Universality of random graphs and rainbow embedding

Asaf Ferber   Rajko Nenadov   Ueli Peter [*]

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Abstract

In this paper we show how to use simple partitioning lemmas in order to embed spanning graphs in a typical member of $\mathcal{G}(n,p)$. Let the *maximum density* of a graph H be the maximum average degree of all the subgraphs of H. First, we show that for $p = \omega(\Delta^{12} n^{-1/2d} \log^3 n)$, a graph $G \sim \mathcal{G}(n,p)$ w.h.p. contains copies of all spanning graphs H with maximum degree at most Δ and maximum density at most d. For $d < \Delta/2$, this improves a result of Dellamonica, Kohayakawa, Rödl and Ruciński. Next, we show that if we additionally restrict the spanning graphs to have girth at least 7 then the random graph contains w.h.p. all such graphs for $p = \omega(\Delta^{12} n^{-1/d} \log^3 n)$. In particular, if $p = \omega(\Delta^{12} n^{-1/2} \log^3 n)$, the random graph therefore contains w.h.p. every spanning tree with maximum degree bounded by Δ. This improves a result of Johannsen, Krivelevich and Samotij.

Finally, in the same spirit, we show that for any spanning graph H with constant maximum degree, and for suitable p, if we randomly color the edges of a graph $G \sim \mathcal{G}(n,p)$ with $(1 + o(1))|E(H)|$ colors, then w.h.p. there exists a *rainbow* copy of H in G (that is, a copy of H with all edges colored with distinct colors).

1   Introduction

A graph G is *universal* for a family of graphs $\mathcal{H}$ (we write G is $\mathcal{H}$-universal), if G contains a copy of every graph $H \in \mathcal{H}$. The construction (explicit and/or randomized) of sparse universal graphs for various families has received a considerable amount of attention (see [1, 2, 3, 4, 6, 7, 9, 10, 11, 12, 13, 15, 19, 21]).

In particular, the probability space $\mathcal{G}(n,p)$ of all graphs on n vertices, in which each pair of vertices forms an edge with probability p independently at random, has been considered in many papers. The problem of finding for which values of p a typical member of $\mathcal{G}(n,p)$ is $\mathcal{H}$-universal for various families of graphs is fundamental in the theory of random graphs.

[*]Institute of Theoretical Computer Science ETH, 8092 Zürich, Switzerland. Emails: asaf.ferber@inf.ethz.ch, rnenadov@inf.ethz.ch and upeter@inf.ethz.ch

Let $\mathcal{H}(n, \Delta, d)$ be the family of all graphs on n vertices with maximum degree at most Δ and with *maximum density* at most d, where the maximum density of a graph G (denoted by $d(G)$) is defined as

$$d(G) = \max\left\{\frac{2|E(H)|}{|V(H)|} : H \subseteq G\right\}.$$

Dellamonica, Kohayakawa, Rödl and Ruciński proved in [15] that for maximum degree $\Delta \geq 3$ and an edge probability $p = \omega\left(n^{-1/\Delta} \log^{1/\Delta} n\right)$, a typical member of $\mathcal{G}(n, p)$ is $\mathcal{H}(n, \Delta, \Delta)$-universal. Recently, Kim and Lee [21] obtained similar bounds for $\Delta = 2$. In the following theorem we show that if $d < \Delta/2$, then the bound in [15] can be further improved.

Theorem 1.1 *Let n be a positive integer, and let $\Delta = \Delta(n) > 1$ and $d = d(n) \geq 2$ be integers. Then for $p = \omega(\Delta^{12} n^{-1/\min\{2d, \Delta\}} \log^3 n)$, a graph $G \sim \mathcal{G}(n, p)$ is w.h.p. $\mathcal{H}(n, \Delta, d)$-universal.*

To prove this theorem, it will be sufficient to prove that it holds for $p = \omega(\Delta^{12} n^{-1/(2d)} \log^3 n)$ as it follows from [15] for the other minimum.

Next, let $\mathcal{H}(n, \Delta, d, g) \subseteq \mathcal{H}(n, \Delta, d)$ denote the family of graphs which additionally have girth at least g (the girth of a graph is the length of its shortest cycle). In our second main result we further restrict ourselves to graphs with girth at least 7, where we obtain better bounds for p.

Theorem 1.2 *Let n be a positive integer, and let $d = d(n)$ and $\Delta = \Delta(n) > 1$ be integers. Then for $p = \omega(\Delta^{12} n^{-1/d} \log^3 n)$, a graph $G \sim \mathcal{G}(n, p)$ is w.h.p. $\mathcal{H}(n, \Delta, d, 7)$-universal.*

Another example of a family of graphs which has attracted the attention of various researchers is the family of *bounded degree* trees. Let $\mathcal{T}(n, \Delta)$ be the family of all forests on n vertices with maximum degree bounded by Δ. Alon, Krivelevich and Sudakov showed in [6] that for fixed $\Delta > 0$ and $0 < \varepsilon < 1$, there exists a constant $c = c(\Delta, \varepsilon)$ such that a typical member of $\mathcal{G}(n, c/n)$ is $\mathcal{T}((1 - \varepsilon)n, \Delta)$-universal. The constant c in this result was further improved in [8]. Later on, Balogh, Csaba and Samotij showed in [9] that $\mathcal{G}(n, c/n)$ is w.h.p. (with high probability) $\mathcal{T}((1-\varepsilon)n, \Delta)$-universal even if an adversary is allowed to delete at most (roughly) half of the edges touching any vertex. Note that universality for spanning trees can not be true for $p = c/n$, as at such a low density the random graph is w.h.p. disconnected. As it turns out, results for spanning subgraphs are much harder to obtain. In the case of the family of *spanning trees* $\mathcal{T}(n, \Delta)$, the best bound known for $\mathcal{G}(n, p)$ to be $\mathcal{T}(n, \Delta)$-universal is $p = \omega(\Delta n^{-1/3} \log^2 n)$, due to Johannsen, Krivelevich and Samotij [19]. The following immediate corollary of Theorem 1.2 improves this bound to $p = \omega(\Delta^{12} n^{-1/2} \log^3 n)$.

Corollary 1.3 *Let n be a positive integer, and let $\Delta = \Delta(n) > 1$ be an integer. Then for $p = \omega(\Delta^{12} n^{-1/2} \log^3 n)$, a graph $G \sim \mathcal{G}(n, p)$ is w.h.p. $\mathcal{T}(n, \Delta)$-universal.*

The proofs of Theorems 1.1 and 1.2 use simple partitioning lemmas for graphs and an embedding technique based on matchings, developed by Alon and Füredi in [5] and by Ruciński in

[22]. Using similar technique, we also managed to obtain a general embedding result in a model of random graphs where each edge is being colored uniformly at random in one color from a given set of colors. This leads us to the second part of our paper.

Let $G \sim \mathcal{G}(n, p)$ and assume that each edge of G is colored uniformly at random with one of the colors from the set $[c] := \{1, \ldots, c\}$. This model is referred to as $\mathcal{G}_c(n, p)$. For a given graph H we say that a typical member of $G \sim \mathcal{G}_c(n, p)$ contains a *rainbow copy* of H, if G contains as a subgraph a copy of H with all the edges colored in distinct colors. In [17], Frieze and Loh showed that for $p \geq (1 + \varepsilon) \log n / n$ and $c = n + o(n)$, a typical member of $\mathcal{G}_c(n, p)$ contains a rainbow Hamilton cycle. Note that their result is asymptotically optimal in both p and the number of colors c. In the following theorem we provide bounds on the edge probability p (do not believed to be optimal), for which given any graph H on n vertices with $\Delta(H) = O(1)$, one can find a rainbow copy of H in a typical member of $\mathcal{G}_c(n, p)$, provided $c = (1 + o(1))|E(H)|$ (c is asymptotically optimal).

Theorem 1.4 *Let $\alpha > 0$, let Δ and d be integers, let n be a sufficiently large integer and let $H \in \mathcal{H}(n, \Delta, d)$. Then $G \sim \mathcal{G}_c(n, p)$ w.h.p. contains a rainbow copy of H, provided that $p \geq n^{-1/d} \log^{5/d} n$ and $c = (1 + \alpha)|E(H)|$.*

We remark that all of our proofs might be easily improved in terms of $\log n$ and Δ factors. Since we believe that our bounds are far from being optimal, we did no effort in optimizing those factors.

