## Isotropic Cosmological Singularities I. Polytropic perfect fluid spacetimes.

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November 7, 2018

#### Abstract

We consider the conformal Einstein equations for  $1 \le \gamma \le 2$  polytropic perfect fluid cosmologies which admit an isotropic singularity. For  $1 < \gamma \le 2$  it is shown that the Cauchy problem for these equations is well-posed, that is to say that solutions exist, are unique and depend smoothly on the data, with data consisting of simply the 3-metric of the singularity. The analogous result for  $\gamma = 1$  (dust) is obtained when Bianchi type symmetry is assumed.

## 1 Introduction

Isotropic singularities are a class of cosmological singularities which are both physically interesting and mathematically tractable (Goode and Wainwright 1985, Tod 1987, Tod 1992). Put simply, an isotropic singularity is one which can be removed by rescaling the spacetime metric with a single function which becomes singular on a spacelike hypersurface.

The motivation for studying these singularities comes from Penrose's Weyl Tensor Hypothesis (WTH) (Penrose 1979, Penrose 1990). Penrose hypothesised that initial singularities, notably the Big Bang, should be different in character from final singularities, as formed in gravitational collapse or the Big Crunch. Specifically (Penrose 1979), he suggested:

I propose that there should be a complete lack of chaos in the initial geometry... This restriction on the early geometry should be something like: the Weyl curvature  $C_{abcd}$  vanishes at any initial singularity.

Penrose expected this condition to imply that the subsequent evolution would necessarily be close to a Friedman-Robertson-Walker (FRW) model. He was led to the above hypothesis by the need for some kind of low entropy constraint on the initial state of the universe, at the same time as the matter content was in thermal equilibrium. He argued firstly that the need for a low entropy constraint follows from the existence of a second law of thermodynamics and secondly that low entropy in the gravitational field must be tied to constraints on the Weyl curvature. Penrose went on to suggest that this constraint on the initial Weyl tensor should be a consequence of an as yet undiscovered quantum theory of gravity.

In order to make mathematical progress with the WTH one must first say exactly what is meant by a 'finite Weyl tensor' singularity, a singularity at which the Weyl tensor is finite while, of necessity, the Ricci tensor is not. There may be more than one plausible way to do this. One strategy, and the one we shall adopt, is first to make the following definition: A spacetime  $(\tilde{M}, \tilde{g}_{ab})$  is said to admit an *isotropic singularity* if there exists a manifold with boundary  $M \supset \tilde{M}$ , a regular Lorentz metric  $g_{ab}$  on M, and a function  $\Omega$  defined on M, such that

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \quad \text{for} \quad \Omega > 0 \tag{1}$$

$$\Omega \to 0 \quad \text{on } \Sigma$$
 (2)

where  $\Sigma$  is a smooth spacelike hypersurface in M, called the *singularity surface*. Since the Weyl tensor with its indices arranged as  $C^a_{bcd}$  is conformally invariant, isotropic singularities form a well-defined class of cosmological singularities with finite Weyl tensor.

Consideration of the conformal Einstein equations near an isotropic singularity, for various matter models, leads naturally to a class of singular Cauchy problems for the unphysical metric  $g_{ab}$ , with data given on  $\Sigma$ , for which one may seek to prove suitable existence and uniqueness theorems. This program was begun in (Tod 1990, 1991), where the problem was treated in the case of a perfect fluid with polytropic equation of state as matter source. There the conformal field equations were written as an evolution system for  $g_{ab}$  with data just the 3-metric of the singularity surface. It was noted in particular that a uniqueness result for these equations would prove an earlier 'Vanishing Weyl Tensor' conjecture,(Tod 1987), which said that, given an equation of state, the Weyl tensor must vanish everywhere in M if it vanishes at an initial isotropic singularity.

Newman (1993a,b) considered cosmologies with an isentropic perfect fluid as source and for which the conformal factor  $\Omega$  remained a smooth coordinate at  $\Sigma$ . This smoothness condition implied that the equation of state approached that for radiation ( $\gamma = 4/3$ ) at the singularity. By taking  $\gamma = 4/3$  throughout  $\tilde{M}$  Newman was able, with a careful choice of field variables, to write the equations in symmetric hyperbolic form but with a singular ' $\frac{1}{t}$ ' forcing term. Again the free data at the singularity consisted of just the 3-metric there. As there was no existence and uniqueness theorem available for the singular evolution equations obtained, a special study was undertaken by Newman, the result appearing in (Claudel and Newman 1998). We shall refer to the existence and uniqueness theorem obtained there as the Newman-Claudel Theorem. The conclusion is that there is exactly one  $\gamma = 4/3$  perfect fluid cosmology with an isotropic singularity for each 3-metric given on  $\Sigma$ .

Our aim in this paper is to show that by imposing a suitable  $\gamma$ -dependent differentiability condition on the conformal factor  $\Omega$ , a similar analysis to that of Newman can be performed for polytropic perfect fluids with index in the range  $1 < \gamma \leq 2$ . It will then follow from the Newman-Claudel Theorem that we have existence and uniqueness of a perfect fluid cosmology for each  $\gamma$  in this range, for each 3-metric on  $\Sigma$ .

The plan of the paper is as follows:

In section 2 we set down our conventions and give a few notes on conformal rescaling.

An account of the theory of singular symmetric hyperbolic systems developed by Claudel and Newman (1998) is given in section 3.

In section 4 we consider cosmologies with  $1 < \gamma < 2$ . Using comoving coordinates as in (Newman 1993a,b) we are able to define a set of variables in terms of which the conformal field equations can be written in a suitable singular symmetric hyperbolic form. The results of (Claudel and Newman 1998) can be applied and the Newman-Claudel theorem then gives existence and uniqueness of solution.

The case  $\gamma = 2$  has to be treated separately and this we do in section 5. We use harmonic rather than comoving coordinates and this enables us once again to write the field equations in the form required in (Claudel and Newman 1998) and to conclude existence and uniqueness of solutions from the Newman-Claudel theorem.

Finally in section 6 we consider the case  $\gamma = 1$ . Here it has not been possible to write the conformal equations in full generality in symmetric hyperbolic form and for this reason we are forced to restrict attention to spatially-homogeneous solutions with Bianchi type symmetry. Subject to this restriction, we are able to prove existence and uniqueness of solutions, given an initial, spatially-homogeneous 3-metric, by an extension of a theorem from (Rendall and Schmidt 1991).

## 2 Conventions and conformal rescaling

Throughout the paper we take the spacetime metric to have signature (+ - - -).

For a metric  $g_{ab}$  our definition of the Riemann tensor is

$$(\nabla_a \nabla_b - \nabla_b \nabla_a) V^c = R^c_{\ dab} V^d \tag{3}$$

while for the Ricci tensor we take

$$R_{ab} = R^c_{\ acb} \tag{4}$$

Now whenever we're considering a rescaling as in (1)-(2), tilded quantities will always refer to the singular, physical spacetime  $(\tilde{M}, \tilde{g}_{ab})$ , while un-tilded quantities will refer to the regular, unphysical spacetime  $(M, g_{ab})$ . For metrics  $g_{ab}$  and  $\tilde{g}_{ab}$  related by (1), the Ricci tensors are related by:

$$\tilde{R}_{ab} = R_{ab} - 2\nabla_a \nabla_b \log \Omega + 2\nabla_a \log \Omega \nabla_b \log \Omega - g_{ab} (\Box \log \Omega + 2\nabla_c \log \Omega \nabla^c \log \Omega)$$
(5)

and the Weyl tensors by:

$$\tilde{C}^a_{\ bcd} = C^a_{\ bcd} \tag{6}$$

If  $\tilde{t}^a$  is a unit vector in  $\tilde{M}$  then  $t^a = \Omega \tilde{t}^a$  is one in M, and if  $\tilde{w}_a$  is a unit covector in  $\tilde{M}$ , then  $w_a = \Omega^{-1} \tilde{w}_a$  is one in M. With respect to a choice of unit timelike vector  $t^a$ , the electric and magnetic parts of the Weyl tensor of  $g_{ab}$  are defined, respectively, by:

$$E_{ab} = C^c_{\ adb} t_c t^d \qquad B_{ab} = \frac{1}{2} \epsilon_{ae}^{\ cd} C^f_{\ bcd} t^e t_f \tag{7}$$

The projection  $h_{ab}$  orthogonal to  $t^a$  is defined by

$$h_{ab} = g_{ab} - t_a t_b \tag{8}$$

If  $t^a$  is hypersurface orthogonal one then calculates that

$$E_{ab} = -{}^{3}R_{ab} + KK_{ab} - K_{ae}K^{e}_{\ b} + \frac{1}{2}h_{a}^{\ r}h_{b}^{\ s}R_{sr}$$

$$+\left(\frac{1}{2}h^{cd}R_{cd}-\frac{1}{3}R\right)h_{ab}\tag{9}$$

where  ${}^{3}R_{ab}$  is the Ricci tensor of  $h_{ab}$ ,  $K_{ab}$  is the second fundamental form of the surfaces orthogonal to  $t^{a}$  and  $K = h^{ab}K_{ab}$ . Also

$$B_{ab} = D_d K_{c(b} e_{a)}^{\ cd} \tag{10}$$

where  $D_a$  is the intrinsic covariant derivative associated with  $h_{ab}$  and  $e_{abc}$  is the volume element associated with  $h_{ab}$ . By (6)-(7) we have

$$\tilde{E}_{ab} = E_{ab} \qquad \tilde{B}_{ab} = B_{ab} \tag{11}$$

It follows from (1) that, if  $\Omega$  vanishes at a surface  $\Sigma$  in M then  $\Sigma$  is a singularity in  $\tilde{M}$  but, from (11),  $\tilde{E}_{ab}$  and  $\tilde{B}_{ab}$  are finite at the singularity. It follows from (10) that if  $\Sigma$  has vanishing second fundamental form  $K_{ab}$  then  $B_{ab}$  is automatically zero at  $\Sigma$  and from (9), subject to the term in the 4-dimensional Ricci tensor being a multiple of  $h_{ab}$ , that  $E_{ab}$  also vanishes at  $\Sigma$  if and only if  $\Sigma$  is a 3-dimensional space of constant curvature. This will be significant in section 5.5.

### 3 Singular evolution equations

The fundamental technique used by Newman (1993b) in tackling the conformal Cauchy problem for the Einstein-perfect fluid equations was to choose certain field variables so that the resulting equations appeared in symmetric hyperbolic form (as in equation (15) below). Although the system was hyperbolic, the unavoidable  $(\frac{1}{t})$  singularity in these equations precluded the use of standard existence and uniqueness theorems as found in e.g. (Kato 1975, Racke 1992). A special study was undertaken by Newman to deal with the singularity, and he was able to prove well-posedness of the Cauchy problem for a class of singular evolution equations into which the  $\gamma = 4/3$  equations fell (Claudel and Newman 1998).

We will show in section 4 that the conformal field equations for other  $\gamma$  in the range  $1 < \gamma \leq 2$  can, after a little algebra, be forced into the class studied in (Claudel and Newman 1998), giving us the result we are looking for.

For ease of reference we now present the main theorem of (Claudel and Newman 1998), both in its abstract form and its application to symmetric hyperbolic systems. Theorem 1 (Newman-Claudel 1998). Let

$$X^{2q+1} \hookrightarrow \ldots \hookrightarrow X^1 \hookrightarrow X^0 \tag{12}$$

be a sequence of continuous dense inclusions of reflexive Banach spaces, and let  $X \equiv X^0$ .

Consider the quasi-linear, first order evolution problem

$$A_t^0(u_t)\frac{d}{dt}u_t = A_t(u_t)u_t + f_t(u_t) + \frac{1}{t}F_t(u_t)u_t, \quad 0 \le t \le T,$$
$$u(0) = u^0$$
(13)

in an open neighbourhood U of  $u^0 \in X$  for families  $\{f_t(x) \in X\}_{(t,x)\in[0,T]\times U}, \{F_t(x) \in B(X)\}_{(t,x)\in[0,T]\times U}$ . (B(X) being the space of bounded operators in X), and a family  $\{A_t(x)\}_{(t,x)\in[0,T]\times U}$  of operators in X such that (i)  $X^{p+1} \subset D((A_t(x))^p)$  for all  $(t,x) \in [0,T] \times U$ ; (ii)  $u^0 \in \bigcap_{i=0}^{2q+1} X^p$ ; (iii)  $F_t(x)u^0 = 0$  for all  $(t,x) \in [0,T] \times U$ .

Let  $U^0 \subset U$  be an open neighbourhood of  $u^0$  in  $X^0$  and, for each  $p \geq 0$ , let  $U^{p+1}$  be an open neighbourhood of  $u^0$  contained in  $U^p$ . Suppose there exists  $\beta \geq 0$  and, for each  $(t, x) \in [0, T] \times U$ , an equivalent renorming  $X_{(t,x)}$  of X, such that

(iv)  $A_t(y) \in G(X_{(t,y)}, 1, \beta)$  for all  $(t, y) \in [0, T] \times U^1$ . (i.e  $A_t(y)$  generates a contraction  $C_0$  semigroup on  $X_{(t,y)}$ );

(v)  $[0,T] \times U \ni x \mapsto || ||_{X_{(t,x)}}$  is Lipschitz continuous;

(vi) for each  $t \in [0,T]$  and all  $p \ge 0$ ,  $U^p \ni x \mapsto f_t(x)$  is in  $Lip(U^p, X^p)$ ,  $U^p \ni x \mapsto F_t(x)|X^p$  is in  $Lip(U^p, B(X^p))$  and  $U^p \ni x \mapsto A_t(x)|X^{p+1}$  is in  $Lip(U^p, B(X^{p+1}, X^p))$ , with t-independent Lipschitz constants in each case; (vii) for all  $p, r : 0 \le r \le p$  one has  $f|[0,T] \times U^{p-r} \in C^r([0,T] \times U^{p-r}, X^{p-r})$ , and that  $[0,T] \times U^{p-r} \ni (t,x) \mapsto F_t(x)|X^{p-r}$  is in  $C^r([0,T] \times U^{p-r}, B(X^{p-r}))$ , and that  $[0,T] \times U^{p-r} \ni (t,x) \mapsto A_t(x)|X^{p-r+1}$  is in  $C^r([0,T] \times U^{p-r}, B(X^{p-r+1}, X^{p-r}))$ .

