - 1] \geqq 0$$

where we used the symmetry of $g_n (g_n(0, x) = g_n(0, -x))$. Now we apply an elementary inequality

$$\cosh(mx) - 1 \leqq \frac{m^2 x^2}{2} \cosh(mx) \qquad \text{for } m, x \in \mathbb{R}$$

to the right hand side to get

$$0 \leqq \hat{g}_n^{(m)}(0) - \hat{g}_n(0) \leqq \sum_x \frac{m^2 x_1^2}{2} g_n(0, x) \cosh(m x_1) = \sum_x \frac{m^2 x_1^2}{2} g_n(0, x)\, e^{m x_1}$$

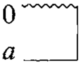
(we again used the symmetry of g_n) and use (2.2). Proofs of (2.17) and (2.18) are similar. □

3. Diagrammatic estimates

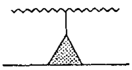
In the previous section, an identity for the ($e^{m x_1}$-weighted) two point function $\tau_p^{(m)}$ was obtained from the corresponding identity for τ_p derived in [27]. Now as in Sect. 3 of [27], we bound the terms appearing in the right hand sides of the inequalities of Prop. 2.1 in terms of the quantities $\bar{T}^{(m)}$, $\bar{W}_\mu^{(m)}$, $\bar{H}_\mu^{(m)}$, which are introduced in the next definition. These quantities form the "e^{mx_1}-weighted" version of the quantities $\bar{T}$, $\bar{W}$, and $\bar{H}$ of [27]. In this section, we extensively use the diagrammatic notation introduced at the beginning of Sect. 2.

Definition 3.1. *For a, a_1 and a_2 in $\mathbb{Z}^d$, we define*

$$B^{(2m)} \equiv \sum_{x \neq 0} (\tau_p^{(m)}(0,x))^2 = 0$$ 

$$T_a^{(m)} \equiv \sum_{x,y} \tau_p^{(m)}(0,x)\,\tau_p(x,y)\,\tau_p(y,a) - \delta_{0,a}\,\tau_p(0,0)^3 = {}^{0}_{a}$$ 

$$W_{a,\mu}^{(m)} \equiv \sum_x |x_\mu|^2\,\tau_p(0,x)\,\tau_p^{(m)}(a,x) = \sum_x {}^{0}_{a}$$  $x\ |x_\mu|^2$

$$H_{a_1,a_2,\mu}^{(m)} \equiv \sum_{x,y,z} |x|^2$$ 

$z+a_1$, $y+a_2$, z, 0, x, y.

We write $T_0^{(m)}$ and $W_{0,\mu}^{(m)}$ simply as $T^{(m)}$ and $W_\mu^{(m)}$, and define

$$\bar{T}^{(m)} \equiv \sup_a T_a^{(m)}, \qquad \bar{W}_\mu^{(m)} \equiv \sup_a W_{a,\mu}^{(m)}, \qquad \bar{H}_\mu^{(m)} \equiv \sup_{a_1,a_2} H_{a_1,a_2,\mu}^{(m)}.$$

We also define

$$W_{a,\mu}^{\prime(m)} \equiv \sum_{x,y} |x_\mu|^2\,p_{0y}\,\tau_p(y,x)\,\tau_p^{(m)}(a,x) = \sum_x {}^{0}_{a}$$  $x\ |x_\mu|^2$,

$$\bar{W}_\mu^{\prime(m)} \equiv \sup_a W_{a,\mu}^{\prime(m)},$$

and write $W_\mu^{\prime(m)}$ for $W_{0,\mu}^{\prime(m)}$. Quantities without the superscript (m) denote those with $m=0$.

Remark. The above $B^{(2m)}$, $T^{(m)}$ and $W_\mu^{(m)}$ are increasing functions of $|m|$. Also $\sum_v p_{0v}^{(m)}$, $\sum_v p_{0v}^{(m)} |v_\mu|^2$ are increasing in $|m|$. These can be seen by rewriting them in terms of $\cosh(mx_1)$ instead of e^{mx_1}, using the symmetry of $\tau_p(0,x)$ or p_{0x}.

Now we can state the following two lemmas:

Lemma 3.2. (*a*) *For $n=0$,*

$$0 \leqq \sum_x h_0^{(m)}(0,x) \leqq \frac{T^{(m)}}{2}$$

and for $n \geqq 1$,

$$\begin{aligned} (3.1) \qquad 0 &\leqq \sum_x h_n^{(m)}(0,x) \\ &\leqq \{\sum_v p_{0v}(B \cdot B^{(2m)})^{1/2} + (\sum_v p_{0v}^2 (B + B^{(2m)}))^{1/2} + 2 \sup_a p_{0a}\} \\ &\quad \cdot (1+T^{(m)})^2 (r^{(m)})^{n-1} \end{aligned}$$

where we defined

$$r^{(m)} \equiv (1+T^{(m)}+\bar{T}^{(m)})\,((\sum_v p_{0v}^{(m)})\,\bar{T}^{(m)}+\sup_v p_{0v}^{(m)}). \tag{3.2}$$

(*b*) *For* $\mu=1,2,\ldots,d$,

$$\begin{aligned} 0 &\leqq \sum_x h_n^{(m)}(0,x)\,|x_\mu|^2 \\ &\leqq \begin{cases} W_\mu^{(m)} & (n=0) \\ W_\mu^{\prime(m)}+10r^{(m)}\,\bar{W}_\mu^{(m)} & (n=1) \\ (2n+1)\left\{\dfrac{n}{2}\sum_v p_{0v}^{(m)}+(n+1)\,r^{(m)}\right\}(1+T^{(m)})^2\,(r^{(m)})^{n-1}\cdot\bar{W}_\mu^{(m)} & \\ \quad +(2n+1)\left[\!\left[\dfrac{n}{2}\right]\!\right](1+T^{(m)})^2\,(r^{(m)})^{n-1}\cdot\bar{W}_\mu^{\prime(m)} & (n\geqq 2). \\ \quad +(2n+1)\left[\!\left[\dfrac{n-1}{2}\right]\!\right](1+T^{(m)})^2\,(r^{(m)})^{n-2}(\sum_v p_{0v}^{(m)})^2\cdot\bar{H}_\mu^{(m)} & \end{cases} \end{aligned} \tag{3.3}$$

In the above, $[\![x]\!]$ *denotes the largest integer which does not exceed* x.

The proof of this lemma is carried out in a way similar to that of Lemma 3.2 of [27]. Before proceeding to the proof, we state another lemma, which gives bounds on some of the quantities appearing in the above lemma.

Lemma 3.3.

$$\bar{T}\leqq T+\sqrt{\frac{3T}{2d}},$$

$$\bar{T}^{(m)}\leqq 2T^{(m)}+4(B\cdot B^{(2m)})^{1/2}+4(B/2d)^{1/2}+2(B^{(2m)})^{1/2}.$$

For $\mu=1,2,\ldots,d$,

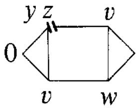
$$\begin{aligned} W_\mu^{\prime(m)} &\leqq \sum_v p_{0v}^{(m)}\,W_\mu^{(m)} \\ &\quad +\{(\sum_v p_{0v}^2\,e^{mv_1}|v_\mu|^2)^{1/2}+(\sum_u p_{0u}^{(m)}\sum_v p_{0v}^{(m)}|v_\mu|^2)^{1/2}\,(B^{(2m)})^{1/2}\}\,(W_\mu^{(m)})^{1/2}, \\ \bar{W}_\mu^{\prime(m)} &\leqq 2\{(\sum_v p_{0v}^{(m)}+1)\,\bar{W}_\mu^{(m)}+\bar{W}_\mu+\sum_v p_{0v}^{(m)}|v_\mu|^2\,(B\cdot B^{(2m)})^{1/2}\}. \end{aligned}$$

Sketch of the Proof of Lemma 3.2. (a) The proof is carried out exactly as in [27]. There we wrote $\sum_x h_n(0,x)$ in the form $\sum_y f(y)\,g(y)$ and used the basic inequality:

$$\sum_x f(x)\,g(x)\leqq \sup_x|f(x)|\sum_x|g(x)| \tag{3.4}$$

to decompose it into its basic unit (i.e. T_a). Here, we have an extra weight factor e^{mx_1} multiplying $h_n(0,x)$, but because of the multiplicative property of the exponential function (i.e. $e^x=e^{x-y}\cdot e^y$) this factor can be expressed as a product of exponentials multiplying each unit, thus giving rise to $T_a^{(m)}$, instead of T_a.

(b) The proof proceeds similarly. Now we have two weight factors: $|x_\mu|^2$ and e^{mx_1} both multiplying $h_n(0,x)$. The idea is to express these two as a sum and a product over the two lines connecting 0 and x in the diagrammatic representation of $h_n(0,x)$. (Recall Fig. 1.) To illustrate the idea, we consider the contribution for $n=1$ from the term in which *neither* of the triangles is a point:

$$\sum_{x,v,w,y,z,u}$$  $x\, e^{mx_1}|x_\mu|^2.$

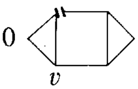
We use the triangle inequality for $|x_\mu|^2$

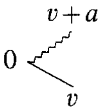
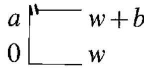
$$|x_\mu|^2 \leqq 3(|v_\mu|^2+|v_\mu-w_\mu|^2+|w_\mu-x_\mu|^2) \tag{3.5}$$

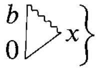
on one hand, and use the multiplicative property of the exponential

$$e^{mx_1}=e^{my_1}\cdot e^{m(z_1-y_1)}\cdot e^{m(u_1-z_1)}\cdot e^{m(x_1-u_1)}$$

on the other. Now use (3.4) twice to bound the resulting expression by a product of each constituent. For example, the term coming from the first term on the right hand side of (3.5) is bounded as (without the factor 3)

$\sum_{x,v}|v_\mu|^2$  $x \quad e^{mx_1}$

$\leqq\Big\{\sup_a\sum_v$  $|v_\mu|^2\Big\}\Big\{\sup_b\sum_{a,w} e^{m(w_1+b_1-a_1)}$  $\Big\}$

$\cdot\Big\{\sum_{b,x}$  $\Big\}.$

The first factor is exactly $W_{-a,\mu}^{(m)}$. For the second term we can use Lemma 3.4 of [27], and the third term is just $T^{(m)}$. Other and higher order terms are treated similarly. We omit the details of these straightforward but tedious calculations. □

Proof of Lemma 3.3. The first inequality is proved in Lemma 3.3 of [27]. The proof of the other inequalities is carried out almost in parallel to that of Lemma 3.3 of [27]. The only subtlety here is that (because of the factor e^{mx_1}) full $\mathbb{Z}^d$-symmetry is absent, and moreover the Fourier transform of $\tau_p^{(m)}$ is not necessarily real. We find it convenient to consider symmetrized quantities, such as $\{T_a^{(m)}+T_{-a}^{(m)}\}/2$ and

$$\tau_{\mathrm{sym}}^{(m)}(0,x)\equiv\tfrac{1}{2}\{\tau_p^{(m)}(0,x)+\tau_p^{(m)}(0,-x)\}=\tau_p(0,x)\cdot\cosh(mx_1). \tag{3.6}$$

For $\bar{T}^{(m)}$, we first write (for $a \neq 0$)

$$\frac{1}{2}(T_a^{(m)} + T_{-a}^{(m)}) = \sum_{x,y} \tau_{\mathrm{sym}}^{(m)}(0,x)\,\tau_p(x,y)\,\tau_p(y,a) \tag{3.7}$$

$$= \int \frac{d^d k}{(2\pi)^d} \hat{\tau}_{\mathrm{sym}}^{(m)}(k)\,(\hat{\tau}_p(k)-1)^2\, e^{-ik\cdot a}$$

$$+ 2\sum_x \tau_{\mathrm{sym}}^{(m)}(0,x) - \delta_{0,x})\,(\tau_p(a,x)-\delta_{a,x}) + \tau_{\mathrm{sym}}^{(m)}(0,a) + 2\tau_p(0,a).$$

We can also write $T^{(m)}$ as

$$T^{(m)} = \int \frac{d^d k}{(2\pi)^d} \{\hat{\tau}_{\mathrm{sym}}^{(m)}(k)\,\hat{\tau}_p(k)^2 - 1\} \geqq \int \frac{d^d k}{(2\pi)^d} \hat{\tau}_{\mathrm{sym}}^{(m)}(k)\,(\hat{\tau}_p(k)-1)^2.$$

Using the Schwartz inequality, (3.7) can be bounded as

$$\leqq T^{(m)} + 2\Big(\sum_{x\neq 0} \tau_p(0,x)^2 \cdot \sum_{x\neq 0} (\tau_{\mathrm{sym}}^{(m)}(0,x))^2\Big)^{1/2} + \tau_{\mathrm{sym}}^{(m)}(0,a) + 2\tau_p(0,a). \tag{3.8}$$

We now use

$$\sum_{x\neq 0} (\tau_{\mathrm{sym}}^{(m)}(0,x))^2 = \frac{1}{2}(B^{(2m)} + B) \leqq B^{(2m)} \tag{3.9}$$

to bound the last three terms.