Notation. Our graph-theoretic notation is standard and follows that of [23]. For a graph G, let $V = V(G)$ and $E = E(G)$ denote its sets of vertices and edges, respectively. For subsets $U, W \subseteq V$, and for a vertex $v \in V$, we denote by $E_G(U)$ all the edges of G with both endpoints in U, by $E_G(U, W)$ all the edges of G with one endpoint in U and one endpoint in W and by $E_G(v, U)$ all the edges with one endpoint being v and one endpoint in U. We write $N_G(v)$ for the neighborhood of v in G and $\deg_G(v)$ for its degree. Moreover, we write $N_G(U)$ for the neighborhood of a set $U \subseteq V$. For any positive integer k and every vertex V we denote the following set as *k-neighborhood* of v:

$$\{v \in V \mid \text{the distance between } u \text{ and } v \text{ is at most } k\}$$

We say that a set $S \subseteq V$ is *k-independent* if and only if (in G) the distance between any two vertices of S is at least $k + 1$.

Given a graph G and a positive constant $d > 0$ we denote by $D_d(G)$ the set of all vertices of G with degree exactly d, by $D_{\leq d}(G)$ the set of all vertices of degree at most d and in a similar way we define $D_{<d}(G)$, $D_{>d}(G)$ and $D_{\geq d}(G)$. When it is clear to which graph G we refer, we just denote it by D_d, $D_{\leq d}$ etc.

Given two graphs H and G, a bijection f from $V(H)$ to $V(G)$ is called an *embedding* of H to G if it maps each edge of H to an edge of G. In case that one assigns colors to the edges of G, an embedding f of H to G is called a *rainbow embedding* if in addition it maps the edges of H into edges with distinct colors in G.

Throughout the paper, wherever we use $\log n$ we refer to the natural logarithm.

2 Preliminaries

2.1 Probabilistic Tools

We will need to employ bounds on large deviations of random variables. We will mostly use the following well-known bound on the lower and the upper tails of the binomial distribution due to Chernoff (see [18]).

Lemma 2.1 *If $X \sim \text{Bin}(n, p)$, then*

- $\Pr\left(X < (1 - a)np\right) < e^{-a^2 np/2}$ *for every $a > 0$;*

- $\Pr\left(X > (1 + a)np\right) < e^{-a^2 np/3}$ *for every $0 < a < 3/2$.*

The proof of the following slightly more general bounds follows directly from the Chernoff bound and is left as an exercise for the reader (see for example Problem 1.7 in [16]).

Lemma 2.2 *Let $p, q \in [0, 1]$ and let $X_1, \ldots, X_n \in \{0, 1\}$ be n indicator variables and $X := \sum_{i=1}^{n} X_i$. If for each $1 \leq i \leq n$*

$$\mathbb{E}[X_i | X_1, \ldots, X_{i-1}] \geq p \quad \text{and} \quad \mathbb{E}[X_i | X_1, \ldots, X_{i-1}] \leq q,$$

then it holds for every $0 < \alpha < 1$ that

$$\Pr[X \geq (1 + \alpha)nq] \leq e^{-\alpha^2 nq/3} \quad \text{and} \quad \Pr[X \leq (1 - \alpha)np] \leq e^{-\alpha^2 np/2}.$$

2.2 Graph-Theoretic Facts

In this section we mention a few facts about graphs which are used extensively throughout the paper.

The first two lemmas consider the existence of k-independent sets in a graph.

Lemma 2.3 *Let G be a graph on n vertices with maximum degree $\Delta \geq 2$ and let $S \subseteq V(G)$ be such that the maximum degree of all vertices in S is at most d (where $d \geq 1$). Then, S contains a set $U \subseteq S$ of size at least $\frac{|S|}{d\Delta^k}$ which is k-independent in G.*

Proof Build U greedily as follows: start with $L := S$ and $U := \emptyset$. In each step add an arbitrary vertex $v \in L$ to U and delete the k-neighborhood of v (including v itself) from L. Since after each addition of a vertex to U we delete at most

4

$$1 + d + d(\Delta - 1) + \ldots + d(\Delta - 1)^{k-1} \le d\Delta^k$$

vertices from L, we obtain the required. □

Lemma 2.4 *Let G be a graph on n vertices with maximum degree $\Delta \ge 2$ and let d be an integer such that $dn \ge 2|E(G)|$. Then, for any integer k, G contains a k-independent set $U \subseteq D_{\le d}(G)$ of size $|U| \ge \frac{n}{(d+1)d\Delta^k}$.*

Proof First, we claim that $|D_{\le d}(G)| \ge \frac{n}{d+1}$. Indeed, let G be a graph which satisfies the conditions of the lemma for some Δ. Using the fact that $|D_{>d}| = n - |D_{\le d}|$, we obtain that

$$dn \ge \sum_{v \in V(G)} \deg_G(v) \ge 0 \cdot |D_{\le d}| + (d+1) \cdot (n - |D_{\le d}|).$$

Therefore, we conclude that $|D_{\le d}| \ge \frac{n}{d+1}$.

Applying Lemma 2.3 we conclude that there exists a k-independent set $U \subseteq D_{\le d}$ in G of size at least

$$|U| \ge \frac{|D_{\le d}|}{d\Delta^k} \ge \frac{n}{(d+1)d\Delta^k},$$

as required. □

A graph G is called *d-degenerate* if every subgraph $G' \subseteq G$ contains a vertex of induced degree at most d. A moment's thought reveals that every graph $H \in \mathcal{H}(n, \Delta, d)$ is d-degenerate (but not vice versa). The following observation follows directly from the definition of d-degenerate graphs.

Observation 2.5 *Let n, Δ and d be positive integers and let H be a d-degenerate graph on n vertices. Then there exists an ordering $(v_1, \ldots, v_n)$ of the vertices of H such that*

$$|N(v_i) \cap \{v_1, \ldots, v_{i-1}\}| \le d$$

for every $2 \le i \le n$.

3 Partitioning Lemmas

In this section we prove some lemmas about partitioning graphs from $\mathcal{H}(n, \Delta, d)$ and $\mathcal{T}(n, \Delta)$. Before that, we define a class of graphs which can be partitioned in a "nice" way, and then we show that $\mathcal{H}(n, \Delta, d)$ and $\mathcal{T}(n, \Delta)$ belong to this class for suitably chosen parameters.

Definition 3.1 *Let n, d and t be positive integers and let ε be a positive number. The family of graphs $\mathcal{F}(n, t, \varepsilon, d)$ consists of all graphs H on n vertices for which the following holds. There exists a partition $V(H) = W_0 \cup \ldots \cup W_t$ such that:*

(i) $|W_t| = \lfloor \varepsilon n \rfloor$,

(ii) $W_0 = N(W_t)$,

(iii) W_t *is 3-independent,*

(iv) W_i *is 2-independent for every* $1 \leq i \leq t - 1$, *and*

(v) *for every* $1 \leq i \leq t$ *and for every* $w \in W_i$, w *has at most* d *neighbors in* $W_0 \cup \ldots \cup W_{i-1}$.

Now, we show that $\mathcal{H}(n, \Delta, d) \subseteq \mathcal{F}(n, 4\Delta^6 \log n + 1, \varepsilon, 2d)$.

Lemma 3.2 *Let* n *be a positive integer, let* $\Delta = \Delta(n) \geq 2$ *and* $d = d(n) \geq 2$ *be integers and let* $\varepsilon_0 = 1/(4\Delta^6)$. *Then for every* $\varepsilon \leq \varepsilon_0$ *we have*

$$\mathcal{H}(n, \Delta, d) \subseteq \mathcal{F}(n, 4\Delta^6 \log n + 1, \varepsilon, 2d).$$

Proof Let $H \in \mathcal{H}(n, \Delta, d)$ and $t = 4\Delta^6 \log n + 1$. We show that $H \in \mathcal{F}(n, t, \varepsilon, 2d)$, for every $\varepsilon \leq \varepsilon_0$.

Using Lemma 2.4, one can find a 4-independent set $U \subseteq D_{\leq d}(H)$ of size

$$|U| \geq \frac{n}{(d+1)d\Delta^4} \geq \varepsilon_0 n.$$

Let $W_t \subseteq U$ be an arbitrary subset of size $\lfloor \varepsilon n \rfloor$, and set $W_0 = N_H(W_t)$ and $H_{t-1} := H \backslash (W_0 \cup W_t)$. We further partition H_i, for $i = t - 1, \ldots, 1$, as follows:

- If $V(H_i) = \emptyset$ then set $W_i := \emptyset$ and $V(H_{i-1}) := \emptyset$.

- Otherwise, $H_i \in \mathcal{H}(|H_i|, \Delta, d)$ and thus by Lemma 2.4 there exists a 2-independent set $U \subseteq D_{\leq d}(H_i)$ of size $|U| \geq \frac{|H_i|}{(d+1)d \cdot \Delta^2} \geq \frac{|H_i|}{2\Delta^4} \geq \varepsilon_0 |H_i|$. Set $W_i := U$ and $H_{i-1} := H_i \setminus W_i$.