Suppose there exists a linear subspace K of X, contained and dense in  $X^p$  for all  $p \ge 0$ , and suppose that for all  $p \ge 0$  there exist  $S^{(p)} \in C^1([0,T], B(X^{p+1}, X^p))$  and operators  $\{E_t^{(p)}(x)\}_{(t,x)\in[0,T]\times U^{p+1}}$  in X such that

(viii)  $S_t^{(p)}: X^{p+1} \to X^p$  is an isomorphism for each p, and an isometry

at t = 0; (ix)  $\frac{d}{dt}S_t^{(p)}x' = [A_t(x), S_t^{(p)}]x' + E_t^{(p)}(x)S_t^{(p)}x'$  for all  $x' \in K$  and all  $(t, x) \in [0, T] \times U^{p+1}$ ; (x)  $[0, T] \times U^{p+1} \ni (t, x) \mapsto E_t^{(p)}(x)$  is  $B(X^p)$ -norm bounded.

Suppose also that there exists an integer  $q \ge 2$  such that

(xi) the spectrum of  $(A_0^0)^{-1}F_0(u^0) \in B(X)$  contains no integer in the interval [1,q];

(xii) 
$$\|((A_0^0)^{-1}F_0)(u^0)\|_{B(X^p)} < q-1 \text{ for all } p: 0 \le p \le q.$$

Lastly suppose that  $A^0: [0,T] \times U \to B(X)$  satisfies (xiii)  $A^0_t(x)$  has range X for all  $(t,x) \in [0,T] \times U$ ; (xiv) there exists  $\alpha > 0$  such that  $\alpha I - A^0_t(x)$  is dissipative on  $X_{(t,x)}$  for all  $(t,x) \in [0,T] \times U$ . (i.e  $||(A^0_t + (\lambda - \alpha)I)x||_{X_{(t,x)}} \ge \lambda ||x||_{X_{(t,x)}} \forall \lambda \ge 0$ ); (xv)  $A^0_t(x)$  leaves  $X^p$  invariant for all  $(t,x) \in [0,T] \times U$  and all  $p \ge 0$ ; (xvi)  $U^p \ni x \mapsto A^0_t(x)|X^p$  is in  $Lip(U^p, B(X^p))$  for all  $t \in [0,T]$  and all  $p \ge 0$ ; (xvii)  $[0,T] \times U^p \ni (t,x) \mapsto A^0_t(x)|X^p$  is in  $C^p([0,T] \times U^p, B(X^p))$  for all  $p \ge 0$ ; (xviii)  $[0,T] \times U^{p+1} \ni (t,x) \mapsto [(A^0_t(x))^{-1}, S^{(p)}_t]A_t(S^{(p)}_t)^{-1}$  is  $B(X^p)$ -norm bounded for all  $p: 0 \le p \le q-1$ .

Let

$$U^{0} = \{(u^{0})' \in U : F_{t}(x)(u^{0})' = 0 \ \forall (t,x) \in [0,T] \times U\}$$
(14)

Then there exists  $T_0 \in (0,T]$  and an open neighbourhood  $\hat{U}^{2q+1} \subset U^{2q+1}$  of  $u^0$ in  $X^{2q+1}$  such that, for each  $(u^0)' \in U^0 \cap \hat{U}^{2q+1}$  the evolution problem (13) has a unique solution  $u' \in \bigcap_{p=0}^q C^p([0,T_0], U^{q-p})$ , and the mapping  $U^0 \cap \hat{U}^{2q+1} \ni (u^0)' \mapsto u' \in C^0([0,T_0], U)$  is continuous with respect to the  $X^{2q+1}$ topology on  $U^0 \cap \hat{U}^{2q+1}$ .

By taking  $X^p = H^{p+s}(\mathbb{R}^m, \mathbb{R}^N)$ , where  $H^r(\mathbb{R}^m, \mathbb{R}^N)$  is the  $L^2$ -type Sobolev space (Adams 1975) of  $\mathbb{R}^N$ -valued functions on  $\mathbb{R}^m$ , and s > m/2, Newman-Claudel are able to apply the above theorem to symmetric hyperbolic PDE, yielding the following result:

Theorem 2 (Newman-Claudel 1998). Consider the system of quasi-

linear PDE

$$A^{0}(u_{t})\partial_{t}u_{t} = A^{j}(u_{t})\partial_{j}u_{t} + f(u_{t}) + \frac{1}{t}F(u_{t})u_{t}, \quad 0 \le t \le T$$
$$u(0) = u^{0}$$
(15)

on  $\mathbb{R}^m$  for an unknown function  $u: [0,T] \mapsto U$ , where U is an open neighbourhood of  $u^0$  in  $X = H^s(\mathbb{R}^m, \mathbb{R}^N)$ , f is an X-valued function on U, and  $A^0, \ldots, A^m, F$  are B(X)-valued functions on U.

Assume

(1)  $A^0(x), \ldots, A^m(x), F(x)$  have the form of  $N \times N$  matrices, and f(x) has the form of an N-component column vector with each matrix or column element a rational function (over  $\mathbb{R}$ ) of the column elements  $x_1, \ldots, x_N$  of x;

(2)  $A^0(x), \ldots, A^m(x)$  are symmetric matrices.

Assume also that the initial data  $u^0$  satisfies

(3)  $F(x)u^0 = 0$  for all  $x \in U$ ;

(4) the range of  $u^0$  in  $\mathbb{R}^N$  is uniformly bounded away from the zero sets of the denominators in (1);

(5)  $A^0(u^0)$  is uniformly positive definite as an  $N \times N$  real matrix-valued function on  $\mathbb{R}^m$ ;

(6) 
$$u^0 \in \bigcap_{r=0}^{2q+1} H^r(\mathbb{R}^m, \mathbb{R}^N)$$

Finally assume there exists an integer  $q \ge 2$  such that

(7) for all  $\xi \in \mathbb{R}^m$  the matrices  $(A^0)^{-1}F(u^0(\xi))$  have no integer eigenvalues in the interval [1,q]; (8)  $\||(A^0)^{-1}F(x^0(\xi))|\|$ 

(8)  $||(A^0)^{-1}F(u^0(\xi))||_{B(\mathbb{R}^N)} < q-1$  holds for  $\xi \in \mathbb{R}^m$ .

Then for initial data  $u^0 \in H^{2q+1+s}(\mathbb{R}^m, \mathbb{R}^N)$ , there exists a unique solution  $u \in \bigcap_{p=0}^q C^p([0, T_0], H^{q-p+s}(\mathbb{R}^m, \mathbb{R}^N))$  of (15), with an  $H^{2q+1+s}(\mathbb{R}^m, \mathbb{R}^N)$ -small change in the initial data  $u^0$  giving rise to a  $C^0([0, T_0], H^s(\mathbb{R}^m, \mathbb{R}^N))$ -small change in u.

## 4 The Cauchy problem for polytropic perfect fluid spacetimes

In this section we consider the conformal Einstein equations for polytropic perfect fluid spacetimes which admit an isotropic singularity, extending the work of Tod (1990, 1991) and Newman (1993a,b). The hard work is to find suitable variables to put a reduced form of the equations into the form (15), and to see that a solution of the reduced equations does give a solution of the Einstein equations. The conclusion will be that for  $1 < \gamma < 2$  the conformal Cauchy problem for the conformal Einstein equations is well posed near  $\Sigma$ , with the free data consisting of just the 3-metric there. The existence and uniqueness results obtained are used to prove the Vanishing Weyl Tensor conjecture described in the Introduction.

#### 4.1 Conformal gauge fixing

Suppose then that  $\Omega$  is the conformal factor relating the physical metric  $\tilde{g}_{ab}$  to the unphysical metric  $g_{ab}$ , by means of

$$\tilde{g}_{ab} = \Omega^2 g_{ab} \tag{16}$$

with  $\Omega = 0$  at  $\Sigma$  in M. Suppose also that the stress-energy-momentum tensor in  $\tilde{M}$  is that appropriate to a polytropic perfect fluid:

$$\tilde{T}_{ab} = (\rho + p)\tilde{u}_a\tilde{u}_b - p\tilde{g}_{ab} \tag{17}$$

$$p = (\gamma - 1)\rho \quad 1 \le \gamma \le 2 \tag{18}$$

To make analytical progress we assume that the rescaled fluid flow given by  $u_a = \Omega^{-1} \tilde{u}_a$  remains regular and non-zero near  $\Sigma$ .

From the work of Newman we know that if one demands that  $\Omega$  be smooth with non-zero gradient at  $\Sigma$  then the Einstein equations imply that the value of the polytropic index must be  $\gamma = 4/3$ , the value for radiation. So if one wants to consider other equations of state then a different differentiability condition on  $\Omega$  is required. To see what this should be, consider the FRW cosmologies, where one may take the scale factor R(t) for  $\Omega$ . With a perfect fluid source, the field equations imply that the behaviour of dR at R = 0 is tied to the value of the polytropic index  $\gamma$  according to

$$dR \sim \tau^{\frac{4-3\gamma}{3\gamma-2}} \text{ as } \tau \to 0$$
 (19)

in terms of conformal time  $\tau$  which is a smooth coordinate in M. In view of this, we will impose the following condition:

At 
$$\Sigma$$
,  $Z \equiv \Omega^{(3\gamma-2)/2}$  is smooth, and  $\nabla_a Z \neq 0$  (20)

It is a result of Goode (1987) that the physical velocity field  $\tilde{u}_a$  must be hypersurface orthogonal, and this will also be true for the unphysical velocity field  $u_a \equiv \Omega^{-1} \tilde{u}_a$ . We know also that  $u_a$  meets  $\Sigma$  orthogonally. Thus near  $\Sigma$  in M

$$u_a = \frac{1}{V}\hat{Z}_{,a}$$
  $V^2 = g^{ab}\hat{Z}_{,a}\hat{Z}_{,b}$  (21)

for some smooth cosmic time function  $\hat{Z}$  on M, with  $\hat{Z} = 0$  on  $\Sigma$ . A natural choice of conformal factor is then given by the following lemma:

**Lemma 3.1** If (20) holds then one may take the conformal factor  $\Omega$  to be a function of the cosmic time defined by the fluid velocity.

Proof We know  $u_a = \psi \nabla_a \hat{Z}$  for some smooth  $\psi$ ,  $\hat{Z}$  in M near  $\Sigma$ , with  $\hat{Z} = 0$  at  $\Sigma$ . Now  $\tilde{g}_{ab} = \hat{\Omega}^2 \hat{g}_{ab}$ , where  $\hat{g}_{ab} = (\Omega^2/\hat{\Omega}^2)g_{ab}$ , and  $\hat{\Omega} = (\hat{Z})^{(1/q)}$   $(q = (3\gamma - 2)/2)$ . So  $\hat{g}_{ab} = (Z/\hat{Z})^{(2/q)}g_{ab}$ , where Z is as in (20). Since  $\nabla_a Z$ ,  $\nabla_a \hat{Z}$  are non-zero in M, one has  $Z = \hat{Z} f$ , for some smooth function f in M, which does not vanish at  $\Sigma$ . Therefore  $f^{(2/q)}$  is smooth at  $\Sigma$ , and  $\hat{g}_{ab}$  is smooth in  $M \square$ 

#### 4.2 Conformal field equations in M

We now make explicit the field equations we wish to solve for  $g_{ab}$  in M.

If we make the conformal gauge choice of Lemma 3.1, with Z as in (20) and  $u_a$ , V as in the unhatted version of (21), then for a physical stress tensor of the form (17) the conformally transformed Einstein-perfect fluid equations for  $g_{ab}$  in M are (Tod 1991):

$$R_{ab} - \frac{4}{3\gamma - 2} \frac{1}{Z} \nabla_a \nabla_b Z + \frac{12\gamma}{(3\gamma - 2)^2} \frac{1}{Z^2} Z_a Z_b (1 - G)$$
  
$$-g_{ab} \left(\frac{2}{(3\gamma - 2)} \frac{1}{Z} \Box Z + \frac{6(2 - \gamma)}{(3\gamma - 2)^2} \frac{V^2}{Z^2} (1 - G)\right) = 0$$
(22)

Here  $Z_a = \nabla_a Z$  and G parametrizes the density according to

$$\rho = \frac{12}{(3\gamma - 2)^2} \frac{V^2 G}{Z^{6\gamma/(3\gamma - 2)}}$$
(23)

This is the field equation in M for which we wish to formulate the Cauchy problem.

Firstly there is information in the contracted Bianchi identity applied to (22). Projecting along and perpendicular to  $u_a$  gives, respectively

$$\Box Z + \frac{(2-\gamma)}{\gamma} \frac{Z^b V_b}{V} + \frac{1}{\gamma} \frac{Z^b G_b}{G} = 0$$
(24)

$$(2-\gamma)\frac{V_a}{V} - (\gamma-1)\frac{G_a}{G} + \left((\gamma-1)\frac{Z^c G_c}{G} - (2-\gamma)\frac{Z^c V_c}{V}\right)\frac{Z_a}{V^2} = 0 \quad (25)$$

where  $V_a = \nabla_a V$ ,  $G_a = \nabla_a G$ .

If  $\gamma \neq 1$  then (25) can be solved to give

$$G = V^P g(Z) \qquad P = (2 - \gamma)/(\gamma - 1) \tag{26}$$

for some function g.

The case  $\gamma = 1$  will be treated shortly. Now we aim to show that g can be set to 1 by a change of cosmic time Z. Multiplying (22) by  $Z^2$  and letting  $Z \to 0$  we get  $G \to 1$ . Also, we see that  $Z^{-1}(1-G)$  has a limit as  $Z \to 0$ . Now multiply (22) by  $ZZ^b$  and let  $Z \to 0$  to find that  $V_b$  is proportional to  $Z_b$  in the limit. This means that V must be constant on  $\Sigma$  and we can take this constant to be 1 wlog. From (26) this entails  $g \to 1$  as  $Z \to 0$ .