$W_\mu^{\prime(m)}$ and $\bar{W}_\mu^{\prime(m)}$ are treated in a similar way as follows. First, for $W_\mu^{\prime(m)}$, by the triangle inequality,

$$W_\mu^{\prime(m)} \leqq \sum_{x,y} p_{0y}^{(m/2)}\,\tau_p^{(m/2)}(y,x)\,\tau_p^{(m/2)}(0,x)\cdot(|x_\mu - y_\mu| + |y_\mu|)\,|x_\mu|.$$

The first term is bounded as

$$\leqq \sum_y p_{0y}^{(m/2)} \sup_y \Big\{\sum_x \tau_p^{(m/2)}(y,x)\,|y_\mu - x_\mu| \cdot \tau_p^{(m/2)}(0,x)\,|x_\mu|\Big\} \leqq \sum_y p_{0y}^{(m/2)}\, W_\mu^{(m)}$$

where in the last step we used Schwarz inequality. The second term is bounded as

$$= \sum_y p_{0y}^{(m/2)} |y_\mu| \Big\{ \sum_{x:x\neq y} \tau_p^{(m/2)}(y,x)\,\tau_p^{(m/2)}(0,x)\,|x_\mu| + |y_\mu|\,\tau_p^{(m/2)}(0,y)\Big\}$$

$$\leqq \sum_y p_{0y}^{(m/2)} |y_\mu|\,(B^{(m)} W_\mu^{(m)})^{1/2} + \Big(\sum_y (p_{0y}^{(m/2)})^2\,|y_\mu|^2\, W_\mu^{(m)}\Big)^{1/2}.$$

As for $\bar{W}_\mu^{\prime(m)}$, we first use the triangle inequality:

$$W_{a,\mu}^{\prime(m)} \leqq 2\sum_{y,x} p_{0y}\,\tau_p(y,x)\,\tau_p^{(m)}(a,x)\,(|y_\mu|^2 + |y_\mu - x_\mu|^2).$$

For the first term, we first use the trivial inequality $p_{xy} \leqq \tau_p(x, y)$ and then use the Schwarz inequality:

$$\leqq 2\sum_y p_{0y}|y_\mu|^2 [\sum_{x:x \neq a, y} \tau_p(y, x)\, \tau_p^{(m)}(a, x) + \tau_p^{(m)}(a, y) + \tau_p(y, a)]$$

$$\leqq 2\sum_y p_{0y}|y_\mu|^2 [\sum_{x:x \neq a, y} \tau_p(y, x)\, \tau_p^{(m)}(a, x)]$$

$$+ 2\sum_y \tau_p(0, y)\, |y_\mu|^2 [\tau_p^{(m)}(a, y) + \tau_p(y, a)]$$

$$\leqq 2\sum_y p_{0y}|y_\mu|^2 (B \cdot B^{(2m)})^{1/2} + 2(W_{a,\mu}^{(m)} + W_{a,\mu}).$$

The second term is simply bounded as

$$\leqq 2\sum_y p_{0y} \cdot \sup_y \{\sum_x \tau_p(y, x)\, \tau_p^{(m)}(a, x)\, |y_\mu - x_\mu|^2\} = 2\sum_y p_{0y} \cdot \bar{W}_\mu^{(m)}. \quad \square$$

4. Proof of Proposition 1.4 for the spread-out model

In this section, we use the results of the previous sections to prove Prop. 1.4 for the spread-out model (model ii). According to Sect. 1.3, this completes the proof of one of our main results, Theorem 1.2.

The structure of the proof of Prop. 1.4 itself is similar to that of Theorem 1.1 and Theorem 1.2 of [27], and is described in the following Sect. 4.1. Actually, the conclusions of Prop. 1.4 can be proven for a wider class of models, whose gaussian quantities (defined below) satisfy the conclusions of the following Lemma 4.1. In this section, we prove Prop. 1.4 for those models, and postpone the proof of Lemma 4.1 to Appendix B.

To state Lemma 4.1, we introduce several definitions. These quantities are defined in terms of a gaussian (simple random walk) theory corresponding to the model, and are relatively easily calculated. For fixed $\{J_b\}_b$, we first define p_G and $\{p_{0x}^{(G)}\}$ so that

$$p_G \cdot \sum_v J_{0v} = 1, \qquad p_{0x}^{(G)} \equiv p_G \cdot J_{0x},$$

and then introduce

$$\hat{D}(k) \equiv p_G \cdot \sum_x J_{0x}\, e^{ik \cdot x}, \qquad \hat{C}(k) \equiv (1 - \hat{D}(k))^{-1}.$$

Note that $\hat{C}$ defines (in k-space) the gaussian propagator (or two point function) of a random walk whose transition probability is given by $p_G \cdot J_{0x} = p_{0x}^{(G)}$. We introduce the e^{mx_1}-weighted gaussian propagator

$$C^{(m)}(0, x) = C(0, x) \cdot e^{mx_1}$$

and define T_G, $T_G^{(m)}$ etc. by replacing τ_p (respectively $\tau_p^{(m)}$) by C (resp. $C^{(m)}$) in the definitions of T, $T^{(m)}$. We also introduce

$$\hat{D}^{(m)}(k) \equiv p_G \cdot \sum_x J_{0x} e^{mx_1} e^{ikx}, \qquad S^{(m)} \equiv d \sum_x p_{0x}^{(G)} e^{mx_1} |x_1|^2.$$

Now we can state our lemma on the gaussian quantities defined above:

Lemma 4.1. *Consider the model* (ii) *of Sect.* 1.1 *which satisfies the conditions of Theorem* 1.2 *on the d-dimensional hypercubic lattice with* $d>6$. *For any* $\varepsilon>0$, *there exists* $L_0(\varepsilon; d, g) \geqq 1$ *such that for all* $L \geqq L_0$ *and for all* $0 \leqq m < \delta L^{-2}$, *the followings are satisfied*:

(4.1) $$\sup_{x \neq 0} p_{0x}^{(G)} e^{mx_1}, \qquad \sum_x (p_{0x}^{(G)} e^{mx_1})^2 \leqq \frac{\varepsilon}{S^{(m)}},$$

(4.2) $$\sup_x p_{0x}^{(G)} e^{mx_1} |x_1|^2, \qquad \sum_x (p_{0x}^{(G)})^2 e^{mx_1} |x_1|^2 \leqq \frac{\varepsilon}{d},$$

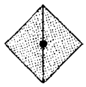
(4.3) $$1 - \hat{D}(k) \geqq \frac{|k|^2}{3\pi^2 d},$$

(4.4) $$0 \leqq C(0,0) - 1 = \int \frac{d^d k}{(2\pi)^d} \frac{\hat{D}(k)}{1 - \hat{D}(k)} \leqq \frac{\varepsilon}{3 S^{(m)}},$$

(4.5) $$T_G \leqq \frac{\varepsilon}{3 S^{(m)}},$$

(4.6) $$W_G = \sum_{\mu=1}^{d} \int \frac{d^d k}{(2\pi)^d} \frac{|\partial_\mu \hat{D}(k)|^2}{(1 - \hat{D}(k))^4} \leqq \frac{\varepsilon}{3},$$

(4.7) $$\int \frac{d^d k}{(2\pi)^d} \frac{|\partial_{\mu\mu} \hat{D}(k)|}{(1 - \hat{D}(k))^3} \leqq \frac{\varepsilon}{2d},$$

(4.8) 0  $|_{\text{gauss}} \leqq \frac{6}{5}$,

(4.9) $$\hat{D}^{(m)}(0) \leqq \tfrac{6}{5},$$

(4.10) $$\int \frac{d^d k}{(2\pi)^d} \left\{ \frac{\hat{D}^{(m)}(k)}{1 - \hat{D}(k)} \right\}^2, \qquad \int \frac{d^d k}{(2\pi)^d} \frac{\hat{D}^{(m)}(k)^2}{(1 - \hat{D}(k))^3} \leqq \frac{\varepsilon}{3 S^{(m)}},$$

(4.11) $$|\mathrm{Im}\, \hat{D}^{(m)}(k)| \leqq \frac{2L^2}{d} m |k_1|,$$

and finally

(4.12) $$L^2 m^2 / 3d \leqq \hat{D}^{(m)}(0) - \hat{D}(0) \leqq L^2 m^2 / d.$$

This lemma on gaussian quantities can be proven along the same line of argument as in [27, Lemma 5.1]. We sketch its proof for completeness in Appendix B.1.

Remark. By $\mathbb{Z}^d$-invariance of $\{J_{0x}\}$, it follows that

$$\sup_x p_{0x}^{(G)} e^{mx_1} |x_\mu|^2 \leqq \sup_x p_{0x}^{(G)} e^{mx_1} |x_1|^2, \tag{4.13}$$

$$\sum_x (p_{0x}^{(G)})^2 e^{mx_1} |x_\mu|^2 \leqq \sum_x (p_{0x}^{(G)})^2 e^{mx_1} |x_1|^2, \tag{4.14}$$

$$\sum_\mu \sum_x p_{0x}^{(G)} e^{mx_1} |x_\mu|^2 \leqq S^{(m)}. \tag{4.15}$$

We use these relations in the following to bound quantities like

$$\sum_x p_{0x}^{(m)} |x_\mu|^2 = \frac{p}{p_G} \sum_x p_{0x}^{(G)} e^{mx_1} |x_\mu|^2$$

which appear in Prop. 2.1 and Lemma 3.3, without further mention.

4.1. General structure of the proof of Proposition 1.4

The proof of Prop. 1.4 for models satisfying the conclusions of Lemma 4.1 is based on following Lemma 4.2 and Prop. 4.3. We also employ an upper bound on p_c/p_G which has been proven in [27]:

$$p_c/p_G \leqq 1 + O(\varepsilon) \leqq 26/25 \tag{4.16}$$

both in the proof of Prop. 1.4 and of Prop. 4.3. (More precisely, it has been proven in [27, Prop. 5.2, eq. (4.10)] that $p_c/p_G \leqq 3$, but this can be easily tightened up as above for models satisfying the conclusions of Lemma 4.1 with small ε.) Similar arguments have already been used by various authors [13, 27, 40], and in particular in [27] exactly the same argument was used to prove the triangle condition. Here the parameter m plays the rôle of p of [27].

Lemma 4.2. *For both models* (i) *and* (ii) *of Sect.* 1.1, $B^{(2m)}$, $T^{(m)}$, $W_{a,\mu}^{(m)}$ *and* $H_{a_1,a_2,\mu}^{(m)}$ (*introduced in Def.* 3.1) *are continuous in* m *for all* $m < m_p$, $p < p_c$ *and for all* $a, a_1, a_2 \in \mathbb{Z}^d$.

Proof. As proved in Appendix A (eq. (A.5)),

$$\tau_p^{(m)}(0, x) \leqq e^{-(m_p - m)\|x\|_\infty}$$

and trivially

$$\tau_p^{(m)}(0, x) = \lim_{m' \to m} \tau_p^{(m')}(0, x)$$

pointwise. The lemma follows immediately from Def. 3.1 by the Dominated Convergence Theorem. □

Proposition 4.3. *Consider the model* (ii) *of Sect.* 1.1 *with* $L \geqq L_0(\varepsilon)$ *sufficiently large so that conclusions* (4.1) *to* (4.12) *of Lemma* 4.1 *are satisfied for sufficiently small* ε ($\varepsilon \leqq 10^{-12}$ *is enough*). *Then for this* ε *and for any fixed* $p \in [p_G, p_c)$ *and for any* $0 \leqq m < \min\{m_p, L^{-1}\}$, P_4 *implies* P_3, *where* P_α *is the statement that the following set of inequalities holds*:

a)

$$B^{(2m)} \leqq \alpha \cdot \frac{\varepsilon}{S^{(m)}}.$$

b)

$$T^{(m)} \leqq \alpha \cdot \frac{\varepsilon}{S^{(m)}}.$$

c) *For* $\mu = 1, 2, \ldots, d$,

$$W_\mu^{(m)} \leqq \alpha \cdot \frac{\varepsilon}{d}.$$

d) *For* $\mu = 1, 2, \ldots, d$ *and for* $\|a\|_\infty \leqq M(m_p - m)^{-1}$

$$W_{a,\mu}^{(m)} \leqq \alpha \cdot K \frac{\varepsilon}{d}.$$

e) *For* $\mu = 1, 2, \ldots, d$ *and for* $\|a_1\|_\infty, \|a_2\|_\infty \leqq M(m_p - m)^{-1}$

$$H^{(m)}_{a_1, a_2, \mu} \leqq \alpha \cdot 10 \frac{\varepsilon}{d}.$$

Here K *is a universal constant independent of* d, ε, m *(determined in the proof;* K *can be taken to be* 10^3 *for* $\varepsilon \leqq 10^{-12}$*), and* M *is a constant which satisfies the conditions of the following remark.*

Remark. In the above, M is chosen so that

$$\tilde{W}_{a,\mu}^{(m)} \leqq \frac{\varepsilon}{d} \quad \text{for } \|a\|_\infty > M(m_p - m)^{-1}$$

and

$$\tilde{H}^{(m)}_{a_1, a_2, \mu} \leqq \frac{\varepsilon}{d} \quad \text{for } \|a_1\|_\infty \quad \text{or} \quad \|a_2\|_\infty > M(m_p - m)^{-1}$$

where $\tilde{W}_{a,\mu}^{(m)}$ and $\tilde{H}^{(m)}_{a_1, a_2, \mu}$ are defined by replacing (in the definitions of $W_{a,\mu}^{(m)}$, $H^{(m)}_{a_1, a_2, \mu}$) $\tau_p(0, x)$, $\tau_p^{(m)}(0, x)$ by their bounds $e^{-m_p \|x\|_\infty}$, $e^{-(m_p - m)\|x\|_\infty}$.

These have the following immediate consequence:

Corollary 4.4. *Under the same assumption as Prop.* 4.3, P_3 *holds for* $p \in [p_G, p_c)$ *and for* $0 \leqq m < \min\{m_p, L^{-1}\}$.

Proof, given Proposition 4.3. If we fix $L \geqq L_0(\varepsilon; d, g)$ and $p \in (p_G, p_c)$, Prop. 4.3 implies that there is a forbidden region in the graph of $(B^{(2m)}, T^{(m)}, W_\mu^{(m)}, \ldots)$ as a function of m. That is, for each m, $(B^{(2m)}, T^{(m)}, W_\mu^{(m)}, \ldots)$ cannot exist in the following region given by the difference of two hypercubes: $\{[0, 4\varepsilon/S^{(m)}] \times [0, 4\varepsilon/S^{(m)}] \times [0, 4\varepsilon/d] \times \ldots\} \setminus \{[0, 3\varepsilon/S^{(m)}] \times [0, 3\varepsilon/S^{(m)}] \times [0, 3\varepsilon/d] \times \ldots\}$. At $m = 0$, we know [27, Prop. 5.2] that P_3 is satisfied, i.e. $(B^{(2m)}, T^{(m)}, W_\mu^{(m)}, \ldots)$ is inside the smaller hypercube. However, as in Lemma 4.2, these quantities are *continuous* functions of m. Therefore P_3 holds. □

Now given in Lemma 4.1, Prop. 1.4 is a direct consequence of Prop. 2.1, Lemma 3.2 and Corollary 4.4.