Using the fact that $\log(1 - x) \leq -x$ for every $0 < x < 1$, we have that

$$t = 4\Delta^6 \log n + 1 = \log n / \varepsilon_0 + 1 \geq -\log n / \log(1 - \varepsilon_0) + 1 = -\log_{1 - \varepsilon_0} n + 1.$$

Since for each i we have that $|V(H_i)| \leq (1 - \varepsilon_0)|V(H_{i+1})|$, and since $t \geq -\log_{1 - \varepsilon_0} n + 1$, it follows that $|V(H_1)| \leq 1$.

Now, let $V(H) = W_0 \cup \ldots \cup W_t$ be the obtained partition and note that each vertex $w \in W_i$ has at most d neighbors in $W_1 \cup \ldots \cup W_{i-1}$ for $2 \leq i < t$ (it follows immediately from the construction). Since all the properties $(i) - (iv)$ of Definition 3.1 follow easily from the construction, it thus remains to show that Property (v) holds. That is, we need to show that every vertex in $w \in W_1 \cup \ldots \cup W_{t-1}$ has at most d neighbors in W_0, and then we conclude that every vertex in $W_1 \cup \ldots \cup W_t$ sends at most $2d$ "back-edges". For this aim, note first that every vertex in

W_t has at most d neighbors in W_0, and that $W_0 = N_H(W_t)$. Therefore, if there exists a vertex $w \in W_1 \cup \ldots \cup W_{t-1}$ with at least $d+1$ neighbors in W_0, then there must exist at least two vertices $x, y \in W_t$ such that $N_H(x) \cap N_H(w) \neq \emptyset$ and $N_H(y) \cap N_H(w) \neq \emptyset$. Therefore, one can find a path of length four between x and y, which clearly contradicts the assumption that W_t is 4-independent. This completes the proof. □

Next, we show that $\mathcal{H}(n, \Delta, d, 7) \subseteq \mathcal{F}(n, 16d^2\Delta^2 \log n + 1, \varepsilon, d)$.

Lemma 3.3 *Let n be a positive integer, let $\Delta = \Delta(n)$ and $d = d(n) \geq 2$ be integers, and let $\varepsilon_0 = 1/(2d^2\Delta^6)$. Then for every $\varepsilon \leq \varepsilon_0$ we have*

$$\mathcal{H}(n, \Delta, d, 7) \subseteq \mathcal{F}(n, 16d^2\Delta^2 \log n + 1, \varepsilon, d).$$

Proof Let $\gamma = \frac{1}{8(d+1)(d-1)\Delta^2} \leq \frac{1}{d^2}$ and observe that for $d \geq 2$

$$\frac{1 - (d+1)(d-1)\Delta^2\gamma}{(d+1)(d-1)\Delta^2} > \gamma \quad \text{and} \quad \frac{d^2(1 - (d+1)(d-1)\Delta^2\gamma)}{d+1} > 1. \tag{1}$$

Let $H \in \mathcal{H}(n, \Delta, d, 7)$ and $t = 16d^2\Delta^2 \log n + 1$. We show that $H \in \mathcal{F}(n, t, \varepsilon, d)$, for every $\varepsilon \leq \varepsilon_0$.

Using Lemma 2.4, we find a 6-independent set $U \subseteq D_{\leq d}(H)$ of size

$$|U| \geq \frac{n}{d(d+1)\Delta^6} \geq \varepsilon_0 n.$$

For a fixed $\varepsilon \leq \varepsilon_0$, let $W_t \subseteq U$ be an arbitrary subset of size $\lfloor \varepsilon n \rfloor$, and set $W_0 = N_H(W_t)$, $X = N_H(W_0)$, and $H_{t-1} := H \setminus (W_0 \cup W_t)$.

For $i = t-1, \ldots, 1$, we iteratively find subsets of vertices $W_i \subseteq V(H_i)$ (and set $H_{i-1} := H_i \setminus W_i$), in such a way that at the end of the process the obtained partition $V(H) = W_0 \cup \ldots \cup W_t$ satisfies Properties $(i) - (v)$ of Definition 3.1.

If $V(H_i) = \emptyset$ then set $W_i := \emptyset$ and $V(H_{i-1}) := \emptyset$. Otherwise, construct W_i as follows:

(1) If there exists a 2-independent set $U \subseteq D_{\leq d-1}(H_i)$ of size $U \geq \gamma|V(H_i)|$, then set $W_i := U$.

(2) Otherwise, pick a 2-independent set $W_i \subseteq D_{\leq d}(H_i) \setminus X$ of size $|W_i| \geq \gamma|V(H_i)|$.

Observe that a vertex can have at most one neighbor in W_0. Otherwise, we would either have that W_t is not 6-independent or that there exists a cycle of length 4 in H, both yielding a contradiction. Therefore, by the definition of W_i, we ensure that Property (v) of Definition 3.1 is satisfied. We now claim that whenever (1) fails, there exists a 2-independent set $U \subseteq D_{\leq d}(H_i)$ of size $|U| \geq \gamma n_i$ (where $n_i = |V(H_i)|$) such that $U \cap X = \emptyset$ as required in (2). We remark that we always consider the graph H_i when we write D_d, $D_{\leq d}$ or $D_{\geq d}$ in the following calculations.

To prove our claim, suppose that there is no 2-independent set $U \subseteq D_{\leq d-1}(H_i)$ of size at least γn_i. First, note that by Lemma 2.3 we have

$$|D_{\leq d-1}| \leq (d-1)\Delta^2 \gamma n_i.$$

Second, since $H_i \in \mathcal{H}(n_i, \Delta, d, 7)$, it follows that

$$dn_i \geq \sum_{v \in V(H_i)} \deg_{H_i}(v) \geq 0 \cdot |D_{\leq d-1}| + (|D_{\leq d}| - |D_{\leq d-1}|) \cdot d + (n_i - |D_{\leq d}|) \cdot (d+1),$$

and therefore

$$|D_{\leq d}| \geq n_i - |D_{\leq d-1}| \cdot d.$$

Using the bound on $|D_{\leq d-1}|$, we get that

$$|D_d| = |D_{\leq d}| - |D_{\leq d-1}| \geq n_i - (d+1)|D_{\leq d-1}| \geq n_i \cdot (1 - (d+1)(d-1)\Delta^2 \gamma). \tag{2}$$

Next, note that if $|X \cap D_d| \leq d|D_d|/(d+1)$, then by Lemma 2.3 there exists a 2-independent set $W_i \subseteq D_d \setminus X$ of size at least

$$|W_i| \geq \frac{|D_d \setminus X|}{d\Delta^2} \overset{(2)}{\geq} \frac{1 - (d+1)(d-1)\Delta^2 \gamma}{d(d+1)\Delta^2} n_i \overset{(1)}{\geq} \gamma n_i,$$

as required. Therefore, assume that $|X \cap D_d| > d|D_d|/(d+1)$. Observe that X is a 2-independent set in H_i, as every vertex in X is a neighbor of a vertex in $W_0 = N_H(W_t)$, W_t is 6-independent and there are no cycles of length at most 6 in H. It thus follows that $N_{H_i}(X) \cap X = \emptyset$ and every vertex in $N_{H_i}(X)$ has exactly one neighbor in X. Therefore,

$$|N_{H_i}(X)| \geq d|X \cap D_d| > d^2|D_d|/(d+1),$$

and it follows from (2) that

$$|N_{H_i}(X)| > \frac{d^2}{d+1} \cdot (1 - (d+1)(d-1)\Delta^2 \gamma)n_i \overset{(1)}{>} n_i,$$

which is not possible. Hence, one can always find a 2-independent set $W_i \subseteq V(H_i)$ of size at least γn_i as required.

Using the fact that $\log(1-x) \leq -x$ for every $0 < x < 1$, we have that

$$t + 1 = 16d^2\Delta^2 \log n + 1 \geq \log n/\gamma + 1 \geq -\log n/\log(1-\gamma) + 1 = -\log_{1-\gamma} n + 1.$$

Since for each i we have that $|V(H_i)| \leq (1-\gamma)|V(H_{i+1})|$, and since $t \geq -\log_{1-\gamma} n+1$, it follows that $|V(H_1)| \leq 1$. Finally, let $V(H) = W_0 \cup \ldots \cup W_t$ be the obtained partition. It follows immediately from the construction that Properties $(i) - (v)$ of Definition 3.1 hold. This completes the proof. $\qquad \square$

4 Proof of Theorem 1.1 and Theorem 1.2

In this section we prove Theorem 1.1 and Theorem 1.2. These theorems follow easily from the following theorem and Lemma 3.3 and 3.2.