Suppose now that we do (16) two different ways:

$$\tilde{g}_{ab} = \Omega^2 g_{ab} = \hat{\Omega}^2 \hat{g}_{ab} \tag{27}$$

then from (23) since these both correspond to the same  $\rho$ ,

$$V^2 G(\hat{Z})^{6\gamma/(3\gamma-2)} = \hat{V}^2 \hat{G} Z^{6\gamma/(3\gamma-2)}$$
(28)

Now use (26) to get

$$\left(\frac{g}{\hat{g}}\right)^{(\gamma-1)/\gamma} = \frac{\hat{V}}{V} \left(\frac{Z}{\hat{Z}}\right)^{6(\gamma-1)/(3\gamma-2)} \tag{29}$$

Finally  $\hat{V}/V = (\hat{\Omega}/\Omega)(d\hat{Z}/dZ)$  so that

$$\frac{d\hat{Z}}{dZ} = \left(\frac{g}{\hat{g}}\right)^{(\gamma-1)/\gamma} \left(\frac{\hat{Z}}{Z}\right)^{2(3\gamma-4)/(3\gamma-2)} \tag{30}$$

Solving (30) for  $\hat{Z}$  with  $\hat{g} = 1$  determines a new choice of unphysical spacetime in which g = 1, and (26) becomes

$$G = V^{(2-\gamma)/(\gamma-1)}$$
(31)

while (24) becomes

$$\Box Z + P \frac{Z^b V_b}{V} = 0 \tag{32}$$

If we let  $h_{ab} = g_{ab} - u_a u_b$  be the metric on the surfaces of constant Z, and  $K_{ab} = \frac{1}{2} \mathcal{L}_u h_{ab}$  the second fundamental form, then we have the identity

$$\Box Z = V\left(K + \frac{\partial V}{\partial Z}\right) \tag{33}$$

where  $K = h^{ab} K_{ab}$ . Therefore (32) can be written as

$$\frac{\partial V}{\partial Z} = (1 - \gamma)K \tag{34}$$

The field equations to be solved are now (22), (31) and (34)

Returning to the case  $\gamma = 1$  we find from (25) that V is a function only of Z. Now we can use a change of cosmic time to set V = 1. To see this note that if  $Z, \hat{Z}$  are two choices of cosmic time with corresponding functions  $\Omega, V$  and  $\hat{\Omega}, \hat{V}$  then

$$\frac{d\hat{Z}}{dZ} = \frac{\hat{V}\hat{\Omega}}{V\Omega} = \frac{\hat{V}}{V} \left(\frac{\hat{Z}}{Z}\right)^2 \tag{35}$$

From (22) we again deduce that  $V \to \text{constant}$  on  $\Sigma$  and we can take this constant to be 1. Now we may put  $\hat{V} = 1$  in (30) and solve for  $\hat{Z}$  to get the required result.

In this case (24) becomes

$$\Box Z + \frac{Z^b G_b}{G} = 0 \tag{36}$$

and thus by (33) one gets

$$\frac{\partial G}{\partial Z} = -GK \tag{37}$$

and the field equations are (22) and (37)

## 4.3 Solving 'reduced' field equations for $1 < \gamma < 2$

If one wishes to give data for (22), (34) at  $\Sigma$  then the equations impose constraints. We've seen that V must be equal to one, and one also gets  $K_{ab} =$  0 from (22) (or see (41) below). The equations impose no constraints on the initial 3-metric  $h_{ab}$ , and this is all the free data at  $\Sigma$ .

The aim now is to cast the equations (22), (34) in a symmetric hyperbolic form for which Theorem 2 applies.

We will use coordinates Z,  $x^i$ , with the  $x^i$  being carried along the integral curves of  $\nabla^a Z$ , on a manifold of the form  $\Sigma \times [0,T]$ .

In these coordinates the metric g assumes the form

$$ds^{2} = \frac{1}{V^{2}}dZ^{2} + h_{ij}dx^{i}dx^{j}$$
(38)

with  $h_{ij}$  negative definite.

Making a 3 + 1 split of equations (22) gives

$$R_{zz} = \frac{2K}{VZ} (4 - 3\gamma)/(3\gamma - 2) + \frac{6}{Z^2} (V^P - 1)/(3\gamma - 2)$$
(39)

$$R_{zi} = \frac{4}{Z} \frac{1}{(3\gamma - 2)} \partial_i \log V \tag{40}$$

$$R_{ij} = \frac{4}{(3\gamma - 2)} \frac{V}{Z} K_{ij} + h_{ij} \left( \frac{2V(2 - \gamma)}{(3\gamma - 2)} \frac{K}{Z} + \frac{6(2 - \gamma)}{(3\gamma - 2)^2} \frac{V^2}{Z^2} (1 - V^P) \right)$$
(41)

The  $\frac{1}{Z^2}$  terms which appear here are undesirable with regard to application of the Newman-Claudel theorem. We therefore introduce a new variable  $\zeta$  according to

$$\zeta = \frac{6(2-\gamma)}{(3\gamma-2)^2} \frac{(V^P - 1)}{Z}$$
(42)

From (34) there follows

$$\frac{\partial \zeta}{\partial Z} = -\frac{1}{Z}\zeta - \frac{1}{Z}\frac{6(2-\gamma)^2}{(3\gamma-2)^2}V^{(P-1)}K$$
(43)

By definition one has

$$R_{ab} = -\frac{1}{2}g^{cd}\partial_{cd}^2 g_{ab} + \partial_{(a}\Gamma_{b)} + H_{ab}$$

$$\tag{44}$$

where

$$H_{ab} = 2g^{ef} \left( g_{c(a} \Gamma^d_{b)f} \Gamma^c_{de} + g_{cd} \Gamma^d_{f[b} \Gamma^c_{e]a} \right) \quad , \quad \Gamma_a = g_{ad} g^{bc} \Gamma^d_{bc} \tag{45}$$

(Note that  $H_{ab}$  is not the magnetic part of the Weyl tensor.) For a metric of the form (38) one therefore has

$$\Gamma_i = \partial_i \log V + h^{mn} (\partial_m h_{ni} - \frac{1}{2} \partial_i h_{mn})$$
(46)

$$\Gamma_z = -\partial_z \log V + \frac{1}{2} h_{ij} k^{ij} \tag{47}$$

where  $k^{ij} = \partial_z h^{ij} \iff K_{ij} = -\frac{1}{2} V h_{im} h_{jn} k^{mn}$ ,  $K = -\frac{1}{2} V (h_{ij} k^{ij})$ ). Also  $H_{ij} = 2b^{kl} \left( h_{ij} \Gamma^n_{ij} \Gamma^m_{jj} + h_{im} \Gamma^n_{ij} \Gamma^m_{jj} \right)$ 

$$H_{ij} = 2h^{kl} \left( h_{m(i}\Gamma_{j)l}^{n}\Gamma_{nk}^{m} + h_{mn}\Gamma_{l[j}^{n}\Gamma_{k]i}^{m} \right)$$
$$- (\partial_{i}\log V)(\partial_{j}\log V) - \Gamma_{ij}^{k}\partial_{k}\log V + \frac{1}{2}V^{2}(\partial_{z}\log V)h_{ki}h_{jl}k^{kl}$$
$$+ \frac{1}{2}V^{2}h_{ki}h_{jl}h_{mn}k^{mk}k^{ln} - \frac{1}{4}V^{2}h_{ki}h_{jl}h_{mn}k^{kl}k^{mn} \qquad (48)$$
$$H_{iz} = (h_{k[j}\partial_{i]}\log V)k^{jk}$$

$$+\frac{1}{2}h_{ij}h_{mn}(h^{pq}k^{mj}-h^{mj}k^{pq})\Gamma^{n}_{pq}$$
(49)

$$H_{zz} = 2(\partial_z \log V)^2 + \frac{1}{2}(\partial_z \log V)h_{mn}k^{mn} - \frac{1}{4}h_{ij}h_{kl}k^{ik}k^{jl} + \frac{1}{V^2}h^{ij}\left((\partial_i \log V)(\partial_j \log V) - \Gamma_{ij}^k\partial_k \log V\right)$$
(50)

Define  $\gamma_{ijk} \equiv \partial_k h_{ij}$ . Then from the definitions (46), (47) one gets

$$\partial_z \Gamma_i = \partial_i \Gamma_z - h_{ik} \partial_j k^{jk} + 2 \partial_i \partial_z \log V - h_{ik} h^{mn} k^{jk} \gamma_{mjn}$$
(51)

Now (34), (47) imply

$$\Gamma_z = \frac{1}{2}(2-\gamma)h_{ij}k^{ij} = P\partial_z \log V$$
(52)

whereby (51) becomes

$$\partial_z \Gamma_i = \left(\frac{\gamma}{2-\gamma}\right) \partial_i \Gamma_z - h_{ik} \partial_j k^{jk} - h_{ik} h^{mn} k^{jk} \gamma_{mjn} \tag{53}$$

We now choose new independent variables in such a way that the field equations for these quantities become symmetric hyperbolic. Define the positive constants E, F, G by

$$E^{2} = \frac{24\gamma(2-\gamma)}{(\gamma-1)^{2}} , \quad F^{2} = \frac{12\gamma}{(\gamma-1)} , \quad G^{2} = \frac{4\gamma}{(2-\gamma)}$$
(54)

and define the following new variables

$$\chi_i \equiv E\left(\partial_i \log V + \frac{1}{2P}\Gamma_i\right) , \quad \nu \equiv F\Gamma_z , \quad \xi_i \equiv G\Gamma_i$$
(55)

In terms of the new variables the field equations can be written as follows:

$$\partial_z h^{ij} = k^{ij} \tag{56}$$

$$\partial_z h_{ij} = -h_{im} h_{jn} k^{mn} \tag{57}$$

$$\partial_z V = \frac{1}{2} (\gamma - 1) V \left( \frac{h_{(mn)} k^{mn}}{(1 - 5\gamma)} - \frac{10\gamma}{(2 - \gamma)(1 - 5\gamma)} \frac{\nu}{F} \right)$$
(58)

$$\partial_z \zeta = \frac{(2-\gamma)}{2(1-5\gamma)} \left( h_{(mn)} k^{mn} - \frac{10\gamma}{(2-\gamma)} \frac{\nu}{F} \right) - \frac{1}{Z} \left( \zeta - \frac{3(2-\gamma)^2}{(3\gamma-2)^2(1-5\gamma)} \left( h_{(mn)} k^{mn} - \frac{10\gamma}{(2-\gamma)} \frac{\nu}{F} \right) \right)$$
(59)

$$-h^{(im)}h^{(jn)}h^{(kl)}\partial_z\gamma_{mnl} = h^{(kl)}\partial_l k^{(ij)} + h^{kl}(h^{im}k^{jn} + h^{jn}k^{im})\gamma_{mnl}$$
(60)

$$-\left(\frac{3\gamma-2}{2\gamma}\right)h^{(ij)}\partial_z\xi_i = \frac{3}{2}\frac{G}{F}h^{(ij)}\partial_i\nu + G\left(\frac{\gamma-2}{2\gamma}\right)\partial_kk^{(jk)}$$

$$+ G\left(\frac{\gamma - 2}{2\gamma}\right)h^{mn}k^{lj}\gamma_{mln} + 2Gh^{ij}\hat{H}_{iz} \\ - \frac{8G}{Z(3\gamma - 2)}h^{(ij)}\left(\frac{\chi_i}{E} - \frac{1}{2PG}\xi_i\right)$$
(61)  
$$V^2h_{(im)}h_{(jn)}\partial_z k^{mn} = h^{(mn)}\partial_m\gamma_{(ij)n} - \frac{2}{G}\partial_{(i}\xi_{j)} \\ + 2V^2k^{km}k^{pr}h_{(im)}h_{(jp)}h_{(kr)} - 2\hat{H}_{ij} \\ + \frac{2V^2}{Z}\left(-\frac{2}{(3\gamma - 2)}h_{(im)}h_{(jn)}k^{mn} - h_{(ij)}\left(\zeta - \frac{h_{(mn)}k^{mn}}{(3\gamma - 2)} + \frac{2(\gamma - 3)}{(3\gamma - 2)(\gamma - 2)}\frac{\nu}{F}\right)\right) \\ \left(\frac{V^2}{2 - \gamma}\right)\partial_z\nu = -\frac{F}{E}h^{(ij)}\partial_i\chi_j + \frac{F}{2PG}h^{(ij)}\partial_i\xi_j \\ + 2Fh^{ij}\left(\frac{\chi_i}{E} - \frac{\xi_i}{2PG}\right)\left(\frac{\chi_j}{E} - \frac{\xi_j}{2PG}\right) + \frac{2V^2}{FP^2}\nu^2 - V^2F\hat{H}_{zz}$$

$$+\frac{V^{2}F}{Z}\left(\left(\frac{3\gamma-2}{2-\gamma}\right)\zeta + \frac{h_{(mn)}k^{mn}}{(3\gamma-2)} + \frac{2(3\gamma-5)}{(2-\gamma)(3\gamma-2)}\frac{\nu}{F}\right)$$
(63)  
$$-h^{(ij)}\partial_{z}\chi_{i} = -\frac{E}{2PF}h^{(ij)}\partial_{i}\nu + \frac{E}{P}h^{(ij)}\hat{H}_{zi}$$
  
$$+\frac{4E}{(3\gamma-2)PZ}h^{(ij)}\left(\frac{\xi_{i}}{2PG} - \frac{\chi_{i}}{E}\right)$$
(64)