Proof of Proposition 1.4. Take $L \geqq L_0(\varepsilon)$ sufficiently large as in Prop. 4.3. Then by Corollary 4.4, we have P_3. Now given P_3 with small ε, Lemmas 3.2 and 3.3 together with (4.13) to (4.15) give

$$0 \leqq \sum_x h_n^{(m)}(0,x) \leqq \begin{cases} O(\varepsilon) & (n=0) \\ O(\varepsilon^n) & (n \geqq 1) \end{cases}, \tag{4.17}$$

$$0 \leqq \sum_x h_n^{(m)}(0,x)\,|x_\mu|^2 \leqq \begin{cases} O(\varepsilon)/d & (n=0,1) \\ n^2\, O(\varepsilon^{n/2})/d & (n \geqq 2) \end{cases}. \tag{4.18}$$

where big-O implies bounds involving computable constants which are independent of d, L, m, p. Now take N sufficiently large so that right hand sides of (2.11) and (2.15) are less than ε/d and that of (2.17) is less than $m^2 \cdot \varepsilon/d$. (The above bounds (4.17) and (4.18) assures that there are such N.) The statements of Prop. 1.4 on $\hat{G}_N^{(m)}$ and $\hat{\Pi}_{\leqq N}^{(m)}$ now follow directly from Prop. 2.1 (writing $10^5\,\varepsilon$ as ε). Those on $\hat{D}^{(m)}(0)$ follow from Lemma 4.1, (4.12). □

Remarks. (Relation to Ornstein-Zernike behaviour)

1. It is expected (and partly proven [15]) that for $p<p_c$, $\tau_p(0,x)$ behaves like

$$\tau_p(0,x) \sim \frac{e^{-m_p l(x)}}{l(x)^{(d-1)/2}} \tag{4.19}$$

at large $|x| \equiv \|x\|_2$ with a suitable norm $l(x)$ ($l(x) \geqq \|x\|_\infty$) which is equivalent (and very close) to $\|x\|_2$. At first glance, this seems to contradict above Corollary 4.4, especially P_3 for $B^{(2m)}$, because the power law decay provided by the denominator of (4.19) is not sufficient to guarantee the convergence of the sum defining $B^{(2m)}$. However, because we have defined $\tau_p^{(m)}(0,x)$ by introducing exponential weight factor (e^{mx_1}) only in *one* direction, the contribution to $B^{(2m)}$ from long distances is

$$\sum_x \frac{e^{-2m_p(l(x)-x_1)}}{l(x)^{(d-1)}}$$

and this sum is convergent (e.g. in $d>3$ for $l(x)=\|x\|_2$) due to the exponentially decaying factor supplied by the difference between $l(x)$ and $|x_1|$.

2. Similar mechanism provides convergence of $T^{(m)}$. In fact, even if we defined $T^{(m)}$ by replacing all of τ_p's by $\tau_p^{(m)}$'s in the definition of T, the result would be still convergent for $d>5$.

3. For $W_\mu^{(m)}$ and $W_{a,\mu}^{(m)}$, however, the situation is different. We cannot replace both τ_p's by $\tau_p^{(m)}$'s [exponential decay factor due to $(l(x)-x_1)$ is not sufficient to guarantee the convergence as $m \nearrow m_p$]. This can be most easily illustrated by gaussian theories, where $W_{\mu=1}^{(m)}$ would *diverge* (as $m \nearrow m_p$) for $d \leqq 7$, if we defined it by replacing both τ_p's by $\tau_p^{(m)}$'s in the definition of W. For a similar reason we have defined $W_{a,\mu}^{(m)}$ by multiplying e^{mx_1} and $|x|^2$ on different legs of W_a. I am grateful to Gordon Slade for calling my attention to these points.

4.2. *Proof of Proposition 4.3*

Now we proceed to the proof of our key proposition, Prop. 4.3. Throughout this section we fix $p \in [p_G, p_c)$ and fix $m \in (0, m_p)$ so that $0<m<\min\{L^{-1}, \delta L^{-2}\}$.

We introduce a symmetrically exponentially-weighted two point function $\tau^{(m)}_{\mathrm{sym}}(0,x)$ according to (3.6), and omit the subscript p in $\hat{\tau}_p(k)$, $\hat{\tau}^{(m)}_p(k)$ etc.

First, if we assume P_4 for sufficiently small ε (it turns out that $\varepsilon \leqq 10^{-12}$ is certainly enough) it follows immediately from the choice of M and Lemma 3.3 and (4.13) to (4.15) that [note, by (4.9), $\sum_x p^{(G)}_{0x} e^{mx_1} \leqq 6/5$, and thus by (4.16), $\sum_x p^{(m)}_{0x} \leqq 5/4$]

$$\bar{T}^{(m)} \leqq 8\sqrt{\varepsilon/S^{(m)}}, \qquad W'^{(m)}_{\mu} \leqq 13\cdot\varepsilon/d, \qquad \bar{H}^{(m)}_{\mu} \leqq 40\cdot\varepsilon/d,$$
$$\bar{W}^{(m)}_{\mu} \leqq 4\cdot K\cdot\varepsilon/d, \qquad \bar{W}'^{(m)}_{\mu} \leqq 30\cdot K\cdot\varepsilon/d.$$

(Here and in the following, the precise values of the coefficients on the right hand sides of these equations are not important. The point here is that they are independent of p, d, L, K or m.) This, together with Lemma 3.2 in turn yields the following bounds.

$$\sum_n \sum_x h^{(m)}_n(0,x) \leqq c_1 \cdot \frac{\varepsilon}{S^{(m)}}, \tag{4.20}$$

$$\sum_n \sum_x h^{(m)}_n(0,x)\,|x_\mu|^2 \leqq c_2 \cdot \frac{\varepsilon}{d}.$$

$$\sum_x h^{(m)}_N(0,x) \leqq c'_1 \cdot (c''_1\,\varepsilon)^{(N+1)/2}, \tag{4.21}$$

$$\sum_x h^{(m)}_N(0,x)\,|x_\mu|^2 \leqq d^{-1}\cdot c'_2 \cdot N^2 \cdot (c''_2\,\varepsilon)^{(N+1)/2}.$$

Here, c_1, c_2 are calculable numerical constants independent of p, d, L, K or m (we can take $c_1 = 14$, $c_2 = 40$). c'_1, c''_1, c'_2 and c''_2 are numerical constants which might depend on K (but not on p, d, L or m).

We use these bounds to control $\hat{\tau}^{(m)}(k)$. We have, from Prop. 2.1 and (4.13) to (4.15),

Lemma 4.5. *Given P_4 for model* (ii) *for sufficiently small ε, we have, for sufficiently large $N \geqq N_0(\varepsilon; m, p)$:*

$$|\hat{G}^{(m)}_N(k) - 1| \leqq c_3 \cdot \frac{\varepsilon}{S^{(m)}}, \qquad |\hat{\Pi}^{(m)}_{\leqq N}(k)| \leqq c_4 \cdot \frac{\varepsilon}{S^{(m)}},$$

$$|\mathrm{Im}\,\hat{G}^{(m)}_N(k)| \leqq c_5 \cdot \frac{\varepsilon}{d}\cdot m|k_1|, \qquad |\mathrm{Im}\,\hat{\Pi}^{(m)}_{\leqq N}(k)| \leqq c_6 \cdot \frac{\varepsilon}{d}\cdot m|k_1|,$$

$$|\partial^s_\mu\,\hat{G}^{(m)}_N(k)| \leqq c_5 \cdot \frac{\varepsilon}{d}, \qquad |\partial^s_\mu\,\hat{\Pi}^{(m)}_{\leqq N}(k)| \leqq c_6 \cdot \frac{\varepsilon}{d},$$

$$0 \leqq \mathrm{Re}(\hat{\Pi}^{(m)}_{\leqq N}(0) - \hat{\Pi}^{(m)}_{\leqq N}(k)) \leqq c_7 \cdot \varepsilon \cdot \frac{|k|^2}{d},$$

$$|\hat{G}^{(m)}_N(0) - \hat{G}_N(0)| \leqq c_8 \cdot \varepsilon \cdot m^2/d, \qquad |\hat{\Pi}^{(m)}_{\leqq N}(0) - \hat{\Pi}_{\leqq N}(0)| \leqq c_9 \cdot \varepsilon \cdot m^2/d$$

where c_3 through c_9 are calculable constants independent of m, p, d, L or K.

Proof. The lemma follows directly from (4.20), (4.21) and Prop. 2.1, and an upper bound on p/p_G, (4.16). For instance, by Prop. 2.1 and (4.20), (4.21)

$$
\begin{aligned}
|\hat{G}_N^{(m)}(k)-1| &\leqq \sum_{n=0}^{N} \sum_x h_n^{(m)}(0,x) + \sum_x \tau_p^{(m)}(0,x) \sum_v p_{0v}^{(m)} \sum_y h_N^{(m)}(0,y) \\
&\leqq c_1\,\varepsilon/S^{(m)} + \chi^{(m)} \sum_v p_{0v}^{(m)}\, c_1' (c_1''\,\varepsilon)^{(N+1)/2}.
\end{aligned}
$$

By taking N sufficiently large (depending on ε) so that the second term becomes less than $\varepsilon/S^{(m)}$, we get the desired bound. Other bounds are proved similarly. □

In the following, we use $O(\varepsilon)$ to denote an upper bound which involves constants independent of m, p, d, L, or K. Now Lemma 4.5 implies the following bounds:

Lemma 4.6. *Under P_4 with sufficiently small ε, $\hat{\tau}_{\mathrm{sym}}^{(m)}(k) \equiv \sum_x \tau_{\mathrm{sym}}^{(m)}(0,x)\, e^{ik\cdot x}$ satisfies*

$$
0 \leqq \hat{\tau}_{\mathrm{sym}}^{(m)}(k) = \operatorname{Re} \hat{\tau}^{(m)}(k) \leqq |\hat{\tau}^{(m)}(k)| \leqq \frac{(1+O(\varepsilon))}{1-\hat{D}(k)}, \tag{4.22}
$$

$$
|\hat{\tau}_{\mathrm{sym}}^{(m)}(k)-1| \leqq (1+O(\varepsilon)) \frac{|\hat{D}^{(m)}(k)|}{1-\hat{D}(k)} + \frac{O(\varepsilon/S^{(m)})}{1-\hat{D}(k)} \leqq \frac{3/2}{1-\hat{D}(k)}. \tag{4.23}
$$

Also

$$
\begin{aligned}
|\partial_\mu^2\, \hat{\tau}(k)| &\leqq (1+O(\varepsilon)) \left\{ \frac{(|\partial_\mu \hat{D}(k)| + O(\varepsilon/d)\,|k_\mu|)^2}{(1-\hat{D}(k))^3} + \frac{|\partial_\mu^2 \hat{D}(k)|}{(1-\hat{D}(k))^2} \right\} \\
&\quad + O\!\left(\frac{\varepsilon}{d}\right) \left\{ \frac{1}{1-\hat{D}(k)} + \frac{1}{(1-\hat{D}(k))^2} + \frac{|\partial_\mu \hat{D}(k)|}{(1-\hat{D}(k))^2} \right\}.
\end{aligned} \tag{4.24}
$$

Proof. By (3.6),

$$
\begin{aligned}
\hat{\tau}_{\mathrm{sym}}^{(m)}(k) &= \sum_x \tau_p(0,x) \cosh(m x_1)\, e^{ik\cdot x} = \sum_x \tau_p(0,x) \cosh(m x_1) \cdot \cos(k\cdot x) \\
&= \sum_x \tau_p^{(m)}(0,x) \cos(k\cdot x) = \operatorname{Re} \hat{\tau}^{(m)}(k).
\end{aligned}
$$

To prove the upper bound of (4.22) we write the numerator and the denominator of (2.5) as follows:

$$
\hat{\tau}^{(m)}(k) = \frac{\hat{G}_N^{(m)}(k)}{1-(p/p_G)\,\hat{D}^{(m)}(k) - \hat{\Pi}_{\leqq N}^{(m)}(k)} \equiv \frac{\hat{G}_N^{(m)}(k)}{\hat{F}^{(m)}(k)}
$$

As for the numerator, we can directly use Lemma 4.5. For the denominator, we first write (note that $\hat{D}^{(m)}(0)$, $\hat{\Pi}_{\leqq N}^{(m)}(0)$ are real)

$$
\begin{aligned}
\operatorname{Re} \hat{F}^{(m)}(k) &= \{1-(p/p_G)\,\hat{D}^{(m)}(0) - \hat{\Pi}_{\leqq N}^{(m)}(0)\} \\
&\quad + (p/p_G) \operatorname{Re}(\hat{D}^{(m)}(0) - \hat{D}^{(m)}(k)) + \operatorname{Re}(\hat{\Pi}_{\leqq N}^{(m)}(0) - \hat{\Pi}_{\leqq N}^{(m)}(k)).
\end{aligned}
$$

As for the first term, for $0<m<m_p$,

$$1-(p/p_G)\,\hat{D}^{(m)}(0)-\hat{\Pi}^{(m)}_{\leqq N}(0)=(\chi^{(m)}/\hat{G}^{(m)}_N(0))^{-1}>0. \tag{4.25}$$

As for the second term,

$$\begin{aligned}\operatorname{Re}(\hat{D}^{(m)}(0)-\hat{D}^{(m)}(k)) &= \sum_x p^{(G)}_{0x}(1-\cos(k\cdot x))\cosh(mx_1)\\ &\geqq \sum_x p^{(G)}_{0x}(1-\cos(k\cdot x))=1-\hat{D}(k).\end{aligned}$$

As for the third term,

$$|\operatorname{Re}(\hat{\Pi}^{(m)}_{\leqq N}(0)-\hat{\Pi}^{(m)}_{\leqq N}(k))|\leqq c_7\cdot\varepsilon\cdot\frac{|k|^2}{d}\leqq 3c_7\cdot\pi^2\cdot\varepsilon(1-\hat{D}(k))$$

where in the last step we used (4.3). As a result,

$$\operatorname{Re}\hat{F}^{(m)}(k)\geqq\left(\frac{p}{p_G}-3c_7\,\pi^2\cdot\varepsilon\right)(1-\hat{D}(k)). \tag{4.26}$$

Thus (because we are considering $p>p_G$)

$$\hat{\tau}^{(m)}_{\mathrm{sym}}(k)\leqq|\hat{\tau}^{(m)}(k)|\leqq\frac{|\hat{G}^{(m)}_N(k)|}{\operatorname{Re}\hat{F}^{(m)}(k)}\leqq\frac{1+O(\varepsilon)}{1-\hat{D}(k)},$$

and the upper bound of (4.22) is proved. Also

$$\begin{aligned}|\hat{\tau}^{(m)}_{\mathrm{sym}}(k)-1| &= \left|\operatorname{Re}\left\{\frac{\hat{G}^{(m)}_N(k)-1+(p/p_G)\,\hat{D}^{(m)}(k)+\hat{\Pi}^{(m)}_{\leqq N}(k)}{\hat{F}^{(m)}(k)}\right\}\right|\\ &\leqq\frac{O(\varepsilon/S^{(m)})}{1-\hat{D}(k)}+(1+O(\varepsilon))\frac{|\hat{D}^{(m)}(k)|}{1-\hat{D}(k)}\end{aligned}$$

and (4.23) is proved.