Theorem 4.1 *Let n and t be positive integers, let $d = d(n) \geq 2$ be an integer, and let $\varepsilon < \frac{1}{2d}$. Then, a graph $G \sim \mathcal{G}(n, p)$ is w.h.p. $\mathcal{F}(n, t, \varepsilon, d)$-universal, provided that $p = \omega\left(\varepsilon^{-1} t n^{-1/d} \log^2 n\right)$.*

In order to prove Theorem 4.1, we use a similar embedding algorithm as the one presented in [21] (and previously in [15]). Let d be a positive integer and ε be a positive constant. Our goal is to show that, whenever a graph G is "good" with respect to some properties, then G is $\mathcal{F}(n, t, \varepsilon, d)$-universal.

Before we state formally what a "good" graph is, we define the following auxiliary bipartite graph. For a graph G, an integer k, a subset $U \subseteq V(G)$ and a collection $\mathcal{L}$ of pairwise disjoint k-subsets of $V(G) \setminus U$, define the bipartite graph $\mathcal{B}(\mathcal{L}, U)$ as follows: the parts are $\mathcal{L}$ and U, and two elements $L \in \mathcal{L}$ and $u \in U$ are adjacent if and only if $L \subseteq N_G(u)$. Now we can define the notion of an (n, t, ε, d)-*good* graph G.

Definition 4.2 *A graph G on n vertices is called (n, t, ε, d)-good if there exists a partition $V(G) = V_0 \cup V_1 \cup \cdots \cup V_t$ with*

$$|V_i| = \frac{\varepsilon n}{16t} \quad \text{for } 1 \leq i \leq t \quad \text{and} \quad |V_0| = (1 - \frac{\varepsilon}{16})n,$$

such that for $p \geq \varepsilon^{-1} t n^{-1/d} \log^2 n$ the following properties hold.

(P1) There exists a set $\mathcal{K} \subset V_0$ of εn vertex-disjoint d-cliques such that for all $U \subset V(G) \setminus V(\mathcal{K})$ with $|U| \leq (p/2)^{-d}/2$, we have

$$\left|\{K_d \in \mathcal{K} \mid V(K_d) \subset N_G(u) \text{ for some } u \in U\}\right| \geq \frac{1}{2^{d+2}} p^d |U| \varepsilon n.$$

(P2) Let $1 \leq k \leq d$, and $\mathcal{L}$ be a collection of pairwise disjoint k-subsets of $V(G)$. If $|\mathcal{L}| \leq (p/2)^{-k}/2$, then for each $i = 1, \ldots, t$ with $V_i \cap (\cup_{L \in \mathcal{L}} L) = \emptyset$, we have that

$$|N_{\mathcal{B}(\mathcal{L}, V_i)}(\mathcal{L})| \geq (p/2)^k |\mathcal{L}| |V_i|/2. \tag{3}$$

If $|\mathcal{L}| \geq (p/2)^{-k} \log^{2(d-1)} n$, then for all U with $|U| \geq (p/2)^{-k} \log^{2(d-1)} n$ and $U \cap (\cup_{L \in \mathcal{L}} L) = \emptyset$, the graph $\mathcal{B}(\mathcal{L}, U)$ has at least one edge.

We first show that a random graph is typically good.

Lemma 4.3 *Let $\varepsilon < \frac{1}{2d}$ and let n be a positive integer. Then, a graph $G \sim \mathcal{G}(n, p)$ is w.h.p. (n, t, ε, d)-good, provided that $p = \omega\left(\varepsilon^{-1} t n^{-1/d} \log^2 n\right)$.*

9

Proof Let $\varepsilon \leq \frac{1}{2d}$, let $p = \omega\left(\varepsilon^{-1}tn^{-1/d}\log^2 n\right)$ and let $G \sim \mathcal{G}(n,p)$. Furthermore, let $q \geq p/2$ be such that $1 - p = (1 - q)^2$, and note that one can expose $G \sim \mathcal{G}(n,p)$ as $G = G_1 \cup G_2$, where G_1 and G_2 are two graphs sampled from $\mathcal{G}(n,q)$ independently (for more details we refer the reader to [18]). We use G_1 to find a family of vertex-disjoint d-cliques, and then G_2 to ensure the properties (P1) and (P2). For a simpler presentation, we assume from now on that q is exactly $p/2$.

First, expose the edges of G_1. Since

$$q = \omega\left(n^{-2/d}(\log n)^{1/\binom{d}{2}}\right),$$

it follows from [20] that G_1 contains w.h.p. $\lfloor n/d \rfloor$ disjoint d-cliques. Let $\mathcal{K}$ be a family of εn vertex-disjoint d-cliques. Next, fix an arbitrary partition $V(G) = V_0 \cup \ldots \cup V_t$ as in Definition 4.2, such that $V(\mathcal{K}) \subset V_0$. Finally, expose G_2. We now show that w.h.p. this partition satisfies Properties (P1) and (P2).

For $U \subseteq V(G) \setminus V(\mathcal{K})$ with $|U| \leq (p/2)^{-d}/2$, let

$$X(U) := \left| \left\{ K_d \in \mathcal{K} \,\middle|\, K_d \subset N_{G_2}(u) \text{ for some } u \in U \right\} \right|.$$

Note that $X(U)$ is the sum of i.i.d. indicator random variables X_L ($L \in \mathcal{K}$), such that $X_L = 1$ iff $L \subseteq N_{G_2}(u)$ for some $u \in U$. Since $|U| \leq (p/2)^{-d}/2$, we have that for each $L \in \mathcal{K}$,

$$\Pr[X_L = 0] = (1 - (p/2)^d)^{|U|} \leq 1 - \frac{|U|p^d}{2^d} + \frac{|U|^2 p^{2d}}{2^{2d}} \leq 1 - \frac{|U|p^d}{2^d}(1 - 1/2) = 1 - \frac{|U|p^d}{2 \cdot 2^d}.$$

(For the first inequality we use the fact that $(1 - a)^b \leq 1 - ab + (ab)^2$ for any positive integer b and $0 < a < 1$).

Therefore, we have that $\Pr[X_L = 1] \geq 2^{-d-1}|U|p^d$, which implies that

$$\mathbb{E}[X(U)] \geq 2^{-d-1}|U|p^d|\mathcal{K}| \geq \frac{\log^{2d} n}{2^{d+1}}|U|.$$

Using Chernoff's bound we obtain that

$$\Pr\left[X(U) < \frac{p^d}{2^{d+2}}|\mathcal{K}||U|\right] \leq 2e^{-\frac{\log^{2d} n}{8 \cdot 2^{d+1}}|U|} \leq \frac{2}{n^{3|U|}}.$$

(The last inequality holds since $d > 1$).

We can therefore upper bound the probability that there exists a set U that violates $(P1)$ by the following union bound

$$\sum_{\ell=1}^{n} \binom{n}{\ell} \frac{2}{n^{3\ell}} = o(1).$$

For property $(P2)$ we first assume that $|\mathcal{L}| \leq (p/2)^{-k}/2$. Note that $X(\mathcal{L}, V_i) := |N_{\mathcal{B}(\mathcal{L}, V_i)}(\mathcal{L})|$ is the sum of i.i.d. indicator random variables X_v (for $v \in V_i$), where $X_v = 1$ iff $L \subset N_{G_2}(v)$

10

for some $L \in \mathcal{L}$. Since $(p/2)^k |\mathcal{L}| \le 1/2$, using the fact that $(1-a)^b \le 1 - ab + (ab)^2/2$ holds for every integer b and any positive constant a for which $ab < 1$ (follows from the binomial formula), we observe that

$$\mathbb{E}[X(\mathcal{L}, V_i)] \ge |V_i| \left(1 - (1 - (p/2)^k)^{|\mathcal{L}|} \right) \ge (1 - 1/4)(p/2)^k |\mathcal{L}| |V_i|.$$

Using Chernoff's bound we obtain that

$$Pr[X(\mathcal{L}, v_i) < (p/2)^k |\mathcal{L}| |V_i|/2] \le \exp\left[-\mathbb{E}[X(\mathcal{L}, v_i)]/36 \right] \le \frac{1}{n^{3d|\mathcal{L}|}},$$

where the last inequality follows as

$$(p/2)^k |V_i| \ge (p/2)^d \cdot \frac{\varepsilon n}{16t} \ge \frac{n \log^{2d} n}{2^{d+4} n} = \omega(d \log n).$$

