# Improved Upper Bounds for Pairing Heaps* 

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#### Abstract

Pairing heaps are shown to have constant amortized time Insert and Meld, thus showing that pairing heaps have the same amortized runtimes as Fibonacci heaps for all operations but Decrease-Key.


## 1 Introduction

Pairing heaps were introduced in 10 as a priority queue data structure modeled after splay trees 18. As in splay trees, pairing heaps do not augment the nodes of the heap with any information, and use simple local restructuring heuristics to perform all operations. Such structures are referred to as self-adjusting. They are easy to code and empirically perform well [20].

While the study of splay trees has been focused on their ability to quickly execute various distributions of operations $1,6,13,15,18,22$, the study of pairing heaps remains stuck at a much earlier stage - tight amortized bounds on the runtimes of all operations in terms of the heap size remain unknown. There is also small body of work studying how pairing heaps work on particular distributions of operations [7, 12, 16].

On the practical side, pairing heaps are in use. They are covered in some elementary and intermediate level data structures texts 23 26. Additionally, one can find on the internet code that has been developed to implement pairing heaps in various languages. Pairing heaps were part of the pre-STL GNU C++ Library, and thus were distributed widely.

[^0]|  |  | Insert and MELD | Decrease-KEY |
| :--- | :---: | :---: | :---: |
| 1986 | Original pairing heap 10 | $O(\log n)$ | $O(\log n)$ |
| 1987 | Stasko and Vitter (variant) 20 | $O(1)$ | Forbidden |
| 1999 | Fredman 9] | $O(\log n)$ | $\Omega(\log \log n)$ |
| 2001 | This Paper | $O(1)$ | $O(\log n)$ |
| 2005 | Pettie 17 | $O\left(2^{2 \sqrt{\log \log n})}\right.$ | $O\left(2^{2 \sqrt{\log \log n})}\right.$ |
| 2010 | Elmasry (variant) 8 | $O(1)$ | $O(\log \log n)$ |

Figure 1: Summary of previous results for pairing heaps and their variants, in chronological order of first appearance. All previous results support $O(1)$ amortized Make-HEAP and $O(\log n)$ amortized Extract-Min. All results are amortized, so the runtimes of the individual operations can not be mixed among the different results. The result of Fredman's $[9$ gives an amortized lower bound for Decrease-Key, given amortized upper bounds for other operations. The result of Stasko and Vitter is for a variant of pairing heaps and does not allow the Decrease-Key operation.

The theoretically leading non-self adjusting priority queue is the Fibonacci heap [11] , and pairing heaps are designed to support the same set of operations as Fibonacci heaps:

- $h=\operatorname{Make}-\operatorname{Heap}():$ Returns an identifier $h$ of a new empty heap.
- $x=\operatorname{Extract}-\operatorname{Min}(h)$ : Removes and returns $x$, the minimum element in heap $h$.
- $p=\operatorname{Insert}(h, x)$ : Inserts $x$ into heap $h$, and returns an identifier $p$ that can be used to manipulate $x$ in the future.
- Delete $(h, p)$ : Removes the item in $h$ identified by $p$.
- Decrease-Key $(h, p, \Delta)$ : Decreases the key value of the item in $h$ identified by $p$ by a nonnegative amount $\Delta$.
- $h=\operatorname{Meld}\left(h_{1}, h_{2}\right)$ : Combines the contents of heaps $h_{1}$ and $h_{2}$ into a new heap with returned identifier $h$. The identifiers to $h_{1}$ and $h_{2}$ are no longer valid.
- $x=$ Find- $\operatorname{Min}(h)$ : Returns the minimum element of the heap $h$.

In the original pairing heap paper [10], all operations except the constant-amortized-time Make-HEAP and Find-Min were shown to take $O(\log n)$ amortized time (see Figure 1 for a comparison of the known bounds on pairing heaps and their variants ). It was conjectured in 10 and empirical evidence was presented by Stasko and Vitter 20 that pairing heaps share the same amortized cost per operation as Fibonacci heaps, which have $O(1)$ Decrease-Key, Insert and Meld operations. However, this possibility was eliminated when it was shown by Fredman 9$]$ that the amortized cost of Decrease-Key must be $\Omega(\log \log n)$ amortized, given $O(\log n)$ amortized costs for the other operations. Recently, Pettie has produced a new analysis that focuses on the DECREASE-KEY operation, where he proves a $O\left(2^{2 \sqrt{\log \log n}}\right)$ amortized bound on Insert, Meld, and Decrease-Key while retaining a $O(\log n)$ upper bound on Extract-Min. The asymptotic cost of Decrease-Key remains unknown as Fredman's $\Omega(\log \log n)$ amortized lower bound and Pettie's $O\left(2^{2 \sqrt{\log \log n}}\right)$ amortized upper bound are the best known amortized bounds.