To prove the lowerbound of (4.22), we first write (denoting the complex conjugate of $\hat{G}^{(m)}_N(k)$ by $\overline{\hat{G}^{(m)}_N(k)}$)

$$\hat{\tau}^{(m)}_{\mathrm{sym}}(k)=\tfrac{1}{2}(\hat{\tau}^{(m)}(k)+\hat{\tau}^{(-m)}(k))=\frac{\operatorname{Re}(\hat{F}^{(m)}(k)\,\overline{\hat{G}^{(m)}_N(k)})}{|\hat{F}^{(m)}(k)|^2}. \tag{4.27}$$

We only have to prove that the numerator is nonnegative. This can be written as:

$$\operatorname{Re}(\hat{F}^{(m)}(k)\,\overline{\hat{G}^{(m)}_N(k)})=\operatorname{Re}\hat{F}^{(m)}(k)\cdot\operatorname{Re}\hat{G}^{(m)}_N(k)+\operatorname{Im}\hat{F}^{(m)}(k)\cdot\operatorname{Im}\hat{G}^{(m)}_N(k). \tag{4.28}$$

As for the first term we have, by Lemma 4.5, (4.26) and (4.3), that

$$\operatorname{Re}\hat{F}^{(m)}(k)\cdot\operatorname{Re}\hat{G}^{(m)}_N(k)\geqq(1-O(\varepsilon))\cdot(1-\hat{D}(k))\geqq\frac{|k|^2}{4\pi^2 d}. \tag{4.29}$$

As for $\operatorname{Im}\hat{F}^{(m)}(k)$, by (4.11), (4.16) and Lemma 4.5,

$$|\operatorname{Im}\hat{F}^{(m)}(k)|=|-(p/p_G)\operatorname{Im}\hat{D}^{(m)}(k)-\operatorname{Im}\hat{\Pi}^{(m)}_{\leqq N}(k)|\leqq\frac{m|k_1|}{d}\left(\tfrac{52}{25}L^2+c_6\,\varepsilon\right)$$

and thus

$$(4.30)\quad |\operatorname{Im} \hat{F}^{(m)}(k)\cdot \operatorname{Im} \hat{G}_N^{(m)}(k)| \leqq \frac{|k_1|^2}{d^2}\cdot c_5\,\varepsilon\cdot m^2\cdot\left(\tfrac{52}{25}L^2+c_6\,\varepsilon\right) \leqq 3c_5\cdot\varepsilon\cdot\frac{|k_1|^2}{d^2}$$

where, in the last step, we used our condition on m: $m \leqq L^{-1}$. Now (4.28) to (4.30) imply

$$\operatorname{Re}(\hat{F}^{(m)}(k)\,\overline{\hat{G}_N^{(m)}(k)}) \geqq \frac{1}{4\pi^2}\frac{|k|^2}{d} - 3c_5\,\varepsilon\frac{|k_1|^2}{d^2} \geqq 0$$

as long as $12\pi^2 c_5\cdot\varepsilon \leqq d$. This proves the lower bound of (4.22).

The bound on $\partial_\mu^2\,\hat{\tau}(k)$ can be obtained as follows. We first write (denoting the partial differentiation with respect to k_μ by the subscript μ)

$$\partial_\mu^2\,\hat{\tau}(k) = \frac{\hat{G}_{\mu\mu}(k)}{\hat{F}(k)} - 2\frac{\hat{G}_\mu(k)\,\hat{F}_\mu(k)}{\hat{F}(k)^2} - \frac{\hat{G}(k)\,\hat{F}_{\mu\mu}(k)}{\hat{F}(k)^2} + 2\frac{\hat{G}(k)\,\hat{F}_\mu(k)^2}{\hat{F}(k)^3}$$

and bound each term using Lemma 4.5. In particular, as for the last term, we write out

$$\hat{F}_\mu(k) = -\frac{p}{p_G}\,\partial_\mu\,\hat{D}(k) - \partial_\mu\,\hat{\Pi}_{\leqq N}(k)$$

and using the symmetry and Taylor's theorem (just as was done after (5.23) of [27]) bound the second term above as

$$|\partial_\mu\,\hat{\Pi}_{\leqq N}(k)| \leqq \frac{c_6\,\varepsilon}{d}\cdot|k_\mu|.$$

(This was essentially done in [27, around (5.22)], and we omit the details.) □

We now proceed to the proof of Prop. 4.3.

Proof of Proposition 4.3. We prove the conclusions one by one. a) First note that

$$B^{(2m)} \equiv \sum_{x\neq 0}(\tau^{(m)}(0,x))^2 = 2\sum_{x\neq 0}(\tau^{(m)}_{\text{sym}}(0,x))^2 - \sum_{x\neq 0}(\tau(0,x))^2.$$

To get the upper bound, we simply omit the second term, and bound the first one as (note that $\tau^{(m)}_{\text{sym}}(0,0)=1$):

$$(4.31)\quad \sum_{x\neq 0}(\tau^{(m)}_{\text{sym}}(0,x))^2 = \int\frac{d^dk}{(2\pi)^d}\,(\hat{\tau}^{(m)}_{\text{sym}}(k)-1)^2$$
$$\leqq \int\frac{d^dk}{(2\pi)^d}\left\{\frac{3}{2}\left((1+O(\varepsilon))\frac{|\hat{D}^{(m)}(k)|}{1-\hat{D}(k)}\right)^2 + 3\left(\frac{O(\varepsilon/S^{(m)})}{1-\hat{D}(k)}\right)^2\right\}$$

and use (4.10) and (4.4), (4.5) of Lemma 4.1. Note that by Schwarz inequality, (4.4) and (4.5) imply the following:

$$(4.32)\quad \int\frac{d^dk}{(2\pi)^d}\left(\frac{1}{1-\hat{D}(k)}\right)^n = 1+O(\varepsilon) \qquad (\text{for } n=1,2,3).$$

b) Using the Fourier transform and the Schwarz inequality,

$$T^{(m)} = \int \frac{d^d k}{(2\pi)^d} \{\hat{\tau}_{\mathrm{sym}}^{(m)}(k)(\hat{\tau}(k)-1)^2 + 2(\hat{\tau}(k)-1)(\hat{\tau}_{\mathrm{sym}}^{(m)}(k)-1)\}$$
$$\leqq \int \frac{d^d k}{(2\pi)^d} \hat{\tau}_{\mathrm{sym}}^{(m)}(k)(\hat{\tau}(k)-1)^2$$
$$+2\left(\int \frac{d^d k}{(2\pi)^d}(\hat{\tau}(k)-1)^2 \int \frac{d^d k}{(2\pi)^d}(\hat{\tau}_{\mathrm{sym}}^{(m)}(k)-1)^2\right)^{1/2}.$$

By Lemma 4.6, the first term is bounded as

$$\leqq \int \frac{d^d k}{(2\pi)^d}\left\{\frac{3}{2}(1+O(\varepsilon))^2 \frac{|\hat{D}(k)|^2}{(1-\hat{D}(k))^3} + 3\frac{O(\varepsilon/S^{(m)})^2}{(1-\hat{D}(k))^3}\right\} \leqq 2\cdot\frac{\varepsilon}{3S^{(m)}}.$$

In the last step, we used (4.10) and (4.32). As for the latter term, we use Lemma 4.6, Lemma 4.1, and (4.32)

$$\int \frac{d^d k}{(2\pi)^d}(\hat{\tau}_{\mathrm{sym}}^{(m)}(k)-1)^2$$
$$\leqq \int \frac{d^d k}{(2\pi)^d}\left\{\frac{3}{2}(1+O(\varepsilon))\left(\frac{|\hat{D}^{(m)}(k)|}{1-\hat{D}(k)}\right)^2 + \left(\frac{O(\varepsilon/S^{(m)})}{1-\hat{D}(k)}\right)^2\right\} \leqq 2\cdot\frac{\varepsilon}{3S^{(m)}}.$$

Thus $T^{(m)} \leqq 2\varepsilon/S^{(m)}$, and (b) is proved.

(c, d, e) Note that $|x_\mu|^2$ appears as a weight factor multiplying $\tau_p(0,x)$ [*not* $\tau_p^{(m)}(a,x)$ etc.] in the definitions of $W_\mu^{(m)}$, $W_\mu'^{(m)}$, $H_{a_1,a_2,\mu}^{(m)}$. So using the Fourier transform we can express these quantities as integrals of product of $\hat{\tau}^{(m)}(k)$ [or $\hat{\tau}^{(m)}(k)-1$] and (appropriate derivatives of) $\hat{\tau}(k)$. Then we can argue exactly as in [27] (using the bounds of Lemma 4.6 on $\hat{\tau}_{\mathrm{sym}}^{(m)}(k)$, $\hat{\tau}_{\mathrm{sym}}^{(m)}(k)-1$). We omit the details, except for the proof of the bound on $W_\mu^{(m)}$, which follows a somewhat different line of argument from that of [27].

c) For $W_\mu^{(m)}$, a direct calculation combined with Lemma 4.5 and Lemma 4.6 yields

$$\text{(4.33)}\quad W_\mu^{(m)} = \int \frac{d^d k}{(2\pi)^d}(-\partial_\mu^2 \hat{\tau}(k))(\hat{\tau}_{\mathrm{sym}}^{(m)}(k)-1) \leqq \int \frac{d^d k}{(2\pi)^d}|-\partial_\mu^2 \hat{\tau}(k)|\cdot|\hat{\tau}_{\mathrm{sym}}^{(m)}(k)-1|$$
$$\leqq \int \frac{d^d k}{(2\pi)^d}\left[(1+O(\varepsilon))\left\{\frac{(|\partial_\mu \hat{D}(k)| + O(\varepsilon/d)|k_\mu|)^2}{(1-\hat{D}(k))^3} + \frac{|\partial_\mu^2 \hat{D}(k)|}{(1-\hat{D}(k))^2}\right\}\right.$$
$$\left. + O\left(\frac{\varepsilon}{d}\right)\left\{\frac{1}{1-\hat{D}(k)} + \frac{1}{(1-\hat{D}(k))^2} + \frac{|\partial_\mu \hat{D}(k)|}{(1-\hat{D}(k))^2}\right\}\right]$$
$$\cdot\left[(1+O(\varepsilon))\frac{|\hat{D}^{(m)}(k)|}{1-\hat{D}(k)} + \frac{O(\varepsilon/S^{(m)})}{1-\hat{D}(k)}\right].$$

Here, contributions from the terms with coefficients $O(\varepsilon/d)$ or $O(\varepsilon/S^{(m)})$ are shown to be of higher order of ε, by Lemma 4.1 and the Schwarz inequality. For example, the product of the first terms of the above two factors are bounded as follows:

$$
\begin{aligned}
(4.34)\qquad & O\left(\frac{\varepsilon}{d}\right)\int\frac{d^d k}{(2\pi)^d}\frac{|\hat{D}^{(m)}(k)|}{(1-\hat{D}(k))^2}\\
&\leqq O\left(\frac{\varepsilon}{d}\right)\left(\int\frac{d^d k}{(2\pi)^d}\frac{|\hat{D}^{(m)}(k)|^2}{(1-\hat{D}(k))^2}\int\frac{d^d k}{(2\pi)^d}\frac{1}{(1-\hat{D}(k))^2}\right)^{1/2}\\
&\leqq O\left(\frac{\varepsilon}{d}\right)\sqrt{O(\varepsilon/S^{(m)})}.
\end{aligned}
$$

This leaves us with two terms, that is

$$
(1+O(\varepsilon))\int\frac{d^d k}{(2\pi)^d}|\hat{D}^{(m)}(k)|\left\{\frac{(|\partial_\mu\hat{D}(k)|+O(\varepsilon/d)\,|k_\mu|)^2}{(1-\hat{D}(k))^4}+\frac{|\partial_\mu^2\hat{D}(k)|}{(1-\hat{D}(k))^3}\right\}.
$$

However, if we use (4.9) to bound $|\hat{D}^{(m)}(k)|\leqq|\hat{D}^{(m)}(0)|\leqq 6/5$, and then use Schwarz inequality to bound the cross term from $(|\partial_\mu\hat{D}(k)|+O(\varepsilon/d)\,|k_\mu|)^2$, this is bounded as

$$
\begin{aligned}
(4.35)\qquad & \leqq(1+O(\varepsilon))\frac{6}{5}\left\{\int\frac{d^d k}{(2\pi)^d}\frac{|\partial_\mu\hat{D}(k)|^2}{(1-\hat{D}(k))^4}+O(\varepsilon/d)^2\int\frac{d^d k}{(2\pi)^d}\frac{|k_\mu|^2}{(1-\hat{D}(k))^4}\right.\\
&+O\left(\frac{\varepsilon}{d}\right)\left(\int\frac{d^d k}{(2\pi)^d}\frac{|\partial_\mu\hat{D}(k)|^2}{(1-\hat{D}(k))^4}\right)^{1/2}\cdot\left(\int\frac{d^d k}{(2\pi)^d}\frac{|k_\mu|^2}{(1-\hat{D}(k))^4}\right)^{1/2}\\
&\left.+\int\frac{d^d k}{(2\pi)^d}\frac{|\partial_\mu^2\hat{D}(k)|}{(1-\hat{D}(k))^3}\right\}.
\end{aligned}
$$

Now in the above, by symmetry and by (4.3), (4.32)

$$
\begin{aligned}
\int\frac{d^d k}{(2\pi)^d}\frac{|k_\mu|^2}{(1-\hat{D}(k))^4}&=\int\frac{d^d k}{(2\pi)^d}\frac{|k|^2/d}{(1-\hat{D}(k))^4}\\
&\leqq 3\pi^2\int\frac{d^d k}{(2\pi)^d}\frac{1}{(1-\hat{D}(k))^3}=3\pi^2(1+O(\varepsilon)),
\end{aligned}
$$

and we can use (4.6) and (4.7) for the rest. As a result, (4.35) is bounded as

$$
\leqq(1+O(\varepsilon))\cdot\frac{6}{5}\cdot\left[\frac{W_G}{d}+O\left(\frac{\varepsilon}{d}\right)^{3/2}+\frac{\varepsilon}{2d}\right]\leqq\frac{3\varepsilon}{2d}.
$$

As a result,

$$
(4.36)\qquad \int\frac{d^d k}{(2\pi)^d}|\partial_\mu^2\,\hat{\tau}(k)|\cdot|\hat{\tau}_{\mathrm{sym}}^{(m)}(k)-1|\leqq 2\frac{\varepsilon}{d}.
$$

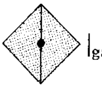
This proves P_3 for $W_\mu^{(m)}$.

d) For $\bar{W}_{\mu}^{(m)}$, we simply write

$$W_{a,\mu}^{(m)} = \int \frac{d^d k}{(2\pi)^d} e^{-ik \cdot a} (-\partial_\mu^2 \hat{\tau}(k)) (\hat{\tau}_{\mathrm{sym}}^{(m)}(k))$$
$$\leqq \int \frac{d^d k}{(2\pi)^d} |-\partial_\mu^2 \hat{\tau}(k)| \cdot |\hat{\tau}_{\mathrm{sym}}^{(m)}(k)|$$

use Lemma 4.6, and evaluate each term as above.

e) The method of [27] can be used here. We have

$$H_{a_1, a_2, \mu}^{(m)} \leqq 4 W_\mu \{(1 + T^{(m)})(1 + 2\bar{T}^{(m)}) + \tfrac{11}{10}\ 0$$  $$|_{\mathrm{gauss}}\} \qquad \square$$

5. Proof of Proposition 1.4 for the nearest-neighbour model

In this section, we prove Prop. 1.4 for the nearest-neighbour model (model (i) of Sect. 1.1), and complete the proof of our main result, Theorem 1.1. Proposition 1.4 for the spread-out model was proven in Sects. 2 through 4, using the continuity (Lemma 4.2) and the "P_4 implies P_3" (i.e., poor bounds imply good estimates) argument (Prop. 4.3) of the quantities $B^{(2m)}$, $T^{(m)}$, $W_\mu^{(m)}$, $W_{a,\mu}^{(m)}$, and $H_{a_1,a_2,\mu}^{(m)}$. We were forced to use all these quantities (which naturally made the proof complicated) in order to avoid quantities which diverge above six dimensions.