Thus, the probability for having sets $\mathcal{L}$ and V_i such that $|N_{\mathcal{B}(\mathcal{L}, V_i)}(\mathcal{L})| < (p/2)^k |\mathcal{L}| |V_i|/2$ can be bounded by

$$t \sum_{\ell=1}^{n} \binom{\binom{n}{k}}{\ell} \frac{1}{n^{3d\ell}} = o(1).$$

Next, assume that $|\mathcal{L}| \ge (p/2)^{-k} \log^{2(d-1)} n$. Observe that each edge in $\mathcal{B}(\mathcal{L}, U)$ is present with probability $(p/2)^k$, hence the probability that there are no edges is bounded by

$$(1 - (p/2)^k)^{|\mathcal{L}||U|} \le \exp\left[-(p/2)^k \cdot |\mathcal{L}||U| \right].$$

Furthermore, for $r, \ell \ge (p/2)^{-k} \log^{2(d-1)} n$, the number of collections of k-subsets $\mathcal{L}$ with $|\mathcal{L}| = \ell$ is at most $n^{k\ell}$, and the number of subsets U with $|U| = r$ is at most n^r. We thus have that

$$\Pr[\exists \mathcal{L}, U \text{ with } |\mathcal{L}| = \ell, |U| = r \text{ and } e(\mathcal{B}(\mathcal{L}, U)) = 0] \le \exp\left[(k\ell + r) \log n - (p/2)^k \ell r \right].$$

Note that

$$(k\ell + r) \log n \le k \cdot (\ell \log n + r \log n) \le 2k \cdot \frac{r\ell (p/2)^k}{\log^{2d-3} n} \le (p/2)^k \ell r/2$$

for n large enough, and hence,

$$\exp\left[(k\ell + r) \log n - (p/2)^k \ell r \right] \le \exp\left[-(p/2)^k \ell r/2 \right] \le \exp\left[-(p/2)^{-k} \log n/2 \right] = o(1).$$

We therefore conclude that the probability for the existence of such sets $\mathcal{L}$ and U without an edge is $o(1)$. $\qquad \square$

Now we want to show that any (n, t, ε, d)-good graph is $\mathcal{F}(n, t, \varepsilon, d)$-universal. Let G be a a (n, t, ε, d)-good graph with a partition $V(G) = V_0 \cup \cdots \cup V_t$ and a clique-set $\mathcal{K}$. We construct an embedding $f : V(H) \to V(G)$ for a given graph $H \in \mathcal{F}(n, t, \varepsilon, d)$ as follows.

Let $H = W_0 \cup \cdots \cup W_t$ be the partition of H that satisfies the conditions $(i)-(v)$ of Definition 4.2. For every $v \in W_t$ let $L(v) := N_G(v) \cap W_0$ denote the neighborhood of v in W_0. Note that since W_t is 3-independent, we have that $L(u) \cap L(v) = \emptyset$ for $u \neq v$. In a first step we choose an arbitrary injective mapping $f_0 : W_0 \to V(\mathcal{K})$ such that for every $w \in W_t$ the vertices in $L(w)$ all map to vertices of the same clique in $\mathcal{K}$. Such a mapping exists as $\mathcal{K}$ consists of $\lfloor \varepsilon n \rfloor$ d-cliques and there are exactly that many sets $L(w)$, each of which contains at most d vertices. Moreover, such a mapping is valid as there can not be edges between $L(u)$ and $L(w)$ for $u \neq w$ (because W_t is 3-independent).

For $i = 1, \ldots, t$, we iteratively construct $f_i : (W_0 \cup \cdots \cup W_i) \to (V_0 \cup \cdots \cup V_i)$ from f_{i-1} as follows. Let $V_i^* := (V_0 \cup \cdots \cup V_i) \setminus \mathrm{Img}(f_{i-1})$. We want to embed W_i to V_i^*. For $w \in W_i$ let $L_i(w) := f_{i-1}(N_H(w) \cap (\cup_{j=0}^{i-1} W_j))$ and let $\mathcal{L}_i := \{L_i(w) \mid w \in W_i\}$. Here it is crucial that W_i is 2-independent and therefore $L_i(w) \cap L_i(w') = \emptyset$ for $w \neq w' \in W_i$. Since a vertex $w \in W_i$ can be mapped only to the vertices in

$$\{v \in V_i^* \mid L_i(w) \subseteq N_G(v)\},$$

we can extend f_{i-1} by a $\mathcal{L}_i$ matching in $B_i := \mathcal{B}(\mathcal{L}_i, V_i^*)$ (recall that in B_i the set $L_i(w) \in \mathcal{L}_i$ is connected to a vertex $v \in V_i^*$ if and only if $L_i(w) \subseteq N_G(v)$). More precisely, for a matching $\mathcal{M}$ which saturates $\mathcal{L}_i$ (an $\mathcal{L}_i$-matching), we define f_i as follows: For $w \in W_0 \cup \cdots \cup W_{i-1}$ let $f_i(w) := f_{i-1}(w)$, and for $w \in W_i$ let $f_i(w) := v$, where $v \in V_i^*$ is the unique vertex such that $\{L_i(w), v\} \in \mathcal{M}$.

As long as we find an $\mathcal{L}_i$-matching for $1 \leq i \leq t$ we clearly construct a valid embedding of H into G. It remains to show that we can find the required matchings.

We first show that for every $1 \leq i \leq t-1$, the auxiliary graph B_i contains an $\mathcal{L}_i$-matching.

Claim 4.4 *For every $1 \leq i \leq t-1$, there exists an $\mathcal{L}_i$-matching in B_i.*

Proof We show that Hall's condition for the existence of an $\mathcal{L}_i$-saturating matching is satisfied. First, we show that $|\mathcal{L}_i| = |W_i| < |V_i^*| - \frac{\varepsilon n}{16}$ for $1 \leq i \leq t-1$. We have

$$|V_i^*| = |V_0 \cup \cdots \cup V_i| - |W_0 \cup \cdots \cup W_{i-1}| = |W_i \cup \cdots \cup W_t| - |V_{i+1} \cap \cdots \cup V_t|$$

and therefore

$$|V_i^*| - |W_i| = |W_{i+1} \cup \cdots \cup W_t| - |V_{i+1} \cap \cdots \cup V_t| \geq |W_t| - \frac{t-i}{16t} \varepsilon n > \frac{15\varepsilon n}{16}.$$

Thus, we have that $|\mathcal{L}_i| = |W_i| \leq |V_i^*| - \frac{15\varepsilon n}{16} < |V_i^*| - \frac{\varepsilon n}{16}$ and the claim therefore follows by Claim 4.5 below. $\square$

Claim 4.5 *For all $U \subseteq \mathcal{L}_i$ that satisfy $|U| \leq |V_i^*| - \frac{\varepsilon n}{16}$, we have*

$$|N_{B_i}(U)| \geq |U|.$$

Proof Let $U = U_0 \cup \ldots \cup U_d$, where

$$U_j := \{L \in U \mid |L| = j\}.$$

If $U_0 \neq \emptyset$, then $N_{B_i}(U) = V_i^*$. Therefore, we may assume that $U_0 = \emptyset$. Pick k such that $|U_k| \geq |U|/d$. We show that the lemma holds for n large enough by distinguish between the following three cases:

Case 1: $|U_k| \leq (p/2)^{-k}/2$. It follows by property $(P2)$ that

$$|N_{B_i}(U)| \geq |N_{B_i}(U_k)| \geq (p/2)^k |U_k||V_i|/2 \geq \frac{\log^{2d} n}{2^{k+1} \cdot 16d}|U| \geq |U|.$$

Case 2: $(p/2)^{-k}/2 \leq |U_k| \leq (p/2)^{-k} \log^{2(d-1)} n$. We fix an arbitrary subset $U_k' \subset U_k$ of size $|U_k'| = (p/2)^{-k}/2$, and by the same argument as in Case 1 we get that

$$|N_{B_i}(U)| \geq |N_{B_i}(U_k')| \geq (p/2)^k |U_k'||V_i|/2 \geq \frac{\log^{2d} n}{2^{k+1} \cdot 16}|U_k'| \geq \frac{\log n}{2^{k+1} \cdot 16d}|U| \geq |U|.$$

Case 3: $|U_k| \geq (p/2)^{-k} \log^{2(d-1)} n$. In this case note that the induced subgraph $B_i[U_k, V_i^* \setminus N_{B_i}(U_k)]$ has no edges. By property $(P2)$ this yields that

$$|V_i^* \setminus N_{B_i}(U_k)| < (p/2)^{-k} \log^{2(d-1)} n = o(\varepsilon n),$$

which implies that $|N_{B_i}(U_k)| \geq |V_i^*| - o(\varepsilon n) \geq |V_i^*| - \frac{\varepsilon n}{16} \geq |U|.$ $\qquad\square$

In the last lemma of this section we show that B_t contains a perfect matching, thus we can complete the embedding of H.