In this work we present a new analysis of pairing heaps that proves, with the exception of the DecreaseKEY operation, pairing heaps share the same asymptotic amortized runtimes per operation as Fibonacci heaps. Specifically, we show the amortized cost of Extract-Min, Delete, and Decrease-Key is $O(\log n)$ and the amortized cost of Make-Heap, Insert, and Meld is $O(1)$. Thus, compared to the original analysis in 10 the amortized upper bound of $O(\log n)$ for the Insert and MELD operations is improved to $O(1)$. Compared to the analysis of Pettie, the $O\left(2^{2 \sqrt{\log \log n}}\right)$ amortized bounds for Insert and Meld are improved to a constant, while Pettie's $O\left(2^{2 \sqrt{\log \log n}}\right)$ amortized bound for Decrease-KEY is tighter than our $O(\log n)$ amortized bound.

It should be noted that Stasko and Vitter in [20] introduced a variant of pairing heaps, the auxiliary twopass method, and proved that this structure supportes constant amortized time Insert. However, their analysis explicitly forbade the DECREASE-KEY operation. Elmasry [8] invented a different variant of pairing heaps that obtains constant amortized Insert and amortized $O(\log \log n)$ Decrease-Key; however Fredman's $\Omega(\log \log n)$ amortized lower bound on DECREASE-KEY does not apply to Elmasry's variant as they do not confirm to his model of generalized pairing heaps.

Here it is shown that pairing heaps are shown to have constant amortized time Insert and Meld, thus showing that pairing heaps have the same amortized runtimes as Fibonacci heaps for all operations but Decrease-Key. The Decrease-Key operation is allowed at an amortized cost of $O(\log n)$.

## 2 Pairing Heaps

A pairing heap is a heap-ordered general tree. Here a min-heap is assumed. The basic operation on a pairing heap is the pairing operation, which combines two pairing heaps into one by attaching the root with the

(a) Remove the root.

(c) The second pairing pass incrementally pairs the right two nodes until a single tree is formed.

(e) Second pairing pass, continued.
(b) The first pairing pass groups the nodes in pairs, and pairs them.

(d) Second pairing pass, continued.

(f) Final heap that results from an Extract-Min.

Figure 2: Illustration of how an Extract-Min is executed on a heap where the root has eight children.
larger key value to the other root as its leftmost child. Priority queue operations are implemented in a pairing heap as follows: Make-Heap creates a new single node heap. Find-Min returns the data in the root of the heap. Meld pairs the roots of the two heaps. Insert creates a new node and pairs it with the root of the heap it is being inserted into. Decrease-Key breaks off the node and its induced subtree from the heap (if the node is not the root), decreases the key value, and then pairs it with the root of the heap. Delete breaks off the node to be deleted and its subtree, performs an Extract-Min on the subtree, and pairs the resultant tree to the root of the heap. Extract-Min is the only non-trivial operation. An Extract-Min removes and returns the root, and then, in pairs, pairs the remaining trees in the resultant forest. Then, the remaining trees from right to left are incrementally paired. See Figure 2 for an example of an Extract-Min executing on a pairing heap. All pairing heap operations take constant actual time, except Extract-Min and Delete, which take time linear in the number of children of the node to be removed. For the purposes of implementation, pairing heaps are typically stored as a binary tree using the leftmost child, right sibling correspondence. Unless otherwise stated, the standard tree terminology will refer to the general tree representation.

## 3 Main Result

### 3.1 Overview

We claim that in a pairing heap the amortized runtimes of Find-Min, Make-Heap, Meld, and Insert are $O(1)$ and Decrease-Key, Delete and Extract-Min are $O(\log n)$. We adopt the convention that $\log$ refers to the binary logarithm.

Let $X=x_{1}, x_{2}, \ldots, x_{m}$ be a sequence of operations to be executed on an initially empty collection of heaps. The remainder of the notation used is implicitly parameterized by a fixed, but arbitrary, $X$. Let $a_{i}$ be the actual time to execute operation $x_{i}$. Let $n_{i}$ be the size of the heap that $x_{i}$ acted upon after the execution of $x_{i}$. We define the actual cost of an operation to be the number of pairings performed plus 1. This cost measure is chosen as the number of parings clearly dominates the asymptotic runtime of each non-constant-cost operation. Let $\hat{a}_{i}$ be the amortized cost that we wish to prove for operation $x_{i}$. Specifically, if $x_{i}$ is an Insert let $\hat{a}_{i}=21$, if $x_{i}$ is an Find-Min let $\hat{a}_{i}=1$, if $x_{i}$ is an Meld let $\hat{a}_{i}=0$, if $x_{i}$ is an Make-Heap let $\hat{a}_{i}=21$, if $x_{i}$ is an Delete let $\hat{a}_{i}=127+144 \log n_{i}$, if $x_{i}$ is an Decrease-Key let $\hat{a}_{i}=26+27 \log n_{i}$ and if $x_{i}$ is an Extract-Min let $\hat{a}_{i}=117 \log \left(n_{i}+1\right)+101$.