Although we can prove Prop. 1.4 for the nearest-neighbour model by carefully following the analysis of the previous section and that of [27], we here present a rather simple proof. We prove the proposition by using the continuity and the "P_4 implies P_3" argument of essentially *one* quantity – the weighted *heptagon* diagram (for $\mu = 1, 2, \ldots, d$):

$$W_{H,\mu}^{(m)} \equiv \sum_{x^{(1)}, \ldots, x^{(6)} \in \mathbb{Z}^d} |x_\mu^{(1)}|^2 \tau_{\mathrm{sym}}^{(m)}(0, x^{(1)}) \tau_p(x^{(1)}, x^{(2)}) \tau_p(x^{(2)}, x^{(3)})$$
$$\cdot \tau_{\mathrm{sym}}^{(m)}(x^{(3)}, x^{(4)}) \tau_{\mathrm{sym}}^{(m)}(x^{(4)}, x^{(5)}) \tau_p(x^{(5)}, x^{(6)}) \tau_p(x^{(6)}, 0). \tag{5.1}$$

This considerably simplifies the proof of the "P_4 implies P_3" property (see Prop. 5.3 below). The price we have to pay is, first, $W_{H,\mu}^{(m)}$ will be finite only for $d > 18$ (this is unsatisfactory in view of the common belief that Theorem 1.1 should hold for $d > 6$), and second, we have to reorganize the analysis of Sect. 3 to fit into the new scheme. Incidentally, for the nearest-neighbour model, the value $d_0 = 48$ for the triangle condition announced in [27] was obtained by a method closely related to the one presented in this section. (Although in [27] a slightly more efficient bound, using a weighted *square* diagram, rather than a weighted heptagon diagram, was used.)

The pattern of the proof is the same as for the spread-out model in Sect. 4. First, in Sect. 5.1, we present a diagrammatic estimate which modifies the analysis of Sect. 3 to fit into our scheme here. Then in Sect. 5.2, we outline the proof, by using the continuity (Lemma 5.2) and the "P_4 implies P_3" argument (Prop. 5.3) of the weighted heptagon diagram. Finally, in Sect. 5.3, we prove Prop. 5.3 and complete the proof of Prop. 1.4. As in the previous section, we use c, c' to denote universal constants which are independent of p, m, d. These may represent different values on different occasions. We also use the big-O notation to denote a bound which involves a universal constant which does not depend on p, m, d.

5.1. Diagrammatic estimates

In this section, we modify the analysis of Sect. 3 to make use of $W_{H,\mu}^{(m)}$. The result, which corresponds to Lemma 3.2, is the following:

Lemma 5.1. *For the nearest-neighbour model* (i) *of Sect.* 1.1, *we have*
 (a) *Same as part* (a) *of Lemma* 3.2
 (b) *For* $\mu=1,2,\ldots,d$,

$$0 \leqq \sum_x h_n^{(m)}(0,x)\,|x_\mu|^2 \leqq \begin{cases} W_\mu^{(m)} & (n=0) \\ 7\cdot e^{|m|}(1+T^{(m)})^2\cdot W_{H,\mu}^{(m)} & (n=1) \\ 25n^2\cdot e^{|m|}(1+T^{(m)})^2\,(r^{(m)})^{n-1}\cdot W_{H,\mu}^{(m)} & (n\geqq 2) \end{cases}$$

where $r^{(m)}$ *is defined in* (3.2).

Remark. It is not difficult to see from the definition of $W_{H,\mu}^{(m)}$ that

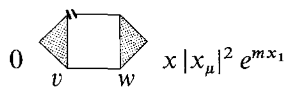
$$W_\mu^{(m)} \leqq \tfrac{1}{6} W_{H,\mu}^{(m)}, \qquad T^{(m)} \leqq \tfrac{1}{2} \sum_{\mu=1}^d W_{H,\mu}^{(m)}, \qquad B^{(2m)} \leqq \sum_{\mu=1}^d W_{H,\mu}^{(m)} \tag{5.2}$$

and thus by Lemma 3.3 that (for $d\geqq 8$)

$$\bar{T}^{(m)} \leqq 5 \sum_{\mu=1}^d W_{H,\mu}^{(m)} + 5\left(\sum_{\mu=1}^d W_{H,\mu}^{(m)}\right)^{1/2}. \tag{5.3}$$

Thus, bounds (a) and (b) of the above Lemma 5.1 can be expressed entirely in terms of $W_{H,\mu}^{(m)}$.

Proof. Part (a) and the $n=0$ case of part (b) are the same as Lemma 3.2. We illustrate the proof of part (b) for $n\geqq 1$ by the simplest case, $n=1$. We proceed in way similar to the proof of part (b) of Lemma 3.2. That is, to evaluate the diagram



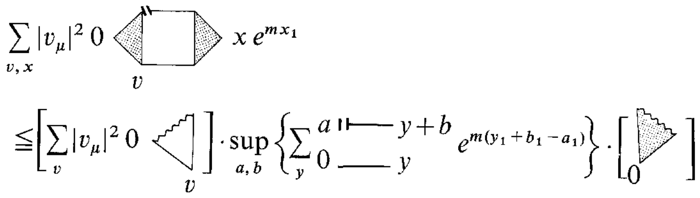
we first use the triangle inequality for $|x_\mu|^2$

$$|x_\mu|^2 \leqq 3(|v_\mu|^2 + |v_\mu - w_\mu|^2 + |w_\mu - x_\mu|^2) \tag{5.4}$$

and then use a basic inequality

$$\sum_x f(x)\,g(x) \leqq \sup_x |f(x)| \sum_y |g(y)| \tag{5.5}$$

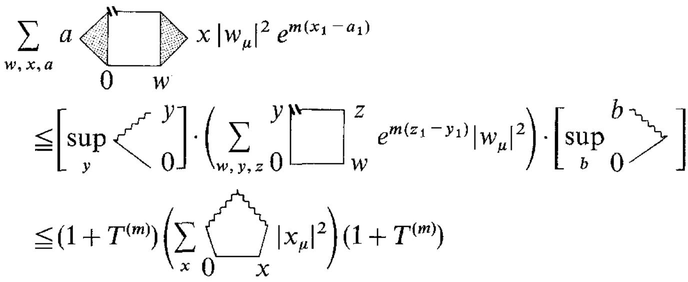
to bound each of the resulting terms. For example, we bound the contribution from the first term of (5.4) as follows:



The first factor (weighted triangle) is bounded by $\frac{1}{3} W_{H,\mu}^{(m)}$, and the last one is nothing but $1+T^{(m)}$. The middle factor is bounded as follows:

$$\sum_{y,z} p_{az}\, \tau_p(z, y+b)\, \tau_p(0,y)\, e^{m(y_1+b_1-a_1)}$$
$$= \sum_{y,z} p_{az}\, e^{m(z_1-a_1)} \cdot \tau_p(z, y+b)\, e^{m(y_1+b_1-z_1)} \cdot \tau_p(0,y)$$
$$\leqq \sum_{y,z:z \neq a} \tau_p(a,z)\, e^{|m|} \cdot \tau_p{}^{(m)}(z, y+b) \cdot \tau_p(0,y) = e^{|m|}\, T_{b-a}^{(m)}$$

where in the second step we used a trivial inequality $p_{az} \leqq \tau_p(a,z)$ and the fact that $p_{az}=0$ unless $\|a-z\|_2=1$. As a result, using $T_a^{(m)}+T_{-a}^{(m)} \leqq 2(1+T^{(m)})$, we can bound the contribution from the first term of (5.4) by $\frac{1}{3} W_{H,\mu}^{(m)} \cdot e^{|m|} \cdot 2(1+T^{(m)})^2$. The second term of (5.4) contributes:



where, in the second step, we bounded the middle factor by the weighted pentagon (which is in turn bounded by $W_{H,\mu}^{(m)}$) by using a trivial inequality

$$p_{0v}^{(m)} = p_{0v}\, e^{mv_1} \leqq \tau_p(0,v)\, e^{mv_1} = \tau_p^{(m)}(0,v). \tag{5.6}$$

The third term of (5.4) contributes the same as the first one.

For $n \geqq 2$, we proceed similarly. We first use the triangle inequality, then use the basic inequality (5.5) to bound each of the resulting terms by its basic units. We omit the details. □

5.2. General structure of the proof of Proposition 1.4 for the nearest-neighbour model

We prove that the weighted heptagon $W_{H,\mu}^{(m)}$ (introduced in (5.1)) is of order d^{-2} for d sufficiently large. This is done by combining the continuity in m

(Lemma 5.2) and the "P_4 implies P_3" argument (Prop. 5.3), as in Sect. 4. We also employ an upper bound on p_c derived in [27, (4.10)]:

$$2dp_c \leqq 1 + O(1/d) \leqq 26/25. \tag{5.7}$$

Lemma 5.2. *For the model* (i) *of Sect.* 1.1, $W_{H,\mu}^{(m)}(\mu=1,2,\dots,d)$ *are continuous in m for all* $|m|<m_p$, $p<p_c$.

Proof. Same as that of Lemma 4.2. □

Proposition 5.3. *Consider the model* (i) *of Sect.* 1.1 *on* $\mathbb{Z}^d$. *There exists* $d_0>6$ *such that for any* $d\geqq d_0$ *and for any fixed* $p\in[1/2d, p_c)$ *and* $|m|\leqq \min\{m_p, d^{-1/2}\}$, P_4 *implies* P_3, *where* P_α *is the statement that the following inequalities hold*:

$$W_{H,\mu}^{(m)} \leqq \alpha\cdot 4\cdot d^{-2} \quad \textit{for } \mu=1,2,\dots,d.$$

Proof of Proposition 1.4, given Lemma 5.2 and Proposition 5.3. As a consequence of the above two and the fact that P_3 holds for $m=0$ (see note added), for $d\geqq d_0$, P_3 holds for $0\leqq m\leqq m_p$, $p<p_c$. Now Prop. 1.4 follows immediately just as in Sect. 4.1, if we combine P_3 and (5.7) with Lemma 5.1 and Prop. 2.1. (The bound on $\hat{D}^{(m)}(0)-\hat{D}(0)$ follows trivially by explicit calculation, using the following (5.12).) □

5.3. *Proof of Proposition 5.3*

We now proceed to the proof of the "P_4 implies P_3" property of $W_{H,\mu}^{(m)}$, Prop. 5.3. We begin by listing some properties of gaussian integrals:

Lemma 5.4. *For the gaussian nearest-neighbour model on* $\mathbb{Z}^d$, *with d sufficiently large, we have*:

$$p_c = 1/2d, \quad \sup_a p_{0a} = p, \quad \sum_x p_{0x} = 2dp, \quad \sum_v p_{0v}^2 = 2dp^2, \tag{5.8}$$

$$\max_a p_{0a}^{(m)} = pe^m, \quad \sum_x p_{0x}^{(m)} = 2dp\left(1+\frac{\cosh m - 1}{d}\right), \tag{5.9}$$

$$S^{(m)} = \cosh m, \tag{5.10}$$

$$\hat{D}(k) = \sum_{\mu=1}^{d} \frac{\cos k_\mu}{d}, \tag{5.11}$$

$$\hat{D}^{(m)}(k) = \hat{D}(k) + \frac{\cosh m - 1}{d}\cos k_1 + i\frac{\sinh m}{d}\sin k_1, \tag{5.12}$$

$$\int \frac{d^dk}{(2\pi)^d}\left(\frac{1}{1-\hat{D}(k)}\right)^n \leqq \frac{11}{10} \quad (0\leqq n\leqq 9). \tag{5.13}$$

We briefly describe the proof in Appendix B.2.

Now, given P_4, (5.7) and the explicit formulas (5.8) to (5.10), we have, just as in the proof of Lemma 4.5:

Lemma 5.5. *Under P_4 for model* (i), *for sufficiently large d and for sufficiently large $N \geqq N_0(d; m, p)$, we have*:

$$|\hat{G}^{(m)}(k)-1|, \quad |\hat{\Pi}^{(m)}_{\leqq N}(k)| \leqq c_1\, d^{-1},$$
$$|\partial^s_\mu \hat{G}^{(m)}(k)|, \quad |\partial^s_\mu \hat{\Pi}^{(m)}_{\leqq N}(k)| \leqq c_2\, d^{-2} \qquad (s=1,2),$$
$$0 \leqq \operatorname{Re}(\hat{\Pi}^{(m)}_{\leqq N}(0)-\hat{\Pi}^{(m)}_{\leqq N}(k)) \leqq c_3\, d^{-1}\frac{|k|^2}{d} \leqq c'_3\, d^{-1}(1-\hat{D}(k)),$$
$$|\hat{G}^{(m)}(0)-\hat{G}(0)|, \quad |\hat{\Pi}^{(m)}_{\leqq N}(0)-\hat{\Pi}_{\leqq N}(0)| \leqq c_4\, d^{-2} m^2,$$

where $c_1, \ldots, c_4$ are universal constants independent of p, m, d.