Lemma 4.6 *There exists a perfect matching in B_t.*

Proof We check Hall's condition for every subset $U \subseteq \mathcal{L}_t$. For sets of cardinality $|U| \leq |V_t^*| - \frac{\varepsilon n}{16}$, Hall's condition follows by Claim 4.5. Therefore, consider only subsets U of cardinality $|U| \geq |V_t^*| - \frac{\varepsilon n}{16}$. Let $U \subseteq \mathcal{L}_t$ be such a subset. Note that by the definition of the partial embedding f_0, every set in U is contained in one of the cliques in $\mathcal{K}$. Suppose first that $|V_t^* \setminus N_{B_t}(U)| \geq (p/2)^{-d}/2$. We fix a subset $Y \subset V_t^* \setminus N_{B_t}(U)$ of size exactly $(p/2)^{-d}/2$. It follows by property $(P1)$ that at least $2^{-d-2} \cdot p^d |Y| \varepsilon n$ of the cliques in $\mathcal{K}$ are completely connected to some vertices in Y. We conclude that

$$|N_{B_t}(V_t^* \setminus N_{B_t}(U))| \geq |N_{B_t}(Y)| \geq 2^{-3}(p/2)^d (p/2)^{-d} \cdot \varepsilon n > \varepsilon n/16,$$

which is not possible since $|U| + |N_{B_t}(V_t^* \setminus N_{B_t}(U))| \leq |V_t^*|$.

Therefore, we conclude that $|V_t^* \setminus N_{B_t}(U)| \leq (p/2)^{-d}/2$. Now, using Property $(P1)$ similarly as above we obtain that

$$|N_{B_t}(V_t^* \setminus N_{B_t}(U))| \geq 2^{-d-3}(p/2)^d \cdot \varepsilon n |V_t^* \setminus N_{B_t}(U)| > |V_t^* \setminus N_{B_t}(U)|.$$

Finally, since

$$|N_{B_t}(U)| + |V_t^* \setminus N_{B_t}(U)| = |V_t^*| = |W_t| \geq |U| + |N_{B_t}(V_t^* \setminus N_{B_t}(U))| > |U| + |V_t^* \setminus N_{B_t}(U)|$$

we get $|N_{B_t}(U)| > |U|.$ $\qquad\square$

5 Proof of Theorem 1.4

In this section we prove Theorem 1.4. Before starting the proof, it will be convenient to introduce the following notation. For any bipartite graph $G = (A \cup B, E)$ with $|A| = |B| = n$ and minimum degree $\delta(G) \geq k$, let $\mathcal{B}_{k-out}^{\ell}(G)$ denote the following set of bipartite graphs: each $D \in \mathcal{B}_{k-out}^{\ell}(G)$ has vertex set $V(D) = V(G)$ and edge set $E(D) \subseteq E$ such that each vertex in A has degree exactly k. Note that we can sample an element from $\mathcal{B}_{k-out}^{\ell}(G)$ uniformly at random by choosing for each $v \in A$ uniformly at random k edges from $E_G(v, B)$.

One of the main ingredients in the proof of Theorem 1.4 is the following simple lemma on the existence of perfect matchings in typical graphs from $\mathcal{B}_{k-out}^{\ell}(G)$.

Lemma 5.1 *Let $\varepsilon > 0$, let n be a sufficiently large integer and let $k = \omega(\log n)$. Then for any bipartite graph $G = (A \cup B, E)$ with $|A| = |B| = n$ and $\delta(G) \geq \frac{n}{2} + \varepsilon n$, a graph D chosen uniformly at random from $\mathcal{B}_{k-out}^{\ell}(G)$ w.h.p. contains a perfect matching.*

Proof Let D be a graph chosen uniformly at random from $\mathcal{B}_{k-out}^{\ell}(G)$. We show that w.h.p. all subsets $S \subset A$ and all subsets $S \subset B$ with $|S| \leq n/2$ satisfy $|S| \leq |N_D(S)|$. It then follows from Hall's theorem (see [23] for more details) that D has a perfect matching.

We first assume that $S \subset A$. Note that $|S| > |N_D(S)|$ implies that there exists a subset $S' \subset B$ of size $|S'| = |S| - 1$ such that $|E_D(S, B \setminus S')| = 0$. Note that in G, since $|S'| \leq n/2$, every vertex $v \in S$ has at least εn neighbors in $B \setminus S'$. Therefore, when choosing the i-th of the k edges incident to v and conditioning on the event that no edge in $E_G(v, B \setminus S')$ has been selected so far, the probability to miss $B \setminus S'$ is at most

$$\frac{\deg_G(v) - \varepsilon n - i + 1}{\deg_G(v) - i + 1} \leq 1 - \varepsilon.$$

Thus,

$$\Pr\left[|S| > |N_D(S)|\right] \leq \Pr\left[\exists S' \subset B | |E_D(S, B \setminus S')| = 0\right]$$
$$\leq \binom{n}{|S| - 1}(1 - \varepsilon)^{|S|k} \leq e^{-\varepsilon|S| \cdot \omega(\log n)},$$

and the probability that such a bad set exists is at most

$$\sum_{s=1}^{n/2} \binom{n}{s} e^{-\varepsilon s \cdot \omega(\log n)} \leq \sum_{s=1}^{n/2} e^{-s \cdot \omega(\log n)} = o(1).$$

Next, assume that $S \subset B$ and observe that in order to have $|S| > |N_D(S)|$, there must exist a set $S' \subset A$ of size $|S| - 1$ such that $|E_D(A \setminus S', S)| = 0$. Note that $|E_G(A, S)| \geq |S| \cdot \left(\frac{n}{2} + \varepsilon n\right)$, $|E_G(S', S)| \leq |S| \cdot |S'| \leq |S| \cdot \frac{n}{2}$ and therefore $|E_G(A \setminus S', S)| \geq |S|\varepsilon n$. Since every edge of G

appears in D with probability at least $\frac{k}{n}$ (but not independently) and since this probability can only decrease if we know that another edge does not appear in D, it follows that

$$\Pr[|S| > |N_D(S)|] \le \Pr\left[\exists S' \subset A | |E_D(A \setminus S', S)| = 0\right]$$
$$\le \binom{n}{|S|-1}\left(1 - \frac{\omega(\log n)}{n}\right)^{|S|\varepsilon n} \le e^{-\varepsilon|S|\cdot\omega(\log n)}$$

as in the previous case. $\qquad\square$

Now we are ready to prove Theorem 1.4.

Proof Our proof is motivated by ideas of Cooper and Frieze [14]. Note that containing a rainbow copy of some fixed graph H is a monotone increasing property and we can therefore fix p to exactly $n^{-1/d}\log^{5/d}n$.

Let Δ and d be positive integers, let n be a sufficiently large integer and let $H \in \mathcal{H}(n, \Delta, d)$. Moreover, let $\bar{d} = \frac{2|E(H)|}{n}$ denote the average degree of H (note that $\bar{d} \le d$ and in fact can be much smaller than d) and let $\alpha > 0$ be some arbitrarily small positive constant. First, we show how to partition H in such a way that will later help us to find a rainbow copy of it in a typical member of $\mathcal{G}_c(n, p)$, where $c = (1 + \alpha)|E(H)|$. For this aim we act as follows. If H contains a set W of $\lceil\frac{\alpha n}{5\log^2 n}\rceil$ isolated vertices (that is, vertices of degree 0 in H), then partition $V(H) = \{w_1\}\cup\ldots\cup\{w_t\}\cup W$ in such a way that for each i, the vertex w_i has at most d neighbors in $\{w_1, \ldots, w_{i-1}\}$. Indeed, such a partition exists since $H' := H - W \in \mathcal{H}(n - |W|, \Delta, d)$, and therefore is d-degenerate, so one can apply Observation 2.5. Otherwise, let x denote the number of vertices of degree larger than 0 and at most $\bar{d}$ in H. Since H contains at most $\frac{\alpha n}{5\log^2 n}$ isolated vertices, the following inequality holds:

$$\bar{d}n = 2|E(H)| \ge x + (\bar{d} + 1)\left(n - \frac{\alpha n}{5\log^2 n} - x\right).$$

Hence, using the fact that n is sufficiently large, we conclude that $x \ge n/(2\bar{d})$. Now, let S be the set consisting of all these vertices. By applying Lemma 2.3 to H and S it follows that there exists a subset $T \subseteq S$, such that T is 2-independent and

$$|T| \ge \frac{|S|}{\bar{d}\Delta^2} \ge \frac{n}{2\bar{d}^2\Delta^2} \ge \left\lceil\frac{\alpha n}{5\log^2 n}\right\rceil$$

for sufficiently large n. Next, let $W \subseteq T$ be an arbitrary subset of size $\lceil\frac{\alpha n}{5\log^2 n}\rceil$, and partition $V(H) = \{w_1\} \cup \ldots \cup \{w_t\} \cup W$ in such a way that for each i, w_i has at most d neighbors in $\{w_1, \ldots, w_{i-1}\}$.