Given this notation, the main theorem can be stated. The time to execute any sequence on an initially empty pairing heap is bounded by the sum of the previously stated amortized times for each operation. Formally,

Theorem 1. $\sum_{i=1}^{m} a_{i} \leq \sum_{i=1}^{m} \hat{a}_{i}$.
The potential method is used to prove this theorem. We define below a potential function $\Phi=$ $\left\langle\Phi_{0}, \Phi_{1}, \ldots, \Phi_{m}\right\rangle$ that is simply a sequence of real numbers. The potential method may be summarized in the following lemma:

Lemma 2. If there exists a sequence $\Phi$ such that for all $i, 1 \leq i \leq m, \hat{a}_{i} \geq a_{i}+\Phi_{i}-\Phi_{i-1}$ and $\Phi_{m}-\Phi_{0} \geq 0$ then $\sum_{i=1}^{m} \hat{a}_{i} \geq \sum_{i=1}^{m} a_{i}$.

This lemma, which summarizes the potential method, may be proved by simple algebraic manipulation. More details on the potential method may be found in [21].

The proof of Theorem 1 proceeds as follows. Section 3.2 is devoted to defining the potential function $\Phi$. This potential function is complex and has several components. We then prove, in a sequence of lemmas, that $\hat{a}_{i} \geq a_{i}+\Phi_{i}-\Phi_{i-1}$ for each type of operation $x_{i}$. Finally, we state in a lemma that $\Phi_{m}-\Phi_{0} \geq 0$. These lemmas, according to Lemma 2 are sufficient to prove Theorem 1. Analysis of InSERT is not presented separately, as Insert is just a Make-Heap followed by a Meld. Similarly, the analysis of Delete is not presented separately, as Delete's can be implemented as a Decrease-Key to negative infinity followed by a Extract-min.

### 3.2 The potential function

For the analysis, a color, black or white, is assigned to every node, and a weight is assigned to those nodes colored white. A node is black if it will remain in the forest of heaps at the end of execution of sequence $X$, and white otherwise. The color of a node never changes, since it is determined as a function of the entire fixed sequence of operations. We say that a white node has a weight of heavy if the number of white nodes in its left subtree in the binary representation is at least the number of white nodes in its right subtree. Roots are always heavy by this definition and every node in a heap of size $n$ can have at most $\log n$ heavy children. We say that the weight of a white node that is not heavy is light.

We say a node has been captured if its parent is black. A captured node must have a DECREASE-KEY performed on it later in the execution sequence before it is involved in any pairings. White captured nodes must have Decrease-Key or a Delete performed on them later in the execution sequence.

The node potential of a white node is the sum of four components: rank potential, weight potential, triple white potential, and capture potential. Let $s(x)$ be the number of white nodes in the induced subtree of $x$ in the binary representation. The rank potential of a white node $x, r(x)$, is $21 \log s(x)$. If a node is white and has immediate right and left siblings that are also white, then the node is referred to as a triple white and has a triple white potential of 0 ; if it is not a triple white it has 6 units of triple white potential. White heavy nodes have a weight potential of 0 , and white light nodes have 6 units of weight potential. We assign captured nodes a capture potential of 0 and non-captured nodes 6 units of capture potential. Black nodes are defined to have 0 rank potential, weight potential and triple-white potential. Thus, the node potential of a black node is only comprised of its capture potential. We also assign each heap a heap potential which is $8-36 \sum_{k=1}^{n} \log k$ where $n$ is the number of white nodes in the heap (note that this could be negative in the middle of the execution of $X$, but is zero at the beginning and positive at the end. See Section 3.3). The potential of a forest of heaps, $\Phi_{i}$, is the sum of the node potentials of the nodes in the heaps and the heap potentials of the heaps as a function of the state of the structure after the execution of $x_{i}$.

It must be noted that the potential function exists for the sole purpose of analysis. To implement a pairing heap neither the potential function, nor its constituents such as node color, need to be stored.

The intuition behind this potential function comes from several places. The rank potential is taken from the original pairing heap analysis [10] and is identical to that used in splay trees 18 . The notion of heavy and light was inspired by a similar idea used in skew heaps and their analysis 19 . The notion of a captured node is due to the fact that a pairing of a black and a white node where the black node has a smaller key value can be paid for by the resultant Decrease-Key or Delete operation that must be performed on the white node before it takes place in any additional pairings. The triple-white potential is a new invention, and was needed to handle a case where locally a mixture of black and white nodes transitions to all white nodes during Extract-Min; this case where the triple-white potential is vital appears as Case 4 of the local analysis of Lemma 7 .