As a result, proceeding just as in the proof of Lemma 4.6, we have:

Lemma 5.6. *Under P_4 with sufficiently large d, for $|m| \leqq d^{-1/2}$ (including $m=0$; note $\hat{\tau}^{(m=0)}_{\mathrm{sym}}(k)=\hat{\tau}(k)$),*

$$|\hat{\tau}^{(m)}_{\mathrm{sym}}(k)| = |\operatorname{Re} \hat{\tau}^{(m)}_p(k)| \leqq |\hat{\tau}^{(m)}_p(k)| \leqq \frac{1+O(d^{-1})}{1-\hat{D}(k)},$$
$$|\partial_\mu \hat{\tau}^{(m)}_{\mathrm{sym}}(k)| = |\operatorname{Re} \partial_\mu \hat{\tau}^{(m)}_p(k)| \leqq |\partial_\mu \hat{\tau}^{(m)}_p(k)| \leqq \frac{10}{11}\cdot\frac{1}{d}\cdot\frac{1}{(1-\hat{D}(k))^2}.$$

Proof. The first bound is proven in exactly the same way as Lemma 4.6. That is, we write

$$|\hat{\tau}^{(m)}(k)| = \frac{|\hat{G}^{(m)}(k)|}{|\hat{F}^{(m)}(k)|} \leqq \frac{|\hat{G}^{(m)}(k)|}{\operatorname{Re} \hat{F}^{(m)}(k)}$$

and use Lemma 5.5 to bound the numerator and the denominator just as was done to prove (4.26).

To prove the second bound, we first write

$$\partial_\mu \hat{\tau}^{(m)}(k) = \frac{\partial_\mu \hat{G}^{(m)}(k)}{\hat{F}^{(m)}(k)} - \frac{\hat{G}^{(m)}(k)\,\partial_\mu \hat{F}^{(m)}(k)}{\hat{F}^{(m)}(k)^2}.$$

For the denominator, by Lemma 5.5, just as in (4.26),

$$|\hat{F}^{(m)}(k)| \geqq \operatorname{Re} \hat{F}^{(m)}(k) \geqq (1-c'd^{-1})(1-\hat{D}(k)).$$

For the numerator, $|\partial_\mu \hat{G}^{(m)}(k)|$ is bounded by Lemma 5.5. $|\partial_\mu \hat{F}^{(m)}(k)|$ is bounded by writing

$$\partial_\mu \hat{F}^{(m)}(k) = 2dp\,\frac{\sin k_\mu}{d} + \delta_{\mu,1}\left\{\frac{\cosh m-1}{d}\sin k_1 + i\,\frac{\sin m}{d}\cos k_1\right\} - \partial_\mu \hat{\Pi}^{(m)}_{\leqq N}(k)$$

and using Lemma 5.5 and (5.7), to get (for $m \leqq d^{-1/2}$)

$$|\partial_\mu \hat{F}^{(m)}(k)| \leqq \frac{26}{25}\,\frac{|\sin k_\mu|}{d} + O(d^{-3/2}) + O(d^{-2}) \leqq \frac{13}{12}\cdot\frac{1}{d}.$$

Combining these, we get the desired bound on $\partial_\mu \hat{\tau}^{(m)}(k)$ for d sufficiently large. □

Given these, we can now prove Prop. 5.3 for the nearest-neighbour model.

Proof of Proposition 5.3 for model (i). We first use the Fourier transform to write,

$$
\begin{aligned}
W_{H,\mu}^{(m)} &= \int \frac{d^d k}{(2\pi)^d} (-\partial_\mu^2 \hat{\tau}_{\mathrm{sym}}^{(m)}(k))\,(\hat{\tau}(k))^5\,(\hat{\tau}_{\mathrm{sym}}^{(m)}(k))^2 \\
&= 2\int \frac{d^d k}{(2\pi)^d} (\partial_\mu \hat{\tau}_{\mathrm{sym}}^{(m)}(k))^2\,(\hat{\tau}(k))^4\,\hat{\tau}_{\mathrm{sym}}^{(m)}(k) \\
&\quad + 4\int \frac{d^d k}{(2\pi)^d} (\partial_\mu \hat{\tau}(k))\,(\partial_\mu \hat{\tau}_{\mathrm{sym}}^{(m)}(k))\,(\hat{\tau}(k))^3\,(\hat{\tau}_{\mathrm{sym}}^{(m)}(k))^2 \\
&\leqq 2\int \frac{d^d k}{(2\pi)^d} (\partial_\mu \hat{\tau}_{\mathrm{sym}}^{(m)}(k))^2\,(\hat{\tau}(k))^4\,\hat{\tau}_{\mathrm{sym}}^{(m)}(k) \\
&\quad + \frac{4}{2}\int \frac{d^d k}{(2\pi)^d} [(\partial_\mu \hat{\tau}(k))^2 + (\partial_\mu \hat{\tau}_{\mathrm{sym}}^{(m)}(k))^2]\,(\hat{\tau}(k))^3\,(\hat{\tau}_{\mathrm{sym}}^{(m)}(k))^2
\end{aligned}
$$

and then use Lemma 5.6 to bound the integrands. We get

$$
W_{H,\mu}^{(m)} \leqq 6 \cdot \left(\frac{11}{10}\right)^2 (1+O(d^{-1}))^5 \frac{1}{d^2} \int \frac{d^d k}{(2\pi)^d} \left(\frac{1}{1-\hat{D}(k)}\right)^9 \leqq \frac{9}{d^2} \cdot \frac{11}{10}.
$$

In the last step, we used (5.13) to bound the integral for large d. □

A. Basic properties of two-point functions

Here we summarize several basic properties of the two point function τ_p and the correlation length $\xi(p)$, which are used in the text. Clear, concise, and beautiful expositions of these and related properties can be found in [16, 22]. Throughout this section, for a given subset $\Lambda \subset \mathbb{Z}^d$, we write $\Lambda^c \equiv \mathbb{Z}^d \setminus \Lambda$. We start with two correlation inequalities which play central rôles.

Proposition A.1. (a) *A simple consequence of Harris-FKG inequality*:

$$
\tau_p(x, y) \geqq \tau_p(x, z)\,\tau_p(z, y) \qquad x, y, z \in \mathbb{Z}^d \tag{A.1}
$$

(b) *Aizenman-Simon type inequality*: *Let Λ be any (nonempty) subset of $\mathbb{Z}^d$ such that $0 \in \Lambda$ and $x \in \Lambda^c$. Then*

$$
\tau_p(0, x) \leqq \sum_{\substack{y \in \Lambda \\ z \in \Lambda^c}} \tau_p(0, y)\,p_{yz}\,\tau_p(z, x). \tag{A.2}
$$

Remark. When dealing with an infinite range model, we find it more convenient to use the above inequality than to use that of Simon-Lieb type.

About the proof. (a) This is a special case of the Harris-FKG inequality [16, 18, 30]. See Sect. 2.1 and (2.22) of [16], or (5.59) of [22].

(b) See Sect. 2.3 of [16]. There Lieb-Simon inequality was derived. The proof of (A.2) proceeds in a similar way, by picking a path (self-avoiding walk) connecting 0, $y \in \Lambda$, $z \in \Lambda^c$ and $x \in \Lambda^c$ (where $\{y, z\}$ is a single bond) for each bond configuration contributing to $\tau_p(0, x)$, and applying van den Berg-Kesten inequality [9]. The corresponding inequality for Ising (and related) models was first derived in [6].

Remark. It is clear from the above proof that a stronger inequality (Lieb-improved version) holds:

$$\tau_p(0, x) \leqq \sum_{\substack{y \in \Lambda \\ z \in \Lambda^c}} \tau_p^{\Lambda^c}(0, y)\, p_{yz}\, \tau_p(z, x)$$

where $\tau_p^A(x, y)$ is defined to be the probability of the event that there exists an occupied path from x to y which does *not* have any cite of A as the endpoints of their bonds.

The "supermultiplicative" (or subadditive for $\ln \tau_p$) property expressed by (A.1) immediately implies:

Proposition A.2. *The mass*

$$\xi(p)^{-1} \equiv m_p \equiv -\lim_{x_1 \to \infty} \frac{\ln \tau_p(0, (x_1, 0, \ldots, 0))}{|x_1|} \tag{A.3}$$

exists ($0 \leqq m_p \leqq \infty$), *and* $\tau_p(0, x)$ *satisfies an a* priori *bound*

$$0 \leqq \tau_p(0, x) \leqq e^{-m_p \|x\|_\infty}. \tag{A.4}$$

Remark. Supermultiplicative property alone does not guarantee the positivity of m_p. Corollary A.4 proves $m_p > 0$ for $p < p_c$.

About the proof. See, e.g., [16] (Sect. 2.3, in particular, Prop. 2.9) or [22] (Sect. 5.2 and Appendix II). (A.4) for x on a coordinate axis follows immediately from the subadditivity. (A.4) for x off the axes is obtained by combining (A.1) [in the form $\tau_p(0, x)^2 \leqq \tau_p(0, x')$] with (A.4), where $x' \equiv 2\|x\|_\infty e_1$ is on the axis. See [22, p. 94] for details. The original "subadditive" argument dates back to [23]. □

Note that (A.4) implies

$$\tau_p^{(m)}(0, x) \leqq e^{-(m_p - m)\|x\|_\infty}. \tag{A.5}$$

Aizenman-Simon inequality (A.2) has the following important consequence. Because it plays the central rôle in the proof of Prop. 1.3, we here present it in a general form and reproduce its proof.

Proposition A.3. *Suppose on* $\mathbb{Z}^d$ *two nonnegative translation-invariant functions* $\sigma(x, y)$ *and* $K(x, y)$ *are given* (i.e. $0 \leqq \sigma(x, y) = \sigma(0, y - x)$, $0 \leqq K(x, y) = K(0, y - x)$ *for* $x, y \in \mathbb{Z}^d$) *and that they satisfy*

$$\sigma(0, x) \leqq \sum_{\substack{y \in \Lambda \\ z \in \Lambda^c}} \sigma(0, y)\, K(y, z)\, \sigma(z, x) \quad \textit{for } 0 \in \Lambda,\ x \in \Lambda^c \tag{A.6}$$

for any (nonempty) finite subset $\Lambda \subset \mathbb{Z}^d$. *Suppose moreover that* $\sum_z K(0,z)\, e^{\delta \|z\|_\infty} < \infty$ *for some* $\delta > 0$ *and that* $\chi \equiv \sum_x \sigma(0,x) < \infty$. *Then*

$$\sigma(0,x) \leqq C \cdot e^{-m\|x\|_\infty}$$

for some finite and positive C and m. (See the proof for a possible choice of C and m.)

Proof. Given (A.6), this is proven following the proof of Corollary 4.2 of [6], which in turn follows the proof of Prop. 3.2 of the same paper. (There, similar results were proven for Ising models.)

Because this proposition is the key in the proof of Prop. 1.3, we reproduce their proof (with minor changes) for the convenience of the reader. The basic idea of the proof can be found in [36, Theorem 1.3, Theorem 5.1]. The following proof (after [6]) is a little more complicated, due to the possible presence of exponentially long-range interactions. We proceed in several steps.

Step 1. We first choose Λ which is suitable for our needs. For this, fix a positive integer R such that

$$(R+1)^d e^{-\delta R/2} \leqq \Big(4 \sum_z K(0,z)\, e^{\delta \|z\|_\infty}\Big)^{-1} \tag{A.7}$$

and

$$\sum_{y:\, \|y\|_\infty \geqq R/2} \sigma(0,y) \leqq \Big(4 \sum_z K(0,z)\, e^{\delta\|z\|_\infty}\Big)^{-1} \tag{A.8}$$

and choose $\Lambda_R \equiv [-R,R]^d \cap \mathbb{Z}^d$, $\Lambda_R^c \equiv \mathbb{Z}^d \setminus [-R,R]^d$. Because $\sum_y \sigma(0,y) < \infty$, and because this is a sum of nonnegative terms,

$$\lim_{R\to\infty} \sum_{y:\, \|y\|_\infty \geqq R/2} \sigma(0,y) = 0$$

and thus the above R (which is finite) exists. In the following, we denote for $z \in \mathbb{Z}^d$, $\operatorname{dist}(z,\Lambda_R) \equiv \inf\{\|y-z\|_\infty \colon y \in \Lambda_R\} = \max\{\|z\|_\infty - R, 0\}$ and, with a slight abuse of notation, $\operatorname{dist}(z,\Lambda_R^c) \equiv \inf\{\|y-z\|_\infty \colon \|y\|_\infty \geqq R\} = \max\{R - \|z\|_\infty, 0\}$. Note that with this choice of R,

$$e^{-\alpha} \equiv \sum_{\substack{z\in\Lambda_R^c \\ y \in \Lambda_R}} \sigma(0,y)\, e^{-\delta\cdot \operatorname{dist}(y,\Lambda_R^c)} K(y,z)\, e^{\delta\|y-z\|_\infty} \leqq \tfrac{1}{2}. \tag{A.9}$$

This can be seen, e.g., by writing

$$\sum_{y\in\Lambda_R} e^{-\delta\cdot\operatorname{dist}(y,\Lambda_R^c)} \sigma(0,y) \leqq \sum_{y:\,\|y\|_\infty \leqq R/2} e^{-\delta\cdot\operatorname{dist}(y,\Lambda_R^c)} + \sum_{y:\,\|y\|_\infty \geqq R/2} \sigma(0,y)$$

and then using above (A.7), (A.8).