where

$$\hat{H}_{zz} = -\frac{1}{4} h_{ij} h_{kl} k^{ik} k^{jl} + \frac{1}{V^2} h^{ij} \left( \frac{\chi_i}{E} - \frac{\xi_i}{2PG} \right) \left( \frac{\chi_j}{E} - \frac{\xi_j}{2PG} \right) 
- h^{ij} \hat{\Gamma}_{ij}^k \left( \frac{\chi_k}{E} - \frac{\xi_k}{2PG} \right)$$
(65)
$$\hat{H}_{iz} = h_{k[j} \left( \frac{1}{E} \chi_{i]} - \frac{1}{2PG} \xi_i \right) k^{jk} 
+ \frac{1}{2} h_{ij} h_{mn} (h^{pq} k^{mj} - h^{mj} k^{pq}) \hat{\Gamma}_{pq}^n$$
(66)
$$\hat{H}_{ij} = 2h^{(kl)} \left( h_{m(i} \hat{\Gamma}_{j)l}^n \hat{\Gamma}_{nk}^m + h_{mn} \hat{\Gamma}_{l[j}^n \hat{\Gamma}_{k]i}^m \right) 
- \left( \frac{\chi_i}{E} - \frac{\xi_i}{2PG} \right) \left( \frac{\chi_j}{E} - \frac{\xi_j}{2PG} \right) - \hat{\Gamma}_{ij}^k \left( \frac{\chi_k}{E} - \frac{\xi_k}{2PG} \right) 
+ \frac{1}{2} V^2 h_{k(i} h_{j)l} k^{km} k^{ln} h_{mn} + \frac{1}{4} V^2 (\gamma - 2) h_{mn} k^{mn} h_{k(i} h_{j)l} k^{kl}$$
(67)

and

$$\hat{\Gamma}_{ij}^{k} = \bar{\Gamma}_{(ij)}^{k} ; \ \bar{\Gamma}_{ij}^{k} = \frac{1}{2} h^{kl} (\gamma_{lij} + \gamma_{jli} - \gamma_{ijl})$$

$$(68)$$

In (58), (59), (62), (63) we have used the first of (52) to partially substitute for  $h_{ij}k^{ij}$ , which will be useful later on. Equation (62) comes from (44) and (41). Equation (63) comes from (44), (39), and the second of (52). Equation (61) comes from (40), (44), and (53) was used to partially substitute for  $\partial_i \Gamma_z$ . Equation (64) comes from (44), (40), and the second of (52) We now seek a solution of the 'reduced' equations (56)-(68), where  $\gamma_{ijk}$ ,  $\chi_i$ ,  $\nu$ ,  $\xi_i$  are to be treated as independent variables, with no reference to their original definitions. We will suppose that the coordinates  $x^i$  are chosen to satisfy the harmonic gauge condition

$$h^{ij}\Gamma^k_{ij} = 0 \tag{69}$$

on  $\Sigma.$ 

The initial data for the reduced equations are as follows:

 $h_{ij}$  is free data at  $\Sigma$ , and is chosen negative definite and symmetric.  $h^{ij}$  is chosen so that  $h^{ij}h_{jk} = \delta_k^i$ , and choose also  $\gamma_{ijk} = \partial_k h_{ij}$ . Since the second fundamental form vanishes at  $\Sigma$  one must have  $k^{ij} = 0$  there, and one also has V = 1. From the first of (52) and (53), (56), (55), (69) there follows  $\nu = \zeta = \chi_i = \xi_i = 0$ 

It is possible, using the formalism of Newman (1993b), to write (56)-(68) explicitly in the following form

$$A^{0}(u)\partial_{z}u = A^{i}(u)\partial_{i}(u) + B(u)u + \frac{1}{Z}C(u)u$$
(70)

where u stands for the field variables written as a vector,  $A^0$  is positive definite and symmetric, the  $A^i$  are symmetric, and  $A^{\alpha}$ , B, C are polynomial in their arguments (see Appendix for details). One has C(u)(u(0)) = 0 for all u, and the eigenvalues of  $((A^0)^{-1}C)(u(0))$  satisfy one of the following

$$\lambda = 0, \quad \lambda = \frac{4(\gamma - 2)}{P(3\gamma - 2)^2}$$
$$\lambda^3 + \frac{3(2 - \gamma)}{(3\gamma - 2)}\lambda^2 + \frac{12(2 - \gamma)(5\gamma - 3)}{(3\gamma - 2)^2(5\gamma - 1))}\lambda + \frac{32(13\gamma - 8)}{(5\gamma - 1)(3\gamma - 2)^3} = 0$$

Hence for  $1 < \gamma < 2$ , no  $\lambda$  is positive. It follows that (56)-(68) satisfy all the requirements of Theorem 2, and thus there exists a solution u unique in a suitable differentiability class. Differentiability is discussed at the end of section 4.4.

One now wishes to know whether the metric g obtained from u via (38) is actually a solution of the conformal Einstein equations (22). i.e. whether the definitions of  $\gamma_{ijk}$ ,  $\chi_i$ ,  $\nu$ ,  $\xi_i$  can be recovered from (56)-(58). The answer is yes, as is shown in the next section.

#### 4.4 Recovering the conformal Einstein equations

We now show that if u is the solution of (56)-(58) as above, then as long as  $h_{(ij)}$  remains negative definite,  $h_{ij}$  is in fact symmetric, and g given by (38) is a solution of the conformal Einstein-perfect fluid equations (22). The key step will be, following Newman (1993b), to use the contracted Bianchi identities to show that  $\nu/F$  and  $\xi_i/G$  agree with the definitions (46)-(47) of  $\Gamma_z$  and  $\Gamma_i$ . Suppose then that  $h_{ij}$  is negative definite. Equation (62) then gives

$$\partial_z (Z^{4/(3\gamma-2)} k^{[ij]}) = 0$$

Hence  $k^{[ij]} = 0$  for all Z. Then (56)-(57) give  $h_{[ij]} = h^{[ij]} = 0$  for all Z. One also gets

$$\partial_z (h^{kj} h_{ji} - \delta_i^k) = -h_{im} (h^{kj} h_{jn} - \delta_n^k) k^{mn}$$

A Gronwall estimate (Racke 1992) therefore gives  $h^{kj}h_{ji} = \delta_i^k$  for all Z. Now (56), (60) imply

$$\partial_z(\gamma_{ijk} - \partial_k h_{ij}) = h_{jm}k^{mn}(\gamma_{ink} - \partial_k h_{in}) + h_{im}k^{mn}(\gamma_{njk} - \partial_k h_{nj})$$

So a Gronwall estimate implies  $\gamma_{ijk} = \partial_k h_{ij}$  for all Z. Define  $\hat{\zeta}$  by

$$\hat{\zeta} = \frac{6(2-\gamma)}{(3\gamma-2)^2} \frac{(V^P - 1)}{Z}$$

then (58), (59) give

$$\partial_z (Z(\zeta - \hat{\zeta})) = 0$$

and hence the definition (52) is recovered. Now write  $\hat{\Gamma}_z = \frac{\nu}{F}$ ,  $\hat{\Gamma}_i = \frac{\xi_i}{G}$ ,  $\psi = \frac{\chi_i}{E}$ . Then (61), (64) imply

$$\partial_z \left( \left( \frac{2-3\gamma}{2\gamma} \right) \hat{\Gamma}_i + 2P\psi_i + \left( \frac{\gamma-2}{2\gamma} \right) \Gamma_i + \frac{2(\gamma-2)}{(\gamma-1)} \partial_i \log V \right) = \\ \partial_i \left( \frac{5}{2} \hat{\Gamma}_z + \frac{(\gamma-2)(5\gamma-1)}{2\gamma(\gamma-1)} \partial_z \log V + \left( \frac{\gamma-2}{4\gamma} \right) h_{ij} k^{ij} \right)$$

and now (58) gives

$$\left(\frac{2-3\gamma}{2\gamma}\right)\hat{\Gamma}_i + 2P\psi_i + \left(\frac{\gamma-2}{2\gamma}\right)\Gamma_i - 2P\partial_i\log V = 0 \tag{71}$$

Note that once we have established  $\hat{\Gamma}_i = \Gamma_i$  then (71) will imply  $\psi_i = \partial_i \log V + \frac{1}{2P} \Gamma_i$  as desired.

From the definition of  $\Gamma_z$  and (58) there follows

$$\partial_z \log V = \left(\frac{\gamma - 1}{2 - \gamma}\right) \hat{\Gamma}_z + \frac{(1 - \gamma)}{2(1 - 3\gamma)} (\hat{\Gamma}_z - \Gamma_z)$$
(72)

while (71) implies

$$\psi_i - \frac{1}{2P}\hat{\Gamma}_i = \frac{(\gamma - 1)}{4\gamma}(\Gamma_i - \hat{\Gamma}_i) + \partial_i \log V$$
(73)

and (64) implies

$$-\partial_z \psi_i = -\frac{1}{2P} \partial_i \hat{\Gamma}_z + \frac{1}{P} \hat{H}_{zi} + \frac{(4P(3\gamma - 2))^{-1}}{Z} \left(\frac{1}{2P} \hat{\Gamma}_i - \psi_i\right)$$
(74)

Equations (72)-(74) and (51) now combine to yield

$$-2\hat{H}_{zi} + \frac{8}{Z(3\gamma - 2)} \left(\psi_i - \frac{1}{2P}\hat{\Gamma}_i\right) = -h_{ik}\partial_j k^{jk} - h_{ik}h^{mn}k^{jk}\gamma_{mjn} + \left(\frac{2 - 3\gamma}{1 - 3\gamma}\right)\partial_i(\Gamma_z - \hat{\Gamma}_z) + \frac{2}{(2 - \gamma)}\partial_i\hat{\Gamma}_z + \frac{(3\gamma - 2)}{2\gamma}\partial_z(\hat{\Gamma}_i - \Gamma_i)$$
(75)

Define a symmetric tensor  $S_{ab}$  by

$$S_{ij} = \frac{V^2}{Z} \left\{ -\frac{2}{(3\gamma - 2)} h_{im} h_{jn} k^{mn} - h_{ij} \left( \zeta + \frac{(2 - \gamma)}{(3\gamma - 2)} \left( \frac{1}{(\gamma - 2)} h_{mn} k^{mn} + \frac{2(3 - \gamma)}{(\gamma - 2)^2} \hat{\Gamma}_z \right) \right) \right\}$$
(76)

$$S_{iz} = \frac{4}{Z} \frac{1}{(3\gamma - 2)} \left( \psi_i - \frac{1}{2P} \hat{\Gamma}_i \right) \tag{77}$$

$$S_{zz} = \frac{1}{Z} \left\{ \left( \frac{(3\gamma - 2)}{2 - \gamma} \right) \zeta + \frac{1}{3\gamma - 2} h_{mn} k^{mn} + \frac{2(3\gamma - 5)}{(2 - \gamma)(3\gamma - 2)} \hat{\Gamma}_z \right\}$$
(78)

Also define  $T_{ab} = S_{ab} - \frac{1}{2}Sg_{ab}$ , where  $S = g^{ab}S_{ab}$ . Then

$$T_{ij} = \frac{V^2}{Z} \left\{ -\frac{2}{(3\gamma - 2)} h_{im} h_{jn} k^{mn} - h_{ij} \left( \frac{2\zeta}{P} - \frac{4}{(3\gamma - 2)} \hat{\Gamma}_z \right) \right\}$$
(79)

$$T_{iz} = \frac{4}{Z} \frac{1}{(3\gamma - 2)} \left( \psi_i - \frac{1}{2P} \hat{\Gamma}_i \right)$$
(80)

$$T_{zz} = \frac{1}{Z} \left\{ \frac{2}{(2-\gamma)} + \frac{4}{(2-\gamma)(3\gamma-2)} \hat{\Gamma}_z \right\}$$
(81)

Equations (62), (64), (63) can now be written, respectively, as

$$S_{ij} = R_{ij} + \partial_{(i}(\hat{\Gamma}_{j)} - \Gamma_{j)}) + (\hat{H}_{ij} - H_{ij})$$

$$S_{iz} = R_{iz} + (\hat{H}_{iz} - H_{iz})$$

$$+ \frac{1}{2} \left( \frac{1+2\gamma}{3\gamma - 1} \right) \partial_i (\hat{\Gamma}_z - \Gamma_z) + \left( \frac{3\gamma - 2}{4\gamma} \right) \partial_z (\hat{\Gamma}_i - \Gamma_i)$$

$$S_{zz} = R_{zz} + (\hat{H}_{zz} - H_{zz}) + \frac{(5\gamma - 1)}{2(3\gamma - 1)} \partial_z (\hat{\Gamma}_z - \Gamma_z)$$

$$+ \left( \frac{1-\gamma}{4\gamma V^2} \right) h^{ij} \partial_i (\hat{\Gamma}_j - \Gamma_j) - \frac{2}{V^2} \left( \frac{\gamma - 1}{4\gamma} \right)^2 h^{ij} (\hat{\Gamma}_i - \Gamma_i) (\hat{\Gamma}_j - \Gamma_j)$$

$$+ \left( \frac{1-\gamma}{V^2 \gamma} \right) h^{ij} (\Gamma_i - \hat{\Gamma}_i) \partial_j \log V + \frac{1}{2} \left( \frac{1-\gamma}{1-3\gamma} \right)^2 (\hat{\Gamma}_z - \Gamma_z)^2 + \frac{2(\gamma - 1)^2}{(2-\gamma)(1-3\gamma)} \hat{\Gamma}_z (\Gamma_z - \hat{\Gamma}_z)$$

$$(84)$$

If (56)-(68) hold, then one calculates the following

$$\nabla_{a}T_{i}^{a} = \frac{2}{(3\gamma - 2)} \frac{V^{2}}{Z} \Biggl\{ \left(\frac{4\gamma - 3}{1 - 3\gamma}\right) \partial_{i}(\Gamma_{z} - \hat{\Gamma}_{z}) + \left(\frac{4\gamma - 3}{2\gamma}\right) \partial_{z}(\Gamma_{i} - \hat{\Gamma}_{i}) + \left(\frac{7\gamma - 1}{3\gamma - 1}\right) \partial_{i} \log V(\hat{\Gamma}_{z} - \Gamma_{z}) + \left(\frac{\gamma - 1}{4\gamma}\right) (\Gamma_{j} - \hat{\Gamma}_{j}) h_{im} k^{mj} + \frac{3}{Z} \frac{(2 - \gamma)(\gamma - 1)}{2(3\gamma - 2)} (\Gamma_{i} - \hat{\Gamma}_{i}) + \left(\frac{\gamma - 1}{4\gamma}\right) \left(\left(\frac{1 + 7\gamma}{1 - 3\gamma}\right) \Gamma_{z} + \frac{(1 - \gamma)(8 - \gamma)}{(2 - \gamma)(1 - 3\gamma)} \hat{\Gamma}_{z}\right) (\Gamma_{i} - \hat{\Gamma}_{i}) \Biggr\}$$