The amortized cost of each operation is now calculated using this potential function. For each operation it is proved that $\hat{a}_{i} \geq a_{i}+\Phi_{i}-\Phi_{i-1}$, that is, the amortized cost of an operation is at least the actual cost plus the change in potential. In order to analyze the change in potential, we analyze the change in each of the components that are summed to make the potential function. For each operation, only a limited number of clearly defined nodes may have their node potential change. These are the nodes that have had their parents change, or white nodes that have had a neighboring sibling or white descendant in the binary representation change.

### 3.3 Global loss

Lemma 3. $\Phi_{m}-\Phi_{0} \geq 0$
Proof. Since the structure is initially empty, $\Phi_{0}=0$. By definition, after the execution of all $m$ operations in $X$, there are no white nodes left in the collection of heaps. Since the only possible negative component of the potential function, the heap potential, requires white nodes to be negative, $\Phi_{m} \geq 0$ and the lemma holds.

### 3.4 Make-heap

Lemma 4. If $x_{i}$ is a Make-HEAP, $a_{i}+\Phi_{i}-\Phi_{i-1} \leq 21$.
Proof. In a Make-Heap, the only change in potential is caused by the introduction of the new one-node heap. The potential of all existing nodes and heaps is unchanged. The actual cost and changes in potential can be accounted for as follows:

Actual cost: 1. No pairings are performed.
Change in rank potential: 0 . The newly created root, if white, has one node in its induced subtree, and thus has $21 \log 1=0$ units of rank potential. If the newly created root is black, black nodes by definition have zero rank potential.

Change in weight potential: 0. If the newly created node is white, it will be heavy and will have 0 units of weight potential. If the newly created node is black, it does not have weight potential.

Change in triple white potential: 6 . The new node is not a triple-white.
Change in capture potential: 6. The new node is not captured.
Change in heap potential: 8. The new heap has a potential of 8 . Summing the actual cost and the change in potential yields the amortized cost: $a_{i}+\Phi_{i}-\Phi_{i-1}=21$.

### 3.5 Meld

Lemma 5. If $x_{i}$ is a Meld, $a_{i}+\Phi_{i}-\Phi_{i-1} \leq 0$.
Proof. The only nodes in a Meld that can change potential are the two old roots, and the (former) leftmost child of the new root. We use $a$ and $b$ to denote the number of white nodes in the heaps being melded.

Actual cost: 2, as 1 pairing is performed.
Change in weight potential: $\leq 6$. Only the old root that lost the pairing can have its weight potential change.

Change in triple white potential: $\leq 0$. Only one node will have its left sibling change, which is a necessary condition for a change of triple-right. Let $x$ be the root that wins the pairing and let $y$ be the root that loses the pairing. Observe that $x$ 's left child will now have $y$ as its left sibling. Observe that the change in triple-white potential $x$ 's left child can only be negative, since $x$ 's left child is not a triple white before the beginning of this operation.

Change in capture potential: $\leq 0$. No gain is possible since no node escapes capture in a MELD. A loss is possible as the new leftmost child of the new root can be captured.

The remainder of the analysis breaks into two cases.
Case 1: At least one of the two heaps being melded only contains black nodes.
Change in rank potential: 0 . No white nodes have any change in their white descendants.
Change in heap potential: -8 . The heap with all black nodes (or one of them if both have only black nodes) has a heap potential of 8 , while the resultant heap has the same potential as the other heap.

Case 2: Each of the heaps being melded contain some white nodes.

Change in rank potential: $\leq 42 \log (a+b)$. The only two nodes that could possibly change rank potential would be the two roots, and their rank potential could rise to be at most $21 \log (a+b)$ each.
Change in heap potential: $\leq-36 \log (a+b)-8$.
The change in heap potential is given by the expression:
$8-36 \sum_{i=1}^{a+b} \log i-8+36 \sum_{i=1}^{a} \log i-8+36 \sum_{i=1}^{b} \log i$
$=-8-36 \sum_{i=1}^{a+b} \log i+36 \sum_{i=1}^{a} \log i+36 \sum_{i=2}^{b} \log i$
$\leq-8-36 \sum_{i=1}^{a+b} \log i+36 \sum_{i=1}^{a+b-1} \log i$ $\leq-8-36 \log (a+b)$.

Summing the actual cost and the change in potential for both Case 1 and Case 2 yields the same amortized cost: $a_{i}+\Phi_{i}-\Phi_{i-1} \leq 0$.

### 3.6 Decrease-key

Lemma 6. If $x_{i}$ is a Decrease-Key, $a_{i}+\Phi_{i}-\Phi_{i-1} \leq 26+27 \log n_{i}$.
Actual cost: 2, since one pairing is performed.
Rank Potential: $\leq 21 \log n_{i}$. The node on which the Decrease-Key is performed could gain as most as $21 \log n_{i}$ in rank potential. The former ancestors of the node in the binary representation could have their rank potential decrease.