Step 2. In the following, we fix x, $y \in \mathbb{Z}^d$ $(y-x \in \Lambda_R^c)$, and analyze $\sigma(x,y)$. We now define a transition probability of a Markov random walk as (the walk itself will be defined in *Step* 3.)

$$p_{wv} \equiv e^{\alpha} \sum_{u: u-w \in \Lambda_R} \sigma(w,u)\, e^{-\delta \cdot \mathrm{dist}(u-w, \Lambda_R^c)} K(u,v)\, e^{\delta \| v-u \|_\infty} \tag{A.10}$$

for $v-w \in \Lambda_R^c$. We rewrite the inequality (A.6) by using the triangle inequality

$$\mathrm{dist}(u-x, \Lambda_R^c) - \|u-v\|_\infty \leqq -\mathrm{dist}(v-x, \Lambda_R)$$

for $u-x \in \Lambda_R$, $v-x \in \Lambda_R^c$, and the above definition of p_{xv} as:

$$\begin{aligned}
\sigma(x,y) &\leqq \sum_{\substack{u: u-x \in \Lambda_R \\ v: v-x \in \Lambda_R^c}} \sigma(x,u)\, K(u,v)\, \sigma(v,y) \\
&= \sum_{v: v-x \in \Lambda_R^c} e^{\alpha} \sum_{u: u-x \in \Lambda_R} \sigma(x,u)\, e^{-\delta \cdot \mathrm{dist}(u-x, \Lambda_R^c)} K(u,v)\, e^{\delta \|u-v\|_\infty} \\
&\quad \cdot \sigma(v,y)\, e^{\delta \cdot \mathrm{dist}(u-x, \Lambda_R^c)} e^{-\delta \|v-u\|_\infty} e^{-\alpha} \\
&\leqq \sum_{v: v-x \in \Lambda_R^c} p_{xv}\, \sigma(v,y)\, e^{-\alpha} e^{-\delta \cdot \mathrm{dist}(v-x, \Lambda_R)}.
\end{aligned}$$

At this stage we can interpret the above sum over v as an expectation with respect to a one step random walk which starts at $\omega(0)=x$ and makes a jump to $\omega(1)=v$ with probability p_{xv}, defined in (A.10). Writing

$$\sum_{v: v-x \in \Lambda_R^c} p_{xv} F(v) \equiv E_1[F(\omega(1))]$$

for an arbitrary function F of v, we can write the above inequality as

$$\sigma(x,y) \leqq E_1[\sigma(\omega(1), y)\, e^{-\alpha - \delta \cdot \mathrm{dist}(\omega(1)-x, \Lambda_R)}]. \tag{A.11}$$

Step 3. We can again apply the inequality (A.6) to $\sigma(\omega(1), y)$ in (A.11), as long as $y-\omega(1) \in \Lambda_R^c$, and can iterate this procedure. For this purpose, for given x, $y \in \mathbb{Z}^d (y-x \in \Lambda_R^c)$, we define a random walk $\omega = (\omega(0), \omega(1), \omega(2), \ldots)$, which is a generalization of the one-step walk in Step 2, according to the following rules: (a) The walk starts at $\omega(0)=x$. (b) If $y-\omega(i) \in \Lambda_R^c$, the walk makes a jump from $\omega(i)$ to $\omega(i+1)$ $[\omega(i+1)-\omega(i) \in \Lambda_R^c]$ with the transition probability $p_{\omega(i), \omega(i+1)}$ [given by (A.10)]. (c) If $y-\omega(i) \in \Lambda_R$, the walk stays at $\omega(i)$, i.e. $\omega(i+1)=\omega(i)$. In other words, the walk stops at the first time t such that $y-\omega(t) \in \Lambda_R$. We define the stopping time τ_N for the N-step walk as either the earliest time $t \in \{1, 2, \ldots, N\}$ for which $y-\omega(t) \in \Lambda_R$ [in this case the walk stops at $\omega(t)$], or N, if $y-\omega(t) \in \Lambda_R^c$ for all $t \in \{1, 2, \ldots, N\}$.

Using this notation, iteration of the above inequality (A.11) gives for $N \geqq 1$

$$\sigma(x,y) \leqq E_N\left[\sigma(\omega(N), y) \exp\left(-\alpha \cdot \tau_N - \delta \sum_{i=1}^{N} \mathrm{dist}(\omega(i)-\omega(i-1), \Lambda_R)\right)\right] \tag{A.12}$$

where we denoted the expectation with respect to the above defined N-step walk by $E_N[\cdot]$.

Step 4. We take N sufficiently large so that $N \cdot R \geqq \|x-y\|_\infty$. In the exponent of (A.12), either $\tau_N = N$ or $\tau_N < N$. If $\tau_N = N$, the quantity inside the expectation is bounded above by

$$\leqq e^{-\alpha N} \leqq \exp\left(-\frac{\alpha}{R}\|x-y\|_\infty\right). \tag{A.13}$$

If $\tau_N < N$, there exists $t \in \{1, 2, \ldots, N\}$ such that $y - \omega(t) \in \Lambda_R$. In this case, $\tau_N = t$, and by the triangle inequality,

$$\sum_{i=0}^{t} \operatorname{dist}(\omega(i)-\omega(i-1), \Lambda_R) \geqq \|\omega(t)-\omega(0)\|_\infty - t \cdot R \geqq \|y-x\|_\infty - (t+1) R$$

and thus the quantity inside the expectation of (A.12) is bounded by

$$\leqq \exp\left(-\min\left\{\frac{\alpha}{R}, \delta\right\} \|x-y\|_\infty + \alpha\right). \tag{A.14}$$

(A.12), (A.13) and (A.14) yield

$$\sigma(x, y) \leqq e^\alpha e^{-\min\left\{\frac{\alpha}{R}, \delta\right\} \|x-y\|_\infty}$$

and we have the desired bound with $C = e^\alpha$, $m = \min\left\{\frac{\alpha}{R}, \delta\right\}$. $\square$

Corollary A.4. *Consider a translation invariant bond percolation model defined in Sect.* 1.1 *with* $\sum_z p_{0z} e^{\delta\|z\|_\infty} < \infty$. *Then* (a) $m_p > 0$ *as long as* $p < p_c \equiv \sup\{p \mid \chi_p < \infty\}$ *and* (b) $m_p \searrow 0$ *as* $p \nearrow p_c$. *Moreover,* (c) *if* $m_p < \delta$,

$$\chi_p^{(m)} \equiv \sum_x \tau_p(0, x) e^{m x_1} \nearrow \infty \qquad \textit{as } m \nearrow m_p.$$

Proof. Part (a) follows directly from Prop. A.3, by replacing σ (respectively K, χ) by τ_p (resp. p, χ_p) [(A.6) follows from (A.2)].

(b) The *a priori* bound (A.4) implies a simple upper bound on m_p (for all $p < p_c$):

$$\chi_p \leqq 1 + 2d \cdot \left(\frac{3}{m_p}\right)^d. \tag{A.15}$$

Because $\chi_p \nearrow \infty$ as $p \nearrow p_c$ [24, 5], taking (a) into account, (A.15) implies $m_p \searrow 0$.

(c) Multiplying (A.2) by $e^{m x_1}$, we get its $e^{m x_1}$-weighted version (as always, $\tau_p^{(m)}(0, x) \equiv \tau_p(0, x) e^{m x_1}$):

$$\tau_p^{(m)}(0, x) \leqq \sum_{y \in \Lambda,\, z \in \Lambda^c} \tau_p{}^{(m)}(0, y)\, p_{yz}^{(m)}\, \tau_p^{(m)}(z, x). \tag{A.16}$$

Fix p such that $m_p<\delta$. Then we have $\sum_z p_{0z}^{(m_p)} e^{\delta' \|z\|_\infty}<\infty$ with (say) $\delta'=(\delta-m_p)/2>0$. Now suppose $\chi_p^{(m)} \leqq C<\infty$ *uniformly* in $m \in [0, m_p)$. Then by the Monotone Convergence Theorem $\chi_p^{(m_p)} \leqq C<\infty$ [note: $\chi_p^{(m)}=\sum_x \tau_p(0,x)\cosh(mx_1)$ is monotone nondecreasing in $|m|$]. Now we can apply Prop. A.3 [with σ, K, χ replaced by $\tau_p^{(m_p)}$, $p^{(m_p)}$, $\chi_p^{(m_p)}$; (A.6) is nothing but (A.16) with $m=m_p$] to conclude $\tau_p^{(m_p)}(0,x) \leqq C' e^{-m'\|x\|_\infty}$ with $0<m'$, $C'<\infty$. This contradicts the definition of m_p. (I am grateful to Gordon Slade for having pointed out an error in an earlier version of the above proof.) □

Remarks. 1. Part (a) was first proven (for the nearest-neighbour model) by Hammersley [24].

2. Divergence of $\xi(p)$ can be proven without relying on the divergence of χ_p, by using "Lieb-improved" AS inequality along the line of argument of [36].

B. Bounds on gaussian quantities

In this appendix, we prove Lemma 4.1 and Lemma 5.4 on the properties of gaussian quantities defined in terms of $\{J_{xy}\}$. The proof of Lemma 4.1 is very much similar to that of Lemma 5.1 of [27]. We concentrate on several points which require a somewhat different treatment from that of [27].

B.1. Proof of Lemma 4.1 (gaussian quantities of the spread-out model)

The proof of Lemma 4.1 proceeds in parallel to that of Lemma 5.1 of [27]. In the following, we write $a(L) \approx b(L)$ when $\lim_{L\to\infty} a(L)/b(L)=1$. We also use big-$O$ to denote upper bounds involving constants independent of L.

First note that by definition,

$$p_{0x}^{(G)}=\frac{L^{-d} g(x/L)}{\sum_x L^{-d} g(x/L)}.$$

As L goes to infinity, the denominator goes to one, because this is nothing but the Rieman sum approximation of the integral $\int d^d y\, g(y) \equiv 1$ (we have rewritten $x/L=y$; recall that we have normalized g in the definition). Now, for $0 \leqq m \leqq \delta \cdot L^{-2}$, from the assumption on the decay of g, (i.e. $g(x) \leqq C e^{-\delta\|x\|_\infty}$)

$$p_{0x}^{(G)} e^{mx_1} \approx L^{-d} g(x/L)\, e^{mx_1} \leqq L^{-d} \cdot C \cdot e^{-(\delta-mL)\|y\|_\infty}=O(L^{-d})$$

(in the second step we wrote $y=x/L$). Also (writing again $y=x/L$)

$$\sum_x (p_{0x}^{(G)} e^{mx_1})^2 \approx L^{-d} \int d^d y \{g(y)\, e^{mLy_1}\}^2 \approx O(L^{-d}).$$

This proves (4.1). For (4.2),

$$\sup_x |x_1|^2 p_{0x}^{(G)} e^{mx_1} \approx \sup_y L^{-d+2} |y_1|^2 g(y) e^{mLy_1} = O(L^{-d+2}),$$

$$\sum_x (p_{0x}^{(G)})^2 e^{mx_1} |x_1|^2 \approx L^{-d+2} \int d^d y g(y)^2 e^{mLy_1} |y_1|^2 = O(L^{-d+2}).$$

The bounds (4.3) to (4.8) are identical to (5.9) to (5.14) of [27] (except that the constants here are a little sharper), and are proven in exactly the same way as in [27]. We are now left with (4.9) to (4.12).

As for (4.9), by the definition of $\hat{D}^{(m)}(k)$, we just use the fact that

$$\hat{D}^{(m)}(0) \approx \frac{\int d^d y g(y) e^{mLy_1}}{\int d^d y g(y)} \to 1 \qquad (\text{as } L \to \infty)$$

for $|m| \leqq \delta \cdot L^{-2}$. For (4.12),

$$\hat{D}^{(m)}(0) - \hat{D}(0) = \frac{L^{-d} \sum_x g(x/L)(\cosh mx_1 - 1)}{L^{-d} \sum_x g(x/L)} \geqq \frac{L^{-d} \sum_x d(x/L) |x_1|^2 (m^2/2)}{L^{-d} \sum_x g(x/L)}$$

$$\approx \int d^d y g(y) |y_1|^2 \cdot \frac{L^2 m^2}{2} = \frac{1}{d} \int d^d y g(y) |y|^2 \cdot \frac{L^2 m^2}{2} = \frac{L^2 m^2}{2d}$$

where in the last step we used the normalization convention (1.8). Furthermore for $|m| \leqq \delta L^{-2}$

$$\hat{D}^{(m)}(0) - \hat{D}(0) \leqq \frac{L^{-d} \sum_x g(x/L) \cosh(mx_1) |x_1|^2 (m^2/2)}{L^{-d} \sum_x g(x/L)}$$

$$\approx \frac{(mL)^2}{2} \int d^d y g(y) |y_1|^2 \cosh(mLy_1) \approx \frac{(mL)^2}{2d} \int d^d y g(y) |y|^2 = \frac{(mL)^2}{2d}.$$

For (4.11), we proceed exactly as we did in (2.21) and (2.22). Using the fact that $p_{0x}^{(G)}$ is even in each $x_1, \ldots, x_d$,

$$|\mathrm{Im}\, \hat{D}^{(m)}(k)| \leqq m|k_1| \sum_x p_{0x}^{(G)} \cosh(mx_1) |x_1|^2$$

$$\approx \int d^d y g(y) |y_1|^2 \cosh(mLy_1) \cdot L^2 \cdot m|k_1| \approx \frac{L^2 m|k_1|}{d}.$$

Lastly for (4.10), we note by Lemma 5.4 of [27] that for sufficiently large L and for $|m| \leqq \delta L^{-2}$

$$|\hat{D}^{(m)}(k)| \leqq 2 \frac{\|\partial^I \{g(y) e^{mLy_1}\}\|_1}{\prod_{v \in I} 2L \left|\sin \frac{k_v}{2}\right|} \approx 2 \frac{\|\partial^I g(y)\|}{\prod_{v \in I} 2L \left|\sin \frac{k_v}{2}\right|}$$

for $I \subset \{1, 2, \dots, d\}$. The numerator is bounded by some constant independent of L, by the assumption on the derivatives of g. Now we can argue just as in [27, Sect. 5.2]. □

B.2. Proof of Lemma 5.4 (gaussian quantities of the nearest-neighbour model)

For the gaussian nearest-neighbour model, we can explicitly calculate most of the quantities in question. (5.8) to (5.12) are proven this way. (5.13) can be proven, e.g., following Sect. 3 of [37]. We here provide the proof of (5.13) and bounds on related quantities following a different line of argument.

Throughout this section, as in Sect. 5, we write

$$\int \frac{d^d k}{(2\pi)^d} \equiv \int_{[-\pi,\pi]^d} \frac{d^d k}{(2\pi)^d}, \tag{B.1}$$

$$\hat{D}(k) = \sum_{\mu=1}^{d} \frac{\cos k_\mu}{d}, \qquad C(0,x) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{e^{ik\cdot x}}{1-\hat{D}(k)}$$

and we define (for nonnegative integers m, n)

$$I_{n,m}(d) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{\hat{D}(k)^m}{(1-\hat{D}(k))^n}.$$

We also write

$$f(d) = a(d) + O(d^{-\alpha})$$

if

$$f(d) = a(d) + b(d) \quad \text{with} \quad |b(d)| \leqq C d^{-\alpha} \tag{B.2}$$

where C is a constant which does not depend on d. (In the following, this C depends on integer parameters m, n.)