$$(85)$$

$$\nabla_{a}T_{z}^{a} = \frac{V^{2}}{Z} \left\{ \left( \frac{\hat{\Gamma}_{z} - \Gamma_{z}}{Z} \right) \left( \frac{2(13\gamma - 8) - 6(V^{P} - 1)(5\gamma^{2} - 4\gamma + 3)}{(1 - 3\gamma)(3\gamma - 2)^{2}} \right) + 2\hat{\Gamma}_{z}(\hat{\Gamma}_{z} - \Gamma_{z}) \frac{(1 - \gamma)(5\gamma - 4)}{(3\gamma - 2)(2 - \gamma)(1 - 3\gamma)} \right\} + \frac{1}{Z} \left\{ \frac{(1 - \gamma)}{(3\gamma - 2)\gamma} h^{kj}(\partial_{j} \log V)(\hat{\Gamma}_{k} - \Gamma_{k}) + \frac{4}{(3\gamma - 2)} \left( \frac{\gamma - 1}{4\gamma} \right)^{2} h^{ij}(\hat{\Gamma}_{i} - \Gamma_{i})(\hat{\Gamma}_{j} - \Gamma_{j}) \right\}$$

$$(86)$$

Now write  $\Delta_{ab} = S_{ab} - R_{ab} = (T_{ab} - \frac{1}{2}Tg_{ab}) - R_{ab}$ . Then by the Bianchi identities one gets

$$g^{cb}\partial_c\Delta_{ab} = \nabla_b T^b_a - \frac{1}{2}\partial_a (T+R) + 2g^{cb}\Gamma^d_{c(a}\Delta_{b)d}$$
(87)

and note that  $T + R = -g^{ab}\Delta_{ab}$ . From (82)-(84)  $\Delta_{ab}$  is given by

$$\Delta_{ij} = \partial_{(i}(\hat{\Gamma}_{j)} - \Gamma_{j)}) + \text{ terms in } (\hat{\Gamma}_{a} - \Gamma_{a})$$
(88)

$$\Delta_{iz} = \frac{1}{2} \left( \frac{1+2\gamma}{3\gamma-1} \right) \partial_i (\hat{\Gamma}_z - \Gamma_z) + \left( \frac{3\gamma-2}{4\gamma} \right) \partial_z (\hat{\Gamma}_i - \Gamma_i) + \text{terms in } (\hat{\Gamma}_a - \Gamma_a)$$
(89)

$$\Delta_{zz} = \frac{(5\gamma - 1)}{2(3\gamma - 1)} \partial_z (\hat{\Gamma}_z - \Gamma_z) + \left(\frac{1 - \gamma}{4V^2\gamma}\right) h^{ij} \partial_i (\hat{\Gamma}_j - \Gamma_j)$$
  
+ terms in  $(\hat{\Gamma}_a - \Gamma_a)$  (90)

Now substituting (85),(86), (88)-(90) into (87) gives

$$g^{cb}\partial_c\Delta_{ib} = \frac{V^2}{2} \left(\frac{5\gamma - 1}{2(3\gamma - 1)}\right) \partial_{iz}^2(\hat{\Gamma}_z - \Gamma_z) + \frac{1}{2} \left(\frac{1 + 3\gamma}{4\gamma}\right) h^{mn} \partial_{im}^2(\hat{\Gamma}_n - \Gamma_n)$$
$$-\frac{3V^2(2 - \gamma)(\gamma - 1)}{(3\gamma - 2)^2} \frac{(\hat{\Gamma}_i - \Gamma_i)}{Z^2} + \text{terms in } (\hat{\Gamma}_a - \Gamma_a), \ Z^{-1}(\hat{\Gamma}_a - \Gamma_a), \ \partial_b(\hat{\Gamma}_a - \Gamma_a)$$
$$+ \frac{V^2}{Z} \left(\frac{2}{3\gamma - 2}\right) \left\{ \left(\frac{3 - 4\gamma}{1 - 3\gamma}\right) \partial_i(\hat{\Gamma}_z - \Gamma_z) + \left(\frac{3 - 4\gamma}{2\gamma}\right) \partial_z(\hat{\Gamma}_i - \Gamma_i) \right\}$$
(91)

$$g^{cb}\partial_{c}\Delta_{zb} = \frac{V^{2}}{4} \left(\frac{5\gamma - 1}{3\gamma - 1}\right) \partial_{z}^{2}(\hat{\Gamma}_{z} - \Gamma_{z}) + \left(\frac{1 + 3\gamma}{8\gamma}\right) h^{mn} \partial_{mz}^{2}(\hat{\Gamma}_{n} - \Gamma_{n}) + \frac{V^{2}}{Z^{2}}(\hat{\Gamma}_{z} - \Gamma_{z}) \left(\frac{2(13\gamma - 8) - 6(V^{P} - 1)(5\gamma^{2} - 4\gamma + 3)}{(1 - 3\gamma)(3\gamma - 2)^{2}}\right) + \text{terms in } (\hat{\Gamma}_{a} - \Gamma_{a}), \ Z^{-1}(\hat{\Gamma}_{a} - \Gamma_{a}), \ \partial_{b}(\hat{\Gamma}_{a} - \Gamma_{a})$$
(92)

The unspecified terms in these equations are polynomial in the stated quantities, with coefficients which are polynomial in the components of u.

One can also calculate  $g^{cb}\partial_c\Delta_{ab}$  directly from (88)-(90), giving

$$g^{cb}\partial_c\Delta_{ib} = \frac{V^2}{2} \left(\frac{1+2\gamma}{3\gamma-1}\right) \partial_{iz}^2(\hat{\Gamma}_z - \Gamma_z) + V^2 \left(\frac{3\gamma-2}{4\gamma}\right) \partial_z^2(\hat{\Gamma}_i - \Gamma_i)$$
$$+ \frac{1}{2}h^{mn}\partial_{mn}^2(\hat{\Gamma}_i - \Gamma_i) + \frac{1}{2}h^{mn}\partial_{mi}^2(\hat{\Gamma}_n - \Gamma_n)$$
$$+ \text{ terms as in (91)}$$
(93)

$$g^{cb}\partial_c\Delta_{zb} = \frac{V^2}{4} \left(\frac{5\gamma-1}{3\gamma-1}\right) \partial_z^2(\hat{\Gamma}_z - \Gamma_z) + \left(\frac{2\gamma-1}{4\gamma}\right) h^{mn} \partial_{mz}^2(\hat{\Gamma}_n - \Gamma_n) + \frac{1}{2} \left(\frac{2\gamma+1}{3\gamma-1}\right) h^{mn} \partial_{mn}^2(\hat{\Gamma}_z - \Gamma_z) + \text{terms as in (92)}$$
(94)

Now we can compare (91) with (93) and (92) with (94) to obtain

$$V^{2}\left(\frac{3\gamma-2}{4\gamma}\right)\partial_{z}^{2}(\hat{\Gamma}_{i}-\Gamma_{i}) = \frac{V^{2}(\gamma-3)}{4(3\gamma-1)}\partial_{iz}^{2}(\hat{\Gamma}_{z}-\Gamma_{z}) + \left(\frac{1-\gamma}{8\gamma}\right)h^{mn}\partial_{mi}^{2}(\hat{\Gamma}_{n}-\Gamma_{n})$$
$$-\frac{1}{2}h^{mn}\partial_{mn}^{2}(\hat{\Gamma}_{i}-\Gamma_{i}) + \frac{V^{2}}{Z}\left\{-\frac{3(2-\gamma)(\gamma-1)}{(3\gamma-2)^{2}}\left(\frac{\hat{\Gamma}_{i}-\Gamma_{i}}{Z}\right)\right\}$$
$$+\frac{2}{(3\gamma-2)}\left(\left(\frac{3-4\gamma}{1-3\gamma}\right)\partial_{i}(\hat{\Gamma}_{z}-\Gamma_{z}) + \left(\frac{3-4\gamma}{2\gamma}\right)\partial_{z}(\hat{\Gamma}_{i}-\Gamma_{i})\right)\right\}$$
$$+ \text{ terms as in (91)}$$
(95)

$$\frac{V^2}{4} \left(\frac{5\gamma - 1}{3\gamma - 1}\right) \partial_z^2 (\hat{\Gamma}_z - \Gamma_z) = \frac{(3 - \gamma)}{8\gamma} h^{mn} \partial_{mz}^2 (\hat{\Gamma}_n - \Gamma_n)$$

$$-\frac{1}{2}\left(\frac{1+2\gamma}{3\gamma-1}\right)h^{mn}\partial_{mn}^{2}(\hat{\Gamma}_{z}-\Gamma_{z}) + \text{ terms as in (93)} + \frac{V^{2}}{Z}\left(\frac{\hat{\Gamma}_{z}-\Gamma_{z}}{Z}\right)\left(\frac{2(13\gamma-8)-6(V^{P}-1)(5\gamma^{2}-4\gamma+3)}{(1-3\gamma)(3\gamma-2)^{2}}\right)$$
(96)

Next we introduce new variables as follows

 $\begin{aligned} \pi_i &= \hat{\Gamma}_i - \Gamma_i, \ \tau = \hat{\Gamma}_z - \Gamma_z, \ \alpha_{ij} = \partial_i (\hat{\Gamma}_j - \Gamma_j), \ \beta_i = \partial_z (\hat{\Gamma}_i - \Gamma_i) \\ \omega_i &= C_1 \partial_i (\hat{\Gamma}_z - \Gamma_z), \ \delta = C_2 \partial_z (\hat{\Gamma}_z - \Gamma_z), \ \eta_i = Z^{-1} (\hat{\Gamma}_i - \Gamma_i), \ \kappa = Z^{-1} (\hat{\Gamma}_z - \Gamma_z) \end{aligned}$ 

where

$$C_1 = C_2 \sqrt{\frac{2\gamma + 1}{2(3\gamma - 1)}}, \quad C_2 = \sqrt{\frac{2\gamma}{3\gamma - 1}}$$

In terms of these new variables (95) becomes

$$-V^{2}\left(\frac{3\gamma-2}{4\gamma}\right)h^{ij}\partial_{z}\beta_{i} = -\frac{(\gamma-3)}{4(3\gamma-1)}\frac{V^{2}}{C_{2}}h^{ij}\partial_{i}\delta - \left(\frac{1-\gamma}{8\gamma}\right)h^{ij}h^{kn}\partial_{k}\alpha_{in}$$
$$+\frac{1}{2}h^{ij}h^{km}\partial_{k}\alpha_{mi} + \text{ order zero terms}$$
$$-\frac{V^{2}}{Z}h^{ij}\left\{-\frac{3(2-\gamma)(\gamma-1)}{(3\gamma-2)^{2}}\eta_{i} + \frac{2(3-4\gamma)}{(3\gamma-2)(1-3\gamma)C_{1}}\omega_{i} + \frac{(3-4\gamma)}{\gamma(3\gamma-2)}\beta_{i}\right\}$$
(97)

and (96) becomes

$$\frac{V^4}{4} \left(\frac{5\gamma - 1}{3\gamma - 1}\right) \partial_z \delta = V^2 C_2 \left(\frac{3 - \gamma}{8\gamma}\right) h^{mn} \partial_m \beta_n - \frac{V^2 C_2}{2C_1} \left(\frac{2\gamma + 1}{3\gamma - 1}\right) h^{mn} \partial_m \omega_n \\
+ \frac{V^4}{Z} \left(\frac{2(13\gamma - 8) - 6(V^P - 1)(5\gamma^2 - 4\gamma + 3)}{(1 - 3\gamma)(3\gamma - 2)^2}\right) \kappa \\
+ \text{ order zero terms}$$
(98)

By the above definitions one also obtains the following

$$\partial_z \pi_i = \beta_i \tag{99}$$

$$\partial_z \tau = (C_2)^{-1} \delta \tag{100}$$

$$\left(\frac{1}{2}h^{im}h^{jn} + \left(\frac{\gamma - 1}{8\gamma}\right)h^{in}h^{jm}\right)\partial_z\alpha_{mn} = \left(\frac{1}{2}h^{im}h^{jn} + \left(\frac{\gamma - 1}{8\gamma}\right)h^{in}h^{jm}\right)\partial_m\beta_n$$

(101)

$$-V^2 h^{ij} \partial_z \omega_j = -\frac{V^2 C_1}{C_2} h^{ij} \partial_j \delta \tag{102}$$

$$-V^2 h^{ij} \partial_z \eta_j = -\frac{V^2}{Z} h^{ij} (-\eta_j + \beta_j)$$
(103)

$$V^4 \partial_z \kappa = \frac{V^4}{Z} ((C_2)^{-1} \delta - \kappa)$$
(104)

The system (97)-(104) is of the form (see Appendix):

$$a^{0}(v)\partial_{z}v = a^{i}(v)\partial_{i}v + b(v)v + \frac{1}{z}c(v)v$$
(105)

where v stands for the variables written as a vector,  $a^0$  is symmetric and positive definite, the  $a^i$  are symmetric, and all coefficients are polynomial in the components of v and those of u (u as in (70)) The eigenvalues of  $(a^0)^{-1}c(0)$  satisfy one of

$$\lambda = 0 , \quad \lambda^2 + \lambda + \frac{8(13\gamma - 8)}{(5\gamma - 1)(3\gamma - 2)^2} = 0$$
$$(1 + \lambda) \left(\lambda + \frac{4(4\gamma - 3)}{(3\gamma - 2)^2}\right) + \frac{12\gamma(2 - \gamma)(\gamma - 1)}{(3\gamma - 2)^3} = 0$$

Hence for  $1 < \gamma < 2$ , no  $\lambda$  is a positive integer.