Weight potential: $\leq 6 \log n_{i}+12$. The node the decrease key is performed on can gain 6 in weight potential if it becomes heavy. Also, on the path $p$ in the binary representation from the node on which the Decrease-Key is to be performed to the root, the removal of the node and its subtree may cause some nodes to change their status from light to heavy or vice versa. Only changing from heavy to light causes a potential gain, and in such nodes the path $p$ goes through the node and its left child in the binary representation. After the Decrease-Key is performed there are only $\log n$ such nodes because the $s(\cdot)$ values on any ancestor-descendent path in the binary representation is non-increasing in general, and decreases by at least a factor of two from a light node to its left child in the binary representation. Thus this gain of 6 can only happen in at most $\log n_{i}+1$ nodes.

Capture potential: $\leq 6$. There can only be a change of 6 in capture potential if the node which the Decrease-Key is performed on was captured.

Triple white: $\leq 6$. Among the node on which the DECREASE-KEY is performed, and its two former siblings to the left and right, a total of 6 units of triple white potential can be gained. The former left child of the new root, will now be the second-leftmost child and could become a triple-white, thus releasing 6 units of potential.

Summing the actual cost and the change in potential yields the amortized cost: $a_{i}+\Phi_{i}-\Phi_{i-1} \leq$ $26+27 \log n_{i}$.

### 3.7 Extract-Min

Lemma 7. Amortized cost of Extract-Min is $117 \log \left(n_{i}+1\right)+101$.
Proof. Actual cost: If there are $c$ children of the root, the actual cost is $c$, since $c-1$ pairings are performed.
As the analysis of Extract-Min is long, it is split into several parts. The first part examines the change of heap potential. Next, we note that the only nodes that can have their node potential change are the old root which is removed, the children of the old root, and the grandchildren of the old root. The second part bounds the changes of node potential of the old root. The third part bounds the changes of triple-white potential of the grandchildren of the root, since this is the only type of node potential change possible in these nodes. The analysis of the node potential change in the children of the old root is the most complicated step, and it is presented in two parts: global and local. In the global part we analyze some potential changes over
the whole structure, while in the local section we analyze how blocks of six children of the root are affected by the remaining potential changes. In the global section we analyze changes in rank potential, gains in weight potential, and gains of capture potential. Part of this analysis, which is a variation of that of that which can be found in the original pairing heap paper 10 and mimimes the original analysis of splay trees [18], is presented separately in Section 3.8. In the local section we analyze losses of weight potential, and changes of triple white potential and losses of capture potential. We let $w$ denote the number of white-white parings in the first pairing pass (Nodes are colored. Pairings involve two nodes. A white-white pairing is a pairing of two white nodes).

Heap potential: $36 \log \left(n_{i}+1\right)$. This is caused by the heap's size being reduced from $n_{i}+1$ to $n_{i}$.
Node potential of root: $\leq 0$. The removal of the root itself causes no potential gain, since it has nonnegative potential

Triple white potential of grandchildren of the old root: $\leq 0$. Some grandchildren of the old root that are white, have white right siblings and no left siblings may, through a pairing that their parent is involved in, acquire a left white sibling and become a triple-white. This can only cause a loss of potential.

## Global changes:

Change of rank potential: $\leq 63 \log n_{i}-42 w$. The derivation of this is a variation of the original pairing heap analysis of 10 . This is topic of Section 3.8 and appears as Lemma 11 .
Gains in weight potential: $\leq 6 \log n_{i}+6$. There are at most $\log n_{i}+1$ heavy children of the root, since the $s(\cdot)$ value of the right sibling of a heavy node decreases by at least a factor of two, and so the potential gain caused by heavy nodes becoming light is at most $6 \log n_{i}+6$.
Gains of capture potential: 0. The Extract-Min operation can cause no increase in capture potential. An increase in the capture potential can only happen when a previously captured node becomes uncaptured. However, since the root bas white, none of the children of the root can have black parents, and thus none were captured.

Local changes: In order to analyze other changes in potential (changes in triple white potential, losses of weight potential caused by a node becoming heavy, and changes in black nodes' potential) we break the $c$ children of the root into blocks of six nodes, excluding the rightmost two nodes. At most 7 nodes can not be included in this analysis, and they could incur a potential gain of up to 12 each for at total of 84 -white node's triple white and weight are bounded by 6 each, and black nodes only have capture potential which is at most 6 . In analyzing these specific potential changes in each block of six, there are seven cases. See Figure 3 for a decision tree showing how to determine which case applies.
Since for each case we are only considering losses in weight potential and capture potential, in some parts we do not explicitly state them and simply assume they are nonpositive. Triple white potential must be considered in each part because we are considering gains, as well as losses, however we show gains are limited to Case 1.

Case 1: There is at least one white-white pairing in the first pairing pass.
Triple white: $\leq 36$. The only gains in potential are the possible gain of 6 units of triple white potential for each white involved in a white-white pairing. This is the only case where a gain in potential, among the components of the potential function under consideration is possible. A gain in triple white potential can only occur when a child of the former root is a triple white, and thus has white left and right siblings. However, this guarantees that any triple white be involved in a white-white pairing and this falls into this case. Thus, increases in triple white potential can not occur in the following cases and will not be considered.