All in all, we have a partition $V(H) = \{w_1\}\cup\ldots\cup\{w_t\}\cup W$ such that $|W| = \lceil\frac{\alpha n}{5\log^2 n}\rceil$ and one of the following holds:

(1) all the vertices of W are isolated in H, or

15

(2) W is 2-independent and consists of non-isolated vertices of degree at most $\bar{d}$.

Note that if (2) holds then

$$|E(W, V \setminus W)| \leq \bar{d}|W| = \frac{2|E(H)|}{n} \cdot \left\lceil \frac{\alpha n}{5 \log^2 n} \right\rceil < \alpha |E(H)|/(2\lceil \log^2 n \rceil), \tag{4}$$

for n large enough.

Now we start to describe the procedure of finding a rainbow copy of H. Let $q \geq p/2$ be such that $1 - p = (1 - q)^2$ and present $G \sim \mathcal{G}(n, p)$ as $G = G_1 \cup G_2$, where G_1 and G_2 are two graphs sampled independently from $\mathcal{G}(n, q)$. We sample a member of $\mathcal{G}_c(n, p)$ by sampling a member of $\mathcal{G}(n, p)$ and randomly coloring exposed edges using c colors.

We find a rainbow embedding of H in $G \sim \mathcal{G}_c(n, p)$ in two phases. In Phase I, we find a rainbow embedding f of $H[\{w_1 \cup \ldots \cup w_t\}]$ with edges which are taken from G_1. If W is as in (1) (that is, all the vertices in W are isolated in H), then we are done. Otherwise, in Phase II we show that one can extend f to a rainbow embedding of H in G, using edges of G_2.

In what follows, we present the exact strategies of Phases I and II and prove that w.h.p. everything works out well.

Phase I: Throughout this phase we maintain a partial rainbow embedding f of H to G_1, a set of *available colors* $\mathcal{C}$ and a set of *available vertices* V'. Initially, set $f = \emptyset$, $\mathcal{C} := [c]$ and $V' := V(G)$. Additionally, we maintain for each vertex $v \in V(G)$ a set $U_v \subseteq V(G)$ such that $U(v) \cap V'$ contains only *unexposed* potential neighbors of v in G_1. Initially, $U_v = V(G) \setminus \{v\}$ for each $v \in V(G)$.

We inductively build the desired partial embedding f as follows. In the first step, let $f(w_1) := v$ for an arbitrary vertex $v \in V'$, and set $V' := V' \setminus \{v\}$. Assume that we have already embedded $\{w_1, \ldots, w_{i-1}\}$ for some $2 \leq i \leq t$ and we wish to embed $w := w_i$. Let $L(w_i) = f(N_H(w_i) \cap \{w_1, \ldots, w_{i-1}\})$ be the set of images of neighbors of w_i which have already been embedded (recall that $|L(w_i)| \leq d$). Let $A_w = V' \cap \left(\cap_{v \in L(w_i)} U_v \right)$ be the set of all available vertices which are still unexposed neighbors of all vertices in $L(w_i)$, and choose an arbitrary subset $S_w \subset A_w$ of size $s := \lceil \alpha n/(4\Delta \log n)^2 \rceil$ (Claim 5.2 shows that throughout Phase I this is indeed possible; that is, A_w is of size at least s). Expose all edges between $L(w_i)$ and S_w, and assign uniformly at random colors to all the obtained edges. Let $x \in S_w$ be a vertex which is connected to all the vertices in $L(w_i)$ and such that all the colors assigned to edges $\{vx \mid v \in L(w_i)\}$ are distinct and belong to $\mathcal{C}$. The existence of such a vertex follows from Claim 5.3 below. We extend f by defining $f(w_i) := x$, update $U_v := U_v \setminus S_w$ for all $v \in L(w_i)$, $V' := V' \setminus \{x\}$ and

$$\mathcal{C} := \mathcal{C} \setminus \{col \in \mathcal{C} \mid \exists v \in L(w_i) \text{ such that } vx \text{ is colored in } col\}.$$

The following two claims show that w.h.p. we manage to find the desired embedding in Phase I.

Claim 5.2 *Throughout Phase I we have that $|A_w| \geq \lceil \alpha n/(4\Delta \log n)^2 \rceil$ for every vertex $w \in V(H)$ which has not been embedded.*

16

Proof The proof of the claim is obtained from the following four observations. First, note that at the beginning of Phase I we have that $U_v = V(G) \setminus \{v\}$ for each $v \in V(G)$. Second, we update U_v only after embedding a vertex w for which $v \in L(w)$ (and then we delete the set S_w which is of size $s = \lceil \alpha n/(4\Delta \log n)^2 \rceil$ from U_v). Third, every vertex v is a member of at most Δ sets $L(w)$ (recall that $\Delta(H) \leq \Delta$). Fourth, note that $|V'| \geq \lceil \alpha n/(5 \log^2 n) \rceil$ throughout Phase I (recall that we do not embed W in this phase).

Therefore, it follows that at any point during Phase I we have

$$|U_v \cap V'| \geq |V'| - 1 - \Delta \cdot \left\lceil \frac{\alpha n}{(4\Delta \log n)^2} \right\rceil,$$

for each vertex $v \in V(G)$. Since $|L(w)| \leq \Delta$, we conclude that

$$|A_w| = |V' \cap \left(\cap_{v \in L(w)} U_v\right)| \geq |V'| - \Delta - \Delta^2 \left\lceil \frac{\alpha n}{(4\Delta \log n)^2} \right\rceil \geq s,$$

for n large enough. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\square$

The next claim states that whenever we wish to embed a vertex w, it has at least one candidate in V'.

Claim 5.3 *Let $w \in V(H) \setminus W$. At the moment we try to embed w there exists with probability $1 - o(1/n)$ a vertex $x \in S_w$ for which the following holds:*

  (i) x is connected to all the vertices in $L(w)$, and

  (ii) all the colors assigned to the edges $\{\{v, x\} : v \in L(w)\}$ are distinct and belong to $\mathcal{C}$.

Proof Let
$$X := |\{v \in S_w \mid L(w) \subseteq N_{G_1}(v)|\}|.$$
Note that X is the sum of i.i.d. indicator random variables X_v (for all $v \in S_w$) for which $X_v = 1$ iff $L(w) \subseteq N_{G_1}(v)$. Clearly, we have that (recall that $|L(w)| \leq d$)

$$\mathbb{E}[X] \geq sq^d \geq \frac{\alpha n}{(4\Delta \log n)^2} \cdot \Omega\left(\frac{\log^5 n}{n}\right) = \Omega(\log^3 n).$$

Applying Chernoff's bound we obtain that

$$\Pr[X \leq \mathbb{E}[X]/2] = e^{-\Omega(\log^3 n)} = o(1/n).$$

Now, note that $|\mathcal{C}| \geq \alpha|E(H)|$ during Phase I. Thus, the probability that for a vertex $x \in S_w$ with $L(w) \subseteq N_{G_1}(x)$, all the edges to $L(w)$ have different colors from $\mathcal{C}$ is at least

$$\frac{\binom{\mathcal{C}}{\ell}}{((1+\alpha)|E(H)|)^\ell} \geq \left(\frac{\alpha|E(H)|}{(1+\alpha)|E(H)|\ell}\right)^\ell \geq \left(\frac{\alpha}{(1+\alpha)d}\right)^d =: \gamma > 0,$$

where $|L(w)| = \ell$. Therefore, if $X \geq \mathbb{E}[X]/2$ then the probability that there is no such x is at most

$$(1 - \gamma)^{|X|} \leq e^{-\gamma|X|} = e^{-\Omega(\log^3 n)} = o(1/n).$$

$\square$

Note that since we embed at most n vertices, applying the union bound we obtain that for every vertex w_i there exists a "good" vertex $x \in S_w$. Now, if W is as in (1) (that is, all the vertices in W are isolated in H), then we are done. Otherwise, we continue to Phase II.

Phase II: Let $V^* := V(G) \setminus f(V(H) \setminus W)$. Our goal is to extend f with a valid embedding of W into V^*, using edges of G_2, in such a way that the resulting embedding is rainbow.