We here prove the following two estimates. The first lemma provides (5.13) as its special case. The method of its proof can be used to generate rigorous asymptotic expansions in d^{-1} for gaussian quantities, and in the Lemma we have given their first order terms. The second one is not necessary for this paper, and is presented here just to give another proof of the $O(1/d)$ bound on W_G. A somewhat related idea was used in [11] to prove $\lim_{d\to\infty} dI_{12}(d) = 1/2$.

Lemma B.1. *For the gaussian nearest-neighbour model* (i) *of Sect.* 1.1, (a) *For a nonnegative integer m,*

$$\frac{(2m-1)!!}{(2d)^m}\left\{1-\frac{c_1}{d}\right\} \leqq \int \frac{d^d k}{(2\pi)^d} |\hat{D}(k)|^{2m} \leqq \frac{(2m-1)!!}{(2d)^m} \tag{B.3}$$

where c_1 is a constant which does not depend on d (but does depend on m).

(b) *$I_{n,0}(d)$ is monotone nondecreasing in n, is monotone nonincreasing in d, and*

$$I_{n,0}(d) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{1}{(1-\hat{D}(k))^n} \geqq 1.$$

(c) *For nonnegative integers m, n, and for* $d \geqq 4n+2$,

(B.4) $$K_{n,m}(d) \equiv \int \frac{d^d k}{(2\pi)^d} \frac{\hat{D}(k)^{2m}}{(1-\hat{D}(k)^2)^n} = \frac{(2m-1)!!}{(2d)^m} \{1+O(d^{-1})\}.$$

Moreover, for $m=0$,

(B.5) $$K_{n,0}(d) = 1 + \frac{n}{2d} + O(d^{-2}).$$

(d) *Consider the same situation as* (c). *We have, for m even,*

(B.6) $$I_{n,m}(d) = \frac{(m-1)!!}{(2d)^{m/2}} \{1+O(d^{-1})\}$$

and, for m odd,

(B.7) $$I_{n,m}(d) = n \cdot \frac{m!!}{(2d)^{(m+1)/2}} \{1+O(d^{-1})\}.$$

Moreover, for $I_{n,0}(d)$,

(B.8) $$I_{n,0}(d) = 1 + \frac{n(n+1)}{4d} + O(d^{-1}).$$

As for quantities which include derivatives, we have

Lemma B.2. *For the gaussian nearest-neighbour model* (i) *of Sect.* 1.1, *the* $|x|^2$*-weighted quantity*

$$W_n(d) \equiv \sum_\mu \int \frac{d^d k}{(2\pi)^d} \frac{|\partial_\mu \hat{D}(k)|^2}{(1-\hat{D}(k))^n}$$

satisfies, for $n > 1 + d/2$,

$$W_n(d) = \frac{1}{2d} + O(d^{-2}).$$

Remark. In particular, for gaussian quantities T_G, W_G defined by replacing τ_p by C in Definition 3.1 we have

(B.9) $$\frac{3}{2d} \leqq T_G \leqq \frac{3}{2d} + O(d^{-2}),$$

(B.10) $$\frac{1}{2d} \leqq W_G \leqq \frac{1}{2d} + O(d^{-2}),$$

and

(B.11) $$\frac{1}{2d} \leqq C(0,0) - 1 = K_{1,0}(d) - 1 = \frac{1}{2d} + O(d^{-2}).$$

In the above, the lower bounds were obtained by rewriting the quantities in terms of $C(0,x)$ in x-space, and by counting the contribution to $C(0,x)$ only from a one-step walk.

Proof of Lemma B.1. (a) By direct computation: just expand $\hat{D}(k)^{2m}$ and evaluate the integral. The upper bound is a trivial case of gaussian inequality.

(b) Here in (b), and in the proof of Lemma B.2, we use v to denote the dimension to avoid confusion with the differential notation. This v should not be confused with the critical exponent. First, by Hölder's inequality,

$$I_{1,0}(v) \leqq (I_{n,0}(v))^{1/n} \leqq (I_{n',0}(v))^{1/n'}$$

for $1 \leqq n \leqq n'$. Also by Jensen's inequality,

$$I_{1,0}(v) \geqq \left(\int \frac{d^v k}{(2\pi)^v} (1-\hat{D}(k)) \right)^{-1} = 1.$$

These prove $I_{n,0}(v) \geqq 1$, and the monotonicity in n. To prove the monotonicity in v of $I_{n,0}(v)$, we employ the formula:

$$\frac{1}{A^n} = \int_0^\infty \frac{dt \cdot t^{n-1}}{(n-1)!} e^{-tA} \qquad \text{for } A \geqq 0$$

to write

$$I_{n,0}(v) = \int_0^\infty \frac{dt \cdot t^{n-1}}{(n-1)!} \left(\int_0^{2\pi} \frac{d\theta}{2\pi} e^{-\frac{t}{v}(1-\cos\theta)} \right)^v. \tag{B.12}$$

Now, in general, for a real random variable X and a probability measure $d\mu(X)$,

$$g(\alpha) \equiv \left(\int d\mu(X)\, e^{-\alpha X}\right)^{1/\alpha} = \| e^{-X} \|_{L^\alpha(d\mu)}$$

is a nondecreasing function of $\alpha \geqq 0$. In (B.12), the integrand of the t-integration is of the form $g(1/v)$, and is monotone nonincreasing in v for each t.

(c) We use c, c', c'' to denote constants which do not depend on d (but do depend on m, n). They many represent different values on different occasions. We also introduce $Y \equiv \hat{D}(k)^2$, and denote the integral over k simply as

$$\langle F(Y) \rangle \equiv \int \frac{d^d k}{(2\pi)^d} F(\hat{D}(k)^2).$$

The lower bound of (B.4) follows immediately, if we note (because $0 \leqq Y \leqq 1$)

$$K_{n,m}(d) = \left\langle \frac{Y^m}{(1-Y)^n} \right\rangle \geqq \langle Y^m \rangle \tag{B.13}$$

and use the lower bound of (B.3). (We can alternatively derive the lower bound by the following argument.)

To get the upper bound, we employ a method which can be used to generate asymptotic expansions in the powers of d^{-1} of the quantities in question. The basic idea is rather simple: $Y = \hat{D}(k)^2$ is a square of a sum of d independent random variables ($\cos k_\mu$) divided by d^2, and thus we can expect $Y \sim d^{-1}$. Hence in the expression (B.13) of $K_{n,m}(d)$, we might be able to neglect Y in the denominator to conclude $K_{n,m}(d) \sim \langle Y^m \rangle \sim d^{-m}$.

To make the above idea rigorous, we first add and subtract $\langle Y^m \rangle$ to write:

$$\text{(B.14)} \quad K_{n,m}(d) = \langle Y^m \rangle + \left\langle \frac{Y^m\{1-(1-Y)^n\}}{(1-Y)^n} \right\rangle = \langle Y^m \rangle + \sum_{l=1}^{n} (-1)^{l+1} \binom{n}{l} \left\langle \frac{Y^{m+l}}{(1-Y)^n} \right\rangle$$

where

$$\binom{n}{l} \equiv \frac{n!}{l!(n-l)!}$$

is the binomial coefficient. The first term gives the main contribution and the rest will be of higher orders in d^{-1} (one factor of Y in the numerator would give rise to one d^{-1}).

Our remaining task is to show that the terms in (B.14) except for the first one are in fact of higher orders in d^{-1}. This is done in several steps. First, we derive a (very rough) uniform (in d) bound on $K_{n,m}(d)$. Because $(1-\hat{D}(k))^{-1}$ is increasing in $\hat{D}(k)$ and $(1+\hat{D}(k))^{-1}$ is decreasing in $\hat{D}(k)$, we can use the FKG inequality to derive (using $|Y| \leqq 1$ first)

$$\begin{aligned} K_{n,m}(d) &\leqq K_{n,0}(d) = \langle (1-\hat{D}(k))^{-n} \cdot (1+\hat{D}(k))^{-n} \rangle \\ &\leqq \langle (1-\hat{D}(k))^{-n} \rangle \, \langle (1+\hat{D}(k))^{-n} \rangle = (I_{n,0}(d))^2. \end{aligned}$$

But, because $1-\hat{D}(k) \geqq 2|k|^2/(d\pi^2)$, $I_{n,0}(d)$ is finite for $d > 2n$. Taking into account the monotonicity of $I_{n,0}(d)$ in d, we thus have a bound (uniform in $d \geqq 2n+1$)

$$K_{n,m}(d) \leqq (I_{n,0}(2n+1))^2 \equiv c_n^2.$$

As the second step, we use this uniform bound to obtain $O(d^{-\alpha})$ bounds. By the Schwartz inequality, and by (B.3), for $d \geqq 4n+2$,

$$\text{(B.15)} \qquad K_{n,m}(d) = \left\langle \frac{Y^m}{(1-Y)^n} \right\rangle \leqq \langle Y^{2m} \rangle^{1/2} \left\langle \frac{1}{(1-Y)^{2n}} \right\rangle^{1/2} \leqq c \cdot d^{-m}$$

where c depends on m, n but not on d.

As the third step, we substitute the above rough bound (B.15) into (B.14), and use (B.3) to bound to resulting $\langle Y^{m+l} \rangle$:

$$\begin{aligned} K_{n,m}(d) &\leqq \frac{(2m-1)!!}{(2d)^m} + \sum_{l=1}^{n} \binom{n}{l} O(d^{-(m+l)}) \\ &\leqq \frac{(2m-1)!!}{(2d)^m} \left\{ 1 + \frac{c'}{d} \right\} \end{aligned}$$

where c' depends on m, n, but not on d. This proves (B.4).

Now it is clear that we can carry out the above procedure to obtain asymptotic expansions in d^{-1} to arbitrary high orders. For example, to the second order, we write

$$K_{n,m}(d)=\langle Y^m\rangle+n\langle Y^{m+1}\rangle+\left\langle\frac{Y^m-Y^m(1-Y)^n-nY^{m+1}(1-Y)^n}{(1-Y)^n}\right\rangle$$

and use (B.15) to bound the last term, together with explicit calculation of $\langle Y^m\rangle$ and $\langle Y^{m+1}\rangle$. For $m=0$ this gives (B.5).

(d) This follows immediately from (c), by writing (by the symmetry of the integral measure)

$$\begin{aligned}I_{n,m}(d)&=\frac{1}{2}\int\frac{d^dk}{(2\pi)^d}\left(\frac{\hat{D}(k)^m}{(1-\hat{D}(k))^n}+\frac{(-\hat{D}(k))^m}{(1+\hat{D}(k))^n}\right)\\&=\sum_{l'=\left[\frac{m+1}{2}\right]}^{\left[\frac{m+n}{2}\right]}\binom{n}{2l'-m}\left\langle\frac{Y^{l'}}{(1-Y)^n}\right\rangle\end{aligned}$$

and using (B.4) to bound the right hand side. For $m=0$, we write

$$I_{n,0}(d)=K_{n,0}(d)+\frac{n(n-1)}{2}\left\langle\frac{Y}{(1-Y)^n}\right\rangle+\sum_{l'=2}^{\left[\frac{n}{2}\right]}\binom{n}{2l'}\left\langle\frac{Y^{l'}}{(1-Y)^n}\right\rangle$$

and use (B.15). This completes the proof of the Lemma B.1. □

Proof of Lemma B.2. We write v for the dimension d to avoid confusion with the differential. The lemma follows immediately from Lemma B.1 and the following *identities*:

(B.16) $$W_n(v)=\frac{1}{n-1}\{I_{n-1,0}(v)-I_{n-2,0}(v)\}\quad\text{for } n\geqq 2,$$

(B.17) $$W_1(v)=\frac{1}{2v}\{I_{1,0}(v)-C(0,2;v)\}.$$

Here in (B.17), $C(0,2;v)$ is the gaussian propagator (B.1) from 0 to $x=(2,0,\dots,0)$. To prove (B.16) and (B.17) we first observe, by the same argument which led to (B.12), that (here we write $t/v=s$)

(B.18) $$W_n(v)=v^{n-1}\int_0^\infty\frac{ds\cdot s^{n-2}}{(n-1)!}e^{-vs}\{I_0(s)\}^{v-1}I_1(s)$$

where

$$I_0(s)\equiv\int_0^{2\pi}\frac{d\theta}{2\pi}e^{s\cos\theta},\qquad I_1(s)\equiv s\cdot\int_0^{2\pi}\frac{d\theta}{2\pi}e^{s\cos\theta}\sin^2\theta$$

are modified Bessell functions. Using the recursion relation

$$I_1(s)=\frac{d}{ds}I_0(s)$$

which follows easily from their definition, and using the integration by parts (for $n \geqq 2$),

$$W_n(v)=v^{n-2}\int_0^\infty \frac{ds\cdot s^{n-2}}{(n-1)!}e^{-vs}\frac{d}{ds}(I_0(s))^v$$

$$=v^{n-2}\left[\frac{s^{n-2}}{(n-1)!}e^{-vs}(I_0(s))^v\right]_0^\infty+\frac{v^{n-1}}{(n-1)!}\int_0^\infty ds\, s^{n-2}e^{-vs}(I_0(s))^v$$

$$-\frac{n-2}{(n-1)!}v^{n-2}\int_0^\infty ds\, s^{n-3}e^{-vs}(I_0(s))^v.$$

For $n \geqq 3$ the first term vanishes and (B.16) is proved. For $n=2$ the last term vanishes, and the first one contributes -1, thus leading to (B.16) again. For $n=1$, we simply rewrite $\sin^2\theta=(1-\cos 2\theta)/2$ in the definition of $I_1(s)$ which occurs in (B.18), and compare the result with expressions for $I_{1,0}$ and $C(0,2;v)$. $\square$

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Note added in proof

In the proof of Propostition 1.4 for the nearest-neighbour model in Section 5.2, we used the fact that $W_{H,\mu} = W_{H,\mu}^{(m=0)} \leqq 12 \cdot d^{-2}$ ($\mu = 1, 2, \ldots, d$). This follows easily from i) continuity of $W_{H,\mu}$ in p, ii) the fact that $W_{H,\mu} \leqq (W_{H,\mu})_{\text{gauss}} \leqq 12 \cdot d^{-2}$ for $p \leqq 1/2d$, and iii) Proposition 5.3 for $m=0$, just as was done in [27] to prove P_3 for quantities T, W, W_a, H_{a_1,a_2}.