Figure 3: Determination of which case applies in the local part of the proof of Lemma 7.

Observe that if Case 1 does not apply all pairings in the first pairing pass are white-black or black-black.

Case 2: There is at least one black-black pairing in the first pairing pass.
Capture potential: $\leq-6$. The black-black pairing(s) causes a loss of at least 6 units of potential.
Observe that if Cases 1-2 do not apply then all pairings in the first pairing pass are blackwhite.

Case 3: At least one of the three white nodes is captured.
Capture potential: $\leq-6$. The capturing of the node(s) causes a capture potential loss of at least 6.

Observe that if Cases 1-3 do not apply then all pairing in the first pairing pass are back-white, and the white nodes win all three first-pairing-pass pairings.

Case 4: All three white nodes participate in the second pairing pass and lose.
Triple white potential: $\leq-6$. Having all three loose the pairings in the second pairing pass causes a loss of potential of 6 , due to the change of status of the middle white node to a triple white.

Observe that if Cases 1-4 do not apply then all pairing in the first pairing pass are back-white, and the white nodes win all three first-pairing-pass pairings and at least one of them win a second-pairing pass pairing. Case 5-7 are based on the weight of such a node.

Case 5: Of one of the three white three nodes that participate in the second pairing pass and win, its weight is light before the operation and heavy after both pairing passes.
Weight potential: $\leq-6$ The light node becomes heavy, as all nodes previously on its right are now in its subtree. Additional nodes that were to the node's left may also be added to its subtree, but this just makes it more heavy. This causes a loss of 6 units of weight potential.

Case 6: Of one of the three white three nodes that participate in the second pairing pass and win, its weight is light before the operation and heavy after the first pairing pass, and light after the second pairing pass.
We may assume no potential gain. This case can only happen $\log n_{i}$ times, because in between the first and second pairing passes there are at most $\log n_{i}$ heavy roots.
Case 7: Observe that if Cases 1-6 do not apply then all pairing in the first pairing pass are back-white, and the white nodes win all three first-pairing-pass pairings and at least one of them win a second-pairing pass pairing, and that all such nodes are heavy at the beginning of the operation. We may assume no potential gain. This case can only happen $\log n_{i}$ times, because there are at most $\log n_{i}$ heavy children of the root.

We now sum together the potential changes covered by these seven cases. Each application of Case 1 causes a potential increase of 36 , and covers at least one white-white pairing. Thus the total gain caused by Case 1 is at most $36 w$. Since there are at least $\left\lfloor\frac{c-2}{6}\right\rfloor \geq \frac{c}{6}-2$ blocks of 6 , and none of Cases $2-6$ have white-white pairings in the first pairing pass, there are at least $\frac{c}{6}-2-w$ blocks of six covered by Cases 2-6. Each of cases 2-5 has a loss of at least 6, and Cases 6-7 have no gain. Since Cases 6 and 7 can each only happen in $\log n_{i}$ blocks, that means the total potential loss of Cases $2-6$ is at least $6\left(\frac{c}{6}-2-w-2 \log n_{i}\right)$.

We now summarize the potential changes discussed, and bring together all parts of our analysis to bound the total potential gain as at most:

$$
\begin{aligned}
& \underbrace{36 \log \left(n_{i}+1\right)}_{\text {Heap potential }}+\overbrace{\underbrace{\text { Rank }}_{63 \log n_{i}-42 w}+\overbrace{6 \log n_{i}+6}^{\text {Weigal }}}^{\text {Weight }} \\
& +\underbrace{\underbrace{\text { Not in block }}_{84}+\overbrace{36 w}^{\text {Case } 1}-\overbrace{6\left(\frac{c}{6}-2-w-2 \log n_{i}\right)}^{\text {Cases 2-7 }}}_{\text {Local }} .
\end{aligned}
$$

This can be simplified, observing $-42 w+36 w+6 w=0$, to give an upper bound on the potential gain of $117 \log \left(n_{i}+1\right)-c+102$. Then summing the actual cost of the Extract-Min operation, $c-1$ with the maximum potential gain yields the amortized cost: $a_{i}+\Phi_{i}-\Phi_{i-1} \leq 117 \log \left(n_{i}+1\right)+101$.

### 3.8 Change in rank potential

In this subsection we examine the change in rank potential in an Extract-Min operation, which was postponed in Section 3.7. The proof of this bound follows in spirit the much simpler related proof in the original pairing heap paper [10], but is adjusted to take into account the presence of node coloring.