The initial data for (97)-(104) is as follows: One has  $\hat{\Gamma}_i = 0$  at Z = 0. Since  $\phi$  is constant at Z = 0 one also has  $\Gamma_i = 0$  there. Hence  $\pi_i = \alpha_{ij} = 0$  initially. Now  $\hat{\Gamma}_z = \nu = 0$  at Z = 0. Since  $k_{ij} = 0$  at Z = 0, (58) implies  $\partial_z V = 0$  there. Hence  $\Gamma_z = 0$  at Z = 0 and so  $\tau = \omega_i = 0$  there. Taking a Taylor expansion of (61), (64) gives  $\partial_z \xi_i = \partial_z \hat{\Gamma}_i = 0$  at Z = 0. The identity (51) then gives  $\partial_z \Gamma_i = 0$  at Z = 0, and so  $\beta_i = 0$  there. A Taylor expansion of (59), (63), (58) gives  $\partial_z^2 V = \frac{1}{P} \partial_z \hat{\Gamma}_z$  at Z = 0. Comparison with (72) then gives  $\partial_z (\hat{\Gamma}_z - \Gamma_z) = 0$  at Z = 0. Thus  $\delta = \kappa = 0$  at Z = 0.

All the initial data for (97)-(104) are therefore zero, and one solution of this system is  $v \equiv 0$ . For sufficiently smooth v this solution is unique by Theorem 2, and we have recovered  $\hat{\Gamma}_a = \Gamma_a$  (An analogue of Theorem 2 can be obtained from Theorem 1 in the case when certain of the coefficients are polynomial in suitably regular known functions). The metric g obtained from u in (70) via (38) is thus a solution of the conformal Einstein-perfect fluid equations (22).

We now discuss the differentiability of solutions to (56)-(68) and (97)-(104). Suppose then that  $\Sigma$  is a smooth paracompact manifold. In order to apply Theorem 1 one must first consider the localised problem, and then use the finite speed of propagation inherent in (70), (105) to globalise (see Claudel and Newman 1998). Specifically, choose a locally finite open cover  $\{O_{\alpha}\}$  of harmonic coordinate charts for the metric  $h_{ij}$  on  $\Sigma$ , with each  $O_{\alpha}$  having compact closure therein. And let  $\{O'_{\alpha}\}$  be another cover for  $\Sigma$  satisfying  $\bar{O}'_{\alpha} \subset O_{\alpha}$ . Suppose that the initial metric lies in the following Sobolev class

$$(h_{ij})_{\alpha} \in H^{2q_{\alpha}+2+s} , s > 3/2$$
 (106)

where

$$q_{\alpha} > \max\{3 + \|(a^0)^{-1}c\|_{B(\mathbb{R}^{24})}, \ 1 + \|(A^0)^{-1}F(0,x)\|_{B(\mathbb{R}^{63})}\}, \ x \in O_{\alpha}$$
(107)

Then Theorem 2 gives that there is a solution  $(u)'_{\alpha}$  of (56)-(58) contained and unique in the following class

$$(u)'_{\alpha} \in \bigcap_{i=0}^{q_{\alpha}} C^{i}([0, T_{q_{\alpha}}], H^{q_{\alpha}+s-i}(O'_{\alpha}))$$
(108)

Now by the definition of v in (105) there follows

$$(v)'_{\alpha} \in \bigcap_{i=0}^{(q_{\alpha}-2)} C^{i}([0, T_{q_{\alpha}}], H^{q_{\alpha}+s-2-i}(O'_{\alpha}))$$
(109)

and (an analogue of) Theorem 2 gives that  $v \equiv 0$ . Hence the metric  $(g_{ab})'_{\alpha}$  obtained from  $(u'_{\alpha})$  is a solution of the conformal Einstein-perfect fluid equations (22), and one has

$$(g_{ab})'_{\alpha} \in \bigcap_{i=0}^{q_{\alpha}} C^{i}([0, T_{q_{\alpha}}], H^{q_{\alpha}+s-i}(O'_{\alpha}))$$
 (110)

with  $(g_{ab})'_{\alpha}$  unique in this class.

Note that if we give  $C^{\infty}$  data  $(h_{ij})_{\alpha}$  at  $\Sigma$  then the evolution given by (108) is

 $C^{\infty}$  on some time interval  $[0, T_{\alpha}]$ . For the data appropriate to the localised problem is  $C_0^{\infty}$  and thus one gets  $(u)'_{\alpha}$  as in (108) for arbitrarily large  $q_{\alpha}$ . For fixed  $\alpha$  the  $T_{q_{\alpha}}$  are positively bounded below by the following standard extension argument: first from (108) one gets by the Sobolev imbedding theorem (Adams 1975) that  $(u)'_{\alpha}$  is  $C^1$  on  $[0, T_{\alpha}]$  for some  $T_{\alpha} > 0$ . Suppose now that  $(u)'_{\alpha} \in C([0, T_1), H^r)$  for some  $r > \frac{5}{2}$ ,  $T_1 < T_{\alpha}$ . Then, by proposition 5.1.E of (Taylor 1991),  $||(u)'_{\alpha}||_{H^r}$  is bounded on  $[0, T_1)$  and  $(u)'_{\alpha}$  can be uniquely extended in  $H^r$  onto  $[0, T_1]$ . Then standard theory for regular symmetric hyperbolic PDE gives an extension of  $(u)'_{\alpha}$  onto a larger interval  $[0, T_2)$  with

$$(u)'_{\alpha} \in \bigcap_{i=0}^{r} C^{i}([T_{1}, T_{2}), H^{s-i}(O'_{\alpha}))$$

Thus  $(u)'_{\alpha}$  can be extended onto  $[0, T_{\alpha}]$  in the above way. It follows by the Sobolev imbedding theorem that  $(u)'_{\alpha}$  is  $C^{\infty}$  on  $[0, T_{\alpha}]$ .

## 5 Solving the Cauchy problem for $\gamma = 2$

The method of section (4) is clearly inadequate for dealing with the cases  $\gamma = 1$  and  $\gamma = 2$  as certain of the coefficients in the field equations diverge at these values of the polytropic index. However, in the case  $\gamma = 2$  one must have  $\Box Z = 0$  by (26), (32), and it turns out that the problem may be successfully attacked using harmonic coordinates in M. Unfortunately in the case  $\gamma = 1$  it has not been possible to write the field equations in symmetric hyperbolic form, and the problem has not been solved in such generality. However in section 6 we are able to solve the problem in the case of Bianchi type spatial homogeneity.

For the case  $\gamma = 2$  first note that the conformal field equations are just

$$\nabla_a \nabla_b Z = Z R_{ab} \tag{111}$$

Again we wish to solve the Cauchy problem on  $M = \Sigma \times [0, T]$ . We will use coordinates  $x^0 = Z$ ,  $x^i$ , i=1,2,3 on M, and in the sequel Greek indices will take the values 0, 1, 2, 3.

Now let  $H^{\alpha} = \Box x^{\alpha}$ , and consider the following reduced equations for  $g_{\mu\nu}$ 

$$ZR^{H}_{\mu\nu} = \nabla_{\mu}\nabla_{\nu}Z - \frac{H^{0}}{g^{00}}\Delta_{\mu\nu}$$
(112)

where

$$R^{H}_{\mu\nu} = R_{\mu\nu} + g_{\alpha(\mu}\partial_{\nu)}H^{\alpha}$$
(113)

$$\Delta_{ij} = \Delta_{i0} = \Delta_{0i} = 0, \quad \Delta_{00} = 1$$
(114)

Note that (112) implies  $R^H = 0$  so that  $G^H_{\mu\nu} = R^H_{\mu\nu}$ Now write  $h_{\alpha\beta\gamma} = \partial_{\gamma}g_{\alpha\beta}$ . Then (112) can be expressed as follows

$$-g^{(ij)}\partial_z h_{\alpha\beta j} = -g^{(ij)}\partial_j h_{(\alpha\beta)0}$$
(115)

$$\partial_z g_{\alpha\beta} = h_{\alpha\beta0} \tag{116}$$

$$\partial_z g^{\alpha\beta} = -g^{\alpha\mu} g^{\beta\nu} h_{\mu\nu0} \tag{117}$$

$$g^{00}\partial_z h_{\alpha\beta0} = -g^{(ij)}\partial_i h_{(\alpha\beta)j} - 2g^{0i}\partial_i h_{(\alpha\beta)0} + 2H_{(\alpha\beta)} + \frac{1}{Z}g^{0\sigma}(-h_{\alpha\beta\sigma} + h_{\beta\sigma\alpha} + h_{\alpha\sigma\beta})$$
(118)

(for  $\alpha$ ,  $\beta$  not both zero)

$$g^{00}\partial_z h_{000} = -g^{(ij)}\partial_i h_{00j} - 2g^{0i}\partial_i h_{000}$$
$$+2H_{00} + \frac{1}{Z} \{ \frac{1}{g^{00}} g^{0i} g^{0k} (h_{ijk} - 2h_{ikj}) + \frac{2}{g^{00}} g^{0k} g^{0i} (h_{0ki} - h_{0ik} - h_{ki0}) + Q \}$$
(119)

where

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$$Q = g^{ij}(h_{ij0} - 2h_{i0j}) - 2g^{0i}h_{00i}$$
(120)

We can write the singular part of (118) as  $\frac{1}{Z}C_1(u)u$  in such a way that the components of  $C_1(u)u$  are

$$\tilde{g}^{00}(h_{\beta0\alpha} + h_{\alpha0\beta} - h_{\alpha\beta0}) + (\tilde{h}_{\betai\alpha} + \tilde{h}_{\alphai\beta} - \tilde{h}_{\alpha\betai})g^{0i}$$
(121)

where the tilded terms are in  $C_1(u)$  and the untilded terms are in u. And the singular part of (119) can be written as  $\frac{1}{Z}C_2(u)u$  with components

$$\frac{1}{\tilde{g}^{00}}(\tilde{h}_{ijk} - 2\tilde{h}_{ikj})\tilde{g}^{ij}g^{0k} + \frac{2}{\tilde{g}^{00}}(\tilde{h}_{0ki} - \tilde{h}_{0ik} - \tilde{h}_{ki0})\tilde{g}^{0i}g^{0k} + \tilde{g}^{ij}(h_{ij0} - 2h_{i0j}) - 2\tilde{g}^{0i}h_{00i}$$
(122)

where the tilded terms are in  $C_2(u)$  and the untilded terms are in u. Clearly equations (115)-(119) are of the form:

$$A^{0}(u)\partial_{z}u = A^{i}(u)\partial_{i}(u) + B(u)u + \frac{1}{Z}C(u)u$$
(123)

where u stands for the field variables written as a vector, the  $A^{\alpha}$  are symmetric with  $A^{0}$  positive definite, and the components of C are those of  $C_{1}$ ,  $C_{2}$  written in an appropriate order.

#### 5.1 Initial data for the reduced equations

For the initial data we first choose a negative definite 3-metric  $h_{ab}$  on  $\Sigma$ . Next choose coordinates  $x^i$  on  $\Sigma$  satisfying the harmonic gauge condition  $h^{jk}\Gamma^i_{jk} = 0$ , and also take  $g_{0i} = 0$ . We know from the field equations that  $g^{00} = 1$ ,  $h_{ij0} = 0$  initially, and hence  $g_{00} = 1$  initially. We make the choices  $h_{\alpha 00}(0) = 0$ , and this ensures  $\Box x^{\alpha} = 0$  at  $\Sigma$ . These choices also ensure that  $(C(u))u(0) = 0 \forall u$ .

## 5.2 The eigenvalues of $(A^0)^{-1}C(0)$

The eigenvalue equation for  $(A^0)^{-1}C(0)$  amounts to the following

$$\lambda h_{\alpha\beta0} = (h_{\beta0\alpha} + h_{\alpha0\beta} - h_{\alpha\beta0}) + g^{0i}(\bar{h}_{\beta i\alpha} + \bar{h}_{\alpha i\beta} - \bar{h}_{\alpha\beta i})$$

(with  $h_{000}$  replaced by  $\bar{g}^{ij}(h_{ij0} - 2h_{i0j})$  in rhs)

$$\lambda h_{\alpha\beta k} = 0$$
$$\lambda g^{0i} = 0$$

(Unbarred quantities are elements of an eigenvector, barred quantities are evaluated at  $\Sigma$  )

So suppose  $\lambda \neq 0$ . Then 1 implies

$$\lambda h_{ij0} = -h_{ij0} + g^{0m} \{ \bar{h}_{jmi} + \bar{h}_{imj} - \bar{h}_{ijm} \}$$

Since the  $x^i$  are harmonic in  $\Sigma$  this implies

$$\lambda \bar{g}^{ij} h_{ij0} = -\bar{g}^{ij} h_{ij0}$$

From 1 one also gets

$$\lambda h_{000} = \bar{g}^{ij} h_{ij0} , \ \lambda h_{i00} = 0$$

It follows that  $\lambda = 0$  or -1. The system (115)-(119) therefore satisfies all the requirements of Theorem 2, and has a solution u for each choice of  $h_{ab}$  on  $\Sigma$ , unique in a suitable differentiability class. Our next aim is to show, as in section (19), that the 4-metric  $g_{ab}$  obtained from u is actually a solution of the full field equations (111).

#### 5.3 Recovering the Einstein equations

Firstly we show that if (115)-(119) are satisfied, then  $g_{\alpha\beta}$  is symmetric and the definition  $h_{\alpha\beta j} = \partial_j g_{\alpha\beta}$  can be recovered. Observe that (118)-(119) imply

$$g^{00}\partial_z h_{[\alpha\beta]0} = -\frac{1}{Z}g^{00}h_{[\alpha\beta]0} - \frac{1}{Z}g^{0i}h_{[\alpha\beta]i}$$

and (115) implies

$$\partial_z h_{[\alpha\beta]j} = 0$$

Thus  $h_{[\alpha\beta]j} = 0$ , and there follows

$$\partial_z (Zh_{[\alpha\beta]0}) = 0$$

Hence  $h_{[\alpha\beta]0} \equiv 0$ , and now (2.101) implies  $g_{\alpha\beta} = g_{\beta\alpha}$ . From (115) one now gets

$$\partial_z (h_{\alpha\beta j} - \partial_j g_{\alpha\beta}) = \partial_j h_{\alpha\beta 0} - \partial_{zj}^2 g_{\alpha\beta} = 0$$

by (116), so that  $h_{\alpha\beta j} = \partial_j g_{\alpha\beta}$ .