For $w \in W$ let $L(w) := f(N_H(w))$ and let $\mathcal{L} = \{L(w) | w \in W\}$. Recall that W is 2-independent and thus all the $L(w)$'s are disjoint. Let $F = (\mathcal{L} \cup V^*, E_F)$ with edge set

$$E_F := \{Lv \mid L \in \mathcal{L}, v \in V^* \text{ and } \forall_{u \in L} uv \notin E(G_1)\}$$

be the ground graph to build a bipartite auxiliary graph $\mathcal{B}(\mathcal{L}, V^*)$. Edges that appeared in G_1 are excluded since we can not color them again. Note that $|\mathcal{L}| = |W| = |V^*|$ and that by the following very rough estimate F satisfies w.h.p. the conditions of Lemma 5.1.

Claim 5.4 *It holds with high probability that $\delta(F) \geq \frac{3}{4}|V^*|$.*

Proof For every $L \in \mathcal{L}$ and $v \in V^*$ the edge $Lv \notin E_F$ if and only if there exists $u \in L$ for which $uv \in E(G_1)$. Since $G_1 \sim G(n, q)$, by applying Chernoff's bound it follows that w.h.p. $\Delta(G_1) \leq 2nq$. Moreover, since for every $L \in \mathcal{L}$ we have that $|L| \leq \bar{d}$, it follows that $deg_F(L) \geq |V^*| - \bar{d}2nq > 3|V^*|/4$. A similar argument shows that we have $deg_F(v) \geq 3|V^*|/4$ for every $v \in V^*$. $\square$

In the following we describe a random process that tries to create a bipartite graph $\mathcal{B}(\mathcal{L}, W) \in \mathcal{B}^\ell_{\lceil \log^2 n \rceil - out}(F)$ by exposing edges from $G_2 \setminus G_1$ and randomly color them. First, let

$$\mathcal{C} := \{col \in [c] \mid \exists \{u, v\} \in E(H \setminus W) \text{ s.t. } \{f(u), f(v)\} \text{ has color } col\}$$

and note that $|\mathcal{C}| \geq \alpha|E(H)|$. Choose an arbitrary ordering $L_1, \ldots, L_{|\mathcal{L}|}$ of the elements in $\mathcal{L}$. Then, in step $1 \leq i \leq |\mathcal{L}|$, set $N_i := N_F(L_i)$ and create $\lceil \log^2 n \rceil$ edges from L_i to vertices in N_i as follows: as long as $|N_{\mathcal{B}(\mathcal{L}, V^*)}(L_i)| < \lceil \log^2 n \rceil$, iteratively pick a vertex $v \in N_i$ uniformly at random, set $N_i := N_i \setminus \{v\}$ and expose all edges from v to vertices in L_i and color them uniformly at random with colors from $[c]$ (note that here the process can fail if at some point $N_i = \emptyset$ while $|N_{\mathcal{B}(\mathcal{L}, V^*)}(L_i)| < \lceil \log^2 n \rceil$). If all the edges are contained in G_2 and if they have distinct colors that are all from the set of available colors $\mathcal{C}$, add $L_i v$ to $\mathcal{B}(\mathcal{L}, V^*)$. At the end of step i remove all the colors used at edges incident to L_i,

$$\mathcal{C} := \mathcal{C} \setminus \{col \in [c] \mid \exists u \in L_i, \exists v \in N_{\mathcal{B}(\mathcal{L}, V^*)}(L_i) \text{ s.t. } uv \text{ has color } col\}.$$

If the process succeeds then every matching M in $\mathcal{B}(\mathcal{L}, V^*)$ is clearly rainbow in the sense that all edges in

$$\{uv \mid \exists Lv \in M \text{ s.t. } u \in L\}$$

have distinct colors that have not been used in the embedding in Phase I. It follows from Claim 5.5 below and Lemma 5.1 that the process succeeds w.h.p. and that the constructed $\mathcal{B}(\mathcal{L}, V^*)$ contains a perfect matching. Finally, such a perfect matching in $\mathcal{B}(\mathcal{L}, V^*)$ extends f into a rainbow embedding of H in G. The following claim therefore completes the proof of Theorem 1.4.

Claim 5.5 *The random process that creates $\mathcal{B}(\mathcal{L}, V^*)$ in Phase II succeeds w.h.p. and it samples uniformly at random from $\mathcal{B}^{\ell}_{\lceil \log^2 n \rceil - out}(F)$.*

Proof Note first that the process can only fail if in some round $1 \le i \le |\mathcal{L}|$ we have that $N_i = \emptyset$ and $|N_{\mathcal{B}(\mathcal{L}, V^*)}(L_i)| < \lceil \log^2 n \rceil$. It therefore suffices to show that in a fixed step $1 \le i \le |\mathcal{L}|$ the process creates with probability $1 - o(1/n)$ the $\lceil \log^2 n \rceil$ required edges. Let

$$X_i := \{v \in N_F(L_i) \mid L_i \subseteq N_{G_2}(v)|\}.$$

Note that $|X_i|$ is the sum of i.i.d. indicator random variables $X_{i,v}$ (for all $v \in N_F(L_i)$) for which $X_{i,v} = 1$ iff $L_i \subseteq N_{G_2}(v)$. Clearly, we have that (recall that $|L_i| \le \bar{d} \le d$)

$$\mathbb{E}[|X_i|] \ge |N_F(L_i)| q^d \ge \delta(F) \cdot \Omega\left(\frac{\log^5 n}{n}\right) \ge \frac{3}{4} \cdot \frac{\alpha n}{5 \log^2 n} \cdot \Omega\left(\frac{\log^5 n}{n}\right) = \Omega(\log^3 n).$$

Applying Chernoff's bound we obtain that

$$\Pr[|X_i| \le \mathbb{E}[|X_i|]/2] = e^{-\Omega(\log^3 n)} = o(1/n). \tag{5}$$

Next, let

$$Y_i := \{v \in X_i \mid \text{all edges in } E(L_i, v) \text{ have distinct colors from } \mathcal{C}\}.$$

Note that $|Y_i|$ is the sum of i.i.d. indicator variables $Y_{i,v}$ (for all $v \in X_i$) for which $Y_{i,v} = 1$ iff all edges in $E(L_i, v)$ have distinct colors from $\mathcal{C}$. Since we have by (4) that $|E(W, V \setminus W)| \le \alpha |E(H)|/(2\lceil \log^2 n \rceil)$ and we remove for each edge in $E(W, V \setminus W)$ at most $\lceil \log^2 n \rceil$ colors from $\mathcal{C}$, the number of available colors in $\mathcal{C}$ is always at least $\alpha |E(H)|/2$. Thus, the probability that for a vertex $v \in X_i$ all the edges to L_i have different colors from $\mathcal{C}$ is at least

$$p_i = \frac{\binom{\mathcal{C}}{\ell}}{((1+\alpha)|E(H)|)^\ell} \ge \left(\frac{\alpha |E(H)|/2}{(1+\alpha)|E(H)|\ell}\right)^\ell \ge \left(\frac{\alpha}{(1+\alpha)2\bar{d}}\right)^{\bar{d}} =: \gamma > 0,$$

where $|L_i| = \ell \le \bar{d}$, and this lower bound for p_i holds independently of all other color assignments in previous steps. Therefore, if $|X_i| \ge \mathbb{E}[|X_i|]/2$, then the expectation of $|Y_i|$ is at least

$$\mathbb{E}[|Y_i|] \ge |X_i| \cdot \gamma = \Omega(\mathbb{E}[|X_i|]) = \Omega(\log^3 n)$$

and it follows from Chernoff's bound that

$$\Pr\left[|Y_i| < \frac{\mathbb{E}[|Y_i|]}{2}\,\middle|\,|X_i| \geq \frac{\mathbb{E}[|X_i|]}{2}\right] = e^{-\Omega(\log^3 n)} = o(1/n). \tag{6}$$

Combining (5) and (6) we conclude that the probability that our process fails is at most

$$\sum_{i=1}^{|\mathcal{L}|} \Pr\left[|Y_i| \leq \lceil \log^2 n \rceil\right] \leq |\mathcal{L}| \cdot o(1/n) = o(1).$$

Finally, since we choose a random ordering of the neighbors of L_i, every $\lceil \log^2 n \rceil$-tuple of neighbors of L_i has the same probability to be part of $\mathcal{B}(\mathcal{L}, V^*)$ and the process therefore samples an element of $\mathcal{B}^\ell_{\lceil \log^2 n \rceil - out}(F)$ uniformly at random. □

□

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