Pairings in an Extract-Min are always performed between a node and its right sibling. Let $\eta$ denote the left node in the pairing, and we adopt the terminology that the pairing is performed on $\eta$. For clarity we introduce notion to separately represent the two nodes before and after the pairing. This notation is described now and illustrated in Figure 4. Let $a_{\eta}=\eta$ and $b_{\eta}$ be the right sibling of $a_{\eta}$; the nodes $a_{\eta}$ and $b_{\eta}$ are paired in the first pairing pass. We will refer to these nodes after the pairing has been completed as $w_{\eta}$ and $l_{\eta}$; the winner of the pairing is $w_{\eta}$ and the loser is $l_{\eta}$. The rank potential increase of the pairing is then

(a) Before pairing, general heap (b) After pairing, general heap view. view.

(c) Before pairing, binary view.
(d) After pairing, binary view.

Figure 4: Illustration of the notation used to analyze a single pairing. Nodes $a_{\eta}$ and $b_{\eta}$ are paired, and $w_{\eta}$ wins the pairing and $l_{\eta}$ loses the pairing. Pre- and post-pairing pictures are presented, in both the general tree and binary views.
simply $r\left(w_{\eta}\right)+r\left(l_{\eta}\right)-r\left(a_{\eta}\right)-r\left(b_{\eta}\right)$. We use $c_{\eta}$ to denote the right sibling of $b_{\eta}$ before the pairing; if there is no such sibling, we imagine $c_{\eta}$ to be a lone black node.

Given this notation, we now bound potential change caused by a single pairing on a node $\eta$.
Lemma 8. In a single pairing the rank potential change is at most $42 \log s\left(a_{\eta}\right)-42 \log s\left(c_{\eta}\right)-42$ if both nodes in the paring are white.

Proof. This bound on the rank potential change can be mathematically derived as follows, starting from the ranks of the two items involved in the pairing:

$$
r\left(w_{\eta}\right)+r\left(l_{\eta}\right)-r\left(a_{\eta}\right)-r\left(b_{\eta}\right)
$$

Observe that $r\left(a_{\eta}\right)=r\left(w_{\eta}\right)$, and cancel them.

$$
\begin{equation*}
=r\left(l_{\eta}\right)-r\left(b_{\eta}\right) \tag{1}
\end{equation*}
$$

Use the definition that $r(x)=21 \log s(x)$.

$$
=21 \log s\left(l_{\eta}\right)-21 \log s\left(b_{\eta}\right)
$$

Add and subtract $21 \log s\left(c_{\eta}\right)+42$.

$$
=21 \log s\left(l_{\eta}\right)+21 \log s\left(c_{\eta}\right)+42-21 \log s\left(b_{\eta}\right)-21 \log s\left(c_{\eta}\right)-42
$$

Use the fact that $s\left(b_{\eta}\right) \geq s\left(c_{\eta}\right)$.

$$
=21 \log s\left(l_{\eta}\right)+21 \log s\left(c_{\eta}\right)+42-42 \log s\left(c_{\eta}\right)-42
$$

Combine first three terms.

$$
\leq 21 \log 4 s\left(l_{\eta}\right) s\left(c_{\eta}\right)-42 \log s\left(c_{\eta}\right)-42
$$

Use the fact that $(x+y)^{2} \geq 4 x y$.

$$
\begin{aligned}
& \leq 21 \log \left(s\left(l_{\eta}\right)+s\left(c_{\eta}\right)\right)^{2}-42 \log s\left(c_{\eta}\right)-42 \\
& \leq 42 \log \left(s\left(l_{\eta}\right)+s\left(c_{\eta}\right)\right)-42 \log s\left(c_{\eta}\right)-42
\end{aligned}
$$

Observe that $s\left(l_{\eta}\right)+s\left(c_{\eta}\right) \leq s\left(a_{\eta}\right)$

$$
\leq 42 \log s\left(a_{\eta}\right)-42 \log s\left(c_{\eta}\right)-42
$$

Lemma 9. In a single pairing the rank potential change is at most $42 \log s\left(a_{\eta}\right)-42 \log s\left(b_{\eta}\right)$.
Proof. The analysis is split into six different types of pairing: white-white, black-black, and four types of black-white depending on whether the white node starts as $a_{\eta}$ or $b_{\eta}$ and ends as $w_{\eta}$ or $l_{\eta}$; see Figure 5. We look at each of these cases separately.

Case 1: White-white. From (1) of Lemma 8 we know the rank potential gain is at most $r\left(l_{\eta}\right)-r\left(b_{\eta}\right)$; observing that $r\left(l_{\eta}\right) \leq r\left(a_{\eta}\right)$ bounds the rank potential gain in this case to be at most $42 \log s\left(a_{\eta}\right)-42 \log s\left(b_{\eta}\right)$.


Case 1


Case 3


Case 5


Case 2


Case 4


Case 6

Figure 5: The six cases in the proof of Lemma 9, in the binary view. The left figure of each case is before pairing $a_{\eta}$ and $b_{\eta}$, and the right is after with $w_{\eta}$ denoting the winner of the pairing and $l_{\eta}$ denoting the loser.