We will now use the Bianchi identities to show that if equations (115)-(119) (or equivalently (112)) are satisfied then we have  $\Box x^{\alpha} = 0$ , and thus (111) is solved.

So suppose (115)-(119) hold. Then one calculates

$$\nabla^{\mu} R^{H}_{\mu\nu} = -\frac{1}{Z} \Big[ (\nabla^{\mu} Z) (R^{H}_{\mu\nu} - R_{\mu\nu}) - \partial_{\nu} H^{0} + g^{\mu\rho} \Big\{ \Delta_{\mu\nu} \partial_{\rho} (H^{0}/g^{00}) - (H^{0}/g^{00}) (\Gamma^{0}_{\rho\mu} \Delta_{\nu0} + \Gamma^{0}_{\rho\nu} \Delta_{0\mu}) \Big\} \Big]$$
(124)

The Bianchi identities now give

$$0 = g^{00} \partial_z^2 H^{\alpha} + 2g^{0i} \partial_{0i}^2 H^{\alpha} + g^{ij} \partial_{ij}^2 H^{\alpha} + \frac{1}{Z} \Big\{ g^{\gamma 0} \partial_{\gamma} H^{\alpha} - g^{\nu \alpha} \partial_{\nu} H^{0} + 2g^{0\alpha} g^{0\rho} ((g^{00})^{-1} \partial_{\rho} H^{0} - (g^{00})^{-2} H^{0} h_{00\rho}) + 2(g^{0\alpha}/g^{00}) (H^{0})^2 - 2g^{\nu \alpha} g^{0\rho} (H^{0}/g^{00}) \Gamma_{\rho\nu}^0 \Big\}$$
(125)

If we now put  $h_{\alpha}^{\beta} = \partial_{\alpha} H^{\beta}$ , then (125) can be written in first order form as follows

$$-g^{ij}\partial_z h^{\alpha}_j = -g^{ij}\partial_j h^{\alpha}_0 \tag{126}$$

$$\partial_{z}H^{\alpha} = h_{0}^{\alpha}$$

$$g^{00}\partial_{z}h_{0}^{\alpha} = -2g^{0i}\partial_{i}h_{0}^{\alpha} - g^{ij}\partial_{i}h_{j}^{\alpha}$$

$$+ \frac{1}{Z} \left\{ -g^{0\gamma}h_{\gamma}^{\alpha} + g^{\nu\alpha}h_{\nu}^{0} + 2g^{0\alpha}g^{0\rho}(-(h_{\rho}^{0}/g^{00}) + (g^{00})^{-2}H^{0}h_{00\rho}) - 2(g^{0\alpha}/g^{00})(H^{0})^{2} + 2g^{\nu\alpha}g^{0\rho}(H^{0}/g^{00})\Gamma_{\rho\nu}^{0} \right\}$$

$$(127)$$

$$(127)$$

This system is symmetric hyperbolic:

$$a^{0}(v)\partial_{z}v = a^{i}(v)\partial_{i}v + b(v)v + \frac{1}{Z}c(v)v$$
(129)

where v stands for  $H^{\alpha}$ ,  $h^{\beta}_{\alpha}$ . We know that  $H^{\alpha} = 0$  at  $\Sigma$ , and now equation (128) implies that  $h^{\alpha}_0 = 0$  at  $\Sigma$  also. Hence all the initial data for (126)-(128) are zero, and one solution is trivial.

The eigenvalues of c(0) satisfy either  $\lambda = 0$  or

$$\lambda_0^\alpha = -h_0^\alpha + \bar{g}^{\nu\alpha} h_\nu^0 - 2\bar{g}^{0\alpha} \bar{g}^{0\rho} h_\rho^0$$

Putting  $\alpha = 0, j$  gives, respectively

$$\lambda h_0^0 = -2h_0^0$$
 ,  $\lambda h_0^i = -h_0^i$ 

Hence  $\lambda \in \{0, -1, -2\}$ . So (126)-(128) is in Newman-Claudel form, and the zero solution is unique in a suitable differentiability class, so that (111) is satisfied.

#### 5.4 Existence, uniqueness, and differentiability

The situation here is very similar to that in section (4). So take covers  $\{O_{\alpha}\}, \{O'_{\alpha}\}$  for  $\Sigma$  as before, and suppose that the initial 3-metric lies in the following class

$$(h_{ij})_{\alpha} \in H^{2q_{\alpha}+2+s}(O_{\alpha}) , \quad s > 3/2$$
 (130)

where

$$q_{\alpha} > \max\{3 + \|(a^{0})^{-1}c(0,x)\|_{B(\mathbb{R}^{96})}, 1 + \|(A^{0})^{-1}C(0,y)\|_{B(\mathbb{R}^{20})}\} \quad x, y \in O_{\alpha}$$
(131)

Then by Theorem 2 there exists a solution u of (112) contained and unique in the following class

$$(u)'_{\alpha} \in \bigcap_{i=0}^{q_{\alpha}} C^{i}([0, T_{q_{\alpha}}], H^{q_{\alpha}+s-i}(O'_{\alpha}))$$
(132)

From the definition of  $H^{\alpha}$ ,  $h^{\beta}_{\alpha}$  it follows that

$$(v)'_{\alpha} \in \bigcap_{i=0}^{(q_{\alpha}-2)} C^{i}([0,T_{q_{\alpha}}], \ H^{q_{\alpha}+s-2-i}(O'_{\alpha}))$$
(133)

and by (an analogue of) Theorem 2 one must have  $v \equiv 0$ . Hence the metric  $g_{ab}$  obtained from u is the unique solution of the Einstein equations (111) with starting metric  $h_{ij}$  on  $\Sigma$ .

We note that, as in section 4,  $C^{\infty}$  data gives rise to a  $C^{\infty}$  solution u.

#### 5.5 The Vanishing Weyl Tensor Conjecture

The results of sections 4-5 can now be used to prove the Vanishing Weyl Tensor conjecture. From (Goode and Wainwright 1985), and as one can see from equation (9) in this case, one knows that the Weyl curvature vanishes at  $\Sigma$  iff the initial 3-metric is of constant curvature, and that any such metric is initial data for some FRW cosmology. The 4-metric of any FRW cosmology is a smooth evolution of smooth data, and hence by the above is the only smooth evolution of the same data. Thus in this case the conjecture is true: for  $1 < \gamma \leq 2$  at least the only cosmologies with an isotropic singularity and vanishing initial Weyl tensor are the FRW models where the Weyl tensor vanishes everywhere.

### 6 Spatially homogeneous $\gamma = 1$ spacetimes

We noted above that for  $\gamma = 1$  the conformal Cauchy problem has not been solved in full generality. However if one restricts attention to certain (Bianchi type) spatially homogeneous cosmologies, the field equations become ODE's and a theorem of Rendall and Schmidt (1991) can be used to obtain the required existence and uniqueness result, as we shall see in this section.

#### 6.1 Bianchi type spacetimes

The Bianchi type spacetimes are a class of spatially homogeneous cosmological models having a 3-parameter Lie group G of isometries. They are classified into 9 types according to the nature of the Lie algebra associated with G (see Wald 1984 for details). In such spacetimes there exists a cosmic time function t, and covector fields  $(e^i)_a$  such that the metric can be written as

$$\tilde{g}_{ab} = \nabla_a t \nabla_b t + \tilde{h}_{ij}(t) (e^i)_a (e^j)_b \tag{134}$$

The  $(e^i)_a$  are preserved under the action of G, and are called *left-invariant* oneform fields.

#### 6.2 Conformal field equations in M

We now consider equations (22), (37) in the case where the physical metric  $\tilde{g}_{ab}$  has Bianchi type symmetry. We may assume wlog that the singularity surface is at Bianchi time t = 0. One must also have that the fluid velocity is orthogonal to the homogeneous hypersurfaces, by the following argument: There exist vector fields  $(\xi_i)^a$  (i = 1, 2, 3) on the Lie group G which are Killing vector fields for  $\tilde{g}_{ab}(t)$ , t > 0. Since the fluid velocity  $\tilde{u}^a$  is geodesic for  $\gamma = 1$  there follows  $\tilde{u}_a(\xi_i)^a = \text{const.}$  in  $\tilde{M}$ . But  $\tilde{u}_a = \Omega u_a$ , with  $u_a$  regular. Letting  $\Omega \to 0$  we see that  $\tilde{u}_a(\xi_i)^a = 0$  in  $\tilde{M}$  as required.

The conformal factor is  $\Omega = Z^2$  and the unphysical metric can be written

$$g_{ab} = \nabla_a Z \nabla_b Z + h_{ij}(Z) (e^i)_a (e^j)_b \tag{135}$$

since V = 1. It follows that  $\Omega = (3t)^{2/3}$  where t is the Bianchi time.

Suppose now that we write equations (22), in an obvious way, as  $G_{ab} = T_{ab}$ . In order that these equations be satisfied at Z = 0 one must have G(0) = 1 and  $K_{ij}(0) = 0$  where  $K_{ij} \equiv \frac{1}{2}\partial_z h_{ij}$ . The initial 3-metric  $h_{ij}(0)$  is free data, as for the other polytropes.

The equations  $G_{ij} = T_{ij}$  take the form

$$\partial_z K_{ij} = {}^3R_{ij} - KK_{ij} + 2K_{im}K_j^m - \frac{4}{Z}K_{ij} - h_{ij}\left(\frac{2K}{Z} + \frac{6(1-G)}{Z^2}\right)$$
(136)

$$\partial_z h_{ij} = 2K_{ij} \tag{137}$$

where  $K = h^{ij} K_{ij}$  and  ${}^{3}R_{ij}$  is the Ricci tensor of the surfaces Z = constant, given by

$${}^{3}R_{ij} = \frac{1}{2}C^{m}_{mk}(C^{r}_{cj}h_{ir} + h_{jr}C^{r}_{ci})h^{kc} - \frac{1}{2}C^{c}_{ki}(C^{k}_{cj} + h_{cm}h^{kl}C^{m}_{lj})$$

$$-\frac{1}{4}C^m_{sk}C^r_{lc}h_{jm}h_{ir}h^{kl}h^{cs}$$
(138)

where  $C_{jk}^i$  are the structure constants of the Lie group G. Now write

$$C = (G_{ab} - T_{ab})Z^a Z^b (139)$$

and

$$C_d = (G_{ab} - T_{ab})h^a_{\ d}Z^b \tag{140}$$

A rather lengthy calculation then shows that if one has a smooth solution  $h_{ij}(Z)$  of (136)-(137), then the following hold:

$$\partial_z C = -D^m C_m - C\left(K + \frac{4}{Z}\right) \tag{141}$$

$$\partial_z C_i = -C_i \left( K + \frac{4}{Z} \right) \tag{142}$$

where  $D_i$  is the derivative operator associated with  $h_{ij}$ .

Now C and  $C_i$  vanish at Z = 0 by the conditions already imposed. If we regard K(Z) as a known smooth function then Theorem 1 of (Rendall and Schmidt 1991) implies that (142) has a unique solution  $C_i$ . Hence  $C_i$  is identically zero, and then in the same way C is identically zero. Our task is therefore to solve equations (37) and (136)-(137).

It will be convenient to work with  $h_{ij}, h^{ij}, G, K_i^{\ j}$  as independent variables, and we also introduce the quantity  $\zeta$  via

$$\zeta \equiv 6(1-G)Z^{-1} \tag{143}$$

The field equations to be solved are then

$$\partial_z h_{ij} = 2h_{(j|k|} K_i^k \tag{144}$$

$$\partial_z h^{ij} = -2h^{(i|k|} K_k^{\ j)} \tag{145}$$

$$\partial_z K_i^{\ j} = h^{jk} ({}^3R_{ik}) - K_m^{\ m} K_i^{\ j} - \frac{4}{Z} K_i^{\ j} - \frac{1}{Z} \delta_i^{\ j} (2K_m^{\ m} + \zeta)$$
(146)

$$\partial_z G = -GK_m^{\ m} \tag{147}$$

$$\partial_z \zeta = -\zeta K_m^m + \frac{1}{Z} (6K_m^m - \zeta) \tag{148}$$

Equations (144)-(148) can be written in matrix form as

$$\frac{du}{dZ} + \frac{1}{Z}Nu = G(u) \tag{149}$$

where  $u = (h_{ij}, h^{ij}, G, K_i^{j}, \zeta)$ , N is a constant matrix, and G is polynomial in the components of u.

The matrix N takes the form

$$N = \left(\begin{array}{cc} O & O\\ O & M \end{array}\right) \tag{150}$$

where the O's stand for blocks of zeros of various sizes and M is a square matrix.

The eigenvalues  $\lambda$  of M satisfy the following

$$-\lambda K_{i}{}^{j} = -4K_{i}{}^{j} - \delta_{i}{}^{j}(2K_{m}{}^{m} + \zeta)$$
(151)

$$-\lambda\zeta = 6K_m^m - \zeta \tag{152}$$

Taking the trace of (151) gives

$$(-\lambda + 10)K_m^{\ m} = -3\zeta\tag{153}$$

and then from (152) there follows

$$(-\lambda + 10)(-\lambda + 1) + 18 = 0, \text{ or } \zeta = 0$$
 (154)

Thus all the eigenvalues of M have strictly positive real part. Now if all the eigenvalues of N had strictly positive real parts, then Theorem 1 of (Rendall and Schmidt 1991) would imply the existence of a unique smooth solution u to equations (149) with the given data. However, in the case where N takes the form (150), with M having eigenvalues with strictly +ve real part, a very elementary modification of their proof gives the same result. It follows that equations (149) have a unique smooth solution u.

It remains to show that if (149) is satisfied then  $h_{ij} = h_{ji}$ ,  $h^{ij}h_{jk} = \delta^i_k$ ,  $K_{ij} \equiv h_{jk}K_i^{\ k} = \frac{1}{2}\partial_z h_{ij}$ , and  $\zeta = 6Z^{-1}(1-G)$ . First note that (144), (145) imply that  $\partial_z h_{[ij]} = \partial_z h^{[ij]} = 0$ , so that  $h_{ij}$ ,  $h^{ij}$ 

stay symmetric if they start that way. Also (144)-(145) imply

$$\partial_z (h^{ij} h_{jk} - \delta^i_{\ k}) = (h^{ij} h_{jm} - \delta^i_{\ m}) K_k^{\ m} - (h^{jm} h_{mi} - \delta^m_{\ k}) K_m^{\ i}$$
(155)

Hence by Gronwall's inequality  $h^{ij}h_{jk} = \delta^i{}_k$  if  $h_{ij}(0)$ ,  $h^{ij}(0)$  are chosen as matrix inverses.