Case 2: Black-black. None of the nodes involved have rank potential and thus there is no rank potential change. As $0 \leq 42 \log s\left(a_{\eta}\right)-42 \log s\left(b_{\eta}\right)$, that gives the result in this case.

Case 3: White-black, white is $a_{\eta}$ and $w_{\eta}$. Since $r\left(a_{\eta}\right)=r\left(w_{\eta}\right)$, there is no potential change. As $0 \leq 42 \log s\left(a_{\eta}\right)-42 \log s\left(b_{\eta}\right)$, that gives the result in this case.

Case 4: White-black, white is $a_{\eta}$ and $l_{\eta}$. Since $r\left(a_{\eta}\right) \geq r\left(l_{\eta}\right)$, there is no potential gain. As $0 \leq$ $42 \log s\left(a_{\eta}\right)-42 \log s\left(b_{\eta}\right)$, that gives the result in this case.

Case 5: White-black, white is $b_{\eta}$ and $w_{\eta}$. The gain is $r\left(w_{\eta}\right)-r\left(b_{\eta}\right)$. Since $r\left(w_{\eta}\right)=21 \log s\left(w_{\eta}\right)=$ $21 \log s\left(a_{\eta}\right)$, the rank potential gain is at most $42 \log s\left(a_{\eta}\right)-42 \log s\left(b_{\eta}\right)$.

Case 6: White-black, white is $b_{\eta}$ and $l_{\eta}$. The gain is is $r\left(l_{\eta}\right)-r\left(b_{\eta}\right)$. As in the white-white case, observing that $r\left(l_{\eta}\right) \leq r\left(a_{\eta}\right)$ bounds the rank potential gain in this case to be at most $42 \log s\left(a_{\eta}\right)-42 \log s\left(b_{\eta}\right)$.

Corollary 10. In a single pairing the rank potential change is at most $42 \log s\left(a_{\eta}\right)-42 \log s\left(c_{\eta}\right)$.
Proof. Follows from Lemma 9 and the fact that $s\left(c_{\eta}\right) \leq s\left(b_{\eta}\right)$.
Lemma 11. The rank potential gain in the Extract-Min operation where there are $w$ white-white pairings in the first pairing pass is at most $63 \log n_{i}-42 w$.

Proof. In an Extract-Min, the root is removed and returned, and then there are two pairing passes. We look at the potential change of each of these events separately. Removing the root causes no gain in potential.

First pairing pass. In the first pairing pass, let $p$ be the total number of pairings and let $\eta_{j}$ denote the item the $j$ th pairing from the left is performed on. Lemma 8 and Corollary 10 bound the potential gain of the first pairing pass to be:

$$
\overbrace{\sum_{\substack{j \in[1, p] \\ \text { is white-white }}}\left(42 \log s\left(a_{\eta_{j}}\right)-42 \log s\left(c_{\eta_{j}}\right)-42\right)}^{\text {from Lemma } 8}+\overbrace{\sum_{\substack{j \in[1, p] \\ \eta_{i} \text { is not white-white }}}\left(42 \log s\left(a_{\eta_{j}}\right)-42 \log s\left(c_{\eta_{j}}\right)\right)}^{\text {from Corollary } 10}
$$

Rearranging and using the fact that the left sum has exactly $w$ elements.

$$
=-42 w+\sum_{j=1}^{p}\left(42 \log \left(s\left(a_{\eta_{j}}\right)\right)-42 \log s\left(c_{\eta_{j}}\right)\right)
$$

Since $c_{\eta_{j}}$ is $a_{\eta_{j+1}}$ the sum telescopes.

$$
\begin{aligned}
& \left.=-42 w+42 \log s\left(a_{\eta_{1}}\right)-42 \log s\left(c_{\eta_{p}}\right)\right) \\
& \leq-42 w+42 \log s\left(a_{\eta_{1}}\right) \\
& \leq-42 w+42 \log n_{i}
\end{aligned}
$$

Second pairing pass. In the second pairing pass, let $q$ be the total number of pairings, and let $\mu_{j}$ denote the item that the $j$ th pairing is performed on. As the pairings are performed incrementally from right-to-left, $\mu_{j}$ is the $j+1$ st node from the right before the beginning of the second pairing pass. Using Lemma 9 bounds the potential gain of the second pairing pass to be:

$$
\sum_{j=1}^{q}\left(42 \log s\left(a_{\eta_{j}}\right)-42 \log s\left(b_{\eta_{j}}\right)\right)
$$

Since $b_{\eta_{j}}$ is $\eta_{j-1}$ the sum telescopes to at most $42 \log s\left(a_{\eta_{q}}\right)$ which is at most $42 \log n_{i}$.
Putting it all together. Summing the upper bounds on rank potential gain of $42 \log n_{i}-42 w$ for the first pairing pass and $42 \log n_{i}$ for the second pairing pass gives the claimed bound of a rank potential gain of at most $63 \log n_{i}-42 w$ for the Extract-Min operation.

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