By (144), (146) one has

$$\partial_z K_{[ij]} = -(K_m^m + 4Z^{-1})K_{[ij]} \tag{156}$$

so that by Theorem 1 of (Rendall and Schmidt 1991)  $K_{[ij]} \equiv 0$  and now (144) implies that  $K_{ij} = \frac{1}{2} \partial_z h_{ij}$ .

Finally, by (147)-(148) there follows

$$\partial_z(\bar{\zeta}-\zeta) = -(Z^{-1}+K)(\bar{\zeta}-\zeta) \tag{157}$$

so that  $\zeta = \overline{\zeta}$ .

We conclude that for each Bianchi type 3-metric  $h_{ij}$  on G, there exists a unique Bianchi type  $\gamma = 1$  cosmology having an isotropic singularity, with  $h_{ij}$  as initial metric in the chosen gauge. In particular, the Vanishing Weyl Tensor conjecture goes through as for  $1 < \gamma \leq 2$ .

## 7 Concluding remarks

The idea of constructing cosmological singularities by means of an infinite conformal transformation provides a mathematical framework for studying the Weyl Tensor Hypothesis of Penrose. In particular one may seek to generate cosmologies with isotropic singularities via a Cauchy problem with data given at the singularity surface. By allowing the conformal factor to be non-smooth at the singularity we have been able to solve this problem for perfect fluid models with equations of state other than that for radiation. For these other equations of state there are as many cosmologies as for  $\gamma = 4/3$  and they are determined by the intrinsic geometry of the singularity surface, with no free data for the matter.

If one wishes to impose the condition  $C^a_{bcd} = 0$  initially then the uniqueness results obtained show that the spacetime geometry must be exactly FRW in a neighbourhood of the big-bang singularity.

One may now ask whether the picture painted above remains the same if one considers other matter models. What, for example, happens to the conjecture regarding vanishing of the Weyl tensor if we are able to give some matter data at the singularity? Indeed, are there matter models for which it is possible to give data at the singularity? This question will be addressed in a second paper in which we shall consider the Cauchy problem for the conformal Einstein-Vlasov equations.

# A Explicit matrix representations for the $1 < \gamma < 2$ field equations

#### A.1 Newman's formalism

Here we present the formalism of Newman (1993b) which allows tensorial evolution systems to be written in matrix form.

Let V be an n-dimensional vector space with dual  $V^*$ . Let V be equipped with a basis, and  $V^*$  with a dual basis. Let h and  $h^*$  be covariant and contravariant tensors over V of rank 2. For any tensor T of total rank r over V, with components  $T^{i_1...}_{...i_r}$  define the  $n^r$  component row and column vectors

$$(\mathbf{T})_{\cdot} = (T^{1\dots}_{\dots 11}, T^{1\dots}_{\dots 12}, \dots, T^{1\dots}_{\dots 1n}, T^{1\dots}_{\dots 21}, \dots, T^{n\dots}_{\dots nn})$$
$$(T)^{\cdot} = \begin{pmatrix} T^{1\dots}_{\dots 11} \\ T^{1\dots}_{\dots 12} \\ \vdots \\ T^{1\dots}_{\dots 21} \\ \vdots \\ T^{n\dots}_{\dots nn} \end{pmatrix}$$

A general matrix will be represented in the form  $M^{\cdot}$ , with  $M^{i}$  the  $i^{\text{th}}$  row and  $M^{\cdot}_{j}$  the  $j^{\text{th}}$  column. For any  $n^{k} \times n^{k}$  square matrix  $M^{\cdot}$ , and each i = $1, 2, \ldots, n \text{ let } {}^{(i)}M^{\cdot}$  denote the matrices formed by the  $1^{\text{st}}, 2^{\text{nd}}, \ldots, n^{\text{th}}$  groups of  $n^{k-1}$  rows of  $M^{\cdot}$  respectively, and let  ${}^{(i)}M^{\cdot}$  be the matrices formed from the  $1^{\text{st}}, 2^{\text{nd}}, \ldots, n^{\text{th}}$  groups of  $n^{k-1}$  columns of  $M^{\cdot}$ . Recall that the tensor product of a  $p \times q$  matrix  $M^{\cdot}$  and an  $r \times s$  matrix  $N^{\cdot}$  may be represented as a  $pr \times qs$  matrix:

$$M^{\cdot} \otimes N^{\cdot} = \begin{pmatrix} M^{1}_{1}N^{\cdot} & M^{1}_{2}N^{\cdot} & \cdots & M^{1}_{q}N^{\cdot} \\ M^{2}_{1}N^{\cdot} & M^{2}_{2}N^{\cdot} & \cdots & M^{2}_{q}N^{\cdot} \\ \vdots & \vdots & \ddots & \vdots \\ M^{p}_{1}N^{\cdot} & M^{p}_{2}N^{\cdot} & \cdots & M^{p}_{q}N^{\cdot} \end{pmatrix}$$

Clearly this representation respects the associativity of tensor products. For each  $k = 1, 2, 3, \ldots$  one may therefore define the  $n^k \times n^k$  matrix

$$(\overset{k}{h})^{\cdot} = \underbrace{(h)^{\cdot} \otimes \ldots \otimes (h)^{\cdot}}_{k \text{ factors}}$$

where

$$(h)^{\cdot} = \begin{pmatrix} h_{11} & h_{12} & \cdots & h_{1n} \\ h_{21} & h_{22} & \cdots & h_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ h_{n1} & h_{n2} & \cdots & h_{nn} \end{pmatrix}$$

If h is a symmetric tensor then  $\binom{k}{h}$  is a symmetric matrix. Inductively one can show that if h and  $h^*$  are tensor inverses in the sense of

$$h^{*ij}h_{jk} = \delta^i_k$$

then  $\binom{k}{h}$  and  $\binom{k}{h^*}$  are matrix inverses:

$$\sum_{\sigma=1}^{n^k} (\overset{k}{h^*})^{\cdot}{}_{\sigma} (\overset{k}{h})^{\sigma}{}_{\cdot} = I^{n^k}$$

 $(h^*)^{\cdot}$  is defined in terms of  $h^*$  by analogy with the preceding definition of  $(h)^{\cdot}$  in terms of h. Moreover, letting  $h_s$  and  $h_s^*$  denote the symmetric parts of h and  $h^*$  one may, for each  $k = 1, 2, 3, \ldots$  define  $(h_s)^{\cdot}$  and  $(h_s^*)^{\cdot}$ . Suppose h and  $h^*$  are tensor inverses and let T be any contravariant tensor of rank r over V. Then, defining a covariant tensor  $T^{\flat}$  of rank r by

$$T_{i_1\dots i_r}^{\flat} = h_{i_1j_1}\dots h_{i_rj_r}T^{j_1\dots j_r}$$

one has a matrix representation of the raising and lowering operations:

$$(T^{\flat})^{\cdot} = \sum_{\sigma=1}^{n^{r}} {\binom{r}{h}}^{\cdot}{}_{\sigma}(T)^{\sigma}, \qquad (T)^{\cdot} = \sum_{\sigma=1}^{n^{r}} {\binom{r}{h^{\ast}}}^{\cdot}{}_{\sigma}(T^{\flat})^{\sigma}$$

There exists a unique symmetric  $n^2 \times n^2$  matrix  $J^{\cdot}$  such that, for every second rank covariant tensor  $T_{ab}$ , the tensor  $\bar{T}_{ab} \equiv T_{ba}$  satisfies

$$(T)^{\cdot} = J^{\cdot}{}_{\sigma}(T)^{\sigma}, \quad (T) \cdot = (T)_{\sigma}J^{\sigma}.$$

In particular, for n = 3 one has

 $J^{\cdot}_{\cdot}$  also plays an analogous role for second rank contravariant tensors.

For each r = 1, 2, ... there exists a unique  $n^r \times n^r$  matrix  $\stackrel{r}{E}$  such that, for any contravariant or covariant tensor field T of rank r, one has

$$(T_s)^{\cdot} = \sum_{\sigma=1}^{n^r} E^{r}_{\sigma} (T)^{\sigma}$$

where  $T_s$  is the symmetric part of T. In particular, for n = 3 one has  $E^{1} = I^{3}$  and

## A.2 The reduced field equations for $1 < \gamma < 2$

If we define the 63-component column vector

$$u = \begin{pmatrix} (h^{*})^{\cdot} \\ (h)^{\cdot} \\ V \\ \zeta \\ (\chi)^{\cdot} \\ (k)^{\cdot} \\ (\gamma)^{\cdot} \\ (\xi)^{\cdot} \\ \nu \end{pmatrix}$$

then the  $63 \times 63$  matrices  $A^{\alpha}(u)$ , C(u) appearing in the reduced equations (70) can be written  $A^{0}(u) =$ 

$$\begin{pmatrix} I^9 & & & & \\ & I^1 & & & & \\ & & I^1 & & & & \\ & & & -(h_s^*)^{\cdot}. & & & \\ & & & V^2(h_s^2)^{\cdot}. & & \\ & & & & V^2(h_s^2)^{\cdot}. & & \\ & & & & & & & \\ & & & & & & \\ & & & & & & \\ &$$

$$\begin{split} C(u) = & \\ \begin{pmatrix} 0 & & \\ 0 & \dots & \\ & 0 & \\ \vdots & -I^1 & 0 & D_1(h_s). & 0 & 0 & D_2I^1 \\ & 0 & D_3(h_s^*)^{\cdot}. & 0 & & 0 & D_4(h_s^*)^{\cdot}. & 0 \\ & -2V^2(h_s)^{\cdot} & 0 & D_5(-2(h_s^2)^{\cdot}. + (h_s)^{\cdot} \otimes (h_s).) & 0 & 0 & D_6(h_s)^{\cdot} \\ & 0 & 0 & 0 & 0 & 0 & 0 \\ & 0 & D_7(h_s^*)^{\cdot}. & 0 & 0 & D_8(h_s^*)^{\cdot}. & 0 \\ & & D_9I^1 & 0 & D_{10}(h_s). & 0 & 0 & D_{11}I^1 \\ \end{pmatrix} \end{split}$$

where

$$D_{1} = \frac{3(2-\gamma)^{2}}{(3\gamma-2)^{2}(1-5\gamma)} \qquad D_{2} = \frac{30\gamma(\gamma-2)F^{-1}}{(3\gamma-2)^{2}(1-5\gamma)}$$
$$D_{3} = \frac{-4P^{-1}}{(3\gamma-2)} \qquad D_{4} = \frac{2EG^{-1}}{P^{2}(3\gamma-2)}$$
$$D_{5} = \frac{2V^{2}}{(3\gamma-2)} \qquad D_{6} = \frac{-4V^{2}(\gamma-3)F^{-1}}{(3\gamma-2)(\gamma-2)}$$
$$D_{7} = \frac{8GE^{-1}}{(2-3\gamma)} \qquad D_{8} = \frac{4P^{-1}}{(3\gamma-2)}$$
$$D_{9} = \frac{V^{2}F(3\gamma-2)}{(2-\gamma)} \qquad D_{10} = \frac{V^{2}F}{(3\gamma-2)} \qquad D_{11} = \frac{2V^{2}(3\gamma-5)}{(2-\gamma)(3\gamma-2)}$$

## A.3 The recovery equations for $1 < \gamma < 2$

Define the 24-component column vector

$$v = \begin{pmatrix} (\pi)^{\cdot} \\ \tau \\ (\alpha)^{\cdot} \\ (\beta)^{\cdot} \\ (\omega)^{\cdot} \\ \delta \\ \eta \\ \kappa \end{pmatrix}$$

then the matrices  $a^{\alpha}(v)$ , c(v) appearing in the recovery equations (105) are as follows  $a^{0}(v) =$ 

$$\begin{pmatrix} I^{3} & & & & \\ & I^{1} & & & \\ & & (h^{-1})^{\cdot}{}_{\sigma} \left(\frac{1}{2}I^{9} + C_{3}J\right)^{\sigma}{}_{\cdot} & & & O \\ & & & C_{4}(h^{-1})^{\cdot}{}_{\cdot} & & \\ & & & -V^{2}(h^{-1})^{\cdot}{}_{\cdot} & \\ & & & C_{5}I^{1} & \\ & & & -V^{2}(h^{-1})^{\cdot}{}_{\cdot} & \\ & & & V^{4}I^{1} \end{pmatrix}$$

where

$$C_3 = \frac{\gamma - 1}{8\gamma} \qquad C_4 = \frac{V^2(2 - 3\gamma)}{4\gamma}$$
$$C_5 = \frac{V^2}{4} \left(\frac{5\gamma - 1}{3\gamma - 1}\right)$$

while  $a^i(v) =$ 

where

$$Q = \frac{1}{2}I^9 + C_3 J$$
$$C_6 = \frac{(3-\gamma)V^2}{4(3\gamma-1)C_2} \quad C_7 = \frac{V^2}{2} \left(\frac{2\gamma+1}{1-3\gamma}\right) \frac{C_2}{C_1}$$

and c(v) =

where

$$C_8 = \frac{V^2(4\gamma - 3)}{(3\gamma - 2)\gamma} \quad C_9 = \frac{2V^2(4\gamma - 3)}{C_1(3\gamma - 2)(1 - 3\gamma)}$$
$$C_{10} = \frac{3V^2(2 - \gamma)(\gamma - 1)}{(3\gamma - 2)^2} \quad C_{11} = V^4 \left\{ \frac{2(13\gamma - 8) + 6(1 - V^P)(5\gamma^2 - 4\gamma + 3)}{(1 - 3\gamma)(3\gamma - 2)^2} \right\}$$

and  $C_1$ ,  $C_2$  were defined in section 4.4.

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